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# The complement of a codimension-$k$ immersion

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## 0. *Introduction*

In [1] Marc Feighn proves the following result: a proper, $C^2$, codimension-1 immersion in a manifold with no first homology separates the ambient space. In [4] Michelangelo Vaccaro proves a related result: the $C^1$-immersed image of a compact $n$-manifold with image a (curved) polyhedron has non-zero $H_n$ with $\mathbb{Z}_2$ coefficients. (In Vaccaro's terminology $f(M)$ is a curved polyhedron if $f$ is smooth and $f(M)$ is the (non-PL) embedded image of a simplicial complex.) Using ideas similar to Feighn's we prove here the following result.

THEOREM. *Let $f: M^n \to Y^{n+k}$ be a proper, $C^2$ immersion. Then both $\pi_k(Y, Y \setminus f(M))$ and $H_k(Y, Y \setminus f(M))$ are non-trivial.*

Feighn's theorem is a corollary of the case $k = 1$ and the long exact sequence of a pair (see § 2, Corollary 2). Alexander duality gives Vaccaro's theorem with arbitrary, not necessarily polyhedral, image but with the additional proviso that the immersion be $C^2$ (see § 2, Corollary 1).

In [4], Vaccaro constructs a PL immersion of $S^2$ which does not separate $\mathbb{R}^3$ (the house with two rooms) showing that the result in this paper cannot in general be extended to the PL category. However, in at the end of § 2 we do prove that when a PL immersion satisfies a certain condition, one which is always satisfied by smooth maps, then $\pi_k(Y, Y \setminus f(M))$ and $H_k(Y, Y \setminus f(M))$ are non-trivial.

## 1. *Definitions and lemmas*

Unless otherwise stated, all maps and manifolds are smooth (at least twice differentiable). We assume manifolds may both have boundary and be non-compact, so we make the following generalized definition.

*Definition.* $f: M \to N$ is proper if

(i) $f^{-1}(K)$ is compact for every compact $K \subset N$,
(ii) $f^{-1}(\partial N) = \partial M$, and
(iii) $f$ is transverse to $\partial M$.

Following Feighn [1], we define the good set of an immersion.

*Definition.* The good set $G(f)$ of an immersion $f$ is the set of all $x \in M$ such that

(i) there is a neighbourhood $V$ of $f(x)$ such that $f^{-1}(V) = \bigcup_{i=1}^{k} U_i$,
(ii) $U_i \cap U_j$ is empty for $i \neq j$,
(iii) $f | U_i$ is an embedding, and
(iv) $f(U_i) = f(U_j)$ for all $i, j$.

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Note that (i), (ii), and (iii) can be satisfied for any $x \in M$. Also it follows from (iv) that each $U_i$ is non-empty and has non-empty intersection with $f^{-1}(x)$. For any $x \in M$, the multiplicity of $f(x)$ is $|f^{-1}(f(x))|$, where $|\cdot|$ denotes cardinality.

LEMMA 1. *If $f: M \to N$ is a proper immersion then the good set, $G(f)$, is open and dense in $M$.*

The proof is in [1].

We will utilize the following construction:

*Definition.* The pullback $P = P(f,g)$ of maps $f: M \to N$ and $g: B \to N$ is the set $P(f,g) = \{(x,y) \in M \times B \mid f(x) = g(y)\}$.

If we define the induced maps $\bar{f}(x,y) = y$ and $\bar{g}(x,y) = x$, then the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\bar{f}} & B \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[f]{} & N \end{array}$$

LEMMA 2. *Let $f: M \to N$ and $g: B \to N$ be transverse. Then*

(i) *$P = P(f,g)$ is a manifold and* $\dim(P) = \dim(M) + \dim(B) - \dim(N)$;

(ii) *for $x \in B$, $|\bar{f}^{-1}(x)| = |f^{-1}(g(x))|$;*

(iii) *if $f$ is an immersion so is $\bar{f}$.*

*Proof.* (i) is proved by standard transversality arguments and (ii) is obvious. For (iii), notice $\nu \in TP$ can be written $\nu = (\nu_1, \nu_2) \in TM \times TB$. Then $T\bar{f}(\nu) = \nu_2$. Assume $\nu \neq 0$. Then $\nu_2 \neq 0$ will prove the lemma.

