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Two-dimensional quasi-stationary flows in gas dynamics

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1. *Introduction.* In this paper we are concerned with the two-dimensional, unsteady flow of an inviscid, polytropic gas whose adiabatic index γ lies between 1 and 3. We recall that comparatively early in the study of gas dynamics we encounter two exact solutions of gas dynamic problems. One, in one-dimensional unsteady flow, is the expansion of a semi-infinite column of gas which is initially at rest behind a piston which, at time $t = 0$, begins to move with constant speed away from the gas. The second, in two-dimensional, steady, supersonic flow, is the Prandtl–Meyer flow round a sharp convex corner. Both of those flows may be regarded as special cases of more general exact solutions which are obtained by the method of characteristics (see, for example, Courant and Friedrichs(1)). On the other hand, each may be obtained directly from the appropriate equations by making use of the fact that, in so far as neither problem contains any characteristic length parameter in its formulation, the principle of dynamic similarity can be used to reduce the system of partial differential equations to one of ordinary differential equations. In the first case the independent variables x and t occur only in the combination x/t and in the second the independent variables x and y occur only in the combination x/y. Interesting and instructive as the derivation of these solutions from such principles may be, it is an unfortunate fact that they are the only non-trivial solutions of the respective equations. This is not altogether surprising as the equations are ordinary with (in this case) a limited number of non-trivially distinct solutions.

The object of the present paper is to examine the *partial* differential equations in two variables which result from seeking similarity solutions in two-dimensional, unsteady, homentropic flow. More specifically, if $u(x, y, t)$, $v(x, y, t)$ and $c(x, y, t)$ are respectively the x and y components of velocity and the local sound speed, we seek the differential equations satisfied by U, V and F as functions of the two variables X and Y, where

$$u(x, y, t) = c_0 U(X, Y), \quad v(x, y, t) = c_0 V(X, Y), \quad c(x, y, t) = c_0 F(X, Y) \tag{1·1}$$

and

$$X = \frac{x}{c_0 t}, \quad Y = \frac{y}{c_0 t}. \tag{1·2}$$

Here c_0 is a constant with the dimensions of a velocity and it is introduced at this stage in order to make all the new variables U, V, F, X, Y dimensionless. As the resulting differential equations are partial, it is to be expected that they will yield a variety of

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physically meaningful solutions. This point of view is complemented by the fact that it is easy to formulate a collection of problems in two-dimensional unsteady gas dynamics without introducing a characteristic length. The expansion into a vacuum of a wedge-shaped column of gas at rest for $t < 0$ is one such example. An intriguing possibility is to consider as another example uniform steady flow in the x-direction at Mach number M in the half-plane $y > 0$. Let us suppose that at $t = 0$ the boundary $y = 0$, $x > 0$ is instantaneously removed, exposing a vacuum. It ought to be possible to obtain the classical Prandtl–Meyer solution as $t \to \infty$ provided $M \geqslant 1$. On the other hand, it should also be possible to demonstrate that no steady state solution exists in the limit as $t \to \infty$ if $M < 1$.

However, it is not our intention at this stage to examine explicit solutions in detail, although work is proceeding in this field and it is hoped that this will form the subject of further papers. Rather, we wish to examine some general properties of the equations which result from the assumption of dynamic similarity. In particular we show how our approach is related to the work of Pogodin, Suchkov and Ianenko (4) and some subsequent Russian work.

2. *The basic equations.* By assuming a polytropic gas with $\gamma \neq 1$ we are able to use c, the local speed of sound, as a dependent variable which is a simple power of the density ρ and also a (different) power of the pressure p and thus can be used to represent either. The equations of continuity and momentum are

$$c_t + uc_x + vc_y + \frac{2}{\gamma - 1}(cu_x + cv_y) = 0,$$

$$u_t + uu_x + vu_y + \frac{\gamma - 1}{2} cc_x = 0,$$

$$v_t + uv_x + vv_y + \frac{2}{\gamma - 1} cc_y = 0,$$

where the subscripts denote partial derivatives. In the new variables defined by (1·1) and (1·2) these become

