JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS Vol. 45, No. 3, June 2010, pp. 623–640
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doi:10.1017/S0022109010000189

Disagreement, Portfolio Optimization, and Excess Volatility

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Abstract

Disagreement, a key factor inducing trading, has been receiving ever increasing attention in recent years. Most research has focused on disagreement about the expected returns. Several authors have shown that if the average belief coincides with the true expected return, in the portfolio context prices are unaffected by disagreement. In this paper we study the pricing effects of disagreement about return variances. We show that i) disagreement about variances has systematic and significant pricing effects—more disagreement leads to higher prices, and ii) prices are very sensitive to the degree of disagreement: Even if the average belief about the variance is constant, tiny fluctuations in the disagreement about the variance lead to substantial price fluctuations. This second result may offer an explanation for the excess volatility puzzle: When small changes in the degree of disagreement occur, they induce relatively large price changes. Yet, the changes in disagreement may be hard to directly detect empirically, leading to apparent "excess volatility."

I. Introduction

Everyday observation reveals that investors hold different beliefs about individual securities and about the market as a whole (this is also true of academics; see Welch (2000)). In a complex and dynamic setting, even if individuals possess the same information, they may differ in their interpretation of this information and consequently hold different beliefs.[1] The link between heterogeneous beliefs and asset pricing is a central issue in financial economics. While heterogeneous beliefs typically greatly complicate the formulation of asset prices, there are a few important cases where the heterogeneity of information has no direct effect on prices. One such situation is the case where prices are fully revealing (see, e.g., Admati (1985)). Even when prices are only partially revealing and there is

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[1]This is in violation of Aumann's (1976) "agreeing to disagree" theorem, and as advocated by Harris and Raviv (1993). Many studies have taken this approach to modeling heterogeneous beliefs (see, e.g., Lintner (1969), Kandel and Pearson (1995), Dumas, Kurshev, and Uppal (2009), Gollier (2007), Fama and French (2007), and David (2008)).

https://doi.org/10.1017/S0022109010000189 Published online by Cambridge University Press

disagreement about the expected returns, Biais, Bossaerts, and Spatt (2010) show that the capital asset pricing model (CAPM) equilibrium pricing is intact (see also DeMarzo and Skiadas (1998) for a general analysis). Moreover, Levy, Levy, and Benita (2006) show that if investors have heterogeneous but unbiased beliefs about the expected returns, the homogeneous-CAPM pricing holds even when prices reveal no information at all.

Most of the literature focuses on disagreement about expected returns and assumes very little or no disagreement on the covariance matrix. The reason for this is that previous papers (e.g., Jorion (1985)) have argued that mean returns are more difficult to estimate than volatilities, and therefore the disagreement on volatilities should be small. Moreover, Merton (1980) argues that the estimator of the expected return does not converge to the true value for a finite estimation period, while the estimator of the variance converges to the true value when the data frequency goes to infinity. This implies that expected returns, rather than volatilities, are the main source of disagreement. Nevertheless, it is hard to imagine that investors are in complete agreement about the volatility. Even with Merton's observation that a constant volatility is easy to estimate, when volatility is stochastic there may be disagreement about the future volatility. Disagreement about volatility may arise from many different sources. For example, investors may estimate the covariances based on different sample periods. Even if investors observe the same sample values but they differ in the *level* of their information, this implies different predictive variances, as in the framework of Barry and Brown (1985).[2] This paper shows that even extremely small levels of disagreement on volatility may have a significant impact on stock prices.

We investigate the case of disagreement about the covariances and find results that are qualitatively different from the results in the case of disagreement about expected returns. Our two main findings are: i) Prices are systematically affected by heterogeneous beliefs about the covariances. This is true even if the beliefs are unbiased (i.e., the average belief about the covariance is identical to its true value). This is in contrast to the results regarding unbiased beliefs about expected returns. When everything else is held constant, the larger the disagreement about an asset's variance, the higher the asset's price. ii) Prices are sensitive to the degree of disagreement about the covariances. A small change in the degree of disagreement may lead to a significant change in price.

These findings suggest a novel explanation for one of the main puzzles in financial economics—the excess volatility puzzle, first discussed by Shiller (1981) and LeRoy and Porter (1981). The excess volatility puzzle has attracted a great deal of attention from financial economists[3] and is eloquently described by Cochrane ((2001), p. 394):

[2]Barry and Brown (1984) analyze the case where the level of information is different across securities and show that this may provide an explanation for the small firm effect. In Barry and Brown (1985) they consider both differential information (across securities) and divergence of opinion (across investors), as well as the interaction between these two factors. In general, both differential information and divergence of opinion can lead to different covariance estimates.

[3]For excellent reviews, see West (1988), LeRoy (1989), and Shiller (2003).

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> The stunning fact about the stock market is its extraordinary volatility. On a typical day, the value of the U.S. capital stock changes by a full percentage point, and days of 2 or 3 percentage point changes are not uncommon. In a typical year it changes by 16 percentage points, and 30 percentage point changes are not uncommon. Worse, most of that volatility doesn't seem to be accompanied by any important news about future returns and discount rates. Thirty percent of the capital stock of the United States vanished in a year and nobody noticed?

The finding that a small change in disagreement about volatility has a large impact on price offers an explanation for the excess volatility puzzle: Small changes in the degree of disagreement are very likely to occur frequently, and these changes induce relatively large price changes. As the changes in disagreement may be hard to directly detect empirically, they may be the source of the *apparent* "excess volatility."

An obvious reason for price changes is the flow of new information into the market. Excess volatility implies price variability that is above and beyond the variability that can be explained by the new information flow. In this paper we propose time-varying disagreement as another source of price variability. Thus, we view price changes as driven by both new information and time-varying disagreement. However, since in this paper we want to investigate the disagreement mechanism and the component of price variation unexplained by the information flow, we focus only on the disagreement mechanism, and we do not model the process of new information arrival.