If $\nu_1 \neq 0$ then $Tf \circ T\bar{g}(\nu) = Tf(\nu_1) \neq 0$ since $f$ is an immersion. But $f \circ \bar{g} = g \circ \bar{f}$, so

$$0 \neq Tf \circ T\bar{g}(\nu) = Tg \circ T\bar{f}(\nu) = Tg(\nu_2).$$

Hence $\nu_2 \neq 0$. ∎

PROPOSITION 1. *Suppose $f: P \to B$ is a proper immersion of a compact one-manifold $P$ to $B$. Then there is a smooth function $\phi: B \to \mathbb{R}$ such that*

(i) *$\phi \circ f$ is a Morse function on $P$;*

(ii) *the endpoints of $P$ are non-critical maxima of $\phi \circ f$.*

*Proof.* Find a proper embedding $B \to \mathbb{R}^{N+}$. Feighn's proposition ([1], §1) proves that for an open and dense set of $w \in \mathbb{R}^N$ condition (i) is true with

$$\phi(\nu) = L_w(\nu) = \|\nu - w\|^2.$$

The proof of Feighn's theorem ([1], §2) gives condition (ii) for some $w \in \mathbb{R}^{N+}$. ∎

## 2. *Main results*

THEOREM 1. *Let $f: M^n \to Y^{n+k}$ be a proper immersion, where $k \geq 1$. Then there is a disk $(D^k, \partial D^k) \subset (Y, Y \setminus f(M))$ which represents a non-trivial class in both $H_k(Y, Y \setminus f(M))$ and $\pi_k(Y, Y \setminus f(M))$. Furthermore $[D^k]$ is not of odd order in $H_k(Y, Y \setminus f(M))$ nor, when $k \geq 2$,*

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*in* $\pi_k(Y, Y\backslash f(M))$. *Also* $[D^k]$ *is not twice any class in* $H_k(Y, Y\backslash f(M))$, *nor the square of any class in* $\pi_k(Y, Y\backslash f(M))$ *if* $k \geqslant 2$.

The proof that follows is quite similar to Feighn's in [1], but more general.

*Proof.* We first prove that $[D^k] \neq 0$ in $\pi_k(Y, Y\backslash f(M))$. The other statements all follow by variations of this proof.

Choose $x \in M$ so the multiplicity, $k = r.2^\alpha$ ($r$ odd), of $f(x)$ has minimal $\alpha$. Let $D_1$ be a $k$-disk embedded in $Y$ which is transverse to $f$, meeting $f(M)$ only at $x$. (Burt Totaro has pointed out to me that we could assume $x \in G(f)$, but we will not need that here.)

If the theorem is false then $D_1$ is homotopic rel $\partial$ to $D_2$, the image of a $k$-disk in $Y\backslash f(M)$, i.e. there is a map $g: B^{k+1} \to Y$ such that $g(\partial B) = D_1 \cup D_2$. We may assume $g$ is smooth and transverse to $f$. We may also assume $g$ misses $\partial Y$. Then $P(f, g)$ is a one-manifold and $\bar{f}^{-1}(\partial B) = \partial P$. The Euler characteristic of $P$,

$$\chi(P) = \tfrac{1}{2}|\partial P| = \tfrac{1}{2}|\bar{f}^{-1}(\partial B)| = \tfrac{1}{2}|f^{-1}(g(\partial B))| = \tfrac{1}{2}|f^{-1}(f(x))| = \tfrac{1}{2}r.2^\alpha.$$

Thus if $\alpha = 0$ we have shown that $P$ does not have integer Euler characteristic, a contradiction. (This case is easy to show using standard intersection number techniques, as well.)

If $\alpha > 0$, we can also calculate $\chi(P)$ by finding a map $\phi: B^{k+1} \to \dot{\mathbb{R}}$ as in Proposition 1. Consider the critical points of $\phi \circ \bar{f}$: $P \to \mathbb{R}$. Defining $i(\text{maximum}) := -1$ and $i(\text{minimum}) := 1$ gives, by elementary Morse theory,

$$\chi(P) = \sum_{x \in \text{crit}(\phi \circ \bar{f})} i(x).$$

For $x \in \text{crit}(\phi \circ \bar{f})$, the points in $\bar{f}^{-1}(\bar{f}(x))$ are all critical points of the same index. Thus $i(x)$ occurs in the expression for $\chi(P)$ with multiplicity $|\bar{f}^{-1}(\bar{f}(x))| = |f^{-1}(g(\bar{f}(x)))|$, which, since we chose $x$ to minimize $\alpha$, is divisible by $2^\alpha$. Hence $\chi(P)$ is a non-zero multiple of $2^\alpha$, contradicting the earlier computation. This proves that $[D_1]$ is not trivial in $\pi_k(Y, Y\backslash f(M))$.