$$(U - X)F_X + (V - Y)F_Y + \frac{\gamma - 1}{2}(FU_X + FV_Y) = 0, \tag{2·1}$$

$$(U - X)U_X + (V - Y)U_Y + \frac{2}{\gamma - 1} FF_X = 0, \tag{2·2}$$

$$(U - X)V_X + (V - Y)V_Y + \frac{2}{\gamma - 1} FF_Y = 0. \tag{2·3}$$

Equations (2·1), (2·2) and (2·3) are the three non-linear equations for U, V and F as functions of X and Y which govern what we shall call two-dimensional quasi-stationary flows. If, in addition, the flow is irrotational, then a velocity potential ϕ exists with $u = \phi_x$, $v = \phi_y$. In view of the dynamic similarity there will exist a dimensionless potential Φ, defined by

$$\phi = c_0^2 t \Phi(X, Y).$$

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The unsteady form of Bernoulli's theorem, namely

$$\phi_t + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + \int \frac{dp}{\rho} = \text{function of } t,$$

becomes

$$\Phi - X\Phi_X - Y\Phi_Y + \tfrac{1}{2}\Phi_X^2 + \tfrac{1}{2}\Phi_Y^2 + \frac{F^2}{\gamma - 1} = \frac{F_0^2}{\gamma - 1}, \tag{2·4}$$

where F_0 is a constant. Further, the relations

$$U = \Phi_X, \quad V = \Phi_Y \tag{2·5}$$

are immediate.

If the values of F_X, F_Y as given by (2·2) and (2·3) are substituted in (2·1), then the equation

$$\{(U-X)^2 - F^2\}U_X + (U-X)(V-Y)(U_Y + V_X) + \{(V-Y)^2 - F^2\}V_Y = 0$$

results. If, in addition, the flow is irrotational, then the equation for Φ, using (2·5), is

$$\{(\Phi_X - X)^2 - F^2\}\Phi_{XX} + 2(\Phi_X - X)(\Phi_Y - Y)\Phi_{XY} + \{(\Phi_Y - Y)^2 - F^2\}\Phi_{YY} = 0, \tag{2·6}$$

where F^2 is given by (2·4). This equation has a certain resemblance to the equation for the potential in two-dimensional steady flow. This resemblance can be strengthened by introducing a new variable Ψ defined by

$$\Psi = \Phi - \tfrac{1}{2}X^2 - \tfrac{1}{2}Y^2. \tag{2·7}$$

Then

$$(\Psi_X^2 - F^2)\Psi_{XX} + 2\Psi_X\Psi_Y\Psi_{XY} + (\Psi_Y^2 - F^2)\Psi_{YY} = \Psi_X^2 + \Psi_Y^2 - 2F^2, \tag{2·8}$$

where

$$\Psi + \tfrac{1}{2}\Psi_X^2 + \tfrac{1}{2}\Psi_Y^2 + \frac{F^2}{\gamma - 1} = \frac{F_0^2}{\gamma - 1}. \tag{2·9}$$

Compared with (2·4) and (2·6), (2·8) and (2·9) eliminate the explicit occurrence of the independent variables in the coefficients, although at the cost of introducing a non-homogeneity or 'right-hand side' into (2·8).

It is not without interest to digress briefly at this stage on the question of whether the flow is irrotational or not. In practice the answer will always be in the affirmative for the type of problems we wish to consider as these will presume a state of rest or of uniform motion for $t < 0$. Kelvin's theorem on the constancy of circulation will then ensure that the velocity vector is curl free for $t \geqslant 0$—at least for any flows which do not involve shock waves. (Let us further note at this stage that if shock waves do exist for $t \geqslant 0$ in initial value problems having no length parameter in their characterization they must originate at $t = 0$ precisely. Otherwise the shock would initiate at some specific time $t = T$ and the quantity $c_0 T$ would be a characteristic length which cannot be related to any length occurring in the formulation of the problem. We are, of course, neglecting diffusive effects since the presence of a viscosity coefficient would automatically introduce a length parameter.) If there is to be any vorticity at all, then, if $\omega = v_x - u_y$, the distribution of vorticity at $t = 0$ must be given by

$$\omega = \frac{c_0 g(\theta)}{r}, \tag{2·10}$$

where r, θ are polar coordinates, since this is the only combination of variables and

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data which has the correct dimensions. However, as stated earlier, we shall be concerned only with the case $\omega = 0$ which guarantees the existence of Φ.

