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General asymptotic distributions for additive arithmetic functions

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1. Let $f(n)$ be a real-valued additive arithmetic function. Let $\alpha(x)$ and $\beta(x) > 0$ be real valued functions, defined for $x \geqslant 2$. Define the frequencies

$$\nu_x(n; f(n) - \alpha(x) \leqslant z\beta(x)) = x^{-1} \sum_{\substack{n \leqslant x \\ f(n) - \alpha(x) \leqslant z\beta(x)}} 1. \tag{1}$$

Since the pioneering paper of Erdös and Kac(4), much work has been done on obtaining conditions which are both necessary and sufficient in order that these frequencies should converge weakly as $x \to \infty$. Generally speaking, it was assumed at the outset either that the function $f(n)$ satisfies some kind of regularity condition (see, for example, Kubilius(7), Fainleib and Levin(5)), or that the renormalizing functions $\alpha(x)$ and $\beta(x)$ satisfy some regularity conditions (see, for example, Levin and Timofeev(8), (9) and Elliott(1), (2)).

In a recent paper, the present author(3) established conditions of a necessary and sufficient nature in order that the frequencies (1) should converge to the improper law, assuming only that $\beta(x) \to \infty$ as $x \to \infty$, and that

$$\sup_{w \leqslant x^2} \beta(w) \leqslant C\beta(x) \quad (x \geqslant 1, C \text{ a constant}).$$

This seems to be the first theorem of such a general nature in which no *a priori* restrictions are made upon the functions $f(n)$ or $\alpha(x)$.

It is our present purpose to strengthen this assumption concerning $\beta(x)$ (only) somewhat, and to consider the possible convergence of the frequencies (1) to laws more general than the improper law. We shall make use of the result of the author's paper(3), already referred to, but not of its method.

THEOREM. *Let* $\beta(x) \to \infty$ *as* $x \to \infty$, *and assume that for each fixed positive number* y, $\beta(x^y)/\beta(x) \to 1$ $(x \to \infty)$. *Assuming further that for some constant* C,

$$\sup_{w \leqslant x} \beta(w) \leqslant C\beta(x).$$

Let

$$\nu_x(n; f(n) - \alpha(z) \leqslant z\beta(x)) \Rightarrow F(z) \quad (x \to \infty). \tag{2}$$

In other words, the frequencies (1) *converge weakly to a limit law* $F(z)$.

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Then there is a constant A so that the independent random variables X_p (p prime) which are defined by

$$X_p = \begin{cases} f(p) - A\log p & \text{with probability } 1/p, \\ 0 & \text{with probability } 1-(1/p), \end{cases} \tag{3}$$

satisfy

$$P\Big(\sum_{p\leqslant x} X_p - \omega(x) \leqslant z\beta(x)\Big) \Rightarrow F(z) \quad (x\to\infty), \tag{4}$$

with the function $\omega(x)$ defined by

$$\omega(x) = \alpha(x) - A\log x.$$

Conversely, assume that for some constant A and function $\omega(x)$ the variables X_p defined in (3) *satisfy the asymptotic condition* (4). *Then if we define $\alpha(x) = \omega(x) + A\log x$ the frequencies in* (2) *will also converge to the limit law $F(z)$ of* (4).

Remarks. (i) The condition that $\sup\beta(w)$ does not exceed $C\beta(x)$ may, in fact, be removed.

(ii) It is easy to see that the random variables $\beta(x)^{-1}X_p$, $2 \leqslant p \leqslant x$, are infinitesimal. Necessary and sufficient conditions in order that the frequencies (4) should converge weakly can therefore be immediately deduced from a well-known theorem of Gnedenko (see the following Lemma 3).

(iii) Since $F(z)$ will be the limit of cumulative sums of (suitably renormalized) random variables, it will belong to the class L of Khinchine.

(iv) There does not seem to be any way to give an explicit, or at any rate easy-to-calculate, representation for the constant A. It can be deduced from (4) of the theorem that there is a representation

$$\alpha(x) = A\log x + \sum_{\substack{p\leqslant x \\ |g(p)|\leqslant\tau\beta(x)}} \frac{g(p)}{p} + C\beta(x) + o(\beta(x)),$$

where $g(p) = f(p) - A\log p$. Apart from the values of the constants τ (>0) and C, this representation is unique up to the addition of a term $o(\beta(x))$. Since, as we shall see presently, $\beta(x) = O((\log x)^{\epsilon})$ for each fixed $\epsilon > 0$, it follows that

$$A = \lim_{x\to\infty} \frac{\alpha(x)}{\log x} \tag{5}$$

will define A uniquely.

