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Helicity of vector fields preserving a regular contact form and topologically conjugate smooth dynamical systems

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Abstract. We compute the helicity of a vector field preserving a regular contact form on a closed three-dimensional manifold, and improve results of Gambaudo and Ghys [Enlacements asymptotiques. *Topology* **36**(6) (1997), 1355–1379] relating the helicity of the suspension of a surface isotopy to the Calabi invariant of the isotopy. Based on these results, we provide positive answers to two questions posed by Arnold in [The asymptotic Hopf invariant and its applications. *Selecta Math. Soviet.* **5**(4) (1986), 327–345]. In the presence of a regular contact form that is also preserved, the helicity extends to an invariant of an isotopy of volume-preserving homeomorphisms, and is invariant under conjugation by volume-preserving homeomorphisms. A similar statement also holds for suspensions of surface isotopies and surface diffeomorphisms. This requires the techniques of topological Hamiltonian and contact dynamics developed by Banyaga and Spaeth [On the uniqueness of generating Hamiltonians for topological strictly contact isotopies. *Preprint*, 2012], Buhovsky and Seyfaddini [Uniqueness of generating Hamiltonians for continuous Hamiltonian flows. *J. Symplectic Geom.* to appear, arXiv:1003.2612v2], Müller [The group of Hamiltonian homeomorphisms in the L^∞-norm. *J. Korean Math. Soc.* **45**(6) (2008), 1769–1784], Müller and Oh [The group of Hamiltonian homeomorphisms and C^0-symplectic topology. *J. Symplectic Geom.* **5**(2) (2007), 167–219], Müller and Spaeth [Topological contact dynamics I: symplectization and applications of the energy-capacity inequality. *Preprint*, 2011, arXiv:1110.6705v2] and Viterbo [On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows. *Int. Math. Res. Not.* (2006), 34028; Erratum, *Int. Math. Res. Not.* (2006), 38748]. Moreover, we generalize an example of Furstenberg [Strict ergodicity and transformation of the torus. *Amer. J. Math.* **83** (1961), 573–601] on topologically conjugate but not C^1-conjugate area-preserving diffeomorphisms of the two-torus to trivial T^2-bundles, and construct examples of Hamiltonian and contact vector fields that are topologically conjugate but not C^1-conjugate. Higher-dimensional helicities are considered briefly at the end of the paper.

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1. *Introduction*

According to Arnold [**Arn86**], 'The asymptotic Hopf invariant is an invariant of a divergence-free vector field on a three-dimensional manifold with given volume element. It is invariant under the group of volume-preserving diffeomorphisms, and describes the 'helicity' of the field, i.e. the mean asymptotic rotation of the phase curves around each other'. If X is a divergence-free vector field on a closed (i.e. compact and without boundary) smooth three-manifold M equipped with a volume form μ, then the two-form $\iota_X\mu$ is closed. Assuming it is exact, one may choose a primitive one-form β_X and define the *helicity* (or *asymptotic Hopf invariant* or *Arnold invariant*) of X to be the real number

$$\mathcal{H}(X) = \int_M \beta_X \wedge d\beta_X.$$

This number does not depend on the choice of β_X with $d\beta_X = \iota_X\mu$. In fact, Arnold gave two more equivalent definitions of helicity, one as an average asymptotic linking number of the flow lines of X, and the other equal to $\int_M g(X, Y)$ where g is some auxiliary Riemannian metric on M and Y is a divergence-free vector field satisfying the relation $\mathrm{curl}_g Y = X$. It is the first construction that we shall use exclusively in this work.

The asymptotic Hopf invariant generalizes the classical Hopf invariant of (the homotopy class of) a map $S^3 \to S^2$. Arnold and Khesin [**AK98**] noted that 'Although the idea of helicity goes back to Helmholtz and Kelvin (see [Kel]), its second birth in magnetohydrodynamics is due to Woltjer [Wol] and in ideal hydrodynamics is due to Moffatt [Mof1], who revealed its topological character (see also [Mor2]). The word 'helicity' was coined in [Mof1] and has been widely used in fluid mechanics and magnetohydrodynamics since then'. In addition to the references cited above, we also recommend [**Ghy07**] for further reading and for more details on the definition and some of its applications. The above publications also establish the basic properties of the helicity invariant and contain additional interesting references; see also §2.