Equation (2·6) is non-linear. It is hyperbolic if

$$(\Phi_X - X)^2 + (\Phi_Y - Y)^2 > F^2,$$

in which case the characteristics are given by solving

$$\frac{dY}{dX} = \frac{(\Phi_X - X)(\Phi_Y - Y) \pm F\sqrt{[(\Phi_X - X)^2 + (\Phi_Y - Y)^2 - F^2]}}{(\Phi_X - X)^2 - F^2}. \tag{2·11}$$

Because of the non-linearity, the characteristics depend on the particular solution considered. Let us consider the special solution for which uniform flow of gas at speed $u = U_0 c_0$ contained in $y \geqslant 0$ is allowed at $t = 0$ to expand into a vacuum in $y < 0$. This solution is merely that of a point-centred simple wave with a uniform velocity in the transverse direction superimposed and it is easily verified that

$$U = U_0, \quad V = \frac{2}{\gamma+1}(Y-1), \quad F = \frac{1}{\gamma+1}\{2+(\gamma-1)Y\}, \quad \Phi = \frac{1}{\gamma+1}(Y-1)^2 + U_0 X \tag{2·12}$$

in the region in which the solution is valid, namely

$$-\infty < X < \infty, \quad -\frac{2}{\gamma-1} \leqslant Y \leqslant 1.$$

In $Y \geqslant 1$, $\Phi = U_0 X$ so that Φ and its first derivatives are continuous. Equation (2·4) shows that in this case

$$F_0^2 = 1 + \frac{\gamma-1}{2} U_0^2. \tag{2·13}$$

Since c_0 is clearly the speed of sound in the uniform stream in $y > 0$, $t < 0$, the significance of (2·13) lies in the fact that it identifies F_0 as the ratio between c_s and c_0 where c_s is the speed of sound in a reservoir of gas at rest before it is accelerated to the steady state in which $u = c_0 U_0$, $c = c_0$. Clearly $F_0 \geqslant 1$ and an important special case occurs when $U_0 = 0$, $F_0 = 1$, corresponding to gas at rest in $y > 0$ which is suddenly allowed to expand into $y < 0$. For simplicity we consider this case in what follows as the more general case $U_0 \neq 0$ may easily be obtained from it.

If the appropriate values from (2·12) are substituted into (2·11), we obtain

$$\frac{dY}{dX} = \frac{FX \pm FX}{X^2 - F^2}$$

for the equations of the characteristics. One family is given by the straight lines $Y =$ constant and the other, after some manipulation, by

$$(\gamma+1)(3-\gamma)X^2 - \{(\gamma-1)Y+2\}^2 = k\{(\gamma-1)Y+2\}^{(\gamma+1)/(\gamma-1)}, \tag{2·14}$$

where k is the parameter of the family. Each characteristic in this family passes through the point $X = 0$, $Y = -2/(\gamma-1)$. The family is sketched in Fig. 1.

Of especial interest is the characteristic which passes through $X = 0$, $Y = 1$. This is the curve

$$(\gamma+1)(3-\gamma)X^2 - \{(\gamma-1)Y+2\}^2 = -(\gamma+1)^{-(3-\gamma)/(\gamma-1)}\{(\gamma-1)Y+2\}^{(\gamma+1)/(\gamma-1)} \tag{2·15}$$

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and its physical significance is that it denotes the subsequent region of influence of a point disturbance placed at the origin at $t = 0$. For the type of problems which we are interested in, it denotes the boundary (or a part of the boundary) between a simple wave solution and a more complicated quasi-stationary flow as given by the appropriate solution of equations (2·1), (2·2) and (2·3).

