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Vicarious Graphs

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Let H be some fixed graph. We call a graph G *vicarious* for H if G is maximal H-free and, for every edge e of G, there is an edge f not in G such that $G-e+f$ is also H-free. We demonstrate various properties of vicarious graphs and several examples are given. It is conjectured that a graph of order n which is vicarious for K_3 has size at most $(1/4+o(1))\binom{n}{2}$.

1. Introduction

Let H be a graph. A graph G is called *vicarious* for H if G is maximal H-free (that is, G contains no copy of H but the addition of any further edge creates a copy) and, for every edge e of G, there is an edge f not in G so that the graph $G-e+f$ (obtained by removing e and adding f) still contains no copy of H. We emphasize that $G-e+f$ is not required to be maximal H-free, only H-free. For a given edge e there might be more than one choice of f such that $G-e+f$ is H-free, but it is convenient to imagine that each e nominates a specific f, called the *vicar* for e. Formally, this means we have a map $v : E(G) \to E(\overline{G})$ such that $G-e+v(e)$ is H-free for every edge e.

We were led to the study of graphs that are vicarious for K_3 by a study of the number of maximal triangle-free graphs. However, it appears that vicarious graphs might exhibit properties which make them interesting in their own right.

A star is vicarious for K_3. But what is the maximum size of a graph of order n that is vicarious for K_3? We conjecture (Conjecture 2.2) that the answer is $(1/4+o(1))\binom{n}{2}$. In the next section we present a class of examples achieving this value, together with upper bounds on the density and on the minimum degree of graphs vicarious for K_3. We subsequently consider the more general case $H=K_r$, and we give some information about what happens when H is a path or a cycle.

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Whereas we do manage to prove some upper bounds on, for example, the size of a graph that is vicarious for K_3, we think they are mostly capable of improvement. For this reason we shall omit the less edifying details of the proofs. In particular we shall assume the reader to be familiar with Szemerédi's lemma and its commonest consequences, as well as with standard probabilistic arguments.

One of the greatest difficulties we have encountered in studying vicarious graphs is that of finding constructions. For example, the original version of this paper included a second construction purporting to be vicarious for K_3, of a nested structure quite different to that given in Section 2, but also of size $(1/4 + o(1))\binom{n}{2}$. However, the construction had a subtle but fatal flaw. This was exposed by a careful referee, whom we gratefully acknowledge.

2. Vicariousness for triangles

Here is a construction of a dense graph that is vicarious for K_3. In fact it is a construction for a class of graphs, insofar as it permits some choice allowing many different non-isomorphic graphs to be built.

Construction 2.1. Split n vertices into two classes, A of order $2\lfloor n/4 \rfloor$ and B which is the remainder. Place $\lfloor n/4 \rfloor$ independent edges inside A. For each vertex of B, join it to exactly one vertex in each of the independent edges of A; thus each vertex of B has degree $\lfloor n/4 \rfloor$. There are many ways to do the joining. For example, choose a subset $L \subset B$ of order $7 \log n$ and a subset M of the edges of A with $|M| = 3 \log n$. For each vertex of L choose a neighbour at random in each edge of A, and for each vertex of $B - L$ choose a neighbour at random in each edge of M. Almost certainly it will be the case that every pair of vertices of B now has a common neighbour in A and every pair of non-adjacent vertices in A has a common neighbour in B. Any way of joining the vertices of $B - L$ to one end of each of the edges in A (not in M) will now produce a graph that is vicarious for K_3 and has size $\lfloor n/4 \rfloor (n - 2\lfloor n/4 \rfloor) + \lfloor n/4 \rfloor = \lfloor n^2/8 \rfloor + \lfloor n/4 \rfloor - \delta$, where $\delta = 1$ if $n \equiv 3 \pmod 4$ and $\delta = 0$ otherwise.

The construction has asymptotic density $1/4$, that is, it contains $(1/4 + o(1))\binom{n}{2}$ edges. We make the following conjecture (mentioned in the Introduction).