We analytically investigate the pricing effects of disagreement about volatilities, focusing on the sensitivity of the price to the level of disagreement. To motivate the analysis and to illustrate the order of magnitude of the pricing effects involved, it is instructive to begin by considering a simple numerical example. This example also provides the intuition for the general results. We take a sample of 100 stocks, randomly selected from the Center for Research in Security Prices (CRSP) data set. We assume that investors agree about the parameters of 99 of these 100 stocks, that is, there is agreement about the expected returns and the standard deviations of these 99 stocks. There is also agreement about the 100×100 correlation matrix. These parameters are taken as their empirical sample values, calculated from the monthly returns in the period January 1996–December 2005. There is disagreement about the parameters of only 1 stock, which we denote by stock A. We analyze 2 cases. In the first case we assume that there is disagreement about the expected return of stock A: Half of the investors estimate the expected return as being $\mu_A(1 + \Delta)$, while the other half estimate it as $\mu_A(1 - \Delta)$, where μ_A is the average belief and Δ is the disagreement factor.[4] In the second case we assume agreement about the expected return, but disagreement about stock A's standard deviation: Half of the investors estimate the standard deviation to be $\sigma_A(1 + \Delta)$, while the other half estimate it as $\sigma_A(1 - \Delta)$.[5] Investors construct optimal mean-variance portfolios based on their expectations, and stock prices

[4]This follows the modeling of disagreement in Levy, Levy, and Benita (2006).

[5]Thus, a given level of Δ implies the same "level of disagreement" for μ_A and for σ_A. For example, $\Delta = 0.1$ implies that half of the investors overestimate μ_A by 10% and the other half underestimate

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are derived from the investors' portfolio weights and the market clearance condition.[6] In order to investigate the effects of time-varying disagreement, we assume that Δ changes over time. Our focus is the sensitivity of prices to Δ, so the exact process followed by Δ is not important: Δ can be mean reverting or follow any other dynamic process. In the following example we assume for simplicity that Δ follows a random walk: $\Delta_{t+1} = \Delta_t \pm 0.005$, where the positive and negative increments have an equal probability of occurring.

Figure 1 illustrates Δ_t and the resulting price of stock A as a function of time. The dashed horizontal line depicts the static price of stock A given no disagreement, that is, in the case $\Delta = 0$, where all investors believe that the expected return is μ_A and that the volatility is σ_A. (Recall that we only model the price changes due to time-varying disagreement. Thus, if there is no disagreement the price is constant.) The solid horizontal line that looks almost straight and is on top of the dashed line represents the first case in which the disagreement is about the expected return. While it is difficult to detect visually, the price does in fact fluctuate a little: Although the no-disagreement price is 10.779, the price with disagreement about the expected return fluctuates in the small range of 10.778–10.779. Thus, disagreement about the expected return has almost no consequence for the stock price. This result also holds if the disagreement is about several or all of the stocks' expected returns and is consistent with the findings of Levy et al. (2006).

The bold solid line in Figure 1 depicts the price in the case of disagreement about stock A's standard deviation for the exact same process of Δ. Note that unlike the case of disagreement about the expected return, disagreement about the standard deviation has a very large impact on the price. Furthermore, the price is quite sensitive to the degree of disagreement. Small changes in disagreement, which may be hard to detect empirically, can lead to substantial volatility in the stock price. Thus, an econometrician who can only observe Graph A of Figure 1 may conclude that the stock price exhibits "excess volatility." Note also that the pricing impact is systematic—the disagreement increases the price relative to the case of no-disagreement. The intuition for these results is given below.

The explanation for the very different price impact of the disagreement about the expected return and the disagreement about the standard deviation is as follows: An investor's demand for a stock is typically approximately linear in his perceived expected return of the stock (see Levy et al. (2006) for more detail). This is graphically depicted in Graph A of Figure 2, which shows the mean-variance optimal investment proportion as a function of the expected return for stock A in the above example with 100 stocks. Thus, if one investor perceives the expected return of stock A as $\mu_A(1 + \Delta)$ and the other investor perceives the expected return as $\mu_A(1 - \Delta)$, their excess demands will almost cancel out, and their total demand will be very similar to that of 2 investors with a homogeneous

this value by 10%. Similarly, $\Delta = 0.1$ implies that half of the investors overestimate σ_A by 10% and the other half underestimate this value by 10%. Note that in principle, $\Delta > 1$ can lead to $\sigma_A < 0$, in which case we would have to technically impose $\sigma_A \geq 0$. However, in practice, the disagreement factor considered is much smaller than 1 (see Figure 1), so that this problem does not arise.

[6]The optimal mean-variance portfolio and the market clearing price are given by equations (1) and (2), see Section II. In this simple example there are only 2 representative investors, that is, $K = 2$, with equal wealth $T^1 = T^2 = T$, and we also assume the same number of shares for all stocks, $n_i = n_j = n$.

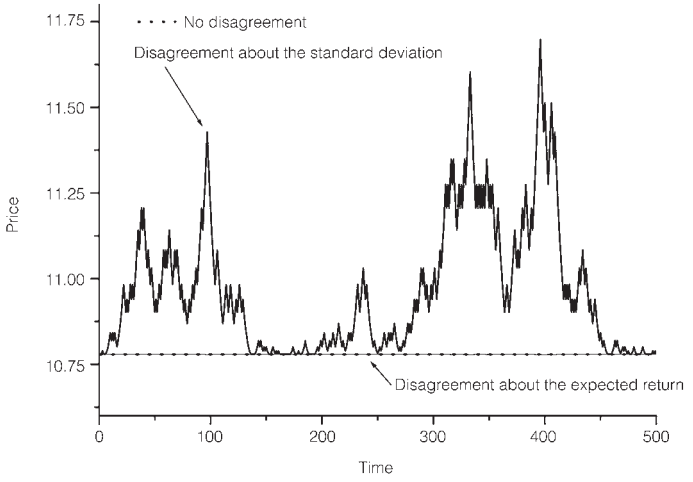
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FIGURE 1

