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Maximal entropy of permutations of even order

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Abstract. A finite invariant set of a continuous map of an interval induces a permutation called its type. If this permutation is a cycle, it is called its orbit type. It has been shown by Geller and Tolosa that Misiurewicz–Nitecki orbit types of period n congruent to 1 (mod 4) and their generalizations to orbit types of period n congruent to 3 (mod 4) have maximal entropy among all orbit types of odd period n, and indeed among all permutations of period n. We further generalize this family to permutations of even period n and show that they again attain maximal entropy amongst n-permutations.

1. *Introduction*

In this paper we find permutations with maximal entropy for continuous maps of the interval. By constructing a family of orbit types of period congruent to 1 (mod 4) Misiurewicz and Nitecki [**5**] showed that the maximal entropy for P_n, the set of permutations of order n, and indeed for C_n, the set of cycles of order n, is asymptotic to $2n/\pi$. Later, Geller and Tolosa [**4**] showed that this family and its natural extension to the remaining odd orbit types actually attains maximal entropy in C_n (and hence in P_n).

If we generalise the definition of this family to permutations of even order, it yields permutations which are products of cycles rather than single cycles. We prove that this family again attains maximal entropy amongst *permutations*.

2. *Preliminaries*

This section introduces some standard notation and well-known results.

Let f be a continuous map of a compact interval I into itself. The *orbit* of a point $x \in I$ is the sequence $\mathrm{Orb}(x) = \{x, f(x), f^2(x), \ldots\}$. If $x = f^m(x)$ for some $m \in \mathbb{N}$, then the orbit is periodic.

A *permutation* on n letters is a bijective map $\theta : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

If θ has the property that $\theta^k(1) = 1$ if and only if $k = n$, then θ is a *cycle*.

We define P_n to be the set of all permutations on n letters and C_n to be the set of all cycles on n letters. We also let $P = \cup_{n\geq 1} P_n$ and $C = \cup_{n\geq 1} C_n$. The *dual* of a permutation $\theta \in P_n$ is the permutation $\overline{\theta} \in P_n$ where $\overline{\theta}(i) = n + 1 - \theta(n + 1 - i)$ for $i = 1, \ldots, n$.

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If f has a periodic orbit of prime period n we can write $\mathrm{Orb}(x) = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$. This induces a cycle $\theta \in C_n$ in the following way: $\theta(i) = j$ if and only if $f(p_i) = p_j$. The cycle θ is called the *orbit type* of $\mathrm{Orb}(x)$. More generally, if $S = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$ is any finite invariant set, we can define the *type* of S to be the permutation $\theta \in P_n$ with $\theta(i) = j$ if and only if $f(p_i) = p_j$.

Definition 1. The *topological entropy* of a map f on a compact interval I is

$$h(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log M_n(\mathcal{U}),$$

where, for any open cover $\mathcal{U}$ of I, $M_n(\mathcal{U})$ is the minimum cardinality of a subcover of

$$\mathcal{U}_n = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-(n-1)}(\mathcal{U})$$

and $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

Definition 2. The *entropy of a permutation* $\theta \in P$ is

$$h(\theta) = \inf\{h(f) : f \text{ is a map with an invariant set of type } \theta\}.$$

We note that for each $\theta \in P$, $h(\overline{\theta}) = h(\theta)$.

If S is an invariant set of type $\theta \in P_n$, then there is a unique map $f_\theta : [1, n] \to [1, n]$ satisfying:

(i) $f_\theta(i) = \theta(i)$ for $i \in \{1, \ldots, n\}$;

(ii) $f_\theta|[i, i+1]$ is linear for $i \in \{1, \ldots, n-1\}$.

The map f_θ is called the *linearisation* of f.

If, for each $i \in \{1, \ldots, n\}$, $f_\theta(i)$ is a local extremum of f_θ then f_θ is said to be *maximodal*.

PROPOSITION 3. *If $\theta \in P$ then $h(\theta) = h(f_\theta)$.*

The *induced matrix* $M(\theta)$ of $\theta \in P_n$ is the $(n-1) \times (n-1)$ matrix with ijth entry given by

$$a_{ij} = \begin{cases} 1, & \text{if } f_\theta([i, i+1]) \supset [j, j+1] \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 4. *If $\theta \in P$ then $h(\theta) = \log \rho(M(\theta))$, where $\rho(M(\theta))$ is the spectral radius of the induced matrix of θ.*

Proofs of Propositions 3 and 4 can be found in Misiurewicz and Szlenk [**6**] and Block and Coppel [**2**] respectively.