It follows almost immediately from the definition that the helicity depends continuously on the vector field in the C^1-topology and is invariant under conjugation by volume-preserving C^1-diffeomorphisms. When viewed as an invariant of the volume-preserving isotopy $\{\phi_X^t\}_{0\le t\le 1}$ generated by the vector field X, the helicity is not continuous with respect to the C^0-topology. We recall two questions posed by Arnold regarding the topological character of the helicity.

Questions 1.1. [**Arn86**]

(i) Is the helicity invariant under conjugation by a volume-preserving homeomorphism? More precisely, if X and Y are (exact) divergence-free vector fields, ϕ is a homeomorphism that preserves the measure induced by μ, and $\{\phi \circ \phi_X^t \circ \phi^{-1}\} = \{\phi_Y^t\}$, does the identity $\mathcal{H}(X) = \mathcal{H}(Y)$ hold?

(ii) If $\{\phi_t\}_{0\le t\le 1}$ is an isotopy of volume-preserving homeomorphisms, can one define a number $\mathcal{H}(\{\phi_t\})$ that extends the definition for smooth isotopies?

The main purpose of this article is to address these questions. As a first step in that direction, we demonstrate in §4 the following.

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THEOREM 1.2. *Suppose that a closed three-manifold M admits a regular contact form α, and equip M with the canonical volume form $\alpha \wedge d\alpha$ induced by α. Let X_H be a strictly contact vector field with contact Hamiltonian function $H = \alpha(X_H)$. Then X_H is exact divergence-free, and*

$$\mathcal{H}(X_H) = \left(4c^2(H) - 3c(H^2)\right) \cdot \mathrm{vol}(M, \alpha \wedge d\alpha),$$

where the integer $\mathrm{vol}(M, \alpha \wedge d\alpha)$ *denotes the total volume and c the average value of a function on M, both with respect to the canonical volume form.*

In a similar vein, Gambaudo and Ghys [**GG97**] showed that the helicity of the suspension of a surface isotopy $\{\phi_t\}$ is proportional to the Calabi invariant of $\{\phi_t\}$. Using different techniques, we improve on their result as follows; see §7 for the relevant definitions.

THEOREM 1.3. *Let D^2 be the unit disk in $\mathbb{R}^2$ with its standard area form ω, and let $\{\phi_t\}_{0\le t\le 1}$ be a smooth area-preserving isotopy of D^2 that is the identity near the boundary ∂D^2. The helicity of the suspension $\tau_* X(\{\phi_t\})$ with respect to the standard volume form dV on $\mathbb{R}^3$ equals twice the Calabi invariant with respect to ω of the time-one map ϕ of the isotopy $\{\phi_t\}$. In fact,*

$$\mathcal{H}\left(X + \frac{\partial}{\partial t}\right) = \mathcal{H}(X) + 2\mathcal{R}\left(X, \frac{\partial}{\partial t}\right) + \mathcal{H}\left(\frac{\partial}{\partial t}\right),$$

and the first and last terms vanish, while $\mathcal{R}(X, \partial/\partial t) = \mathrm{Cal}(\phi)$.

Here $\mathcal{R}(\cdot\,,\cdot)$ denotes the relative helicity defined below in §2. In particular, the helicity is not C^0-continuous with respect to the isotopy $\{\phi_t\}$, because the Calabi invariant is not C^0-continuous with respect to ϕ (or $\{\phi_t\}$) [**GG97**]. The latter, by definition, is the real number

$$\mathrm{Cal}(\phi) = \int_0^1 \int_{D^2} F_t \omega \, dt, \tag{1}$$

where $F : [0, 1] \times D^2 \to \mathbb{R}$ is the unique normalized smooth Hamiltonian function generating the Hamiltonian isotopy $\{\phi_t\} = \{\phi_F^t\}$, and depends only on the time-one map ϕ.

See also §7 for the definition of the double suspension of a pair of surface isotopies $\{\phi_1^t\}$ and $\{\phi_2^t\}$ with time-one maps ϕ_1 and ϕ_2, respectively. Gambaudo and Ghys proved that the helicity of this double suspension depends linearly on the Calabi invariants of ϕ_1 and ϕ_2. In §7 we calculate the following formula.