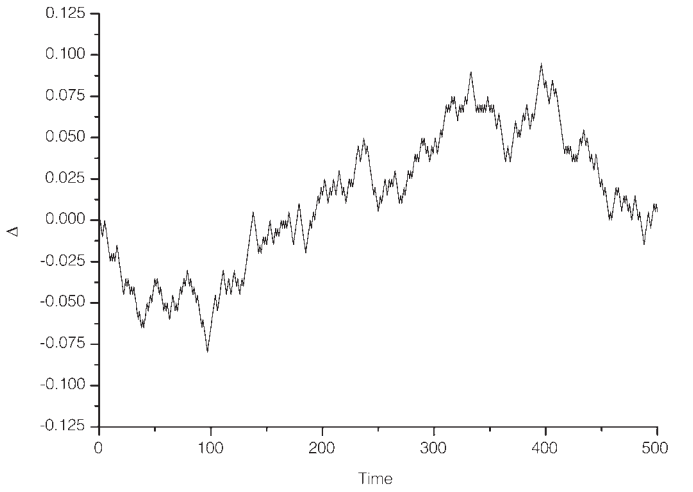
Disagreement and Price of Stock A

In Figure 1 we randomly select 100 stocks. There is full agreement about the parameters of 99 of the stocks, and disagreement of degree Δ about 1 stock, stock A. We assume that the level of disagreement follows a random walk, as shown in Graph B. Graph A shows the price of stock A when there is no disagreement (dashed line), when the disagreement is about stock A's expected return (light solid line), and when the disagreement is about stock A's standard deviation. While disagreement about the expected return has almost no price impact, disagreement about the standard deviation has a large and systematic impact, leading to apparent "excess" volatility of the stock price.

Graph A. Disagreement and the Stock Price



Graph B. Random Walk



perception of an expected return of μ_A.[7] Thus, the disagreement about the expected return has almost no pricing effects.

[7]Assuming that all 4 investors invest the same amount of capital in the stock market. When the number of investors is large, it suffices to assume that beliefs are uncorrelated with the investors' capital.

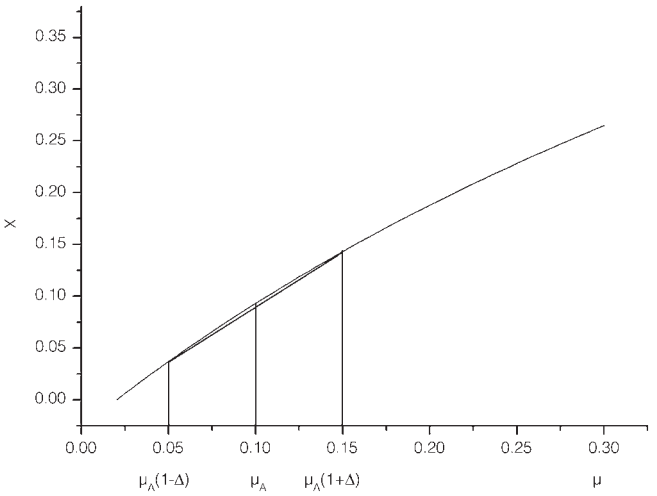
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FIGURE 2

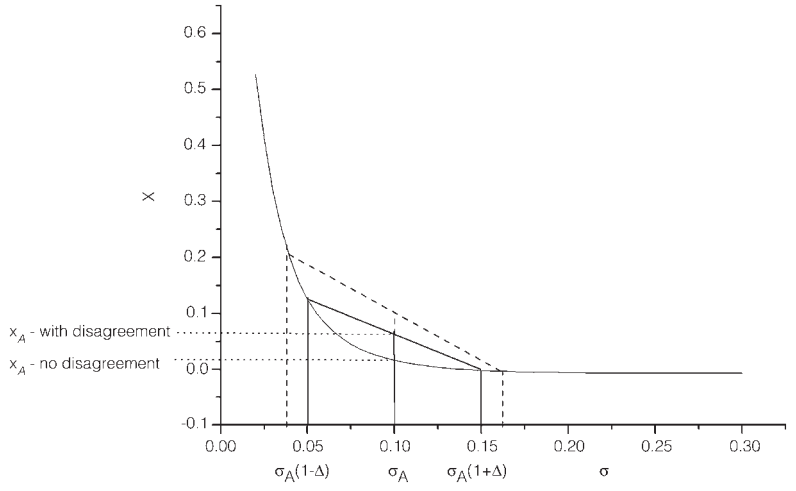
Optimal Investment Proportion as a Function of μ and σ

In Figure 2 the optimal investment proportion of stock A is a function of its perceived expected return and standard deviation. Stock A is held as part of the optimal mean-variance portfolio, together with the 99 other stocks randomly drawn from CRSP. The parameters of these stocks are taken as the empirical monthly values during the period January 1996–December 2005. The optimal investment proportion, x, is almost linear in μ (Graph A). In contrast, it is a highly nonlinear function of σ (Graph B). The nonlinearity of x in σ explains the large and systematic price impact of disagreement about the volatility.

Graph A. Demand as a Function of μ



Graph B. Demand as a Function of σ



In contrast, an investor's demand for a stock is highly nonlinear in his belief about the stock's standard deviation. This is demonstrated graphically by Graph B of Figure 2, drawn for the same parameters as in Graph A. Intuition for this nonlinearity is given below, and it is explained in detail in Section III. The result of this nonlinearity is that the stock price is affected systematically and significantly by disagreement about the standard deviation. The total demand for stock A of 2

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investors, one with belief $\sigma_A(1+\Delta)$ and the other with belief $\sigma_A(1-\Delta)$, is significantly higher than the total demand of 2 investors with homogeneous belief σ_A. Thus, disagreement about the standard deviation leads to an increase in the price relative to the homogeneous-belief case. This is consistent with the analysis of Yan (2010), who shows that if the demand is convex in the factor about which there is disagreement (or bias), the bias induces a systematic price increase. Figure 2 also reveals that the price may be very sensitive to the degree of disagreement (see the dashed lines for an illustration of this point). This is the explanation for the price dynamics in Figure 1: The time-varying disagreement about the stock's standard deviation induces the observed price fluctuations. Again, we stress that this basic idea is independent of the exact fashion in which disagreement changes over time. Thus, in this paper we do not make any assumptions regarding the process of Δ, but rather we focus on the analysis of the sensitivity of the stock price to the level of disagreement, which is the source of the induced excess volatility.