Conjecture 2.2. *A graph of order n that is vicarious for K_3 has at most $(1/4 + o(1))\binom{n}{2}$ edges.*

The evidence for this conjecture is regrettably limited. What we are able to offer is weak upper bounds on the density and minimum degree of a graph that is vicarious for K_3. A couple of simple but useful observations are highlighted in the next lemma.

Lemma 2.3. *Let G be a graph of order n that is vicarious for K_3. Then*

- *each vicar is vicar for either one or two edges of G, and*
- *the neighbourhood of any vertex in G spans at most n vicars.*

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Proof. Let $f = v(e)$ for some edge e of G. Neither G nor $G - e + f$ have any triangles but $G + f$ does. Thus $G + f$ has a unique triangle and it contains e. But any other edge for which f is vicar must lie in the same triangle, so f is vicar for at most two edges.

Let x be a vertex in G and f be a vicar whose end-vertices are both neighbours of x. These end-vertices together with x span a triangle in $G + f$. Thus $f = v(e)$ for some edge e incident with x, and so the number of vicars in the neighbourhood of x is at most the degree of x in G. □

Note that the first part of Lemma 2.3 implies that a vicarious graph G requires at least $e(G)/2$ vicars. Below we show that the number of vicars in such a graph is close to the number of edges, *i.e.*, all except a small fraction of the edges nominate a vicar chosen by no other. In order to show it we use Szemerédi's lemma. Given a vicarious graph G of order n, let G_v be the graph whose edges are vicars: that is, $E(G_v) = v(E(G))$. Given $\epsilon > 0$, take a partition of $V(G)$ into at least $1/\epsilon$ parts which is a Szemerédi partition with error parameter ϵ simultaneously for G and G_v.

We call an edge of $G \cup G_v$ *invisible* if it lies within a class of the partition, or it lies between two classes spanning an irregular pair in either G or G_v, or it is an edge of G (respectively G_v) which lies between two classes spanning a regular pair in G (respectively G_v) of density less than ϵ. The total number of invisible edges is at most ϵn^2 if n is large.

Lemma 2.4. *A graph G of order n that is vicarious for K_3 has at least $e(G) - o(n^2)$ vicars.*

Proof. Let T be the set of vicars which are vicar for two edges. By Lemma 2.3, the total number of vicars is $e(G) - |T|$. We distinguish three denominations of vicar in T: those which are vicar for some invisible edge, those which are themselves invisible, and those which are visible and are vicar for two visible edges. Since there are at most ϵn^2 invisible edges, there can be at most $2\epsilon n^2$ vicars of the first two denominations.

Suppose there is a vicar f of the third denomination, so $f = v(e) = v(e')$ where f, e and e' are all visible. Then there are three classes A, B, C of the Szemerédi partition of $V(G)$, each pair of which is regular both in G and G_v, so that f lies between A and B, e lies between A and C and e' lies between B and C. The definition of visibility means that the pair (A, B) has positive density in G_v and the pairs (A, C) and (B, C) have positive density in G. By a standard property of regular partitions, some vertex $x \in C$ has at least ϵn neighbours (in G) in each of A and B. But then (by the property of regular partitions) there are more than n edges of G_v lying between the neighbours of x in A and its neighbours in B, which contradicts Lemma 2.3.

Therefore there are no vicars of the third denomination, so $|T| \leqslant 2\epsilon n^2$ and the lemma is proved. □

Using these lemmas we can obtain a bound on the density of G that is at least better than the trivial bound $1/2$.

Theorem 2.5. *Let G be a graph of order n that is vicarious for K_3. If n is large then $e(G) \leqslant 0.367\binom{n}{2}$.*

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Proof. Let G have $\alpha n^2/2$ edges and let $d(x)$ be the degree of the vertex x in G. Then

$$\sum_{xy\in E(G)} \big(d(x)+d(y)\big) = \sum_x d(x)^2 \geqslant \frac{1}{n}\left(\sum_x d(x)\right)^2 = \alpha^2 n^3,$$

where the first sum is over all edges of G. So there is an edge xy for which $d(x)+d(y) \geqslant 2\alpha n$.