THEOREM 1.4. *The helicity of the double suspension $X(\{\phi_1^t\}, \{\phi_2^t\})$ with respect to the standard volume form dV on the three-sphere S^3 equals*

$$\mathcal{H}(X(\{\phi_1^t\}, \{\phi_2^t\})) = 2\pi^2 \cdot (\mathrm{Cal}(\phi_1) + \mathrm{Cal}(\phi_2) + 2\pi^2).$$

In both the case of a vector field preserving a regular contact form and the case of the suspension of a surface isotopy, the helicity is an invariant of the generating Hamiltonian *function* rather than the vector field or isotopy. This allows an extension

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of the invariant to isotopies of volume-preserving homeomorphisms, and enables one to show that conjugation by volume-preserving homeomorphisms does not alter the helicity, provided the isotopies and homeomorphisms can be described as lifts from a surface. This is explained in greater detail below. As an immediate corollary to Theorem 1.2, we have the following result.

COROLLARY 1.5. *Let M be a closed three-manifold with a regular contact form α, and let H_i be a uniformly Cauchy sequence of smooth basic functions on M such that the corresponding strictly contact isotopies $\{\phi^t_{H_i}\}$ are uniformly Cauchy as well. Then the sequence $\mathcal{H}(X_{H_i})$ of real numbers converges.*

Definition 1.6. Let M be a closed three-manifold with a regular contact form α. The *helicity* of a *continuous strictly contact isotopy* $\{\phi_t\}$ with unique continuous contact Hamiltonian function H is defined to be

$$\mathcal{H}(\{\phi_t\}) = \lim_{i\to\infty} \mathcal{H}(X_{H_i}) = \left(4c^2(H) - 3c(H^2)\right) \cdot \mathrm{vol}(M, \alpha \wedge d\alpha),$$

where H_i is a uniformly Cauchy sequence of smooth basic functions with $\{\phi^t_{H_i}\} \to \{\phi_t\}$ uniformly.

The definition of a continuous strictly contact isotopy [**BS12**] is given in §8, where we also prove the following result.

THEOREM 1.7. *Let M be a closed three-manifold with a regular contact form α. If a homeomorphism ϕ of M is the uniform limit of a sequence of strictly contact diffeomorphisms, then for any two smooth or continuous strictly contact isotopies $\{\phi_t\}$ and $\{\psi_t\} = \{\phi^{-1} \circ \phi_t \circ \phi\}$ that are conjugated by ϕ, we have the identity $\mathcal{H}(\{\phi_t\}) = \mathcal{H}(\{\psi_t\})$.*

Regarding suspensions of continuous surface isotopies, Theorems 1.3 and 1.4 imply the following corollary.

COROLLARY 1.8. *Suppose that H_i is a uniformly Cauchy sequence of smooth Hamiltonian functions on the disk $(D^2, \partial D^2)$ and that the corresponding Hamiltonian isotopies $\{\phi^t_{H_i}\}$ converge uniformly to a continuous isotopy $\{\phi_t\}$ of $(D^2, \partial D^2, \omega)$. Then the helicities $\mathcal{H}(\tau_* X(\{\phi^t_{H_i}\}))$ with respect to the standard volume form dV on $\mathbb{R}^3$ of the suspensions $\tau_* X(\{\phi^t_{H_i}\})$ converge to twice the Calabi invariant of $\{\phi_t\}$ with respect to the area form ω.*

Moreover, if F_i is another uniformly Cauchy sequence of smooth Hamiltonian functions and the corresponding Hamiltonian isotopies $\{\phi^t_{F_i}\}$ converge uniformly to a continuous isotopy $\{\psi_t\}$ of $(D^2, \partial D^2, \omega)$, then the helicities $\mathcal{H}(X(\{\phi^t_{H_i}\}, \{\phi^t_{F_i}\}))$ with respect to the standard volume form dV on S^3 of the double suspensions $X(\{\phi^t_{H_i}\}, \{\phi^t_{F_i}\})$ converge to the number $2\pi^2 \cdot (\mathrm{Cal}(\{\phi_t\}) + \mathrm{Cal}(\{\psi_t\}) + 2\pi^2)$.

Definition 1.9. The *helicity* of the *suspension of a continuous Hamiltonian isotopy* $\{\phi_t\}$ of $(D^2, \partial D^2, \omega)$ is defined to be twice the Calabi invariant of $\{\phi_t\}$. The *helicity* of the *double suspension of two continuous Hamiltonian isotopies* $\{\phi_t\}$ and $\{\psi_t\}$ of $(D^2, \partial D^2, \omega)$ is defined to be the number $2\pi^2 \cdot (\mathrm{Cal}(\{\phi_t\}) + \mathrm{Cal}(\{\psi_t\}) + 2\pi^2)$.