(v) As an application of the theorem we note that with the same conditions on $\beta(x)$ as in the theorem, there will be a function $\alpha(x)$ so that

$$\nu_x(n; f(n) - \alpha(x) \leqslant z\beta(x)) \to \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{z} e^{-w^2/2}\,dw \quad (x\to\infty),$$

if and only if there is a constant A so that the function $g(p) = f(p) - A\log p$ satisfies

$$\sum_{\substack{p\leqslant x \\ |g(p)|>\epsilon\beta(x)}} \frac{1}{p} \to 0, \quad \frac{1}{\beta^2(x)} \sum_{\substack{p\leqslant x \\ |g(p)|\leqslant\epsilon\beta(x)}} \frac{g^2(p)}{p} \to 1 \quad (x\to\infty),$$

for each fixed $\epsilon > 0$. When these conditions are satisfied one may choose

$$\alpha(x) = A\log x + \sum_{\substack{p\leqslant x \\ |g(p)|\leqslant\tau\beta(x)}} \frac{g(p)}{p}$$

for any fixed positive number τ.

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This corollary can be deduced by means of the aforementioned theorem of Gnedenko.

(vi) It is easy to see that we can, without loss of generality, regard $\beta(x)$ as a measurable function of x. This we shall do.

This concludes our remarks.

2. *Some preliminary results.* Regarding $\beta(x)$ as a measurable function of x, it can be proved that, as $x \to \infty$,

$$\beta(x^y)/\beta(x) \to 1$$

uniformly for all values of y in any bounded interval of the form $u \leqslant y \leqslant v$, with $0 < u < v < \infty$. We can then construct a continuous increasing function $r = r(x)$, so that $0 < r \leqslant x$, and as $x \to \infty$,

$$\frac{\log r}{\log x} \to 0, \quad \frac{\beta(w)}{\beta(x)} \to 1 \quad \text{(uniformly for } r \leqslant w \leqslant x\text{)}. \tag{6}$$

We shall assume such a function r to be defined for $x \geqslant 2$, say.

From the hypothesis on $\beta(x)$, if $0 < \epsilon < 1$, and x is sufficiently large, say $x \geqslant x_0$, then

$$\beta(x) \leqslant (1+\epsilon)\,\beta(x^{1/2}).$$

Proceeding by induction

$$\beta(x) \leqslant (1+\epsilon)^2\,\beta(x^{1/2^2}) \leqslant \ldots \leqslant (1+\epsilon)^k\,\beta(x^{1/2^k}),$$

and so on, provided only that $2^k \log x_0 \leqslant \log x$. Choosing the largest possible value of k, and letting $x \to \infty$, yields

$$\limsup_{x\to\infty} \frac{\log \beta(x)}{\log \log x} \leqslant \frac{\log(1+\epsilon)}{\log 2}.$$

It follows that
$$\beta(x) = O((\log x)^\epsilon) \tag{7}$$
for any fixed positive value of ϵ.

We shall need the following result from the author's paper (3).

LEMMA 1. *Define the improper distribution*

$$I(z) = \begin{cases} 1 & \text{if} \quad z \geqslant 0, \\ 0 & \text{if} \quad z < 0. \end{cases}$$

Assume that $\beta(x) \to \infty$ *as* $x \to \infty$, *and that*

$$\sup_{w \leqslant x} \beta(w) \leqslant C\beta(x) \quad (x \geqslant 2), \quad \limsup_{x\to\infty} \{\beta(x^2)/\beta(x)\} < 2.$$

Then the conclusion of Theorem 1 *is certainly true in the case that* $F(z) = I(z)$. *Moreover, in the decomposition*

$$\alpha(x) = A \log x + \omega(x)$$

the function $\omega(x)$ *satisfies*

$$\omega(x^y) = \omega(x) + o(\beta(x)) \quad (x \to \infty, y > 0 \text{ fixed}).$$

Remarks. The hypothesis which $\beta(x)$ needs to satisfy in Lemma 1 is weaker than stated in the main theorem.

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In particular, it follows that when Lemma 1 is applicable, and the frequencies (2) converge to $I(z)$, then for each fixed $\epsilon > 0$,

$$\sum_{\substack{p \leqslant x \\ |g(p)| > \epsilon\beta(x)}} \frac{1}{p} \to 0 \quad (x \to \infty).$$

This ends our remarks.