If $\theta, \phi \in P$ we say that θ *forces* ϕ and we write $\theta \vdash \phi$ if every map which has an invariant set of type θ also has an invariant set of type ϕ.

PROPOSITION 5. *The relation $\vdash$ is a partial preorder on P and a partial order on C.*

That is, $\vdash$ is reflexive and transitive on P and in addition is anti-symmetric on C. The anti-symmetry property is due to Baldwin [**1**].

Definition 6. The permutation θ is *forcing-maximal* on a subset S of P if $\theta \in S$ and, for all $\phi \in S$, if $\phi \vdash \theta$ then $\theta \vdash \phi$.

If $\theta, \phi \in P$ and $\theta \vdash \phi$ then, from Definition 2, clearly $h(\theta) \geq h(\phi)$.

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3. *Misiurewicz–Nitecki permutations of even period*

Let n be even and $k = n/2$. We define the permutation θ_n of period n by

$$\theta_n : j \mapsto \begin{cases} k - j + 1, & \text{if } 1 \le j \le k \text{ and } j \text{ is odd} \\ j - k, & \text{if } k + 1 \le j \le n \text{ and } j \text{ is odd} \\ k + j, & \text{if } 1 \le j \le k \text{ and } j \text{ is even} \\ 3k - j + 1, & \text{if } k + 1 \le j \le n \text{ and } j \text{ is even.} \end{cases}$$

It is straightforward to verify that this mapping is a permutation. We note the following general features of θ_n.

(i) f_{θ_n} has a local minimum at $x = 1$.

(ii) θ_n is maximodal and has all maximum values above all minimum values.

(iii) f_{θ_n} has a global minimum at $j = k$ if k is odd and at $j = k + 1$ if k is even.

(iv) f_{θ_n} has a global maximum at $j = k + 1$ if k is odd and at $j = k$ if k is even.

(v) If $n = 4p + 2$ and p is odd, θ_n is a product of disjoint 2-cycles of the form

$$\begin{aligned} &(1\ k)(3\ k-2)\dots(\tfrac{k-1}{2}\ \ \tfrac{k+3}{2}) \\ &\quad (2\ k+2)(4\ k+4)\dots(k-1\ n-1) \\ &\qquad (k+1\ n)(k+3\ n-2)\dots(\tfrac{3k-1}{2}\ \ \tfrac{3k+3}{2}). \end{aligned}$$

If $n = 4p + 2$ and p is even, θ_n is a product of disjoint cycles of the form

$$\begin{aligned} &(1\ k)(3\ k-2)\dots(\tfrac{k-3}{2}\ \ \tfrac{k+5}{2})(\tfrac{k+1}{2}\ \ \tfrac{k+1}{2}) \\ &\quad (2\ k+2)(4\ k+4)\dots(k-1\ n-1) \\ &\qquad (k+1\ n)(k+3\ n-2)\dots(\tfrac{3k-3}{2}\ \ \tfrac{3k+5}{2})(\tfrac{3k+1}{2}\ \ \tfrac{3k+1}{2}). \end{aligned}$$

Each factor in the product is a 2-cycle apart from the factors $(\frac{k+1}{2}\ \ \frac{k+1}{2})$ and $(\frac{3k+1}{2}\ \ \frac{3k+1}{2})$ which are fixed points.

If $n \equiv 0 \pmod 4$, θ_n is a product of disjoint 4-cycles of the form

$$(1\ k\ n\ k+1)(3\ k-2\ n-2\ k+3)\dots(k-1\ 2\ k+2\ n-1).$$

The general 'shape' of the graph of f_{θ_n} for $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$ is illustrated in Figure 1 below.