3. *Connexion with work of Pogodin, Suchkov and Ianenko.* A somewhat different approach to these problems is made by Pogodin, Suchkov and Ianenko (4), referred to subsequently as PSI. Their method is essentially to seek so-called 'travelling wave'

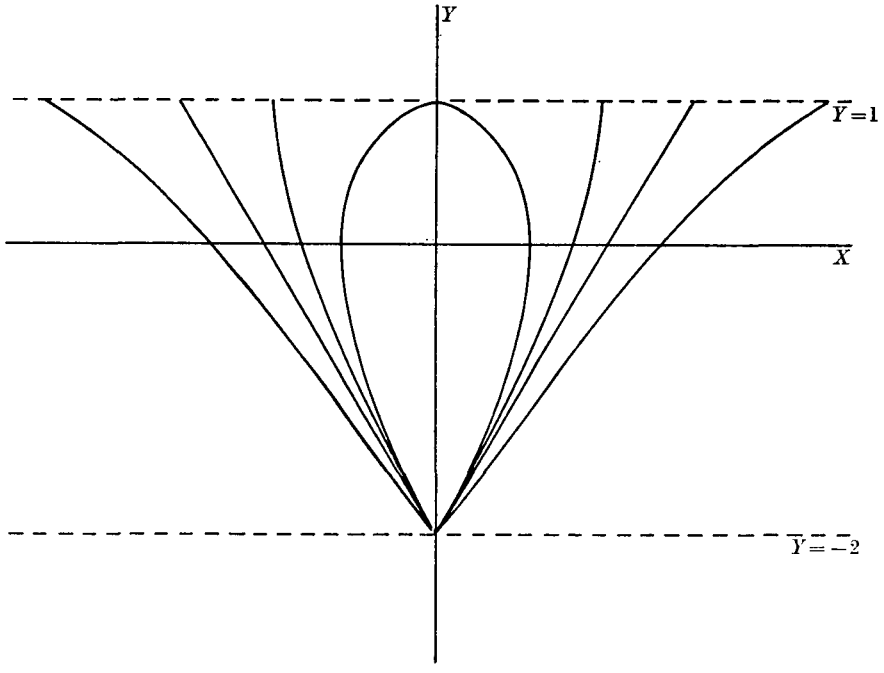


Fig. 1. Characteristics drawn for $\gamma = 2$.

solutions of the gas dynamic equations. Their general approach is much broader than that used here, but this very generality prevents progress to particular solutions except those considered here. (A somewhat similar analysis has recently been made by Cumberbatch and Varley (2).) Consequently we shall briefly survey and recast the PSI approach adapted to the particular problems of two-dimensional unsteady flow possessing dynamic similarity as described above in section 1.

Basically (and in this context) the PSI approach starts from the simple observation that if U, V and F are functions of two variables X and Y as in (1·1), then there will exist a functional relationship between F, U and V arising from the elimination of X and Y—specifically F may be thought of as a function of U and V. The more general PSI point of view is to regard any solution in which $c(x, y, t)$ is a function of $u(x, y, t)$ and $v(x, y, t)$ as a special case embedded in the complete family of solutions. However, when the three independent variable x, y and t are reduced to two, X and Y, then the functional relationship becomes obligatory. This is the case in which we are interested.

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In order to derive the equation satisfied by F as a function of U and V, we first obtain expressions for F_U and F_V through the relationship

$$(U_X V_Y - V_X U_Y)dF = (F_X V_Y - V_X F_Y)dU + (U_X F_Y - F_X U_Y)dV.$$

Substituting for F_X, F_Y from (2·2), (2·3) and using the irrotationality condition $U_Y = V_X$, we obtain the important equations

$$X - U = \frac{2}{\gamma - 1} F F_U, \tag{3·1}$$

$$Y - V = \frac{2}{\gamma - 1} F F_V. \tag{3·2}$$

Equation (2·1) now gives

$$(F_U U_X + F_V V_X) F_U + (F_U U_Y + F_V V_Y) F_V = \kappa^2 (U_X + V_Y), \tag{3·3}$$

where, following Suchkov (5) and for later convenience, we set $\kappa = \frac{1}{2}(\gamma - 1)$. Equation (3·3) is homogeneous in U_X, U_Y, V_X, V_Y and these quantities may therefore be replaced by $Y_V, -X_V, -Y_U, X_U$ respectively provided a non-degenerate solution in which the Jacobian $\partial(U, V)/\partial(X, Y)$ is neither zero nor infinite is considered. The latter four quantities are easily obtained from (3·1) and (3·2) and after some manipulation we obtain

$$(\kappa^2 - F_V^2) F_{UU} + 2F_U F_V F_{UV} + (\kappa^2 - F_U^2) F_{VV} = \frac{\kappa}{F}\{(F_U^2 + F_V^2)(1 - \kappa) - 2\kappa^2\} \tag{3·4}$$

the equation used (though without explanation) by Suchkov (5).

