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Plasma dynamics with electron-inertia effects

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Some aspects of plasma dynamics with electron-inertia effects are considered. The behaviour of (i) one-dimensional steady current sheets, (ii) two-dimensional steady reconnection models, (iii) one-dimensional current-sheet evolution, (iv) linear tearing-mode evolution and (v) nonlinear magnetic-island evolution in this regime is investigated. Stability of the two-dimensional plasma in this regime is then discussed, and *necessary* conditions for linear stability in the Liapunov sense for steady states are then given.

1. Introduction

Many magnetic field reconnection theories are based on the collisional magnetohydrodynamic equations (Shivamoggi 1985). However, the high-temperature plasmas in fusion systems and the magnetosphere have been shown to be collisionless so considerable work has been done on collisionless reconnection processes (see e.g. Coppi 1964; Coppi *et al.* 1966, 1981; Schindler 1974; Drake and Lee 1977; Galeev 1978; Wesson 1990; Drake and Kleva 1991; Ottaviani and Porcelli 1993, 1995; Drake *et al.* 1994). The reconnection rate has been found to increase by an order of magnitude in the electron-inertia regime (Wesson 1990; Ottaviani and Porcelli 1995).

In a collisionless plasma, electron inertia leads to decoupling of the plasma motion from that of the magnetic field lines, limits the electron current and prevents it from becoming unbounded as the resistivity $\eta \to 0$. In the electron-inertia-dominated regime, conservation of the magnetic flux is replaced by conservation of the generalized electron-fluid momentum.

We consider here some aspects of plasma dynamics with electron-inertia effects. We investigate the behaviour in this regime of the following:

(i) one-dimensional steady current sheets:
(ii) two-dimensional steady reconnection models;
(iii) one-dimensional current-sheet evolution;
(iv) linear tearing-mode evolution;
(v) nonlinear magnetic-island evolution.

Stability of a two-dimensional plasma in this regime is then discussed, and *necessary* conditions for linear stability in the Liapunov sense for steady states are given.

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2. Generalized Ohm's law with electron-inertia effects

Generalization of Ohm's law to include electron inertia leads to (with the usual notation)

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J}+\frac{1}{\omega_{pe}^2}\left[\frac{d\mathbf{J}}{dt}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{J}\right]. \tag{1}$$

The absence of the electron pressure gradient term in Ohm's law (1) is known to lead to a current-layer thickness of the order of the electron skin depth d_e:

$$d_e\equiv\frac{c}{\omega_{pe}}.$$

The electron pressure-gradient term, on the other hand, is found (Aydemir 1992; Zakharov *et al.* 1993) to broaden the current layer to a width of the order of the ion-sound radius ρ_s:

$$\rho_s\equiv\frac{c_s}{\Omega_i}.$$

Putting

$$\mathbf{v}=\boldsymbol{\nabla}\phi\times\hat{\mathbf{i}}_z,\qquad \mathbf{B}=\boldsymbol{\nabla}\psi\times\hat{\mathbf{i}}_z,\qquad \mathbf{J}=J\hat{\mathbf{i}}_z, \tag{2}$$

(1) becomes

$$\frac{\partial}{\partial t}(\psi-d_e^2\nabla^2\psi)+\left(\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)(\psi-d_e^2\nabla^2\psi)=\eta J. \tag{3}$$

3. The steady current sheet: Parker problem

In order to determine the structure of the magnetic field and the velocity field in the electron-inertia-controlled regime, it proves useful to reconsider a stagnation-point flow at a current sheet separating plasmas of opposite magnetizations (Parker 1973), as described by (3). The field lines are straight and parallel to the current sheet. Here, pure resistive magnetic field annihilation without reconnection occurs.

Specifically, consider a unidirectional magnetic field

$$\mathbf{B}=B(x)\,\hat{\mathbf{i}}_y, \tag{4}$$

which is carried towards a neutral sheet at $x=0$ by a stagnation-point flow (see Fig. 1),

$$\mathbf{v}=k(-x\hat{\mathbf{i}}_x+y\hat{\mathbf{i}}_y). \tag{5}$$

Equation (3) can now be written as

$$E-kxB+d_e^2kx\frac{d^2B}{dx^2}=\eta\frac{dB}{dx}. \tag{6}$$

On scaling x with respect to some reference length L, (6) becomes

$$\hat{E}-xB+\sigma x\frac{d^2B}{dx^2}=\mu\frac{dB}{dx}, \tag{7}$$

where

$$\hat{E}=\frac{E}{kL},\qquad \sigma=\frac{d_e^2}{L^2},\qquad \mu=\frac{\eta}{kL^2}.$$