Let X be the set of neighbours of x and let Y be the neighbours of y. Neither X nor Y contains an edge of G. Any vicar f in X has to be a vicar for some edge incident with x, since $G+f$ has a unique triangle, so there are at most n vicars in X or in Y. By Lemma 2.4 the total number of edges plus vicars is $(\alpha+o(1))n^2$, and only $o(n^2)$ of these lie inside X or Y. Therefore

$$(\alpha+o(1))n^2 \leqslant \binom{n}{2} - \binom{d(x)}{2} - \binom{d(y)}{2} \leqslant (1-2\alpha^2)\frac{n^2}{2},$$

and so $2\alpha^2+2\alpha-1 \leqslant o(1)$, or $\alpha \leqslant (\sqrt{3}-1)/2 + o(1) < 0.367$ for large n. □

For the minimum degree of G we can get a bound which is slightly better than that implied by Theorem 2.5.

Theorem 2.6. *Let G be a graph of order n that is vicarious for K_3. If n is large then $\delta(G) \leqslant 14n/43 < 0.3256n$.*

Proof. Suppose to the contrary that $\delta(G) > 14n/43$. Since G is maximal triangle-free, the graph $G \cup G_v$ contains a triangle, with vertices $\{a,b,c\}$ say. If ab is an edge of G then a and b have no common neighbour in G and, by Lemma 2.3, if ab is a vicar then a and b have at most one common neighbour. Therefore we can choose disjoint subsets A, B, C in $V(G)$, each of order $\delta(G)-4$, so that every element of A (respectively B, C) is a neighbour of a (respectively b, c). By removing elements from each set if necessary, we may assume that $|A|$ is odd, that $14n/43 - 5 \leqslant |A| < 14n/43$, and that $|A| = |B| = |C|$. Let $Z = V(G) - A - B - C$; then $|Z| \leqslant n/43 + 15$.

There are no edges of G within A, B or C, so a vertex $x \in A$, say, has at least $13n/43 - 15$ neighbours in $B \cup C$. We say that x *shuns* a set of vertices if it has at most $3n/43 - 15$ neighbours in the set. Suppose that x shuns neither B nor C. Let B_x be the neighbours of x in B and C_x its neighbours in C, where we suppose $|B_x| \geqslant |C_x| \geqslant 3n/43 - 15$. Then $|B_x| \geqslant 13n/43 - 15 - |C_x|$. By Lemma 2.3 there are at most n vicars within each of the sets A, B, C and $B_x \cup C_x$. By Lemma 2.4 the total number of pairs of vertices joined neither by an edge of G nor by a vicar is at most $(15/43 + o(1))\binom{n}{2}$. Therefore

$$\frac{15}{86} + o(1) \geqslant \frac{1}{n^2}\left(\binom{|A|}{2} + \binom{|B|}{2} + \binom{|C|}{2} + |B_x||C_x|\right) \geqslant \frac{3}{2}\frac{14^2}{43^2} + \frac{10}{43}\frac{3}{43} + o(1),$$

which is false if n is large. So x must shun B or C. But it cannot shun both because it has at least $13n/43 - 15$ neighbours in $B \cup C$.

It follows that every vertex in any of the three sets A, B and C shuns exactly one of the other two sets. We say that A *favours* B if fewer than half of its elements shun B. Since the three sets are of odd order, each favours exactly one of the other two. So there must

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be a pair of sets in which one favours the other but the favour is not returned; we may suppose that A favours B but B does not favour A.

Let A' be the set of vertices in A that shun B and B' the set of vertices in B that shun A. Then $2|A'| < |A| = |B| < 2|B'|$. Recall that $|A| \geqslant 14n/43 - 5$, so $|A - A'| > 7n/43 - 3$. Since each vertex of B' has at most $3n/43 - 15$ neighbours in A, it is joined to fewer than $3|A - A'|/7$ of the vertices of $A - A'$. Since $|B'| > |B|/2$, the average number of neighbours in B of a vertex of $A - A'$ is at most $5|B|/7 < 10n/43$. But each vertex of $A - A'$ has at least $13n/43 - 15 - (3n/43 - 15) = 10n/43$ neighbours in B. This is a contradiction and it completes the proof. □