The mean-variance framework employed in this paper is very general, in the sense that it provides a good approximation to expected utility maximization for a wide spectrum of preferences, even when the return distributions are not normally distributed (see Levy and Markowitz (1979)). However, one can obtain insights about the general results derived in this paper and about the picture revealed in Figure 2 by considering the special case of a single risky asset, constant absolute risk aversion (CARA) preferences, and normal return distributions.[8] Consider a 2-period setup with 0 interest rate. There is a risky asset with supply Q, and there are K investors with utility $u(c) = -e^{-\gamma c}$, where γ is the coefficient of the absolute risk aversion, and c is time 1 consumption. Investor k believes that the asset's payoff is distributed $\tilde{P}_1 \sim N(\mu_k, \sigma_k^2)$. Then, as is well known, utility maximization leads the demand of investor k to be $Q_k = (\mu_k - P)/(\gamma\sigma_k^2)$, where P is the asset's period 0 price. Assume that the distribution of beliefs about μ_k and σ_k are uncorrelated. Then, the average demand is

$$\mathrm{E}[Q_k] \quad = \quad \frac{\mathrm{E}[\mu_k] - P}{\gamma}\mathrm{E}[\sigma_k^{-2}],$$

and the market clearing price is

$$P \quad = \quad \mathrm{E}[\mu_k] - \frac{\gamma Q}{K\mathrm{E}[\sigma_k^{-2}]}.$$

This equation for the price leads to several insights. First, unbiased heterogeneity of beliefs about μ does not affect the price. In other words, a mean-preserving spread in μ_k has no pricing effects. In contrast, heterogeneity about σ does affect the price even if the average belief remains constant. The more heterogeneous the beliefs about σ, the larger $\mathrm{E}[\sigma_k^{-2}]$ becomes, and the higher the price. This is consistent with the general mean-variance optimization case shown in Figure 2.

The same idea holds in the case where the asset is not held in isolation, but in the presence of background risk. For example, consider the case where the

[8] We are grateful to the referee for suggesting this analysis.

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terminal wealth of investor k is given by $\tilde{W}_k = Q_k(\tilde{P}_1 - P) + \tilde{M}$, where $\tilde{M}$ represents the background market risk, which is perceived by investor k to have correlation ρ_k with the risky asset. Investor k's demand is given in this case by

$$Q_k = \frac{\mu_k - P}{\gamma\sigma_k^2} - \frac{\rho_k\sigma_M}{\sigma_k},$$

and the equilibrium price is

$$P = \mathrm{E}[\mu_k] - \frac{\gamma Q}{K\mathrm{E}[\sigma_k^{-2}]} - \frac{\gamma\sigma_M\mathrm{E}[\rho_k\sigma_k^{-1}]}{\mathrm{E}[\sigma_k^{-2}]}.^{9}$$

Again, a mean-preserving spread in μ_k does not affect the price, while a mean-preserving spread in σ_k does. The analysis in Section III shows that similar results hold in the general case with multiple assets and any nondecreasing preferences.

The rest of the paper is organized as follows. In Section II we formulate the model. The theoretical results are presented in Section III. Section IV provides a numerical analysis of the magnitude of the pricing effects. Section V provides concluding remarks.

II. The Model

Consider an economy with N risky assets and one risk-free asset. There are K investors, who may have different beliefs regarding the stocks' parameters. Denote investor k's estimate of the expected excess return of stock i by μ_i^k, and his estimate of the covariance between the returns of stocks i and j by $\sigma_{i,j}^k$. The investors employ their parameter estimates to construct optimal mean-variance portfolios. Thus, investor k holds stock i in the following proportion in his risky asset portfolio:

$$x_i^k = \frac{\sum_{j=1}^{N} V_{i,j}^k \mu_j^k}{\sum_{h=1}^{N}\sum_{j=1}^{N} V_{h,j}^k \mu_j^k}, \tag{1}$$

where V^k is the inverse of the covariance matrix Σ^k estimated by investor k (see Merton ((1972), eq. (41)) and Levy (1973)). Note that μ_j^k is the expected return in excess of the risk-free rate. The market clearing price for asset i is given by

$$P_i = \frac{1}{n_i}\sum_{k=1}^{K} T^k x_i^k, \tag{2}$$

[9] Recall that the CARA investor facing normal returns is maximizing

$$\mathrm{E}[\tilde{W}_k] - \frac{1}{2}\gamma\mathrm{Var}[\tilde{W}_k] = Q_k(\mu_k - P) + \bar{M} - \frac{1}{2}\gamma\left[Q_k^2\sigma_k^2 + \sigma_M^2 + 2Q_k\sigma_M\sigma_k\rho_k\right].$$

Deriving this expression with respect to Q_k and equating to 0 leads to the above expression for the optimal demand. The price is obtained by taking expectations of both sides of the equation for the optimal demand and recalling that $\mathrm{E}[Q_k] \cdot K = Q$. In this analysis the exposure to the background risk is assumed to be given exogenously. Very similar results are obtained if the investor can vary and optimize this exposure.

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where n_i is the number of outstanding shares of firm i, and T^k is the total investment of investor k in the stock market. Levy et al. (2006) study the case of heterogeneous beliefs about the expected returns, that is, the case where μ_i^k is different across investors, but the beliefs about the covariance matrix are homogeneous. They show that in this case prices are the same as in a market with homogeneous beliefs equal to the average belief, $\mu_i^{\text{avg}} \equiv (1/K)\sum_{k=1}^{K} \mu_i^k$.[10] Thus, in this case prices are insensitive to changes in the degree of disagreement as long as the average belief remains unchanged.

In this study we investigate the case of heterogeneous beliefs about the covariance matrix. Thus, we assume homogeneous beliefs about the expected excess returns, $\mu_i^k = \mu_i$ for all investors, but different beliefs about the covariance matrix Σ^k (and therefore about its inverse, V^k) across investors. Of course, the most realistic case is the one where investors may disagree about both the covariances and the expected returns. We intentionally restrict the present analysis to disagreement only about the covariances in order to focus on the pricing effects of this type of disagreement. Furthermore, as disagreement about the expected returns has no significant pricing effects, the results presented here also reflect the results in the case of disagreement about both the expected returns and the standard deviations.