Proof of Lemma 1. Lemma 1 may be found as theorem 2 in the author's paper (3). Our present definition of $\omega(x)$ differs slightly from the one used in that reference. However, this difference is not essential, since any renormalizing function $\alpha(x)$ in (2) is uniquely determined to within the addition of a function which is $o(\beta(x))$ as $x \to \infty$.

LEMMA 2. *Let $h(p)$ be a real-valued function defined for primes p in the range $2 \leqslant p \leqslant w$. Define independent random variables Y_p, for these same primes, by*

$$Y_p = \begin{cases} h(p) & \text{with probability } 1/p, \\ 0 & \text{with probability } 1 - 1/p. \end{cases}$$

Let x be a real number, $x \geqslant w$.

Then there is a positive absolute constant c so that

$$\nu_x(n; \sum_{p \mid n, p \leqslant w} h(p) \leqslant z) = P(\sum_{p \leqslant w} Y_p \leqslant z) + O\left(\exp\left(-c\frac{\log x}{\log w}\right)\right)$$

uniformly for $2 \leqslant w \leqslant x$, and all real numbers z, and $h(p)$, $2 \leqslant p \leqslant x$. The constant implied in the O-term is absolute.

Proof. This lemma is a form of Kubilius' basic representation theorem. An exposition is given in his monograph ((7), chapter 2).

We shall need one theorem from the theory of probability proper.

LEMMA 3. *Let*

$$X_{nk}, \quad k = 1, \ldots, k_n \quad (n = 1, 2, \ldots)$$

denote a double array from random variables, independent for each fixed value of n with varying values of k. Suppose that as $n \to \infty$ the distributions

$$P\left(\sum_{k=1}^{k_n} X_{nk} - A_n \leqslant z\right)$$

converge weakly for suitably chosen constants A_n.

Then there are functions $M(u)$ and $N(u)$, defined for $u < 0$ and $u > 0$ respectively, so that for almost all u

$$\sum_{k=1}^{k_n} P(X_{nk} > u) \to -N(u), \quad \sum_{k=1}^{k_n} P(X_{nk} < -u) \to M(u),$$

both as $n \to \infty$.

If the limit distribution is of class L then these asymptotic relations are satisfied for every negative (respectively positive) value of u.

Proof. This lemma is but one part of the result of Gnedenko referred to above. A complete proof of his theorem can be found in the monograph (6) of Gnedenko and Kolmogorov. See, in particular, theorem 2 of chapter 4.

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Remark. For the particular case of the normal law as limit law, and which is relevant to the remark (v) following the statement of the theorem, we refer to (6), chapter 5, theorem 2.

3. *Proof of the theorem, beginning with* (2) *implies* (4). The proof is carried out in a number of improving steps. We first associate with $\alpha(x)$ a number A.

Let $\delta(x)$ be a positive valued function, and let $\delta(x) \to \infty$ as $x \to \infty$. Then it follows from the hypothesis (2) of the theorem that for each fixed $\epsilon > 0$

$$\nu_x(n; |f(n) - \alpha(x)| > \epsilon\delta(x)\beta(x)) \to 0 \quad (x \to \infty).$$

Assuming for the moment that

$$\sup_{w \leqslant x} \delta(w)\beta(w) = O(\delta(x)\beta(x)), \quad \limsup_{x \to \infty} \{\delta(x^2)\beta(x^2)/\delta(x)\beta(x)\} < 2,$$

we can apply Lemma 1, and deduce that there is a constant A, and a function $\omega_1(x)$ so that

$$\alpha(x) = A \log x + \omega_1(x),$$

and so that

$$\omega_1(x) = \omega_1(x^{1/2}) + o(\delta(x)\beta(x)).$$

We have already pointed out that $\beta(x) = O((\log x)^{1/4})$, so that if we further require that $\delta(x) = O((\log x)^{1/4})$ then we shall have that

$$|\omega_1(x)| \leqslant |\omega_1(x^{1/2})| + C_1(\log x)^{1/2}$$

holds for all $x \geqslant 2$, say, and a suitably chosen constant C_1. Replacing x by $x^{1/2}$, $x^{1/2^2}$, ... and so on, we prove by induction that

$$|\omega_1(x)| \leqslant C_1 \sum_{2^k \leqslant \log x} (2^{-k} \log x)^{1/2} + o(1) = O((\log x)^{1/2}).$$

Hence as $x \to \infty$, $\alpha(x) = A \log x + o(\log x)$, and no matter what the choice of the function $\delta(x)$, provided that it satisfies the above (weak) requirements on its rate of growth, the value of the constant A is uniquely determined.