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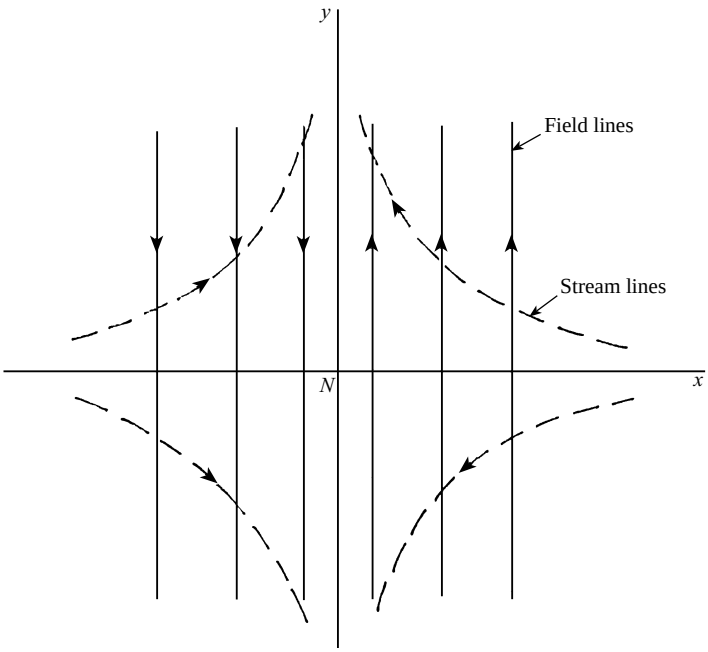


Figure 1. Neutral sheet in a stagnation-point flow.

The boundary condition is

$$B = 0 \quad (x = 0). \tag{8}$$

In the limit as μ and $\sigma \to 0$, (6) gives the outer ideal solution satisfying

$$\hat{E} - xB^{(0)} \approx 0, \tag{9}$$

from which

$$B^{(0)} \approx \frac{\hat{E}}{x}, \tag{10}$$

which does not satisfy the boundary condition (8) at $x = 0$. It is actually singular there! Physically, with the outer solution (10), the magnetic field increases because the field lines are convected into the diffusion region by the converging flow, but cannot escape because they are permanently frozen in the fluid. So, they pile up against the y axis, and the field strength increases without bound. However, even a small electron inertia near the current sheet allows the magnetic energy to be converted into electron kinetic energy, and a small resistivity near the current sheet allows the field lines to diffuse and escape with the outflow of fluid, so that the aforementioned singular behaviour disappears.

Thus, near $x = 0$, the resistivity and electron inertia become important, and the inner solution satisfies the following specialized version of (7):

$$\hat{E} + \sigma x \frac{d^2 B^{(i)}}{dx^2} \approx \mu \frac{dB^{(i)}}{dx}. \tag{11}$$

Equation (11) exhibits the following different behaviours, depending on the relative magnitudes of μ and σ:

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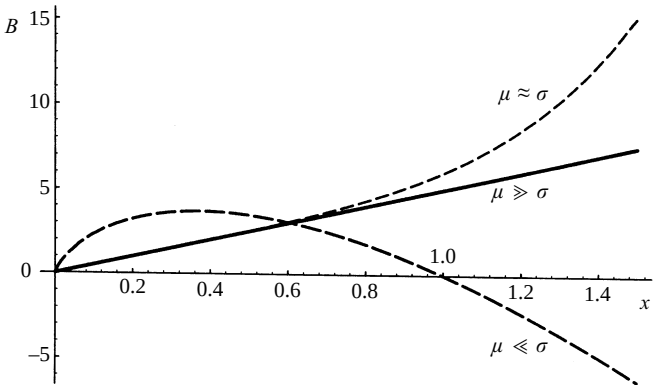


Figure 2. Magnetic field profiles in the various regimes.

(i) $\mu \gg \sigma$:

$$\mu \frac{dB^{(i)}}{dx} \approx \hat{E},$$

or

$$B^{(i)} \approx \frac{\hat{E}}{\mu} x; \tag{12a}$$

(ii) $\mu \ll \sigma$:

$$\sigma x \frac{d^2 B^{(i)}}{dx^2} \approx -\hat{E},$$

or

$$B^{(i)} \approx -\frac{E}{\sigma} x \ln x; \tag{13a}$$

(iii) $\mu \approx \sigma$:

$$\sigma x \frac{d^2 B^{(i)}}{dx^2} - \mu \frac{dB^{(i)}}{dx} \approx \hat{E},$$

or

$$B^{(i)} \approx \frac{\hat{E}}{\mu} x + A x^{\mu/\sigma + 1}. \tag{14a}$$

The magnetic stream functions corresponding to (12a)–(14a) are

(i) $$\psi \approx \frac{\hat{E}}{2\mu} x^2; \tag{12b}$$

(ii) $$\psi \approx \frac{\hat{E}}{4\sigma} x^2 (1 - 2 \ln x); \tag{13b}$$

(iii) $$\psi \approx \frac{\hat{E}}{2\mu} x^2 + A x^{\mu/\sigma + 2}. \tag{14b}$$

Observe that (12)–(14) are well behaved at $x = 0$ and satisfy the boundary condition (8) there. Note that the current density at the neutral plane ($x = 0$) increases as the resistivity (or μ) decreases (Fig. 2).

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2. Two-dimensional steady plasma flow in a nearly hyperbolic magnetic field

One approach to developing steady-state models for magnetic reconnection is to construct solutions to the governing equations with some specific properties imposed on either the plasma flow or the magnetic field to obtain the reconnection topology. In the Parker problem considered in Sec. 3, one starts from a given plasma flow and then constructs a magnetic field compatible with this flow and relevant to reconnection.

Alternatively, one may start from a given magnetic field relevant to the reconnection and then construct the plasma flow compatible with this magnetic field (Habbal and Tuan 1979; Shivamoggi 1985).