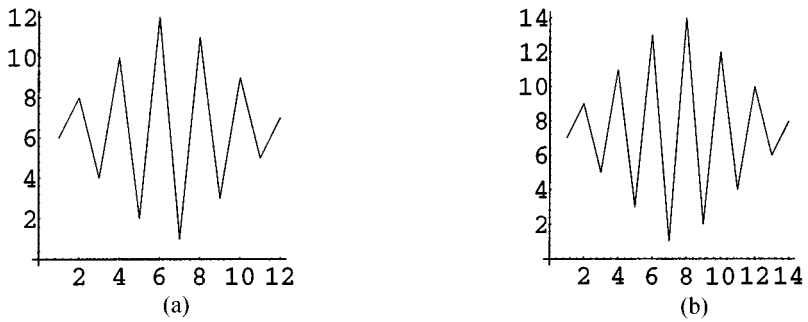


FIGURE 1. (a) Graph of f_{θ_n} when $n = 12$. (b) Graph of f_{θ_n} when $n = 14$.

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Geller and Tolosa [**4**] showed that θ_n actually attains maximal entropy for cycles of period n odd. The main result of this paper is that θ_n also attains maximal entropy for n even but in P_n rather than C_n.

THEOREM 7. *For n even, the self-dual permutation θ_n has maximal entropy among all permutations of length n.*

We will now consider the induced matrix $A = M(\theta_n)$ of θ_n.

PROPOSITION 8. *Let n be even, $k = n/2$ and $A = M(\theta_n)$.*

$$A_{j\,r} = 1 \Leftrightarrow \begin{cases} k - j + 1 \le r \le k + j, & \text{if } j \le k-1 \text{ and } j \text{ is odd} \\ k - j \le r \le k + j - 1, & \text{if } j \le k-1 \text{ and } j \text{ is even} \\ 1 \le r \le 2k - 1, & \text{if } j = k \\ j - k \le r \le 3k - j - 1, & \text{if } j \ge k+1 \text{ and } j \text{ is odd} \\ j - k + 1 \le r \le 3k - j, & \text{if } j \ge k+1 \text{ and } j \text{ is even.} \end{cases}$$

In the case $n \equiv 0 \pmod 4$ the matrix A is symmetric, but for the case $n \equiv 2 \pmod 4$ the matrix A has the property that

$$A_{j\,r} = A_{n-r\,\,j}.$$

In this case we say that A is 'quasi-symmetric'.

We can now use these properties to convert information about rows of A into information about columns of A. Given $a, b \in \mathbb{R}$ with $1 \le a \le b \le n-1$ we define $\langle a, b\rangle$ to be the column with ith element $(1 \le i \le n-1)$

$$\langle a, b\rangle_i = \begin{cases} 1, & \text{if } a \le i \le b \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 9. *For $n \equiv 0$ (mod 4) the column $A^{(j)}$ is given by*

$$\begin{array}{ll} \langle k - j + 1, k + j\rangle, & \text{when } j \le k-1 \text{ is odd} \\ \langle k - j, k + j - 1\rangle, & \text{when } j \le k-2 \text{ is even} \\ \langle 1, 2k - 1\rangle, & \text{when } j = k \\ \langle j - k, 3k - j - 1\rangle, & \text{when } j \ge k+1 \text{ is odd} \\ \langle j - k + 1, 3k - j\rangle, & \text{when } j \ge k+2 \text{ is even.} \end{array}$$

When $n \equiv 2$ (mod 4) the column $A^{(j)}$ is given by

$$\begin{array}{ll} \langle k - j, k + j - 1\rangle, & \text{when } j \le k-1 \text{ is odd} \\ \langle k - j + 1, k + j\rangle, & \text{when } j \le k-1 \text{ is even} \\ \langle 1, 2k - 1\rangle, & \text{when } j = k \\ \langle j - k + 1, 3k - j\rangle, & \text{when } j \ge k+1 \text{ is odd} \\ \langle j - k, 3k - j - 1\rangle, & \text{when } j \ge k+1 \text{ is even.} \end{array}$$

Proof. This follows directly from Proposition 8 since A is symmetric in the first case and quasi-symmetric in the second. □

COROLLARY 10. *The matrix A has jth column sum*

$$|A^{(j)}| = \begin{cases} 2j, & \text{if } j < k \\ n - 1, & \text{if } j = k \\ 2(n - j), & \text{if } j > k. \end{cases}$$

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4. *Idea of the proof of the theorem*

The proof of the main theorem for n even is very similar to that of Geller and Tolosa [**4**] for n odd. As stated in Proposition 4, the entropy of a permutation θ can be defined in terms of the spectral radius of the induced matrix $M(\theta)$. So we need to show that if $\theta \in P_n$ then $\rho M(\theta) \leq \rho M(\theta_n)$.