Define the additive function $g(n)$ by

$$g(p) = f(p) - A \log p$$

and $g(p^m) = g(p)$ if $m = 2, 3, \ldots$. Thus $g(n)$ is a *strongly-additive* arithmetic function. Define the function $\omega(x)$ by

$$\omega(x) = \alpha(x) - A \log x.$$

As our next step we shall prove

LEMMA 4. *As* $x \to \infty$,

$$\nu_x(n; g(n) - \omega(x) \leqslant z\beta(x)) \Rightarrow F(z).$$

Proof. The characteristic function of the frequency

$$\nu_x(n; f(n) - \alpha(x) \leqslant z\beta(x))$$

is

$$x^{-1} \exp(-it\alpha(x)/\beta(x)) \sum_{n \leqslant x} \exp(itf(n)/\beta(x)).$$

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Let $\phi(t)$ denote the characteristic function of $F(z)$. Define the additive function $h(n) = f(n) - A \log n$. Let ϵ be a temporarily fixed real number, $0 < \epsilon < 1$. Then if $\epsilon x < n \leqslant x$, $\log n = \log x + O(1)$, so that as $x \to \infty$,

$$\sum_{\epsilon x < n \leqslant x} \exp(ith(n)/\beta(x)) = \exp(-it A \log x/\beta(x)) \sum_{\epsilon x < n \leqslant x} \exp(itf(n)/\beta(x)) + o(x)$$

and thus

$$\limsup_{x \to \infty} \left| \exp(-it\omega(x)/\beta(x)) \frac{1}{x} \sum_{n \leqslant x} \exp(ith(n)/\beta(x)) - \phi(t) \right| \leqslant \epsilon.$$

Since ϵ can be chosen arbitrarily small

$$\sum_{n \leqslant x} \exp\left(\frac{it}{\beta(x)} (h(n) - \omega(x)) \right) \to \phi(t),$$

or, what is equivalent to this,

$$\nu_x(n; h(n) - \omega(x) \leqslant z\beta(x)) \Rightarrow F(z) \quad (x \to \infty).$$

We shall now prove that for each fixed $\epsilon > 0$,

$$\nu_x(n; |h(n) - g(n)| > \epsilon\beta(x)) \to 0 \quad (x \to \infty).$$

Lemma 4 will then follow easily.

Let P be a (large) positive integer. Consider those integers n not exceeding x which are divisible by the square of a prime exceeding P. Their frequency does not exceed

$$x^{-1} \sum_{p > P} \left[\frac{x}{p^2} \right] \leqslant \sum_{p > P} \frac{1}{p^2} < P^{-1}.$$

Let q be a prime not exceeding P. The number of integers not exceeding x which are divisible by a power q^m, with $m > P$ is at most

$$\sum_{m > P} \left[\frac{x}{q^m} \right] \leqslant xq^{-P}.$$

The frequency of those integers n which are divisible by any such prime-power q with $q \leqslant P$, is at most

$$\sum_{q \leqslant P} q^{-P} \leqslant 2^{-P}P.$$

Consider now any integer n, not exceeding x, and which has not been counted in either of the two frequencies just estimated. For such an integer n we shall have

$$|h(n) - g(n)| \leqslant \sum_{p^m \| n} |h(p^m) - g(p^m)| \leqslant (\text{constant dependent on } P)$$

the summation being restricted to run over those prime-powers p^m with $p \leqslant P$ and $m \leqslant P$. Since $\beta(x) \to \infty$ as $x \to \infty$, we have proved that

$$\limsup_{x \to \infty} \nu_x(n; |h(n) - g(n)| > \epsilon\beta(x)) \leqslant P^{-1} + 2^{-P}P.$$

Letting $P \to \infty$ we complete the proof of Lemma 4.

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LEMMA 5. *Define the function*

$$s(u) = \limsup_{x\to\infty} \sum_{\substack{p\leqslant x \\ |g(p)|>u\beta(x)}} \frac{1}{p}.$$

Then

(i) *There is a positive number* u_0 *for which* $s(u_0) < \infty$.

(ii) *As* $u \to \infty$, $s(u) \to 0$.