Consider a two-dimensional steady-state nearly hyperbolic magnetic field configuration in the (x, y) plane surrounding an X-type neutral point, given by the magnetic potential

$$\psi = \alpha(\kappa x^2 - y^2) + \tfrac{1}{12} J_0 (a x^4 + d x^2 y^2 + b y^4), \tag{15}$$

where

$$J_0 \equiv 2\alpha(1-\kappa),$$

and κ, a, b and d are positive constants. The magnetic field components are then given by

$$B_x = \frac{\partial \psi}{\partial y} = -2\alpha y + \tfrac{1}{6} d J_0 x^2 y + \tfrac{1}{3} b J_0 y^3,$$

$$B_y = -\frac{\partial \psi}{\partial x} = -2\alpha\kappa x - \tfrac{1}{6} d J_0 x y^2 - \tfrac{1}{3} a J_0 x^3. \tag{16}$$

Ampère's law then gives for the current density

$$J_z = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = J_0 [1 - (a + \tfrac{1}{6} d) x^2 - (b + \tfrac{1}{6} d) y^2], \tag{17}$$

which peaks at the neutral point. This is to be understood because the neutral point is also a stagnation point of the flow, so that the plasma remains in the vicinity of this point for a long time, and consequently undergoes acceleration longer in this region.

The electron momentum-balance equation

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\omega_{pe}^2} \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{J} + \eta \mathbf{J}, \tag{18}$$

in conjunction with the incompressibility condition

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{19}$$

then leads to

$$\left(\hat{\kappa} x + \frac{d J_0}{12 \hat{\alpha}} x y^2 + \frac{a J_0}{6 \hat{\alpha}} x^3 \right) \frac{\partial v_y}{\partial y} + \left(y - \frac{d J_0}{12 \hat{\alpha}} x^2 y - \frac{b J_0}{6 \hat{\alpha}} y^3 \right) \frac{\partial v_y}{\partial x}$$

$$= \left[\frac{y}{x} + \left(\frac{a J_0}{3 \hat{\alpha} \hat{\kappa}} + \frac{d J_0}{6 \hat{\alpha}} - \frac{d J_0}{12 \hat{\alpha} \hat{\kappa}} \right) x y - \frac{b J_0}{6 \hat{\alpha}} \frac{y^3}{x} \right] v_y + \frac{J_0 \eta}{2 \hat{\alpha}} \left[(b + \tfrac{1}{6} d) \frac{y^2}{x} - (a + \tfrac{1}{6} d) x \right], \tag{20}$$

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where

$$\hat{\alpha} \equiv \alpha + d_e^2 J_0 (b + \tfrac{1}{6} d),$$

$$\hat{\kappa} \equiv \frac{\alpha\kappa - d_e^2 J_0 (a + \tfrac{1}{6} d)}{\alpha + d_e^2 J_0 (b + \tfrac{1}{6} d)}.$$

Equation (20) may be solved by using the method of characteristics:

$$\frac{dy}{\hat{\kappa} x + \dfrac{aJ_0}{6\hat{\alpha}} x^3 + \dfrac{dJ_0}{12\hat{\alpha}} x y^2} = \frac{dx}{y - \dfrac{bJ_0}{6\hat{\alpha}} y^3 - \dfrac{dJ_0}{12\hat{\alpha}} x^2 y}$$

$$= \frac{dv_y}{\left[\dfrac{y}{x} + \left(\dfrac{aJ_0}{3\hat{\alpha}\hat{\kappa}} + \dfrac{dJ_0}{6\hat{\alpha}} - \dfrac{dJ_0}{12\hat{\alpha}\hat{\kappa}}\right) xy - \dfrac{bJ_0}{6\hat{\alpha}} \dfrac{y^3}{x}\right] v_y + \dfrac{J_0 \eta}{2\hat{\alpha}} \left[(b + \tfrac{1}{6} d) \dfrac{y^2}{x} - (a + \tfrac{1}{6} d)\, x \right]}. \tag{21}$$

The first equality gives the expected result

$$\psi_e \equiv \psi - d_e^2 \nabla^2 \psi = \hat{\alpha}(\hat{\kappa} x^2 - y^2) + \tfrac{1}{12} J_0 (a x^4 + d x^2 y^2 + b y^4) = \text{const.} \tag{22}$$

The second equality gives, in the neighbourhood of the neutral point,

$$\frac{dv_y}{dy} - \frac{y v_y}{\psi_e/\hat{\alpha} + y^2} \approx \frac{\dfrac{J_0 \eta}{2\hat{\alpha}} (b + \tfrac{1}{6} d) \dfrac{\psi_e}{\hat{\alpha}}}{\psi_e/\hat{\alpha} + y^2}, \tag{23}$$

where we have taken

$$a = b\hat{\kappa} + \tfrac{1}{6} d(\hat{\kappa} - 1)$$

without any loss of the essential physics.

We obtain from (23),

$$v_y \approx -\frac{J_0 \eta}{2\hat{\alpha}} (b + \tfrac{1}{6} d)\, y, \tag{24a}$$

and we have from (19)

$$v_x \approx \frac{J_0 \eta}{2\hat{\alpha}} (b + \tfrac{1}{6} d)\, x. \tag{24b}$$

This flow is therefore given by the stream function

$$\phi \approx \frac{J_0 \eta}{2\hat{\alpha}} (b + \tfrac{1}{6} d)\, xy, \tag{25}$$

according to which the streamlines form systems of rectangular hyperbolae, the flow being perpendicular to the field along the x and y axes, as required by the assumed external symmetry. Furthermore, at the neutral point, we also have a stagnation point of the flow.† Observe that the electron-inertia effects ($d_e^2 \neq 0$) lead to a stronger flow.