To this end, we will construct a cone $\mathcal{P}$ in $\mathbb{R}^{n-1}$ which is invariant under A and allows us to show that the spectral radius of A is indeed maximal. Firstly, however, we will utilise some results concerning maximality which will allow us to reduce the class of permutations that we will need to consider.

If θ_n is not maximal, then there exists $\theta \in P_n$ such that θ has maximal entropy in P_n and $h(\theta) > h(\theta_n)$. Let $B = M(\theta)$. The following result is due to Misiurewicz and Nitecki [**5**, Lemma 11.9].

PROPOSITION 11. *If θ is a permutation of length n, then its induced matrix $M = M(\theta)$, of order $(n-1) \times (n-1)$, has jth column sum*

$$|M^{(j)}| \leq \min\{2j, 2(n-j)\}.$$

This allows us to establish the following corollary.

COROLLARY 12. *Let n be even and $k = n/2$. If $\theta \in P_n$ is entropy maximal among all permutations of length n, with induced matrix $B = M(\theta)$, then the jth column sum $|B^{(j)}|$ of B satisfies*

$$|B^{(j)}| \leq \begin{cases} 2j, & \text{if } j < k \\ n-1, & \text{if } j = k \\ 2(n-j), & \text{if } j > k. \end{cases}$$

5. *Construction of the cone $\mathcal{P}$ and its invariance under A*

Let n be even and $k = n/2$. We denote the vector $\mathbf{u} \in \mathbb{R}^{n-1}$ by

$$\mathbf{u} = (u_1, u_2, \ldots, u_{k-1}, u_k, u_{n-(n-k-1)}, \ldots, u_{n-2}, u_{n-1}).$$

Definition 13. We define the *cone* $\mathcal{P}$ to be the set of all $\mathbf{u} \in \mathbb{R}^{n-1}$ for which the following condition holds:

$$0 \leq u_1 = u_{n-1} \leq u_2 = u_{n-2} \leq \cdots \leq u_{k-1} = u_{n-(k-1)} \leq u_k.$$

Thus a vector $\mathbf{u}$ in the cone peaks at its centre element u_k, and then drops away equally on either side of the peak.

Now let $\mathbf{u} \in \mathcal{P}$. We define $U(i, j)$ as

$$U(i, j) = \sum_{p=i}^{j} u_p.$$

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PROPOSITION 14. *Let* $\mathbf{u} \in \mathcal{P}$, $A = M(\theta_n)$ *and* $\mathbf{x} = \mathbf{u}A$.

If $n \equiv 0$ *(mod 4), then*

$$\begin{aligned} x_i &= U(k-i+1, k+i), && \text{if } i < k \text{ and } i \text{ is odd} \\ x_i &= U(k-i, k+i-1), && \text{if } i < k \text{ and } i \text{ is even} \\ x_i &= U(1, 2k-1), && \text{if } i = k \\ x_{n-i} &= U(k-i, k+i-1), && \text{if } i < k \text{ and } i \text{ is odd} \\ x_{n-i} &= U(k-i+1, k+i), && \text{if } i < k \text{ and } i \text{ is even.} \end{aligned}$$

If $n \equiv 2$ *(mod 4), then*

$$\begin{aligned} x_i &= U(k-i, k+i-1), && \text{if } i < k \text{ and } i \text{ is odd} \\ x_i &= U(k-i+1, k+i), && \text{if } i < k \text{ and } i \text{ is even} \\ x_i &= U(1, 2k-1), && \text{if } i = k \\ x_{n-i} &= U(k-i+1, k+i), && \text{if } i < k \text{ and } i \text{ is odd} \\ x_{n-i} &= U(k-i, k+i-1), && \text{if } i < k \text{ and } i \text{ is even.} \end{aligned}$$

Proof. The formulae for the components of $\mathbf{x}$ follow directly from the formulae in Proposition 9 describing the matrix A. □

We will now show that $\mathbf{x} \in \mathcal{P}$.

PROPOSITION 15. *The cone* $\mathcal{P}$ *is invariant under* $A = M(\theta_n)$.

Proof. Let $n \equiv 0$ (mod 4). We assume $\mathbf{u} \in \mathcal{P}$ and will show that $\mathbf{x} = \mathbf{u}A \in \mathcal{P}$. It is obvious that x_i and $x_{n-i} \geq 0$ for each $i \leq k$. Firstly, we show that $x_i = x_{n-i}$ for $i < k$.