We can imitate this proof to show $0 \neq [D_1] \in H_k(Y, Y\backslash f(M))$. If the theorem is false then $D_1$ is homologous to $D_2$, this time by a singular $k$-chain in $Y\backslash f(M)$. In order to utilize the same type of differential topology techniques we use a proposition proved in Fenn [2], p. 10.

PROPOSITION 2. *Let* $\Sigma^{k+1}$ *be a singular chain in* $Y$. *Then* $\Sigma$ *may be represented by a map* $g$ *of a simplicial complex* $X^{k+1}$ *to* $Y$, *and* $X^{k+1}$ *has the property that* $X^{k+1}\backslash X^{k-1}$ *is a smooth manifold.*

Using transversality and induction on the skeleta of $X$, we can homotope $g$ so that $g(\text{neighbourhood}(X^{k-1}))$ misses $f(M)$ entirely. Let $\tilde{g}$ be an $\epsilon$-approximation to $g$ which is smooth off a smaller neighbourhood of $X^{k-1}$ and which is transverse to $f$. This is well defined since $\tilde{g}$ is smooth wherever any intersections occur.

The original arguments in the homotopy case now apply. $P(f, \tilde{g})$ is a 1-manifold properly immersed in $X\backslash\text{neighbourhood}(X^{k-1})$. One count of the Euler characteristic gives $r.2^{\alpha-1}$, the other a multiple of $2^\alpha$. Thus $[D_1] \neq 0 \in H_k(Y, Y\backslash f(M))$.

The rest of the theorem follows similarly. Suppose that $[D_1]$ is of order $2j+1$ in $\pi_k(Y, Y\backslash f(M))$. Map $D^k \to D_1$ as the standard branched cover with $2j+1$ sheets and branch set not containing $x$. Then this $D^k$ represents the $(2j+1)$th power of $[D_1]$.

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Since this is trivial there is a ball $B$ which determines a 1-manifold $P$ with $(2j+1).r.2^\alpha$ endpoints, but whose Morse theoretic Euler characteristic is some multiple of $2^\alpha$. A similar argument works for homology.

Suppose $[D_1] = [D_2]^2 \in \pi_k(Y, Y \setminus f(M))$. Make $D_2$ transverse to $f(M)$. Each point in $D_2 \cap f(M)$ lifts via $f^{-1}$ to a multiple of $2^\alpha$ points. Let $l.2^\alpha$ be the total number of points in $f^{-1}(D_2)$. Then $f^{-1}$ of a representative of $[D_2]^2$ has $2l.2^\alpha$ points. Since $[D_1] = [D_2]^2$, there is a homotopy between their representatives which, in the usual manner, yields a 1-manifold with $(2l+r).2^\alpha$ endpoints, but again its Morse theoretic Euler characteristic contradicts this. The corresponding claim for homology is proved similarly. ∎

COROLLARY 1 (Generalized Vaccaro's Theorem). *Let $f: M^n \to Y$ be a proper immersion of a boundaryless manifold $M$. Then $\check{H}^n_c(f(M)) \neq 0$, and $\check{H}^n_c(f(M); \mathbb{Z}_2) \neq 0$, where $\check{H}^n_c$ refers to Čech cohomology with compact support.*

*Proof.* Since $M$ is boundaryless we may assume that $Y$ is, too. Let $g: Y \to \mathbb{R}^s$ be a proper embedding. Then $g \circ f: M \to \mathbb{R}^s$ is a proper immersion and $\check{H}^n_c(g \circ f(M)) = \check{H}^n_c(f(M))$. Theorem 1 and Alexander duality now show that

$$0 \neq H_{s-n}(\mathbb{R}^s, \mathbb{R}^s \setminus g \circ f(M)) \cong \check{H}^n_c(g \circ f(M)).$$