The two proofs just given appear to be quite crude applications of Lemmas 2.3 and 2.4, but we offer a construction indicating that in order to prove $e(G) \leqslant (1/4 + o(1))\binom{n}{2}$, or even to prove that the minimum degree is at most $n/4 + o(n)$, a finer analysis of the rôle of invisible edges would be needed, rather than just more careful calculation. Suppose that $G \cup G_v$ has the following form. Let $1/4 < \alpha < 1/3$. Let the vertices of G lie in 5 classes A, B, C, D, E where $|A| = |B| = \alpha n$, $|C| = |D| = (1 - 3\alpha)n$ and $|E| = (4\alpha - 1)n$. Suppose that A and C span a complete bipartite graph in G, as do B and D, A and E, and B and E. Likewise suppose that the pairs A, D and B, C span complete bipartite graphs in G_v. We also allow some edges of G_v between C and D. Apart from the edges mentioned, all further edges of $G \cup G_v$ are assumed invisible. This construction presents no apparent contradiction to Lemma 2.3. In order not to contradict Lemma 2.4 we need as many vicars as original edges, and this happens provided $|C||D| \geqslant |A||E| + |B||E|$. The value $\alpha = 2 - \sqrt{3} = 0.267$ satisfies this requirement, and gives a graph of minimum degree αn (and density greater than α).

3. Vicariousness for complete graphs

Though the case $H = K_3$ remains unresolved, we offer some general remarks about the case $H = K_r$.

Any graph that is vicarious for K_t can be made vicarious for K_r, $r > t$ by joining every vertex to every vertex in a copy of K_{r-t}. Applying this to Construction 2.1 gives graphs of density $1/4 + o(1)$ that are vicarious for K_r, $r \geqslant 3$. But we can do better for K_4, and hence for K_r, $r \geqslant 4$, as follows

Construction 3.1. Assume that n is a multiple of 4 and take a matching on n vertices. Split the edges into two classes of $n/4$ edges each. For each pair of matching edges with one taken from each class, add three of the four possible edges between them, the choice being made randomly.

It is not hard to see that this construction, akin to Construction 2.1, (almost certainly) produces a graph that is vicarious for K_4 and has density $3/8 + o(1)$.

As for upper bounds on the density of a graph vicarious for K_r, Turán's theorem shows that it cannot exceed $(1 - 1/(r - 1) + o(1))$. We can improve on this only slightly.

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Theorem 3.2. *Let $r \geqslant 3$ be a natural number. Then there are is a positive number ϵ_r, such that a graph of order n that is vicarious for K_r has at most $(1-1/(r-1)-\epsilon_r)\binom{n}{2}$ edges.*

Proof. By the stability theorem of Erdős and Simonovits [1, 2, 4] there is an $\epsilon_r > 0$ such that, if G is a graph of order n, size at least $(1-1/(r-1)-\epsilon_r)\binom{n}{2}$ and not containing K_r, then there is an $(r-1)$-partite Turán graph T on the same vertex set, with G differing from T in at most $n^2/32r^3$ edges. We shall show that such a graph G cannot be vicarious for K_r and this will prove the theorem.

Suppose, on the contrary, that G is vicarious for K_r and let f be a vicar. Then $G+f$ contains copies of K_r; let K be one such. If e is any edge of G such that $f = v(e)$ then K is not contained in $G-e+f$, and so e must be an edge of K. Therefore every vicar is vicar for fewer than $\binom{r}{2}$ edges of G. Since the density of G exceeds $1/2$, the density of the vicars is at least $1/r^2$.

Let U be the set of vertices incident with at least $n/4r$ edges that are in T but not G. Then $|U| \leqslant n/4r^2$. At most $n^2/4r^2$ vicars can be incident with U and at most $n^2/32r^3$ are edges of T, so some vicar f lies in one of the classes (say C) of T and is not incident with U. Since f is not incident with U, all but at most $n/2r$ of the vertices not in C are joined in G to both ends of f. Therefore f lies in at least $(n/r)^{r-2}/2r$ copies of K_r for which every edge (except f) lies in T and every edge incident with f lies in G. An edge of T not in G that lies in one of these K_rs is not incident with f and so is contained in at most $(n/r)^{r-4}$ of these K_rs. Hence at most $(n/r)^{r-2}/32r$ of these K_rs are not contained in $G+f$.