There are, of course, many different ways to model disagreement about the covariance matrix. Here we focus on the simplest way, which is to assume disagreement about the standard deviations only (i.e., we assume agreement about the correlations). The analysis can be extended to the more complex case of disagreement about the correlations as well.

III. Systematic Pricing Effects: Analytical Results

Consider the simplest form of heterogeneity possible: Half of the investors believe that $\sigma_i^{\text{Inv1}} = \sigma_i(1+\Delta)$, while the other half believe that $\sigma_i^{\text{Inv2}} = \sigma_i(1-\Delta)$. Figure 2 illustrates that in the numerical example investigated, the optimal demand for a stock is highly nonlinear in the stock's estimated standard deviation and covariances. Such nonlinearity implies that the aggregate demand of the 2 investor types indicated previously will generally be very different from the demand of 2 homogeneous "average" investors who believe that the standard deviation is σ_i (see Graph B of Figure 2). In general terms, the difference in the demands for stock i between the heterogeneous and homogeneous cases above is

$$(3) \quad \Delta x_i \equiv x_i\left(\sigma_i(1+\Delta)\right) + x_i\left(\sigma_i(1-\Delta)\right) - 2x_i\left(\sigma_i\right) \approx \frac{\partial^2 x_i}{\partial \sigma_i^2}\sigma_i^2\Delta^2,$$

[10] This statement is mathematically precise only when the number of risky assets and the number of investors approach infinity, for the case where beliefs are uncorrelated with wealth, and for some restrictions on the covariance structure. However, by numerical analysis Levy et al. (2006) show that this result is a very good approximation even for relatively modest numbers of investors and assets, and for general covariance structures.

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where this approximation is derived from a Taylor expansion of the demands.[11] Thus, the second derivative $(\partial^2 x_i)/(\partial \sigma_i^2)$ is key in determining the pricing effects of disagreement. As will be illustrated below, this term is typically large, implying that even small changes in the degree of disagreement can lead to significant price changes.

It is instructive to begin investigating the term $(\partial^2 x_i)/(\partial \sigma_i^2)$ by first analyzing the special case of independent stocks. In this case the inverse covariance matrix elements are $V_{i,i} = 1/\sigma_i^2$ and $V_{\substack{i,j \\ i \neq j}} = 0$, and equation (1) becomes

$$x_i = \frac{\frac{\mu_i}{\sigma_i^2}}{\sum_{j=1}^{N} \frac{\mu_j}{\sigma_j^2}}. \tag{4}$$

(For simplicity we omit the investor-specific index k in this analysis and only reintroduce it later when aggregating demands across all investors.) It is evident from equation (4) that the optimal investment proportion depends on the standard deviation nonlinearly, and therefore we expect disagreement about the standard deviation to affect prices. The magnitude of the pricing effects is directly linked to the degree of nonlinearity of x_i in σ_i, as shown by equation (3). Differentiating x_i with respect to σ_i, we obtain[12]

$$\frac{\partial x_i}{\partial \sigma_i} = -\frac{2x_i}{\sigma_i} + \frac{2x_i^2}{\sigma_i} = \frac{2}{\sigma_i}\left(x_i^2 - x_i\right). \tag{5}$$

Note that if the excess returns are positive ($\mu_i > 0$), then $0 < x_i < 1$ (see equation (4)), which implies that $(\partial x_i)/(\partial \sigma_i)$ is negative, as one would intuitively expect. The second derivative is

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2}{\sigma_i^2}\left(x_i - x_i^2\right) + \frac{2}{\sigma_i}\left(-\frac{\partial x_i}{\partial \sigma_i} + 2x_i \frac{\partial x_i}{\partial \sigma_i}\right).$$

Employing equation (5) yields

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2}{\sigma_i^2}\left(x_i - x_i^2\right) + \frac{2}{\sigma_i}\left(\frac{2x_i}{\sigma_i} - \frac{2x_i^2}{\sigma_i}\right)\left(1 - 2x_i\right),$$

or

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{1}{\sigma_i^2}\left(6x_i - 14x_i^2 + 8x_i^3\right). \tag{6}$$

---

[11] $x_i(\sigma_i + \Delta\sigma_i) \cong x_i(\sigma_i) + \Delta\sigma_i \frac{\partial x_i}{\partial \sigma_i} + \frac{\Delta^2 \sigma_i^2}{2} \frac{\partial^2 x_i}{\partial \sigma_i^2} + \cdots$

$x_i(\sigma_i - \Delta\sigma_i) \cong x_i(\sigma_i) - \Delta\sigma_i \frac{\partial x_i}{\partial \sigma_i} + \frac{\Delta^2 \sigma_i^2}{2} \frac{\partial^2 x_i}{\partial \sigma_i^2} - \cdots.$

[12] $\frac{\partial x_i}{\partial \sigma_i} = \left(\left(-\frac{2\mu_i}{\sigma_i^3} \cdot S\right) + \frac{2\mu_i}{\sigma_i^2}\frac{\mu_i}{\sigma_i^3}\right)\frac{1}{S^2}$, where S is the denominator of equation (4) (i.e., $S \equiv \sum_{j=1}^{N}(\mu_j/\sigma_j^2)$). Equation (5) is obtained by employing $x_i = (\mu_i/\sigma_i^2)(1/S)$.

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Several points are noteworthy. First, as anticipated, the second derivative is nonzero, which means that disagreement will affect prices. Second, the pricing effects can be large. To see this, let us analyze the price change implied by equation (6). Plugging the expression for the second derivative in equation (6) into equation (3) yields

$$\Delta x_i \approx \frac{\partial^2 x_i}{\partial \sigma_i^2} \cdot \sigma_i^2 \cdot \Delta^2 = \left(6x_i - 14x_i^2 + 8x_i^3\right) \cdot \Delta^2.$$

The *relative* price change is given by

$$\frac{\Delta x_i}{x_i} \approx \left(6 - 14x_i + 8x_i^2\right) \cdot \Delta^2. \tag{7}$$

To get an idea of the implied pricing effect, consider, for example, a stock with $x_i = 0.01$. Let us assume a degree of disagreement of $\Delta = 0.10$. Note that this implies rather little disagreement: If the average belief is, for example, $\sigma_i = 0.10$, the value of $\Delta = 0.10$ implies that half of the investors estimate $\sigma_i = 0.1(1 + 0.1) = 0.11$, while the other half estimate $\sigma_i = 0.1(1 - 0.1) = 0.09$. By equation (7), the relative price change in this case is 0.0586 or 5.86%.[13] In other words, the disagreement increases the average investment proportion by 5.86% relative to the case with no disagreement. If the beliefs are uncorrelated with the wealth invested in risky assets, T_k, this corresponds to a 5.86% price increase (see equation (2)). (If the invested wealth *is* correlated with the beliefs, the investment proportion becomes even more sensitive to the beliefs, and the price effects are even further magnified.)