For a discussion of Alexander duality in this context see Spanier [3], theorem 6·3. Now $[D_1]$, the non-zero class in $H_{s-n}(\mathbb{R}^s, \mathbb{R}^s \setminus g \circ f(M))$, is not twice any other class, so $0 \neq [D_1] \pmod 2 \in H_{s-n}(\mathbb{R}^s, \mathbb{R}^s \setminus g \circ f(M); \mathbb{Z}_2)$ as well. Alexander duality now shows that $\check{H}^n_c(g \circ f(M); \mathbb{Z}_2) \neq 0$. ∎

COROLLARY 2 (Generalized Feighn's Theorem). *Let $f: M^n \to Y^{n+k}$ be a proper immersion.*

*If $\pi_k(Y)$ is trivial then $\pi_{k-1}(Y \setminus f(M))$ is not trivial. If $H_k(Y)$ is trivial then neither $H_{k-1}(Y \setminus f(M))$ nor $\pi_{k-1}(Y \setminus f(M))$ is trivial. A non-zero element is, of course, given by some linking $S^{k-1}$.*

*Proof.* The proof is just a diagram chase on

$$\begin{array}{ccccc} \pi_k(Y) & \longrightarrow & \pi_k(Y, Y \setminus f(M)) & \longrightarrow & \pi_{k-1}(Y \setminus f(M)) \\ \downarrow & & \downarrow & & \downarrow \\ H_k(Y) & \longrightarrow & H_k(Y, Y \setminus f(M)) & \longrightarrow & H_{k-1}(Y \setminus f(M)) \end{array}$$

where the vertical maps are Hurewicz homomorphisms. Note that the middle vertical map is a surjection. ∎

The requirement that the immersion $f$ be $C^2$ seems a bit unnatural, and perhaps it is. A proof for $C^1$ immersions would certainly be welcome, but, surprisingly, a proof for the PL (or TOP) categories is impossible. There are examples of surfaces PL-immersed in $\mathbb{R}^3$ which do not separate $\mathbb{R}^3$. See Feighn [1] and especially Vaccaro [4] for a discussion of this. Some results are possible in the PL category, however. Vaccaro's paper has one theorem about when a PL immersion acts like a smooth map in terms of its separation properties. Basically, it is a requirement of $C^1$ triangulability of the image. Here we present another sufficient condition.

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THEOREM 2. *Let $f: M^n \to Y^{n+k}$ be a proper* PL *immersion. Let $\alpha(x)$ be the maximal $\alpha$ such that the multiplicity of $f(x)$ is $r \cdot 2^\alpha$. Suppose an absolute minimum for $\alpha(x)$ is achieved on $G(f)$. (One can show that this is always true for smooth maps.) Then the conclusion and corollaries to Theorem* 1 *hold in this case as well. In particular, $\pi_k(Y, Y \setminus f(M))$ and $H_k(Y, Y \setminus f(M))$ are not trivial.*

*Proof.* The proof is isomorphic to the proof of Theorem 1, but in the PL category. Pick $x \in f(M)$ so that $x$ minimizes $\alpha$ and $f^{-1}(x) \subseteq G(f)$, i.e. $x$ is in the interior of the $n$-skeleton of $f(M)$. If the dual $k$-disk, $D_1$, at $x$ were trivial in $\pi_k(Y, Y \setminus f(M))$ then there would be a map $g$ of a $(k-1)$-ball $B$ to $Y$ with $g(\partial B) \cap f(M)$ empty. Approximate $g$ by a PL map $\tilde{g}$ which is in general position with respect to every subcomplex of $f(M)$. Then the pullback $P(f, \tilde{g})$ is a 1-manifold with $r \cdot 2^\alpha(x)$ endpoints, hence $\chi(P) = r \cdot 2^{\alpha(x)-1}$.

On the other hand, counting simplices shows that the number of 0-simplices and the number of 1-simplices are multiples of $2^{\alpha(x)}$, implying $\chi(P)$ is a multiple of $2^{\alpha(x)}$, contradicting the earlier calculation. Hence $D_1$ is non-trivial in $\pi_k(Y, Y \setminus f(M))$.

The homology claims are proved analogously. ∎

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