Remark. Since $s(u)$ is a non-increasing function of u for $u \geqslant u_0$, $\lim s(u)$ $(u \to \infty)$ must exist.

Proof. Define the function

$$\sigma(x, u) = \sum_{\substack{p\leqslant x \\ |g(p)|>u\beta(x)}} \frac{1}{p}.$$

Assume (contrary to (i)), that for each $u > 0$,

$$\limsup_{x\to\infty} \sigma(x, u) = \infty.$$

Let c be a positive real number.

Define a sequence of real numbers $x_1 < x_2 < \ldots$, one for each positive integer k, so as to satisfy the following conditions:

$$(a)\ \sigma(x_k, k) \geqslant c, \quad (b)\ x_0 = 1, \quad x_k \geqslant 1 + x_{k-1}, \quad (c)\ x_k > x_{k-1}^2,$$

all of these conditions for $k = 1, 2, \ldots$. Our (temporary) hypothesis concerning $\sigma(x, u)$ ensures that we can fulfil them. In terms of this sequence define the function

$$\delta(x) = k \quad \text{if} \quad x_k \leqslant x < x_{k+1} \quad (k = 1, 2, \ldots).$$

Then if x lies in the interval $[x_k, x_{k+1})$, we must have $x^2 < x_{k+1}^2 < x_{k+2}$, so that

$$\delta(x^2)/\delta(x) \leqslant \left(1 + \frac{1}{k}\right).$$

Clearly, therefore,

$$\limsup_{x\to\infty} \{\delta(x^2)\beta(x^2)/\delta(x)\beta(x)\} < 2 \quad (\text{in fact} = 1).$$

Moreover, $\delta(x)$ is unbounded with x, so that as $x \to \infty$

$$\nu_x(n; f(n) - \alpha(x) \leqslant z\delta(x)\beta(x)) \Rightarrow I(z).$$

But from Lemma 1 we deduce that

$$\sum_{\substack{p\leqslant x \\ |g(p)|>\delta(x)\beta(x)}} \frac{1}{p} \to 0 \quad (x \to \infty).$$

In particular, as $k \to \infty$,

$$\sigma(x_k, k) \to 0$$

and this contradicts the assumption made at the beginning of the proof of the present lemma.

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Hence there is a value of u so that

$$s(u) = \limsup_{x\to\infty} \sigma(x, u) \leqslant c.$$

This proves (i), and since c may be chosen arbitrarily small, (ii) as well.

Lemma 5 is now proved.

Define the distributions

$$F_x = F_x(z) = \nu_x(n; g(n) - \omega(x) \leqslant z\beta(x)),$$

$$P_x = P_x(z) = P(\sum_{p\leqslant x} X_p - \omega(x) \leqslant z\beta(x)).$$

LEMMA 6. *Any sequence of distributions P_x, with $x \to \infty$, is compact.*

Remarks. This lemma asserts that there is a subsequence of x-values for which the P_x converge weakly.

Proof. Let ϵ be a real number, $0 < \epsilon < 1$. Let z and u be further real numbers, $u > 0$.

Define the function

$$g_\epsilon(n) = \sum_{p|n,\, p\leqslant x^\epsilon} g(p)$$

for integers n in the range $1 \leqslant n \leqslant x$.

From Lemma 2 with $h(p) = g(p)$, $w = x^\epsilon$,

$$\begin{aligned} F_x(z) - F_x(-z) &\leqslant \nu_x(n; |g(n) - g_\epsilon(n)| > u\beta(x)) + \nu_x(n; |g_\epsilon(n) - \omega(x)| \leqslant (z+u)\beta(x)) \\ &\leqslant \nu_x(n; |g(n) - g_\epsilon(n)| > u\beta(x)) + P(|\sum_{p\leqslant x} X_p - \omega(x)| \leqslant (z+u)\beta(x)) + O(e^{-c/\epsilon}). \quad (8) \end{aligned}$$

If $|g(n) - g_\epsilon(n)| > u\beta(x)$ then n must be divisible by a prime p in the range $x^\epsilon < p \leqslant x$ for which $|g(p)| > \epsilon u\beta(x)$. The first of the three expressions which appear in this final line is therefore not more than $s(u\epsilon) + o(1)$, as $x \to \infty$, assuming only that $\epsilon u \geqslant u_0$ so that we may apply Lemma 5.