Third, note that a moderate change in the disagreement can lead to a large price change. In the previous example, if the disagreement grows from $\Delta = 0.1$ to $\Delta = 0.2$ (i.e., now half of the investors believe that $\sigma_i = 0.12$ while the other half believe $\sigma_i = 0.08$), the relative price change grows from 5.86% to 23.44% $((6 - 14 \times 0.01 + 8 \times 0.01^2) \times 0.2^2 = 0.2344)$. Thus, the small increase in the disagreement causes a price increase of 16.6% (the relative increase from $1.0586P$ to $1.2344P$, where P is the original price).

Finally, in the range that is typically relevant, $0 < x < \frac{3}{4}$, the derivative in equation (6) is positive, which means that the disagreement induces average demands (and prices) that are *higher* relative to the case of homogeneous beliefs (see equation (3)).[14,15]

We turn now to the more realistic and more complex case of correlated assets. In the case of correlated assets, the inverse covariance matrix element $V_{i,j}$

[13] Employing equation (7), we have $\Delta x_i / x_i \approx \left(6 - 14 \times 0.01 + 8 \times 0.01^2\right) \times 0.1^2 \approx 0.0586$.

[14] For a reasonably large number of stocks with positive excess returns, we expect the xs to be fairly small (because they are positive and add up to 1). The derivative in equation (6) is positive in the range $0 < x < \frac{3}{4}$. The reason that for $\frac{3}{4} < x$ an increase in the disagreement leads to a price decrease is that in the case of independent stocks with positive expected excess returns, the investment proportion in any asset is bounded from above by 1. Thus, if x is large, an increase in Δ causes the investor with belief $\sigma_i(1 - \Delta)$ to increase his investment proportion (which is already close to 1) by less than the decrease in the investment proportion of the investor with the belief that $\sigma_i(1 + \Delta)$.

[15] As the average proportions add up to 1, this means that the disagreement about stock i induces somewhat lower average demands for the other stocks $j \neq i$.

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can be written as $V_{i,j} = (1/(\sigma_i\sigma_j))\delta_{i,j}$, where $\delta_{i,j}$ is the i,j element of the inverse correlation matrix. The mean-variance optimal investment proportions can be written as

$$x_i = \frac{\frac{1}{\sigma_i}\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}{S_1}, \tag{8}$$

where the normalization factor S_1 is given by

$$S_1 \equiv \sum_{h=1}^{N}\frac{1}{\sigma_h}\sum_{j=1}^{N}\delta_{h,j}\frac{\mu_j}{\sigma_j}.$$

The derivative of x_i with respect to σ_i is given in this general case by

$$\frac{\partial x_i}{\partial \sigma_i} = -\frac{1}{S_1}\left(\frac{1}{\sigma_i^2}\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j} + \delta_{i,i}\frac{\mu_i}{\sigma_i^3}\right) + \frac{1}{S_1^2}\frac{1}{\sigma_i^3}\left(\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_i+\mu_j}{\sigma_j}\right)\left(\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}\right). \tag{9}$$

This can be rewritten as

$$\frac{\partial x_i}{\partial \sigma_i} = \frac{x_i^2}{\sigma_i}\left(1 + \frac{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_i}{\sigma_j}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}\right) - \frac{x_i}{\sigma_i}\left(1 + \frac{\frac{\mu_i}{\sigma_i}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}\right). \tag{10}$$

Note that when assets are uncorrelated, $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$, and thus the brackets above are equal to 2, and equation (10) reduces to equation (5), the first derivative when assets are uncorrelated. Differentiating equation (10) with respect to σ_i yields

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2x_i - 4x_i^2 + 2x_i^3}{\sigma_i^2} + \frac{4\mu_i x_i}{\sigma_i^3 S_1}M_i + \frac{2\mu_i^2}{\sigma_i^4 S_1^2}\left(\sum_{j=1}^{N}\frac{\delta_{i,j}}{\sigma_j}\right)M_i - \frac{4\mu_i}{\sigma_i^3 S_1}M_i, \tag{11}$$

where

$$M_i \equiv x_i\sum_{j=1}^{N}\frac{\delta_{i,j}}{\sigma_j} - \frac{1}{\sigma_i}.$$

The derivation of equation (11) is given in the Appendix, which also shows that for the case of uncorrelated assets, equation (11) reduces to equation (6).

Equation (11) reveals that in the general case of correlated assets, the second derivative is nonzero and highly nonlinear in the standard deviation, which means that disagreement will affect prices. Equation (11) also suggests that the

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pricing effects will depend on stock correlations. However, from equation (11) it is hard to determine the magnitude of the pricing effects induced by volatility disagreement. It is also unclear whether the correlations tend to increase or decrease the pricing effects. In order to address these questions and to quantitatively estimate the pricing effects of disagreement, in the next section we employ numerical analysis with empirically estimated stock parameters.

IV. Numerical Analysis

This section presents a numerical analysis that provides an estimate of the magnitude of the pricing effects induced by volatility disagreement in a realistic setting with a large number of correlated assets and the empirically estimated parameter values.

Our stock universe is composed of 100 stocks randomly selected from the CRSP database, with means and covariances empirically estimated from the 120 monthly observations between January 1996 and December 2005. The monthly risk-free rate is taken as 0.327% (for an annual rate of 4%, similar to the historical long-run average annual T-bill rate).