† This means that the electron-inertia effects vanish at the neutral point, where even a small but finite resistivity becomes important and leads to the breaking and reconnection of the magnetic field lines!

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5. Electron-inertial and resistive evolution of a current sheet

Consider the electron-inertial and resistive evolution of a one-dimensional current sheet. This problem is governed by

$$\frac{\partial}{\partial t}\left(J - d_e^2 \frac{\partial^2 J}{\partial x^2}\right) = \eta \frac{\partial^2 J}{\partial x^2}, \tag{26}$$

$$J = 2J_0\,\delta\left(\frac{x}{d_e}\right) \quad (t = 0). \tag{27}$$

On Fourier transforming with respect to x, according to

$$\left.\begin{aligned} J_k(t) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ikx} J(x,t)\,dx, \\ J(x,t) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{ikx} J_k(t)\,dk, \end{aligned}\right\} \tag{28}$$

(26) and (27) give

$$\frac{dJ_k}{dt} + \frac{\eta k^2}{1 + d_e^2 k^2} J_k = 0 \tag{29}$$

$$J_k = \left(\frac{2}{\pi}\right)^{1/2} J_0\, d_e \quad (t = 0). \tag{30}$$

The solution of (29) and (30) is

$$J_k(t) = \left(\frac{2}{\pi}\right)^{1/2} J_0\, d_e \exp\left(-\frac{\eta k^2}{1 + d_e^2 k^2} t\right). \tag{31}$$

Fourier inversion of (31) gives (see the appendix)

$$J(x,t) = J_0\, e^{-\eta t/d_e^2}\left[2\delta\left(\frac{x}{d_e}\right) + \sum_{n=0}^{\infty} \frac{(\eta t/d_e^2)^{n+1}}{2^n n!(n+1)!}\left(\frac{|x|}{d_e}\right)^n e^{-|x|/d_e} \times \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!m!}\left(\frac{d_e}{2|x|}\right)^m\right]. \tag{32}$$

In the electron-inertia-dominated regime, (32) leads to

$$J(x,t) \approx J_0\, e^{-\eta t/d_e^2}\left[2\delta\left(\frac{x}{d_e}\right) + \frac{\eta t}{d_e^2} e^{-|x|/d_e} + \ldots\right]. \tag{33}$$

Thus, in the electron-inertia-dominated regime, the effect of collisional resistivity is to cause an exponential damping of the current sheet that is being smoothed by the electron-inertia effects. (The effect of a lower-order operator, taken as a perturbation, on the evolution described by the higher-order (higher by order one) operator was shown by Whitham (1959) to be equivalent to an exponential damping.)

6. Linear tearing modes

Let us consider the electron-inertia effects on the linear evolution of the tearing modes (Furth *et al.* 1963). This problem was also considered by Ottaviani and Porcelli (1995), who used the analytical solution method of Furth *et al.* (1963).

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Here we shall reformulate this problem using the variational method due to Hazeltine and Strauss (1978) and Hazeltine and Ross (1978), which leads to a more convenient analytical dispersion relation.

Consider the dynamics in the thin resistive layer near the magnetic neutral surface where the electron-inertia effects become important. The set of equations governing two-dimensional plasma dynamics in this layer are the generalized Ohm's law (1), written in the form

$$\frac{\partial \psi_e}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\psi_e=\eta\nabla^2\psi, \tag{34}$$

along with the equation of motion of the plasma (in the usual notation),

$$\rho\frac{d\mathbf{v}}{dt}=-\boldsymbol{\nabla}p+\mathbf{J}\times\mathbf{B}. \tag{35}$$

Operating with $\hat{\mathbf{i}}_z\cdot\boldsymbol{\nabla}\times$ on (35), and using (2), we obtain

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\right)\nabla^2\phi=-\hat{\mathbf{i}}_z\cdot[\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}(\nabla^2\psi)]. \tag{36}$$

Let us write

$$\left.\begin{aligned}&\psi=\psi_0(x)+\tau_H\psi_1(x)\,e^{\gamma t}\cos ky, \qquad \psi_0'=-B_*F,\\ &\phi=\frac{1}{kB_*}\phi_1(x)\,e^{\gamma t}\sin ky,\end{aligned}\right\} \tag{37}$$

where

$$\tau_H\equiv\frac{\rho^{1/2}}{kB}.$$

Equations (34) and (36) then give, on linearizing in ψ_1 *and* ϕ_1,

$$\omega[\psi_1-d_e^2(\psi_1''-k^2\psi_1)]-(F-d_e^2F'')\,\phi_1=\epsilon(\psi_1''-k^2\psi_1), \tag{38}$$

$$-\omega(\phi_1''-k^2\phi_1)=F(\psi_1''-k^2\psi_1)-F''\psi_1, \tag{39}$$

where primes denote differentiation with respect to x, the distances have been non-dimensionalized using a reference length L, and

$$\epsilon\equiv\frac{\tau_H}{\tau_R}, \qquad \omega\equiv\gamma\tau_H, \qquad \tau_R\equiv\frac{L^2}{\eta}.$$

Near the magnetic neutral surface $x=0$, where $F\approx x$, and the flow and field gradients become large, (38) and (39) become

$$\omega(\psi_1-d_e^2\psi_1'')-x\phi_1=\epsilon\psi_1'', \tag{40}$$

$$\omega\phi_1''=-x\psi_1''. \tag{41}$$

If we do not use the 'constant-ψ' approximation, the analytical solution method of Furth *et al.* (1963) applied to (40) and (41) leads to a complicated dispersion relation (Ottaviani and Porcelli 1995). We therefore use a variational method due to Hazeltine and Strauss (1978) and Hazeltine and Ross (1978) to develop a more convenient analytical dispersion relation for the tearing mode.