Let λ and τ be positive real numbers, and define new independent random variables Z_p, by

$$Z_p = \begin{cases} X_p & \text{if } |X_p| \leqslant \lambda\beta(x), \\ 0 & \text{if } |X_p| > \lambda\beta(x). \end{cases}$$

Clearly, as $x \to \infty$,

$$P(\exists X_p \neq Z_p, x^\epsilon < p \leqslant x) \leqslant \sum_{\substack{x^\epsilon < p\leqslant x \\ |g(p)| > \lambda\beta(x)}} \frac{1}{p} \leqslant s(\lambda) + o(1),$$

if $\lambda \geqslant u_0$. Moreover,

$$\begin{aligned} P(|\sum_{x^\epsilon < p\leqslant x} Z_p| > \tau\beta(x)) &\leqslant \tau^{-2}\beta(x)^{-2}\operatorname{expect}((\sum_{x^\epsilon < p\leqslant x} Z_p)^2) \\ &= O\left(\tau^{-2}\lambda^2\left(\sum_{x^\epsilon < p\leqslant x} \frac{1}{p}\right)^2\right) = O\left(\left(\frac{\lambda \log 1/\epsilon}{\tau}\right)^2\right). \end{aligned}$$

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Therefore

$$P(|\sum_{p\leqslant x^{\epsilon}} X_p - \omega(x)| \leqslant (z+u)\beta(x)) \leqslant P(|\sum_{p\leqslant x} X_p - \omega(x)| \leqslant (z+u+\tau)\beta(x)) + O\left(\left(\frac{\lambda \log 1/\epsilon}{\tau}\right)^2\right) + s(\lambda) + o(1), \quad (9)$$

as $x \to \infty$.

Let $0 < \epsilon_1 < 1$. If we choose z suitably, then for all large values of x,

$$F_x(z) - F_x(-z) > 1 - \epsilon_1.$$

This is assured by (2). We next choose ϵ so small that the term $O(e^{-c/\epsilon})$ is $< \epsilon_1$. We then choose u so large that $s(u\epsilon) + o(1)$ is $< \epsilon_1$. Next choose λ so large that $s(\lambda) + o(1)$ is $< \epsilon_1$. Finally we choose τ so large that $O((\lambda \log 1/\epsilon)^2 (\tau^{-2}))$ is $< \epsilon_1$. In this way we define a number $z_0 = z + u + \tau$, so that for all sufficiently large values of x,

$$P(|\sum_{p\leqslant x} X_p - \omega(x)| \leqslant z_0\beta(x)) > 1 - 5\epsilon_1.$$

Since ϵ_1 may be chosen arbitrarily small, Lemma 6 is proved.

Remark. For future use we note another result which has been established in the course of the proof of this lemma. Let $\rho(F, G)$ denote the Lévy-distance between the distributions $F(z)$ and $G(z)$. This is defined to be the infimum of all positive numbers h for which the inequalities

$$F(z-h) - h \leqslant G(z) \leqslant F(z+h) + h$$

hold uniformly for all real numbers z.

The above proof establishes that as $x \to \infty$

$$\rho(F_x, P_x) \leqslant c_3\left(e^{-c/\epsilon} + \frac{u}{\epsilon} + \tau + \left(\frac{\lambda \log 1/\epsilon}{\tau}\right)^2 + \sum_{\substack{x^{\epsilon} < p \leqslant x \\ |g(p)| > u\beta(x)}} \frac{1}{p} + \sum_{\substack{x^{\epsilon} < p \leqslant x \\ |g(p)| > \lambda\beta(x)}} \frac{1}{p} + o(1)\right). \quad (10)$$

Here the number c_3 does not depend upon ϵ, u, τ, or λ, and we have replaced u in the above account by u/ϵ.

LEMMA 7. *The function $s(u)$ is well defined for every positive u.*

Proof. Assume to the contrary that for some $u > 0$, we have

$$\limsup_{x\to\infty} \sigma(x, u) = \infty.$$

Let $x_1 < x_2 < \ldots$ be an unbounded sequence of real numbers for which $\sigma(x_\nu, u) \to \infty$. By Lemma 6 there will be a subsequence of this sequence on which the distributions $P_x(z)$ will converge weakly. Without loss of generality we shall denote this subsequence by $x_1 < x_2 < \ldots$ also. Since

$$\lim_{k\to\infty} P(\sum_{p\leqslant x_k} X_p - \omega(x_k) \leqslant z\beta(x_k))$$

exists in the usual probability sense, it follows from Gnedenko's theorem (Lemma 3) that for almost all values of u_1,

$$\lim_{k\to\infty} \sigma(x_k, u_1)$$

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exists and is finite. If we choose one such value u_1 which also satisfies $u_1 < u$, then we can deduce that

$$\limsup_{k\to\infty} \sigma(x_k, u)$$

is finite. But now we have reached a contradiction.