See §9 for the definition of a continuous Hamiltonian isotopy and its Calabi invariant.

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THEOREM 1.10. *Suppose that the suspensions of two smooth or continuous Hamiltonian isotopies* $\{\phi_t\}$ *and* $\{\psi_t\}$ *of* $(D^2, \partial D^2, \omega)$ *are conjugated by a homeomorphism of the form* $(x, t) \mapsto (\varphi(x), t)$*, where* φ *is an area-preserving homeomorphism of* D^2 *that is the identity near the boundary of the disk. Then their helicities necessarily coincide. The same holds for topologically conjugate double suspensions, provided the conjugating homeomorphism of* S^3 *is of the above product form on the two solid tori which in Hopf coordinates are given by* $\{\eta \leq \pi/4\}$ *and* $\{\eta \geq \pi/4\}$.

As a motivation for studying helicity, we mention the following interesting problem in hydrodynamics, referring to [**Arn86, AK98**] for details. The mathematical model for fluid dynamics that we consider describes the hydrodynamics of an incompressible inviscid homogeneous fluid filling a simply-connected manifold M or, in other words, the (volume-preserving) flow of a divergence-free vector field X on M. Let g be some auxiliary Riemannian metric, and define the (magnetic) energy of X with respect to g by $E(X) = \int_M g(X, X)$. The group $\mathrm{Diff}(M, \mu)$ of volume-preserving diffeomorphisms acts on the Lie algebra of divergence-free vector fields by $X \mapsto \phi_* X$. Consider the problem of minimizing the functional E on the (adjoint) orbit $\{\phi_* X \mid \phi \in \mathrm{Diff}(M, \mu)\}$ of a fixed vector field X. For general X, there need not be a minimizing (smooth) vector field. If there is not, can the energy be made arbitrarily small? For generic X, the answer is no. Arnold showed in [**Arn86**] that

$$E(\phi_* X) \geq C \cdot |\mathcal{H}(X)|, \tag{2}$$

where C is some positive constant that depends on the metric g. The helicity is invariant under the action of volume-preserving diffeomorphisms, and is independent of the metric g. For generic X, the helicity does not vanish, and the above inequality gives a lower bound for the magnetic energy on the orbit of X. Arnold also proved that the critical points of E restricted to a fixed orbit are precisely those divergence-free vector fields that commute with their curl, including, in particular, Beltrami fields, i.e. eigenfields of the curl operator. The Hopf field on the three-sphere is an example; see §5. Beltrami fields with respect to some Riemannian metric are Reeb vector fields of some contact form, and vice versa [**EG00**]. We will review contact geometry in §3. Similar problems in hydrodynamics are discussed in the book by Arnold and Khesin [**AK98**].

Regarding Theorem 1.2, we point out that strictly contact vector fields are generalizations of the aforementioned Reeb vector fields, and occur quite naturally in the present context. A vector field is strictly contact if and only if it is divergence-free and contact, and strictly contact vector fields are those vector fields that commute with the Reeb vector field. If the contact form is regular, i.e. the Reeb vector field induces a free S^1-action on M, then strictly contact vector fields are precisely the lifts of Hamiltonian vector fields on the quotient of M by the Reeb flow. Similarly, in Theorems 1.3 and 1.4, we consider lifts of isotopies of the disk to the solid two-torus or the three-sphere.

In the first part of the paper, our methods are elementary and use the calculus of differential forms and the geometry of (regular) contact and symplectic manifolds. Section 2 reviews the definition of helicity and establishes its most important basic

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properties. In §3 we review the contact geometry of (regular) contact manifolds, and in §4 the proof of Theorem 1.2 is given. Section 5 discusses the case of the three-sphere, which is of greatest interest. In §6 homotopies rel endpoints are considered, and §7 is concerned with suspensions of surface isotopies.

The second part of the paper comprises §§8 and 9. We use the tools from topological Hamiltonian and contact dynamics developed in [**BS11, BS12, Mül08a, Mül08b, MO07, MS11, Vit06**] to address Arnold's questions.

In the final part of the paper, §§10 and 11 take up the question of the existence of diffeomorphisms and vector fields that are topologically conjugate but not C^1-conjugate. The proofs use the uniqueness theorems and transformation laws of topological Hamiltonian and contact dynamics. Section 12 is devoted to higher-dimensional helicities.