To estimate the magnitude of the pricing effects induced by volatility disagreement, we perform the following analysis: First, we calculate the optimal mean-variance investment proportions given our set of 100 stocks (via equation (1)). Then we select 1 stock out of the set of 100 and assume that there is disagreement of degree Δ about its volatility (i.e., half of the investors estimate the stock's ex ante volatility as $\sigma_i(1+\Delta)$, while the other half estimate it as $\sigma_i(1-\Delta)$, where σ_i is the empirical sample value for the 1996–2005 period). We assume that the investor's total investment in the stock market, T_k, is uncorrelated with his belief about stock i's standard deviation. We record the aggregate investment proportion in the stock for different values of Δ. For each value of Δ we report the value of the relative price-change $(\Delta P_i/P_i)$, where P_i is the price with no disagreement and ΔP_i = Price with disagreement – P_i. (Note that as we assume that beliefs are uncorrelated with wealth, the stock price is proportional to the average investment proportion.) We perform this analysis for each of the 100 stocks. The results are reported in Table 1. The mean relative price change as a function of Δ is shown in Figure 3.

First, note that the pricing effects are large. For example, consider the case of $\Delta = 0.15$ (bold row in Table 1). Note that $\Delta = 0.15$ does not constitute a very large degree of disagreement: If the average belief about the stock's standard deviation is, for example, 20%, Δ = 0.15 implies that half of the investors believe that $\sigma = 20\%(1.15) = 23\%$, while the other half believe that $\sigma = 20\%(0.85) = 17\%$. This rather mild degree of disagreement implies an *average* price increase of 8.8% relative to the no-disagreement case. More importantly, the price is very sensitive to the degree of disagreement. An increase of Δ from Δ = 0.15 to Δ = 0.20 induces an average price increase of about 6.6% (see Figure 3).[16]

[16]If we denote the no-disagreement price by P, with $\Delta = 0.15$ we have an average price of $1.088P$ and with $\Delta = 0.20$ we have an average price of $1.160P$ (see Table 1 and Figure 3). The relative price change between these two cases is $((1.160P)/(1.088P)) - 1 = 0.066$.

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TABLE 1

Price Impact of Volatility Disagreement

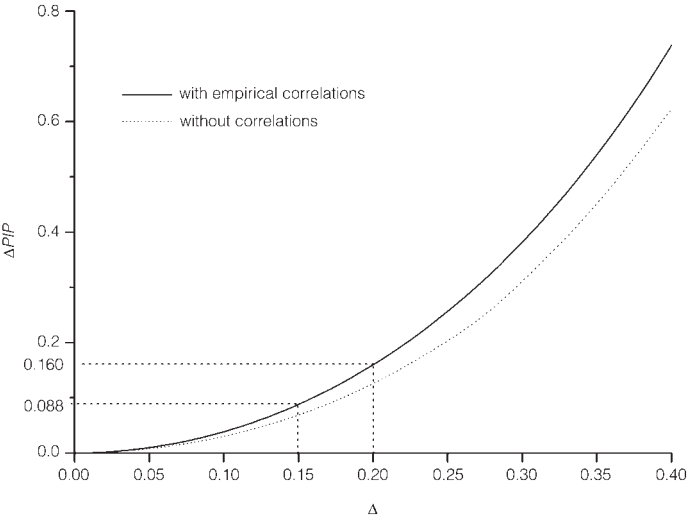
In Table 1 we randomly select 100 stocks from the CRSP database. Their expected returns, standard deviations, and correlations are estimated from the 120 monthly returns during the period January 1996–December 2005. We assume Δ disagreement about 1 stock: Half of the investors estimate the stock's ex ante standard deviation as $\sigma(1 + \Delta)$, while the other half estimate it as $\sigma(1 - \Delta)$, where σ is the empirical value. We record the price impact, $\Delta P/P$, for the stock. This is repeated for each of the 100 stocks, and the distribution properties of the price impact are reported below.

Δ	Mean	Median	75th Percentile	Std. Dev.
0.00	0.000	0.000	0.000	0.000
0.05	0.010	0.003	0.006	0.046
0.10	0.038	0.011	0.025	0.184
0.15	**0.088**	**0.024**	**0.057**	**0.423**
0.20	0.160	0.044	0.105	0.774
0.25	0.256	0.071	0.170	1.254
0.30	0.382	0.105	0.255	1.888
0.35	0.540	0.149	0.364	2.709
0.40	0.738	0.203	0.505	3.762
0.45	0.982	0.272	0.679	5.108
0.50	1.282	0.357	0.895	6.839

FIGURE 3

Average Price Impact

In Figure 3 we show the average price impact, $\Delta P/P$, as a function of the disagreement, Δ. The solid line describes the case of correlated assets, where the empirical correlations are employed. A disagreement factor of $\Delta = 0.15$ implies an average price increase of 8.8% relative to the no-disagreement price. If Δ increases from $\Delta = 0.15$ to $\Delta = 0.20$, the stock price increases on average by 6.6%. The dashed line shows the price impact if the correlations are assumed to be 0. Thus, the correlations magnify the pricing effects of disagreement about the volatility.



Note that these numbers reflect the *average* price impact. There is a large degree of cross-sectional diversity in the price impact of disagreement. This is exhibited by Table 1, which provides not only the mean price impact, but also the median and the 75th percentile. For example, for $\Delta = 0.15$ the mean price impact is 8.8%, and it is much larger than the median price impact (only 2.4%), and even the value at the 75th percentile (5.7%) (bold row in Table 1). Thus, the

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distribution of price impact is very skewed, with a small number of assets having a very large price impact.

From equation (7) it is evident that the correlations between the assets affect the pricing effects. However, it is difficult to determine from this equation whether the correlations tend to magnify or attenuate the pricing effects. In order to address this question, we repeat the above analysis, but this time assuming that the stocks are uncorrelated (all the other parameters are taken as the empirical values, as before). The average relative price change as a function of Δ for the case of zero correlations is depicted by the dashed line in Figure 3. The figure reveals that the average relative price change with correlations is higher than the price change without correlations. Thus, the correlations tend to magnify the pricing effects induced by disagreement about the volatility.