Thus there are at least $(n/r)^{r-2}/4r$ copies of K_r in $G+f$. Let K be one of these. Let $f = v(e)$ for some e and let x be an end-vertex of e that is not in f. Now x is contained in at most n^{r-3} copies of K_r in $G+f$, so if n is large (which we may assume) then there is another copy K' of K_r which does not contain e. This contradicts the definition of a vicar, and completes the proof. □

4. Vicariousness for other graphs

We are primarily interested in dense vicarious graphs. For some H the only graphs that are vicarious for H are very sparse. For example, if $H = 2K_2$ consists of two independent edges, then the only graphs that are maximal H-free are stars or a triangle together with isolated vertices, of which stars (except $K_{1,3}$) are not vicarious for H. If $H = K_{1,2}$ is a path of length two, then a maximal H-free graph consists of a matching plus (if n is odd) an isolated vertex; of these examples the ones with n odd are the only ones vicarious for H. If H is a path of length three then the components of a maximal H-free graph are stars and triangles (perhaps with an isolated vertex); a few triangles plus an isolate is also vicarious for H. A cycle plus an isolated vertex is vicarious for the star $H = K_{1,3}$. We do not know of a graph H for which there is no graph vicarious for H.

We conclude by considering the case where $H = C_{2k+1}$ is an odd cycle. In this case the only graphs of interest are non-bipartite, since a bipartite graph that is maximal C_{2k+1}-free is complete bipartite and certainly not vicarious. There are dense non-bipartite C_{2k+1}-free graphs; for example, three copies of $K_{n/6,n/6}$ each with one vertex identified with a vertex

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of a triangle. Any large non-bipartite C_{2k+1}-free of large minimum degree must have a cut-vertex, by a theorem of Erdős, Faudree, Gyárfás and Schelp [3], who proved that if G is large, 2-connected, non-bipartite and $\delta(G) > 2n/(2k+3)$ then G contains C_{2k+1}.

Here is a way to construct a graph of order n that is vicarious for C_5, and whose minimum degree is approximately $n/12$.

Construction 4.1. Let $a_1b_1c_1a_2b_2c_2$ be a 6-cycle. Take $n/12$ copies of this 6-cycle, together with three disjoint sets of other vertices A, B, C each of order $n/6$. For each vertex $x \in A$ and for each 6-cycle, join x to either a_1 or a_2, the choice being made randomly. Likewise join each vertex of B to either b_1 or b_2 and each vertex of C to either c_1 or c_2.

A feature of these graphs that is crucial to their being vicarious for C_5 is that each pair $x, y \in A$ have many common neighbours in the 6-cycles and each pair a_i, a_j in different 6-cycles have many common neighbours in A. This yields the following 'über-principle', that (almost surely) holds for any such graph G: given an integer $\ell \geqslant 1$, any two vertices $x, y \in A$ and any two of the 6-cycles, there is a path of length 2ℓ between x and y that meets neither of the two given 6-cycles. This path has vertices alternately in A and in the other 6-cycles; its existence can readily be seen.

Any C_5 in the graph so constructed must contain an edge of a 6-cycle, and it is easily checked that there are none – that is, G is C_5-free. The über-principle shows that G is maximal C_5-free; if we add a diagonal a_1a_2 to some 6-cycle then we can find $x, y \in A$ with $xa_1, ya_2 \in E(G)$ and, since x and y are joined by a path of length 2 avoiding the 6-cycle, there is a C_5. Joining two vertices of A makes a C_5 by the über-principle. Joining $x \in A$ to $z \in B$ makes a C_5 because there is some 6-cycle in which $a_1x, b_2z \in E(G)$, and so on. Joining a_i in some 6-cycle to, say, $z \in B$ gives a C_5 because a_ix is an edge for some $x \in A$ and there is some other 6-cycle for which both xa_1 and zb_1 are edges. Joining a_i in one 6-cycle to a_j in another forms a C_5 by the über-principle applied to their neighbours. Joining a_1 to b_2, say, in another cycle forms a C_5 $a_1c_2wc_2b_2$ for some $w \in C$, where the two c_2s lie in different C_6s. All cases are accounted for in this way except the addition of a non-diagonal edge within a 6-cycle, which clearly makes a C_5.