It should be emphasized that the requirement that the solution be non-degenerate is a very real one and that equation (3·4) is not satisfied by the simple wave type solution (2·12), despite the fact that in this solution F is a function of V, namely

$$F = 1 + \kappa V. \tag{3·5}$$

On the other hand, it is extremely instructive to consider what information (3·4) supplies if a quasi-stationary flow of the type governed by (3·4) is bounded by a simple wave as given by (2·12). The line in the U, V-plane which bounds the quasi-stationary flow is a segment of the line $U = U_0$. On this line $F_V = \kappa$ from (3·5) and consequently the coefficient of F_{UU} vanishes in (3·4). Equation (3·4) may therefore be thought of as an ordinary differential equation for F_U as a function of V on $U = U_0$, $-1/\kappa \leqslant V \leqslant 0$. The variables separate and on integration

$$F_U = \kappa \left[\frac{1 + \kappa}{1 - \kappa} + \left\{ C^2 - \frac{1 + \kappa}{1 - \kappa} \right\} \{1 + \kappa V\}^{(1-\kappa)/\kappa} \right]^{\frac{1}{2}}, \tag{3·6}$$

where κC is the value of F_U at $U = U_0$, $V = 0$. When a wedge-shaped column of gas originally at rest and making an angle $\frac{1}{2}\pi - \alpha$ with a rigid wall is suddenly allowed to flow out into a vacuum, it was shown by Suchkov (5) that $C = \tan\alpha$ and he went on to determine a remarkably simple exact solution for the case where

$$\tan^2\alpha = \frac{1 + \kappa}{1 - \kappa} = \frac{\gamma + 1}{3 - \gamma},$$

in which F is a linear function of U and V.

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Returning to (3·6) and setting $U_0 = 0$ for comparison with earlier work, we may use (3·1) and (3·2) to obtain a parametric form for the curve in the X, Y-plane separating the quasi-stationary flow from the simple wave. This form is

$$X = (1+\kappa V)\left[\frac{1+\kappa}{1-\kappa} + \left\{C^2 - \frac{1+\kappa}{1-\kappa}\right\}\{1+\kappa V\}^{(1-\kappa)/\kappa}\right]^{\frac{1}{2}}, \tag{3·7}$$

$$Y = 1+(\kappa+1)V. \tag{3·8}$$

In the case where we examine a point disturbance located at the origin at $t = 0$ for the solution (2·12) with $U_0 = 0$ it is clear that $X = 0$, $Y = 1$ when $U = V = 0$. Hence $C = 0$ and elimination of V between (3·7) and (3·8) with $C = 0$ gives (2·15) as before. More generally, we see that the parametric equations (3·7) and (3·8) are identical with equation (2·14). The striking simplicity of Suchkov's result can now be interpreted from Fig. 1 as the case of (2·14) for which $k = 0$ and the characteristics of (2·6) reduce to straight lines. Two different approaches to the same problem are therefore integrated.

4. *Isobar property.* In two-dimensional, steady, irrotational flow it is a simple and well-known result that the lines of constant pressure meet any straight streamline at right angles. In the case of quasi-stationary flows there is an analogous result which, though not as strong as that for the steady case, is nevertheless of considerable interest in the qualitative discussion of such flows. The result is contained in the following theorem.