Lemma 7 is therefore proved.

It is now convenient to recall that to each number x we have associated a further number $r = r(x)$. (See the remarks of § 2.)

Let ϵ be a real number, $0 < \epsilon < \frac{1}{2}$. By Lemma 7 there are real numbers x_0, and c_4, so that $\sigma(x, \epsilon^2) < c_4$ for all real numbers $x \geqslant x_0$. Let m_0 be a positive integer. Then, if x is sufficiently large,

$$\sum_{m=0}^{m_0} \sum_{\substack{x^{\epsilon^{m+1}} < p \leqslant x^{\epsilon^m} \\ |g(p)| > \epsilon^3 \beta(x)}} \frac{1}{p} \leqslant \sigma(x, \epsilon^2) \leqslant c_4.$$

Hence if m_0 is chosen suitably in terms of ϵ and c_4, there will be at least one integer m in the range $0 \leqslant m \leqslant m_0$, for which

$$\sum_{\substack{x^{\epsilon^{m+1}} < p \leqslant x^{\epsilon^m} \\ |g(p)| > \epsilon^3 \beta(x)}} \frac{1}{p} \leqslant \frac{c_4}{(m_0 + 1)} < \epsilon.$$

For ease of notation we shall denote such a specially chosen value of m, by m itself.

Define a sequence of real numbers

$$w_1 < w_2 < \ldots < w_k < \ldots$$

so that $w_k \to \infty$ as $k \to \infty$, and the above construction can be carried out with $\epsilon = 2^{-k}$, for all $x \geqslant w_k$. It is convenient also to require that if $x \geqslant w_k$, then $x^{\epsilon^{m+1}} \geqslant r(x)$.

To each number x we attach a further number $y = y(x)$ as follows; we define

$$y(x) = x^{\epsilon^m} \quad \text{with} \quad \epsilon = 2^{-k}, \quad \text{for} \quad w_k \leqslant x < w_{k+1} \quad (k = 1, 2, \ldots).$$

Note that the integer m may well depend upon ϵ, and so k. This will not matter in what follows. We also define a positive function

$$\gamma = \gamma(x) = 2^{-k}, \quad \text{for} \quad w_k \leqslant x < w_{k+1} \quad (k = 1, 2, \ldots).$$

Thus $\gamma(x)$ is non-increasing, and $\gamma(x) \to 0$ as $x \to \infty$.

Our definition of these last two functions $y(x)$ and $\gamma(x)$, allows us to assert that for all sufficiently large values of x,

$$\sum_{\substack{y^\gamma < p \leqslant y \\ |g(p)| > \gamma^3 \beta(x)}} \frac{1}{p} \leqslant \gamma,$$

where it is to be understood that y and γ are functions of x.

We now appeal to our hypothesis concerning the growth rate of $\beta(x)$. This allows us to assert that if x is sufficiently large,

$$\beta(x) \leqslant 2\beta(w)$$

uniformly for $r \leqslant w \leqslant x$, and in particular, $\beta(x) \leqslant 2\beta(y)$ (because $r \leqslant y \leqslant x$).

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Therefore for all large values of x,

$$\sum_{\substack{y^\gamma < p \leqslant y \\ |g(p)| > 2\gamma^2\beta(y)}} \frac{1}{p} \leqslant \gamma.$$

We now make use of the remark (10). We replace $(x, \epsilon, u, \tau, \lambda)$ by $(y, \gamma, \frac{1}{2}\gamma^2, \gamma, \frac{1}{2}\gamma^2)$ respectively, and deduce that

$$\rho(F_y, P_y) \leqslant c_3\left(o(1) + \tfrac{1}{2}\gamma + \left(\frac{\gamma^2 \log 1/\gamma}{\gamma}\right)^2 + 2\gamma + o(1)\right) = o(1), \quad (x \to \infty).$$

Since $F_y \Rightarrow F(z)$ as x (and so y) $\to \infty$, we can assert that $P_y \Rightarrow F(z)$ as $x \to \infty$.