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On Fourier transforming the variables according to

$$\left.\begin{aligned} q_k &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-ikx} q(x)\,dx, \\ q(x) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{ikx} q_k\,dk, \end{aligned}\right\} \tag{42}$$

we obtain from (40) and (41)

$$\omega\psi_{1k} - i\frac{d\phi_{1k}}{dk} = -(\epsilon + \omega d_e^2)\,k^2\psi_{1k}, \tag{43}$$

$$-\omega k^2\phi_{1k} = i\frac{d}{dk}(k^2\psi_{1k}). \tag{44}$$

Introducing

$$J_k \equiv k^2\psi_{1k}, \tag{45}$$

we have from (43) and (44)

$$\frac{d}{dk}\left(\frac{1}{\omega k^2}\frac{dJ_k}{dk}\right) - \left(\epsilon + \omega d_e^2 + \frac{\omega}{k^2}\right)J_k = 0. \tag{46}$$

Observe that (46) is self-adjoint, so that we may construct a variational formalism for it. Indeed, (46) corresponds to extremizing the bilinear functional

$$S = \int_{-\infty}^{\infty}\left[\frac{1}{\omega k^2}\left(\frac{dJ_k}{dk}\right)^2 + \left(\epsilon + \omega d_e^2 + \frac{\omega}{k^2}\right)J_k^2\right]dk. \tag{47}$$

In order to obtain an analytical dispersion relation, let us use a trial function for J_k:

$$J_k = e^{-\mu k^2/2}. \tag{48}$$

Equation (47) then becomes

$$S = \pi^{1/2}\left(\frac{\mu^{3/2}}{\omega} + \frac{\epsilon + \omega d_e^2}{\mu^{1/2}} - 2\omega\mu^{1/2}\right). \tag{49}$$

We then determine μ and ω by imposing the conditions

$$S = 0, \tag{50}$$

$$\frac{\partial S}{\partial \mu} = 0, \tag{51}$$

which lead to

$$\frac{3\mu^2}{2\omega} - \frac{\epsilon + \omega d_e^2}{2} - \omega\mu = 0, \tag{52}$$

$$\frac{\mu^2}{\omega} + \epsilon + \omega d_e^2 - 2\omega\mu = 0. \tag{53}$$

We obtain from (52) and (53) the dispersion relation

$$\omega^3 - d_e^2\omega - \epsilon = 0. \tag{54}$$

If ω_1, ω_2 and ω_3 are the three roots of (54) then we have

$$\omega_1\omega_2\omega_3 = \epsilon > 0. \tag{55}$$

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We now have two regimes depending on whether the discriminant

$$D = -\tfrac{1}{27}d_e^6 + \tfrac{1}{4}c^2 \tag{56}$$

is positive or negative.

(i) $D > 0$ (*resistive regime*). This corresponds to one real root, say ω_1, which, from (55), is positive, and we have the collisional tearing-mode instability ($\omega_1 \propto \eta^{1/3}$) (Furth *et al.*, 1963; Coppi *et al.*, 1977).

(ii) $D < 0$ (*electron-inertial regime*). This corresponds to three real roots, at least one of which, say ω_1, from (55), is positive, and we have the collisionless tearing-mode instability ($\omega_1 \propto d_e$) (Coppi 1964; Coppi *et al.* 1966).

The unstable tearing modes, as is well known, cause the magnetic field lines to tear, reconnect and form an island structure at the magnetic neutral surface. The linear theory ceases to be valid when the width of the magnetic island becomes comparable to the thickness of the reconnection layer.

7. Nonlinear magnetic island evolution

Let us now consider the electron-inertia effects on the nonlinear evolution of the magnetic island produced by the tearing-mode instability (Rutherford 1973).

Taking the curl of the equation of motion of the plasma,

$$\rho\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}p + \mathbf{J}\times\mathbf{B}, \tag{35}$$

and ignoring the inertia term, we obtain

$$(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{J} \approx 0. \tag{57}$$

Using (2), (57) implies the constancy of the current density on the magnetic surfaces:

$$J = J(\psi). \tag{58}$$

Writing

$$\psi(x,y,t) = \psi_0(x) + \tilde{\psi}(y,t), \tag{59}$$

(3) becomes

$$\frac{\partial\psi}{\partial t} + d_e^2\frac{\partial\tilde{J}}{\partial t} + \frac{\partial\phi}{\partial y}\frac{\partial\psi_0}{\partial x} + d_e^2\frac{\partial\phi}{\partial y}\frac{\partial J_0}{\partial x} = -\eta(\tilde{J} - J_0). \tag{60}$$