In the two appendices, we prove a proposition from §6 and compute the helicity of strictly contact vector fields on the three-torus.

2. *Helicity of divergence-free vector fields*

Let M be a closed smooth three-manifold equipped with a volume form μ. For the time being, assume $H^2(M) = 0$. By Cartan's formula, if a smooth vector field X on M is divergence-free, i.e. if the Lie derivative satisfies $\mathcal{L}_X \mu = 0$, then the two-form $\iota_X \mu$ is closed, where ι denotes interior multiplication of a differential form by a vector field. By our hypothesis, there exists a one-form $\beta = \beta_X$ with $d\beta_X = \iota_X \mu$, called a *primitive* of the two-form $\iota_X \mu$. The helicity of X is defined to be the real number

$$\mathcal{H}(X) = \int_M \beta_X \wedge d\beta_X = \int_M \beta_X(X) \cdot \mu. \tag{3}$$

This definition does not depend on the choice of the primitive β of $\iota_X \mu$. Indeed, suppose that β' is another one-form satisfying $d\beta' = \iota_X \mu = d\beta$. Then $\beta - \beta'$ is closed, and we have

$$\int_M \beta \wedge d\beta - \int_M \beta' \wedge d\beta' = \int_M (\beta - \beta') \wedge d\beta = \int_M d(\beta \wedge (\beta - \beta')) = 0$$

by Stokes' theorem. For example, one can choose $\beta_X = G\delta(\iota_X \mu)$ using the Hodge decomposition with respect to some auxiliary Riemannian metric. The second equality in (3) follows from the fact that interior multiplication is an anti-derivation, and $\beta \wedge \mu$ vanishes for dimension reasons. For later reference, we formalize this argument in the following obvious lemma.

LEMMA 2.1. *Given a p-form σ and a q-form τ on a smooth manifold M, the $(p+q+1)$-forms $d\sigma \wedge \tau$ and $\sigma \wedge d\tau$ coincide up to sign and an exact form. More precisely, $[d\sigma \wedge \tau] = (-1)^{p+1}[\sigma \wedge d\tau]$. In particular, if M is closed and $p + q = \dim M - 1$, then*

$$\int_M d\sigma \wedge \tau = (-1)^{p+1} \int_M \sigma \wedge d\tau.$$

If $p + q > \dim M$, then $\iota_X \sigma \wedge \tau = (-1)^{p+1} \sigma \wedge \iota_X \tau$ for any vector field X.

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If $H^2(M)$ is non-zero, the helicity invariant is defined on the Lie subalgebra of divergence-free vector fields X such that $\iota_X\mu$ is exact. Such vector fields are sometimes referred to as *exact* in the literature. There is a homomorphism from the Lie algebra of divergence-free vector fields into the $(\dim M - 1)$st cohomology group of M, defined by $X \mapsto [\iota_X\mu]$ (the *flux* of X), and its kernel consists precisely of the exact vector fields. We refer to [**Ban97**] for more information on this important homomorphism. It is shown in [**Mül11**] that if the volume form is induced by a regular contact form, this kernel contains all (divergence-free) contact vector fields.

The helicity is a quadratic form on the space of exact divergence-free vector fields. For X and Y exact, define the *relative helicity* by

$$\mathcal{R}(X, Y) = \int_M \beta_X \wedge d\beta_Y = \int_M \beta_Y \wedge d\beta_X$$

which, owing to Lemma 2.1, is independent of the choices of β_X and β_Y. The relative helicity $\mathcal{R}$ is symmetric and $\mathbb{R}$-bilinear, and we have the obvious identities $\mathcal{H}(X) = \mathcal{R}(X, X)$ and

$$\mathcal{H}(X \pm Y) = \mathcal{H}(X) \pm 2\mathcal{R}(X, Y) + \mathcal{H}(Y). \tag{4}$$

In particular,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{H}(X + \epsilon Y) = 2\mathcal{R}(X, Y),$$

or $d\mathcal{H}(X) = 2\mathcal{R}(X, \cdot)$, and for any non-zero X there exists an exact divergence-free vector field Y such that $\mathcal{R}(X, Y)$ is non-zero. Thus, the helicity of a C^1-generic (exact divergence-free) vector field does not vanish.