V. Conclusion

Levy, Levy, and Benita (2006) show that, under some mild conditions, disagreement about the expected returns does not affect asset prices. The implication of this result is that the market portfolio, M, is equal to the mean-variance optimal tangency portfolio, T, even if investors disagree about the expected returns (as long as the beliefs are unbiased, that is, the average belief about the expected return is equal to the actual value). This result of $T = M$ implies that the capital asset pricing model (CAPM) holds under heterogeneous unbiased beliefs about the expected returns.[17]

This paper analyzes the implication of disagreement about the standard deviations. In contrast to the results regarding expected returns, disagreement about the standard deviations has systematic and significant pricing effects. This result stems from the fact that while the optimal investment proportion is approximately linear in the perceived expected return (explaining the no-impact result of expected return disagreement), the optimal investment proportion depends very nonlinearly on the perceived standard deviation. Thus, in general, if there is disagreement about the assets' volatilities, prices change relative to the case of no-disagreement, we have $T \neq M$, and the CAPM equilibrium does not hold.

Prices are typically very sensitive to the degree of disagreement about volatility. Thus, a small change in the degree of disagreement can lead to a large price change. This suggests a new explanation for the excess volatility puzzle: The small changes in disagreement are hard to directly detect empirically, leading to apparent "excess volatility" of stock prices.

Note that this explanation for excess volatility holds at the level of the individual stock, as well as at the market level—one can view the market as a single asset in the context of international portfolio optimization.

[17] Whether the resulting tangency portfolio has only positive portfolio weights, as required by the CAPM, is a separate question. See, for example, Nielsen (1988), (1992), Levy (2007), and Levy and Roll (2010), for a discussion of this issue.

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Appendix. Derivation of Equation (11)

Recall that the mean-variance optimal investment proportion can be written as

$$x_i = \frac{\frac{1}{\sigma_i}\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}{S_1},$$

where $$S_1 \equiv \sum_{h=1}^{N}\frac{1}{\sigma_h}\sum_{j=1}^{N}\delta_{h,j}\frac{\mu_j}{\sigma_j}.$$

The first derivative of x_i with respect to σ_i is given by equation (10)

$$\frac{\partial x_i}{\partial \sigma_i} = \frac{x_i^2}{\sigma_i}\left(1+\frac{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_i}{\sigma_j}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}\right)-\frac{x_i}{\sigma_i}\left(1+\frac{\frac{\mu_i}{\sigma_i}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}\right).$$

Let us denote the first bracketed term as R_i and the second bracketed term as T_i. Thus, equation (10) can be rewritten as

$$\frac{\partial x_i}{\partial \sigma_i} = \frac{x_i^2}{\sigma_i}R_i - \frac{x_i}{\sigma_i}T_i. \tag{A-1}$$

To obtain the derivative of equation (10) with respect to σ_i, we first differentiate R_i and T_i:

$$\frac{\partial R_i}{\partial \sigma_i} = \frac{\frac{\mu_i}{\sigma_i^2}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}(R_i-2), \tag{A-2}$$

$$\frac{\partial T_i}{\partial \sigma_i} = \frac{\frac{1}{\sigma_i^2}}{\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}(T_i-1-\mu_i). \tag{A-3}$$

Now we can differentiate equation (10) with respect to σ_i:

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2x_i\sigma_i\frac{\partial x_i}{\partial \sigma_i}-x_i^2}{\sigma_i^2}R_i+\frac{x_i^2}{\sigma_i}\frac{\partial R_i}{\partial \sigma_i}-\frac{\sigma_i\frac{\partial x_i}{\partial \sigma_i}-x_i}{\sigma_i^2}T_i-\frac{x_i}{\sigma_i}\frac{\partial T_i}{\partial \sigma_i}. \tag{A-4}$$

Plugging equations (A-2) and (A-3) into equation (A-4) yields

$$\begin{aligned}\frac{\partial^2 x_i}{\partial \sigma_i^2} = {} & \frac{1}{\sigma_i^2}x_i(T_i-x_iR_i)+\frac{1}{\sigma_i}\frac{\partial x_i}{\partial \sigma_i}(2x_iR_i-T_i)\\ & +\frac{1}{\sigma_i^3\sum_{j=1}^{N}\delta_{i,j}\frac{\mu_j}{\sigma_j}}\left[\mu_i x_i^2(R_i-2)-x_i(T_i-1-\mu_i)\right].\end{aligned} \tag{A-5}$$

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Substituting R_i, T_i, and equation (10) back into equation (A-5), we obtain equation (11):

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2x_i - 4x_i^2 + 2x_i^3}{\sigma_i^2} + \frac{4\mu_i x_i}{\sigma_i^3 S_1} M_i + \frac{2\mu_i^2}{\sigma_i^4 S_1^2} \left(\sum_{j=1}^{N} \frac{\delta_{i,j}}{\sigma_j} \right) M_i - \frac{4\mu_i}{\sigma_i^3 S_1} M_i,$$

where $$M_i \equiv x_i \sum_{j=1}^{N} \frac{\delta_{i,j}}{\sigma_j} - \frac{1}{\sigma_i}.$$

Finally, note that equation (11) is equivalent to equation (6) when assets are uncorrelated. To see this, notice that when assets are uncorrelated, we have

$$M_i = \frac{1}{\sigma_i}(x_i - 1). \tag{A-6}$$

Plugging equation (A-6) back into equation (A-5) and recalling that when assets are uncorrelated $x_i = (\mu_i/\sigma_i^2)\,(1/S_1)$, we obtain

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{2x_i - 4x_i^2 + 2x_i^3}{\sigma_i^2} + \frac{4x_i^3}{\sigma_i^2} - \frac{4x_i^2}{\sigma_i^2} + \frac{2x_i^3}{\sigma_i^2} - \frac{2x_i^2}{\sigma_i^2} - \frac{4x_i^2}{\sigma_i^2} + \frac{4x_i}{\sigma_i^2}, \tag{A-7}$$

and after collecting terms, equation (A-7) gives us equation (6):

$$\frac{\partial^2 x_i}{\partial \sigma_i^2} = \frac{1}{\sigma_i^2}\left(6x_i - 14x_i^2 + 8x_i^3\right).$$

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