As for finding vicars for the edges, we can take $v(a_1b_1) = a_1b_2$ and, if $x \in A$, we can take $v(xa_1) = xa_2$. Thus every edge has a vicar and G is vicarious for C_5.

The construction just described can be adapted to a general paradigm for building graphs that are vicarious for C_{2k+1}. Unlike the previous graph, the graphs built below do not have large minimum degree, but they do have positive density. The rôle of the 6-cycle is performed by a small fixed graph that we call a 'C_{2k+1}-gadget'.

Construction 4.2. A graph J with two special vertices is called a C_{2k+1}-*gadget* if the following hold.

- G contains no C_{2k+1}.
- The path of least odd length between the special vertices has length $2k-1$.
- Every non-special vertex is joined by a path of odd length at most $2k-3$ to one of the special vertices.

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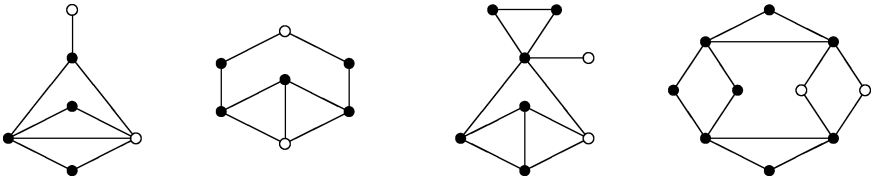


Figure 1. Four gadgets (see text): special vertices are shown hollow.

- For every pair u, v of distinct non-adjacent vertices of F, there is a u–v path of length $2k$, or there is a path from u to one special vertex and a disjoint path from v to the other special vertex, so that the path lengths total $2k-\ell$ for some even $\ell \geqslant 4$.
- For every pair u, v of vertices of F, there is a path joining u to a special vertex and another path joining v to a special vertex (the paths need not be disjoint and the special vertices can be the same) whose lengths total $2k-\ell$ for some even $\ell \geqslant 2$.
- For every edge e of F, one of the two preceding properties fails for some pair u, v when F is replaced by $F-e$.

Given a gadget J, take $n/2|J|$ disjoint copies of it, together with a set A of $n/2$ other vertices. Join each vertex $x \in A$ to one special vertex in each gadget, chosen at random.

This process produces a graph of order n which, by the first two properties of the gadget, is C_{2k+1}-free. The über-principle together with the properties preceding the last mean that the graph is maximal C_{2k+1}-free. Finally, we can find a vicar for every edge; if e is an edge of J then the last property gives a vicar uv either within a single copy of J or between two copies, and of course any edge from A to a special vertex can take as its vicar the edge to the other special vertex in the same gadget.

Clearly, the usefulness of this construction hangs on whether suitable gadgets can be found. We do not know whether C_{2k+1}-gadgets exist for every value of k. Figure 1 shows a sample of known gadgets: a C_5-gadget of order 6, a 2-connected C_5-gadget of order 7, a C_7-gadget of order 8 and a 2-connected C_9-gadget of order 10. The fact that they act as claimed can be verified by a tedious but mechanical check.

5. Conclusion

Having introduced the notion of vicariousness for graphs, we have given some constructions but have not settled the question of whether these constructions are optimal. Apart from Conjecture 2.2, there are several further questions that arise naturally from the work above. We do not present these in the form of conjectures because we have no clear view as to what answers to expect.

Questions

- Is there at least one vicarious graph for every graph H?
- For non-bipartite H, are there large graphs of positive density that are vicarious for H?

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- Does the density of the densest graph vicarious for K_r tend to one as $r \to \infty$?
- Does the density of the densest graph vicarious for C_{2k+1} tend to zero as $k \to \infty$?

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