Near $x = 0$, let us write

$$\psi_0(x) = \tfrac{1}{2}Ax^2. \tag{61}$$

Dividing (60) by Ax, and averaging over y at $\psi = \text{const}$, we obtain

$$\sigma\tilde{J} + \tilde{J} = J_0 - \frac{1}{\eta}\frac{\left\langle\dfrac{\tilde{\psi}_t}{(\psi-\tilde{\psi})^{1/2}}\right\rangle_y}{\left\langle\dfrac{1}{(\psi-\tilde{\psi})^{1/2}}\right\rangle_y} = f(t), \quad \text{say}, \tag{62}$$

where

$$\sigma \equiv \frac{d_e^2}{\eta}.$$

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According to the formulation sketched in Sec. 6, the electron-inertia effects dominate the linear evolution if $d_e > \eta^{1/3}$. On the other hand, according to (62), the electron-inertia effects dominate the nonlinear Rutherford evolution if $d_e > \eta^{1/2}$. Therefore the dominance of the electron-inertia effects in the whole linear/nonlinear evolution is guaranteed if

$$d_e > \eta^{1/3} > \eta^{1/2} \quad \text{or} \quad \eta < \min(1, d_e^2). \tag{63}$$

From (62), we have

$$\tilde{J} = \int_0^t \frac{1}{\sigma} e^{-(t-\tau)/\sigma} f(\tau)\, d\tau. \tag{64}$$

Putting

$$\tilde{\psi}(y,t) = \tilde{\psi}_1(t) \cos ky \tag{65}$$

and asymptotically matching across the inner layer to the outer ideal solution, we obtain

$$\Delta_1' \psi_1 \equiv \tilde{\psi}_1'(d_e) - \tilde{\psi}_1'(-d_e) = -2\left\langle \cos ky \left(\int_{-\infty}^{\infty} \tilde{J}\, dx \right) \right\rangle_y. \tag{66}$$

Using (64), (66) becomes

$$\Delta_1' \tilde{\psi}_1 = \frac{4}{\eta (2A)^{1/2}} \int_{\psi_{\min}}^{\infty} \left\langle \frac{\int_0^t \frac{1}{\sigma} e^{-(t-\tau)/\sigma} \tilde{\psi}_\tau\, d\tau}{(\psi - \tilde{\psi})^{1/2}} \right\rangle_y \frac{\left\langle \frac{\cos ky}{(\psi - \tilde{\psi})^{1/2}} \right\rangle_y}{\left\langle \frac{1}{(\psi - \tilde{\psi})^{1/2}} \right\rangle_y} d\psi, \tag{67}$$

from which we obtain

$$\tilde{\psi}_1^{1/2} = \frac{1}{\beta} \int_0^t \frac{1}{\sigma} e^{-(t-\tau)/\sigma} \tilde{\psi}_1'\, d\tau, \tag{68}$$

where

$$\beta \equiv \frac{\eta \Delta_1' (2A)^{1/2}}{4C}, \qquad C \equiv \int_{-1}^{\infty} \frac{\left\langle \frac{\cos ky}{(W - \cos ky)^{1/2}} \right\rangle_y^2}{\left\langle \frac{1}{(W - \cos ky)^{1/2}} \right\rangle_y} dW > 0.$$

Putting

$$\tilde{\psi}_1 = \quad ^2, \tag{69}$$

(68) becomes

$$(2 \quad - \sigma\beta) \quad ' - \beta\sigma = 0, \tag{70}$$

the solution of which is

$$2 \quad - \sigma\beta \ln \quad = \beta t + \text{const}, \tag{71a}$$

or

$$\left(\frac{\quad}{\quad_0} \right)^{\sigma} = e^{2(\quad - \quad_0)/\beta - t}, \tag{71b}$$

where

$$_0 \equiv (t = 0).$$

Observe that in the resistivity-dominated regime (small σ), (71) give the Rutherford (1973) result

$$\tilde{\psi}_1(t) \propto t^2. \tag{72}$$

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On the other hand, in the electron-inertia-dominated regime (large σ), (71) give

$$\tilde{\psi}(t) \propto e^{-2t/\sigma} \tag{73}$$

which shows the collapse of the magnetic island in this regime.

It is pertinent to note in this context the numerical simulations of Drake and Kleva (1991), which showed the collapse of a current layer in the electron-inertia regime, when the magnetic-island width becomes comparable to the plasma skin depth.

8. Two-dimensional plasma dynamics: linear stability in the Liapunov sense

The set of equations governing two-dimensional plasma dynamics with electron-inertia effects are the generalized Ohm's law (1),

$$\frac{\partial \psi_e}{\partial t} + \partial(\psi_e, \phi) = 0, \tag{74}$$

along with the momentum-balance equation for the plasma,

$$\frac{\partial \zeta}{\partial t} + \partial(\zeta, \phi) + \partial(\psi, J) = 0, \tag{75}$$

where we have scaled the distances with respect to the skin depth d_e, ζ is the vorticity,

$$\boldsymbol{\nabla} \times \mathbf{v} = \zeta \hat{\mathbf{i}}_z,$$

and

$$\partial(a, b) \equiv \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

This system has the following non-canonical Hamiltonian representation (in the notation of Benjamin 1984):

$$\mathscr{H}[\phi] = \frac{1}{2}\iint_{\mathscr{V}} (\phi\zeta + \psi J - J^2)\, dx\, dy, \tag{76}$$

where $\mathscr{V} \subseteq \mathbb{R}^2$ is the appropriate two-dimensional domain on which the plasma flow and the magnetic field evolve.