Let i be odd. Then by Proposition 14,

$$x_i = U(k-i+1, k+i) \quad \text{and} \quad x_{n-i} = U(k-i, k+i-1).$$

Thus,

$$\begin{aligned} x_{n-i} - x_i &= U(k-i, k+i-1) - U(k-i+1, k+i) \\ &= u_{k-i} + U(k-i+1, k+i-1) - U(k-i+1, k+i-1) - u_{k+i} \\ &= u_{k-i} - u_{k+i} \\ &= 0, \quad \text{since } \mathbf{u} \in \mathcal{P}. \end{aligned}$$

Now let i be even. Then

$$\begin{aligned} x_{n-i} - x_i &= U(k-i+1, k+i) - U(k-i, k+i-1) \\ &= U(k-i+1, k+i-1) + u_{k+i} - u_{k-i} - U(k-i+1, k+i-1) \\ &= u_{k+i} - u_{k-i} \\ &= 0, \quad \text{since } \mathbf{u} \in \mathcal{P}. \end{aligned}$$

Hence $x_{n-i} = x_i$.

Secondly, we show that $x_i \leq x_{i+1}$ for $i < k$. Let i be odd. Then by Proposition 14

$$\begin{aligned} x_i &= U(k-i+1, k+i), \quad i < k, \\ x_{i+1} &= U(k-i-1, k+i), \quad i < k-1, \\ x_k &= U(1, 2k-1). \end{aligned}$$

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So

$$\begin{aligned} x_{i+1} - x_i &= U(k-i-1, k+i) - U(k-i+1, k+i) \\ &= u_{k-i-1} + u_{k-i} + U(k-i+1, k+i) - U(k-i+1, k+i) \\ &\geq 0 \end{aligned}$$

and

$$x_k - x_{k-1} = U(1, 2k-1) - U(2, 2k-1) = u_1 \geq 0.$$

Now let i be even. Then

$$\begin{aligned} x_i &= U(k-i, k+i-1), \quad i < k-1, \\ x_{i+1} &= U(k-i, k+i+1), \quad i+1 < k-1. \end{aligned}$$

So,

$$\begin{aligned} x_{i+1} - x_i &= U(k-i, k+i+1) - U(k-i, k+i-1) \\ &= U(k-i, k+i-1) + u_{k+i} + u_{k+i+1} - U(k-i, k+i-1) \\ &= u_{k+i} + u_{k+i+1} \\ &\geq 0. \end{aligned}$$

So in all cases $x_i \leq x_{i+1}$ as required.

The proof for the case $n \equiv 2 \pmod 4$ is completely analogous and is omitted. □

6. *Conclusion of the proof*

For the remainder of the proof we will let n be even, $A = M(\theta_n)$ and $B = M(\theta)$, where θ is as in Corollary 12.

PROPOSITION 16. *If* $\mathbf{u} \in \mathcal{P}$ *then* $\mathbf{u}A \geq \mathbf{u}B$.

Proof. We need to show that for each $j \in \{1, \ldots, n-1\}$,

$$\mathbf{u}A^{(j)} \geq \mathbf{u}B^{(j)}.$$

By Corollary 12 and Corollary 10 we know that $|A^{(j)}| \geq |B^{(j)}|$ for each j. Thus column j of B contains no more unit elements than column j of A. What we are looking for then, is which matrix picks off the biggest elements of $\mathbf{u}$ for each j; in other words, which matrix is most centered. Proposition 9 shows which rows contain a one for each column of A.

From Proposition 14 we have that for $n \equiv 0 \pmod 4$, $x_j = \mathbf{u}A^{(j)}$ is given by

$$\mathbf{u}A^{(j)} = \begin{cases} U(k-j+1, k+j), & \text{if } j < k,\ j \text{ odd} \\ U(k-j, k+j-1), & \text{if } j < k,\ j \text{ even} \\ U(1, 2k-1), & \text{if } j = k \\ U(k-(n-j), k+(n-j)-1), & \text{if } j > k,\ j \text{ odd} \\ U(k-(n-j)+1, k+(n-j)), & \text{if } j > k,\ j \text{ even.} \end{cases}$$