THEOREM. *If, in an irrotational, quasi-stationary flow, there is a fixed radial streamline, then at any time $t > 0$ the lines of constant pressure (the isobars) meet this line at right angles. In particular, if the gas is expanding into a vacuum and is bounded laterally by a straight line through the origin, the front of the gas meets the line at right angles.*

By a quasi-stationary flow in the enunciation of the above is meant a flow in which F is a function of U and V satisfying (3·4) with associated transfer to the X, Y-plane by means of (3·1) and (3·2). The proof is not hard. If the line is $y = x\tan\alpha$ in the original x, y-plane, then clearly $V = U\tan\alpha$ on $Y = X\tan\alpha$ which, by (3·1) and (3·2), means that

$$F_V = F_U \tan\alpha \tag{4·1}$$

at the line. Now the slope of an $F =$ constant line is

$$-\frac{F_X}{F_Y} = -\frac{F_U U_X + F_V V_X}{F_U U_Y + F_V V_Y} = -\frac{U_X\cos\alpha + V_X\sin\alpha}{U_Y\cos\alpha + V_Y\sin\alpha}$$

at the line by (4·1). If we now use the fact that $U_Y = V_X$ by irrotationality, we see that this expression becomes $-(dU/dS)/(dV/dS)$, where S is arc length along the line, and this expression is $-\cot\alpha$. Accordingly, the line of constant F intersects the line $y = x\tan\alpha$ at right angles.

This result has some interest when compared with a particular problem in incompressible flow of similar quasi-stationary type. Garabedian (3) has shown that, when a two-dimensional wedge is plunged symmetrically and with constant velocity into an incompressible medium at rest, the angle which the free surface makes with the wedge side must lie between 0 and $\frac{1}{4}\pi$. For a compressible fluid the above theorem would seem

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to indicate that such an angle must be $\frac{1}{2}\pi$. This, however, would be a wrong interpretation since the compression engendered by such a motion would cause a shock wave and thereby automatically destroy the assumption of irrotationality on which the above analysis is based. Nevertheless, in any (not necessarily symmetric) similar problem in which the occurrence of shock waves is avoided it is clear that the contact angle is $\frac{1}{2}\pi$. The assumption of compressibility is vital and in no circumstances can such qualitative results be said to include those for an incompressible fluid.

5. *Unsteady Prandtl–Meyer expansions.* An important class of quasi-stationary problems is that in which there is flow round a sharp convex corner located at the origin. The classical steady case of the supersonic Prandtl–Meyer flow has already been mentioned. There is no corresponding solution when the flow upstream is subsonic as such a problem is not well-posed. Expanding this point further so as not to appear to take refuge in a mathematical technicality without any physical motivation, we may observe that the reason why such a problem is not well-posed is that any steady-state problem is the limit as $t \to \infty$ of some unsteady problem and, in the case of subsonic flow round a sharp convex bend, no such limit exists. It is not possible to use such arguments referring to lack of well-posed data for initial value problems and, as a typical example, we may think of the problem of gas at rest contained in $y > 0$ allowed to expand at $t = 0$ into a vacuum in $y < 0$ but with the semi-infinite line $y = 0$, $-\infty < x \leqslant 0$ retained as a rigid barrier. Not only is this physically a well-posed problem but, in the absence of any length parameter, it must mathematically possess a solution in the field of quasi-stationary flows. The nature of the flow near the origin is of extreme importance and we now investigate this. We shall use a number of different approaches which lead essentially to the same result.

In the first place we put (2·6) into polar co-ordinates R, θ where $X = R\cos\theta$, $Y = R\sin\theta$ and seek a solution of the form

$$\Phi = Rf(\theta) + o(R).$$

It is actually somewhat easier (and formally closer to the steady Prandtl–Meyer expansion) to use (2·8) along with (2·7) and seek a function

$$\Psi = Rf(\theta) + o(R). \tag{5·1}$$

The equation for Ψ in polar coordinates is

$$\Psi_R^2\Psi_{RR} + \frac{2\Psi_R\Psi_\theta}{R^2}\Psi_{R\theta} + \frac{\Psi_\theta^2}{R^4}\Psi_{\theta\theta} - \frac{\Psi_R\Psi_\theta^2}{R^3} = F^2\left(\Psi_{RR} + \frac{\Psi_R}{R} + \frac{\Psi_{\theta\theta}}{R^2} - 2\right) + \Psi_R^2 + \frac{\Psi_\theta^2}{R^2}.$$