Hamilton's equations for the canonical variables ψ_e and ζ are be written as

$$\left(\frac{\partial \psi_e}{\partial t}, \frac{\partial \zeta}{\partial t}\right) = \mathbf{J}\left(\frac{\delta \mathscr{H}}{\delta \psi_e}, \frac{\partial \mathscr{H}}{\delta \zeta}\right)^T, \tag{77}$$

which are just (74) and (75)! Here $(\cdot\,, \cdot)$ is a row vector, $(\cdot\,, \cdot)^T$ is the corresponding column vector, $\delta\mathscr{H}/\delta q$ is the variational derivative of $\mathscr{H}$ with respect to q, and $\mathbf{J}$ is the non-invertible symplectic operator given by

$$\mathbf{J} = \begin{pmatrix} 0 & \partial(\psi_e, \cdot) \\ \partial(\psi_e, \cdot) & \partial(\zeta, \cdot) \end{pmatrix}. \tag{78}$$

The skew-symmetry property of $\mathbf{J}$ is described by

$$\langle \mathbf{u}, \mathbf{J}\mathbf{v} \rangle = -\langle \mathbf{J}\mathbf{u}, \mathbf{v} \rangle, \tag{79}$$

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for column vectors $\mathbf{u}$ and $\mathbf{v}$, where $\langle \mathbf{u}, \mathbf{v} \rangle$ is the L^2 inner product given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \iint_{\mathscr{V}} \mathbf{u}\mathbf{v}^T \, dx \, dy.$$

Next, note from (74) and (75), that the functional

$$\mathscr{C}[\phi] = \iint_{\mathscr{V}} [F(\psi_e) + \zeta G(\psi_e)] \, dx \, dy \tag{80}$$

is conserved for any functions F and G.

Furthermore, $\mathscr{C}$ is also a Casimir invariant for this problem, because we have, from (78) and (80),

$$\mathbf{J}\left(\frac{\delta \mathscr{C}}{\delta \psi_e}, \frac{\delta \mathscr{C}}{\delta \zeta}\right) = \begin{pmatrix} \partial(\psi_e, G(\psi_e)) \\ \partial(\psi_e, F'(\psi_e) + \zeta G'(\psi_e)) + \partial(\zeta, G(\psi_e)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equilibrium state (denoted by an overbar) is a constrained conditional extremum of $\mathscr{H}$, and is therefore a stationary point of the functional

$$\mathscr{I}[\phi] = -\mathscr{H}[\phi] + \mathscr{C}[\phi], \tag{81}$$

which becomes, on using (76) and (80),

$$\mathscr{I}[\phi] = \iint_{\mathscr{V}} \{\tfrac{1}{2}[(\boldsymbol{\nabla}\phi)^2 + (\boldsymbol{\nabla}\psi)^2 + (\nabla^2\psi)^2] + F(\psi_e) + \zeta G(\psi_e)\} \, dx \, dy. \tag{82}$$

The equilibrium state is therefore given by

$$\delta \mathscr{I} = \iint_{\mathscr{V}} \{[-\bar{\phi} + G(\bar{\psi}_e)] \, \delta\zeta + [-\bar{J} + F'(\bar{\psi}_e) + \bar{\zeta} G'(\bar{\psi}_e)] \, \delta\psi_e\} \, dx \, dy = 0, \tag{83}$$

from which

$$-\bar{\phi} + G(\bar{\psi}_e) = 0, \tag{84}$$

$$-\bar{J} + F'(\bar{\psi}_e) + \bar{\zeta} G'(\bar{\psi}_e) = 0. \tag{85}$$

From (84) and (85), we have

$$\bar{\zeta} = [\bar{J} - F'(\bar{\psi}_e)] \, \bar{\psi}_e'(\bar{\phi}). \tag{86}$$

Next, consider the second variation

$$\delta^2 \mathscr{I} = \iint_{\mathscr{V}} \{(\boldsymbol{\nabla}\delta\phi)^2 - (\boldsymbol{\nabla}\delta\psi)^2 - (\nabla^2\delta\psi)^2 + [F''(\bar{\psi}_e) + \bar{\zeta}\bar{\phi}''(\bar{\psi}_e)] \, (\delta\psi_e)^2 + 2\bar{\phi}'(\bar{\psi}_e) \, \delta\zeta \, \delta\psi_e\} \, dx \, dy, \tag{87a}$$

which may be rewritten as

$$\delta^2 \mathscr{I} = \iint_{\mathscr{V}} \{\{1 - [\bar{\phi}'(\psi_e)]^2\} \, (\boldsymbol{\nabla}\delta\psi_e)^2 - (\nabla^2\delta\psi)^2 - (\boldsymbol{\nabla}\nabla^2\delta\psi)^2 + [F''(\bar{\psi}_e) + \bar{\zeta}\bar{\phi}''(\bar{\psi}_e) + \bar{\phi}'(\bar{\psi}_e) \, \nabla^2\phi'(\bar{\psi}_e)] \, (\delta\psi_e)^2\} \, dx \, dy. \tag{87b}$$

A sufficient condition for linear stability in the Liapunov sense (Arnold 1989) is that $\delta^2\mathscr{I}$ be positive-definite (or negative-definite) for all suitably smooth perturbations $\delta\phi$ and $\delta\psi$. In the present case (thanks to the negative-definite terms introduced in (87*b*) by the electron-inertia effects), it is not possible to