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whilst for $n \equiv 2 \pmod 4$

$$\mathbf{u}A^{(j)} = \begin{cases} U(k-j, k+j-1), & \text{if } j < k,\ j \text{ odd} \\ U(k-j+1, k+j), & \text{if } j < k,\ j \text{ even} \\ U(1, 2k-1), & \text{if } j = k \\ U(k-(n-j)+1, k+(n-j)), & \text{if } j > k,\ j \text{ odd} \\ U(k-(n-j), k+(n-j)-1), & \text{if } j > k,\ j \text{ even.} \end{cases}$$

Thus in both cases

$$\mathbf{u}A^{(j)} = \begin{cases} \text{the sum of the } 2j \text{ largest elements of } \mathbf{u}, & \text{if } j < k \\ \text{the sum of all } n-1 \text{ elements of } \mathbf{u}, & \text{if } j = k \\ \text{the sum of the } 2(n-j) \text{ largest elements of } \mathbf{u}, & \text{if } j > k. \end{cases}$$

But by Corollary 12,

$$B^{(j)} \text{ has } \begin{cases} \text{at most } 2j \text{ nonzero entries (all ones)}, & \text{if } j < k \\ \text{at most } n-1 \text{ nonzero entries (all ones)}, & \text{if } j = k \\ \text{at most } 2(n-j) \text{ nonzero entries (all ones)}, & \text{if } j > k. \end{cases}$$

It follows that, for each j, $\mathbf{u}A^{(j)} \geq \mathbf{u}B^{(j)}$. □

Proof of the theorem. Let $\mathbf{w} = (1, 1, \ldots, 1) \in \mathbb{R}^{n-1}$. Then $\mathbf{w} \in \mathcal{P}$ and by Proposition 15, $\mathbf{w}A \in \mathcal{P}$. It follows that $\mathbf{w}A^m \in \mathcal{P}$ for $m \geq 1$.

Let $\theta \in P_n$ be entropy maximal in P_n with induced matrix B. We are required to show that $h(\theta_n) \geq h(\theta)$. This is equivalent to showing that $\log \rho(A) \geq \log \rho(B)$.

Now by Proposition 16, $\mathbf{w}A \geq \mathbf{w}B$. Now assume inductively that $\mathbf{w}A^m \geq \mathbf{w}B^m$. Then

$$\begin{aligned} \mathbf{w}A^{m+1} &= (\mathbf{w}A^m)A \\ &\geq (\mathbf{w}A^m)B \quad (\text{since } \mathbf{w}A^m \in \mathcal{P}) \\ &\geq (\mathbf{w}B^m)B \quad (\text{by inductive assumption and } B \text{ is non-negative}) \\ &= \mathbf{w}B^{m+1}. \end{aligned}$$

For any non-negative square matrix we can write

$$\|M\| = \sum_{i,j} M_{ij} = |\mathbf{w}M|.$$

The spectral radius $\rho(M)$ of M satisfies

$$\rho(M) = \lim_{m \to \infty} \|M^m\|^{1/m}.$$

Thus

$$\begin{aligned} \log \rho(A) &= \log \lim_{m \to \infty} \|A^m\|^{1/m} \\ &= \log \lim_{m \to \infty} |\mathbf{w}A^m|^{1/m} \\ &\geq \log \lim_{m \to \infty} |\mathbf{w}B^m|^{1/m} \\ &= \log \lim_{m \to \infty} \|B^m\|^{1/m} \\ &= \log \rho(B). \end{aligned}$$

So θ_n has maximal entropy. □

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7. *A general formula*

We now give a unified definition for permutations with maximal entropy for any n. Let $k = n/2$ and let a be equal to the fractional part of $n/2$. We define the permutation θ_n of period n by

$$\theta_n : j \mapsto \begin{cases} k - j + 1 + (-1)^{((n+1)/2)}a, & \text{if } 1 \le j \le k \text{ and } j \text{ is odd} \\ j - k + (-1)^{((n-1)/2)}a, & \text{if } k+1 \le j \le n \text{ and } j \text{ is odd} \\ k + j + (-1)^{((n+1)/2)}a, & \text{if } 1 \le j \le k \text{ and } j \text{ is even} \\ 3k - j + 1 + (-1)^{((n-1)/2)}a, & \text{if } k+1 \le j \le n \text{ and } j \text{ is even.} \end{cases}$$

It can be easily verified that this definition is equivalent to the one given in §3 for n even (since here $a = 0$) and to that given by Geller and Tolosa in [**4**] for n odd.

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