Substituting (5·1) in this and retaining only the most significant terms, we obtain

$$\frac{2ff'^2}{R} + \frac{f'^2f''}{R} - \frac{ff'^2}{R} = F^2\left(\frac{f}{R} + \frac{f''}{R}\right),$$

or

$$(f'^2 - F^2)(f'' + f) = 0.$$

We reject $f'' + f = 0$ since this merely gives $\Phi = lX + mY$ (uniform flow) and this is the solution when there is no corner at the origin. Thus $f'^2 = F^2$ and, in addition, since all the gas in the as yet undisturbed region is at rest with $c = c_0$ so that we may take

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$\Phi = 0$ there, the necessary continuity conditions to be satisfied by Φ mean that $F_0 = 1$ in (2·4) or (2·9). Hence from (2·9) and (5·1) we must have at the origin

$$\tfrac{1}{2}Q^2 + \frac{F^2}{\gamma-1} = \frac{1}{\gamma-1}, \tag{5·2}$$

where $$Q^2 = U^2 + V^2 = f^2 + f'^2.$$

From these equations $$\frac{\gamma-1}{2}f^2 + \frac{\gamma+1}{2}f'^2 = 1$$

and the appropriate solution of this is

$$f = \sqrt{\left(\frac{2}{\gamma-1}\right)}\sin\left\{\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)}(\beta-\theta)\right\}, \quad f' = -\sqrt{\left(\frac{2}{\gamma+1}\right)}\cos\left\{\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)}(\beta-\theta)\right\},$$

where β is a constant. Since $F^2 = f'^2 \leqslant Q^2$ this flow cannot exist unless the velocity of the gas is sonic or supersonic and it seems reasonable to suppose that, since the Mach line leaves the barrier $y = 0, x \leqslant 0$ normally when the flow is sonic, the above solution can be said to start from this line and this fixes $\beta = \frac{1}{2}\pi$. The flow is subsonic in $\frac{1}{2}\pi < \theta < \pi$. The maximum value of f then occurs when

$$\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)}\left(\frac{\pi}{2}-\theta\right) = \frac{\pi}{2}, \quad \text{or} \quad -\theta = \frac{\pi}{2}\left(\sqrt{\left(\frac{\gamma+1}{\gamma-1}\right)} - 1\right) = \theta^* \quad \text{say.}$$

The picture which emerges is therefore mathematically identical with the classical Prandtl–Meyer flow, although the interpretation is different. The maximum angle of deflexion beyond the horizontal is θ^* as in the steady case and the speed along the boundary $c = 0$ is $\sqrt{[2/(\gamma-1)]}c_0$. This must be interpreted also as the actual speed with which the leading gas particle leaves the corner and proceeds along the straight line $\theta = \theta^*$. This is slower than the escape speed of $[2/(\gamma-1)]c_0$ with which particles on $x > 0$, $y = 0$ at $t = 0$ begin to move into the vacuum before being influenced by the corner.

An alternative way of looking at this is from (3·1) and (3·2). If at the point $X = 0$, $Y = 0$ there is a relationship between F, U and V, then

$$FF_U = -\frac{\gamma-1}{2}U, \quad FF_V = -\frac{\gamma-1}{2}V$$

or $$\tfrac{1}{2}(U^2+V^2) + \frac{F^2}{\gamma-1} = \text{constant}.$$

The earlier argument can once again be invoked to identify the constant in terms of the initial conditions as in (5·2).

Finally we can use the following argument. Since

$$\phi = c_0^2 t\Phi\left(\frac{r}{c_0 t}, \theta\right),$$

where $r^2 = x^2 + y^2$, we require, for the velocity to be finite at $r = 0$, that

$$\phi = c_0 r f(\theta) + o\left(\frac{r}{c_0 t}\right)c_0^2 t g(\theta).$$

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As $t \to \infty$ for fixed r terms other than the first tend to zero. But the first term must be the classical Prandtl–Meyer flow since this is the only steady solution of this form (other than the trivial uniform flow) of the full two-dimensional equations. Thus we have the result that as time increases the flow near the corner must become more and more like a classical Prandtl–Meyer flow.

Although the particular case considered above was that of expansion from a state of rest round a semi-infinite line, it is clear that in all problems involving flow round a sharp convex corner the basic arguments used above will apply and that as $t \to \infty$ the conventional pattern associated with the Prandtl–Meyer flow will develop near the corner.

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