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give any *sufficient* conditions for linear stability in the Liapunov sense. (This is to be contrasted with the corresponding situation without the electron-inertia effects, namely the MHD case (Holm *et al.* 1985).) Nonetheless, one may give necessary conditions for linear stability in the present case, from (87*b*), which are

$$1-[\bar{\phi}'(\bar{\psi}_e)]^2 \geqslant 0, \tag{88a}$$

or

$$F''(\bar{\psi}_e)+\bar{\zeta}\bar{\phi}''(\bar{\psi}_e)+\bar{\phi}'(\bar{\psi}_e)\nabla^2\bar{\phi}'(\bar{\psi}_e) \geqslant 0. \tag{89a}$$

The inequalities (88*a*) and (89*a*) may be rewritten as

$$\left(\frac{\partial\bar{\phi}}{\partial\bar{\psi}}\right)^2 \leqslant \left(1-\frac{\partial\bar{J}}{\partial\bar{\psi}}\right)^2, \tag{88b}$$

or

$$\frac{\partial\bar{J}/\partial\bar{\psi}}{1-\partial\bar{J}/\partial\bar{\psi}} \geqslant 0 \quad \text{or} \quad 0 \leqslant \frac{\partial\bar{J}}{\partial\bar{\psi}} < 1. \tag{89b}$$

Of course, (89*b*) is not satisfied by the usual plasma confinement systems. On the other hand, (88*b*) implies the stabilizing nature of even moderately super-Alfvénic flows. This appears to indicate the absence of dynamo action (Moffatt 1989) in the electron-inertia regime: there is no increase in magnetic energy at the expense of kinetic energy, even if the flow is super-Alvénic.

9. Discussion

In this paper, some aspects of plasma dynamics have been considered with the inclusion of electron-inertia effects. The electron inertia limits the electron current and prevents it from becoming unbounded as $\eta \to 0$. The electron-inertia effects have been shown here to enhance the instability of the plasma, and, when very strong, to lead to collapse of the magnetic island. It may be noted that Drake and Kleva (1991) and Drake *et al.* (1994) have argued that the current sheet should become unstable well before its collapse occurs, so that the resultant enhanced transport of electron parallel momentum would prevent the collapse of the current sheet.

Furthermore, at high temperatures, the resistive-layer thickness is less than the ion gyroradius, so that the ions are not magnetized and are not treated correctly in the magnetohydrodynamic model, necessitating the inclusion of kinetic effects in Ohm's law. The set of questions governing the plasma dynamics in this regime has been given by Schep *et al.* (1994).

Finally, a few remarks are in order about the relationship of the two-dimensional magnetic reconnection formulations discussed in this paper with the three-dimensional real-world situation. The magnetic reconnection process has been studied in the literature largely for cases where one of the space coordinates is ignorable. In these cases, magnetic reconnection process can be discussed in terms of topological arguments based on the two-dimensional geometry (like neutral lines, separatrices, etc.). Besides, the two-dimensional symmetry ensures that all magnetic surfaces remain intact, as described by the existence of a flux function ψ such that a given field line lies in a flux surface $\psi = \text{const}$. A two-dimensional model appears to be a reasonable approximation to a three-dimensional situation when a precise direction of ignorability is

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defined and maintained over very long distances (Greene 1988). However, Schindler *et al.* (1988) pointed out that a two-dimensional model exhibits structural instability in three dimensions. Nonetheless, the magnetic reconnection process in three dimensions can still be understood via local violation of magnetic field line conservation due to an electric field parallel to the magnetic field in this region (Schindler *et al.* 1988).

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Appendix

Fourier inversion of (31) leads to

$$J(x,t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{1/2} J_0 \, d_e \exp\left(-\frac{\eta k^2}{1+d_e^2 k^2} t\right) e^{ikx} \, dk, \tag{A 1}$$

which may be rewritten as

$$J(x,t) = \frac{J_0 \, d_e}{\pi} e^{-\eta/d_e^2} \int_{-\infty}^{\infty} \exp\left(\frac{\eta t/d_e^2}{1+d_e^2 k^2}\right) e^{ikx} \, dk,$$

and, on expansion,

$$J(x,t) = \frac{J_0 \, d_e}{\pi} e^{-\eta t/d_e^2} \int_{-\infty}^{\infty} \left[1 + \sum_{n=0}^{\infty} \frac{(\eta t/d_e^2)^{n+1}}{(n+1)!} \frac{1}{(1+d_e^2 k^2)^{n+1}}\right] e^{ikx} \, dk. \tag{A 2}$$

Integration gives

$$J(x,t) = \frac{J_0}{\pi} e^{-\eta t/d_e^2} \left[2\pi\delta\left(\frac{x}{d_e}\right) + \sum_{n=0}^{\infty} \frac{(\eta t/d_e^2)^{n+1}}{(n+1)!} \frac{(2\pi)^{1/2}}{2^n n!} \left(\frac{|x|}{d_e}\right)^{n+1/2} K_{n+1/2}\left(\frac{|x|}{d_e}\right)\right],$$

or

$$J(x,t) = J_0 \, e^{-\eta t/d_e^2} \left[2\delta\left(\frac{x}{d_e}\right) + \sum_{n=0}^{\infty} \frac{(\eta t/d_e^2)^{n+1}}{(n+1)!} \frac{1}{2^n n!} \left(\frac{|x|}{d_e}\right)^n \right.$$
$$\left. \times \, e^{-|x|/d_e} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)! m!} \left(\frac{d_e}{2|x|}\right)^m\right], \tag{A 3}$$

$K_n(x)$ being the modified Bessel function of the second kind.

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