From Lemma 3 we deduce that for a certain function $N(u)$, and almost all positive numbers u,

$$\sum_{\substack{p \leqslant y \\ g(p) > u\beta(y)}} \frac{1}{p} \to -N(u) \quad (y \to \infty).$$

Consider now any sufficiently large number x. Then if ϵ is a positive real number, $0 < \epsilon < u$, and $u + \epsilon$ is a continuity point of $N(\ )$,

$$\sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} \geqslant \sum_{\substack{p \leqslant y \\ g(p) > (u+\epsilon)\beta(y)}} \frac{1}{p} = -N(u+\epsilon) + o(1) \quad (x \to \infty).$$

Likewise, for the same (large) value of x, we can find a number w, so that if $y = y(w)$, $r = r(w)$, then $r \leqslant x \leqslant y \leqslant w$. Hence if $u - \epsilon$ is a continuity point of $N(\ )$,

$$\sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} \leqslant \sum_{\substack{p \leqslant y \\ g(p) > (u-\epsilon)\beta(y)}} \frac{1}{p} = -N(u-\epsilon) + o(1) \quad (x \to \infty).$$

We conclude that

$$\limsup_{x \to \infty} \sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} - \liminf_{n \to \infty} \sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} \leqslant -(N(u-\epsilon) - N(u+\epsilon)).$$

Therefore if u is a point of continuity of $N(\ )$, and $\epsilon \to 0+$ suitably, we arrive at

$$\lim_{x \to \infty} \sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} = -N(u).$$

A similar result can be obtained involving the function $M(u)$ and the values of

$$g(p) < u\beta(x)$$

for negative values of u.

Let u be a point of continuity of $N(\ )$, and let ϵ and v be positive real numbers, $0 < \epsilon < 1$, $0 < v < u$, so that $u + v$ is also a continuity point of $N(\ )$. Then as $x \to \infty$,

$$\begin{aligned} \sum_{\substack{x^\epsilon < p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} &\leqslant \sum_{\substack{p \leqslant x \\ g(p) > u\beta(x)}} \frac{1}{p} - \sum_{\substack{p \leqslant x^\epsilon \\ g(p) > u\beta(x)}} \frac{1}{p} \\ &\leqslant -N(u) + o(1) - \sum_{\substack{p \leqslant x^\epsilon \\ g(p) > (u+v)\beta(x^\epsilon)}} \frac{1}{p} = N(u+v) - N(u) + o(1) \quad (x \to \infty), \end{aligned}$$

this second step being valid since $\beta(x^\epsilon)/\beta(x) \to 1$ as $x \to \infty$.

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By means of this result, and a similar one involving the values of $g(p) < u\beta(x)$ ($u < 0$), we deduce that

$$\limsup_{x\to\infty} \sum_{\substack{x^\epsilon < p \leqslant x \\ |g(p)| > u\beta(x)}} \frac{1}{p} \leqslant \{N(u+v) - N(u)\} + \{M(u) - M(u-v)\}.$$

We let $v \to 0$ and deduce that for almost all $u > 0$, and therefore for all $u > 0$,

$$\sum_{\substack{x^\epsilon < p \leqslant x \\ |g(p)| > u\beta(x)}} \frac{1}{p} \to 0 \quad (x \to \infty).$$

We can now return to our remark (10) once more, and set this time $u = \epsilon^2$, $\lambda = \epsilon$, $\tau = \epsilon^{\frac{1}{2}}$, and deduce that

$$\limsup_{x\to\infty} \rho(F_x, P_x) \leqslant c_3(e^{-c/\epsilon} + 2\epsilon^{\frac{1}{2}} + \epsilon(\log 1/\epsilon)^2).$$

Since ϵ may be chosen arbitrarily small, $\rho(F_x, P_x) \to 0$, as $x \to \infty$, or what is equivalent,

$$P(\sum_{p \leqslant x} X_p - \omega(x) \leqslant z\beta(x)) \Rightarrow F(z) \quad (x \to \infty).$$

This is what we wished to prove.

We have now proved that the hypothesis (2) implies the hypothesis (4). In the other direction the above result that for each pair ϵ, u

$$\sum_{\substack{x^\epsilon < p \leqslant x \\ |g(p)| > u\beta(x)}} \frac{1}{p} \to 0 \quad (x \to \infty),$$

will still be available to us, and so will be the inequality (10). We can therefore deduce at once that $\rho(F_x, P_x) \to 0$ as $x \to \infty$, and we are done.

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