If we want to emphasize the dependence on the volume form μ, we write $\mathcal{H}(X; \mu)$ and denote the bilinear form by $\mathcal{R}(X, Y; \mu)$. However, the definitions depend on the choice of volume form on M only up to scaling and a volume-preserving change of coordinates. Recall that by Moser's argument, two volume forms μ and ν on M are isotopic if and only if the total volumes of M with respect to μ and ν coincide. Thus, up to scaling by a non-zero constant c, μ is isotopic to ν; this means that there exists a diffeomorphism ϕ (which is isotopic to the identity) such that $\phi^*\nu = c\mu$.

LEMMA 2.2. *If μ is a volume form on M, ϕ an orientation-preserving diffeomorphism, and X an exact divergence-free vector field with respect to the volume form $\phi^*\mu$, then ϕ_*X is exact divergence-free with respect to μ, and $\mathcal{H}(\phi_*X; \mu) = \mathcal{H}(X; \phi^*\mu)$. If c is a non-zero constant, then we have $\mathcal{H}(X; c\mu) = c^2\mathcal{H}(X; \mu)$. More generally, if f is a nowhere-vanishing smooth function on M and X is an exact divergence-free vector field with respect to the volume form $f\mu$, then the vector field fX is exact divergence-free with respect to μ, and the identity $\mathcal{H}(X; f\mu) = \mathcal{H}(fX; \mu)$ holds. Analogous statements hold for the relative helicity $\mathcal{R}(X, Y)$. In particular, both $\mathcal{H}$ and $\mathcal{R}$ are invariant under the action of volume-preserving diffeomorphisms on (exact) divergence-free vector fields.*

We note that the flow of ϕ_*X is the conjugation $\phi \circ \phi_X^t \circ \phi^{-1}$ of the flow ϕ_X^t of X by ϕ, and the flow of fX is related to the flow of X via the formula

$$\phi_{fX}^t(x) = \phi_X^{\tau(t,x)}(x).$$

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Here the smooth function $\tau : \mathbb{R} \times M \to \mathbb{R}$ solves the following ordinary differential equation with initial condition $\tau(0, x) = 0$ for all $x \in M$:

$$\frac{d}{dt}\tau(t, x) = f(\phi_X^{\tau(t,x)}).$$

Proof. It is straightforward to check the well-known identity

$$\phi^*(\iota_{\phi_* X}\mu) = \iota_X(\phi^*\mu). \tag{5}$$

Therefore, if β_X is a primitive of $\iota_X\phi^*\mu$, then $(\phi^{-1})^*\beta_X$ is a primitive of $\iota_{\phi_* X}\mu$. The first claim now follows from the change-of-variables formula. The other identities are proved similarly. □

We may consider the helicity as an invariant of the volume-preserving isotopy $\{\phi_X^t\}_{0\le t\le 1}$ generated by the vector field X, i.e. $d/dt\ \phi_X^t = X \circ \phi_X^t$, and ϕ_X^0 is the identity. There is also a flux homomorphism defined for volume-preserving isotopies [**Ban97**] and, in fact, the flux of $\{\phi_X^t\}$ by definition equals the flux of its infinitesimal generator X. Thus, if $H^2(M) \neq 0$, the helicity is defined for exact volume-preserving isotopies, i.e. those in the kernel of the flux map.

By Hodge theory, the helicity depends continuously on the vector field X, provided we equip the Lie algebra of divergence-free vector fields with the C^1-topology. As previously noted, with respect to the C^0-topology, the helicity does not depend continuously on the isotopy $\{\phi_X^t\}$ generated by X.

The helicity is also defined for M compact and connected with non-empty boundary, provided M is simply-connected and X is tangent to the boundary of M. One can also define the helicity for compact connected embedded submanifolds of $\mathbb{R}^3$ with non-empty boundary, if the divergence-free (with respect to the standard volume form dV on $\mathbb{R}^3$) vector field X is again tangent to the boundary. In the latter case, however, the helicity does depend on the embedding into $\mathbb{R}^3$; see §7. We refer to [**AK98, GG97**] for details.

The definition of $\mathcal{H}(X)$ generalizes in an obvious fashion to time-dependent vector fields $\{X_t\}$. Suppose that the closed two-forms $\iota_{X_t}\mu$ are exact for all $0 \le t \le 1$. By Hodge theory, after choosing an auxiliary Riemannian metric on M, we may choose the primitives β_t satisfying $d\beta_t = \iota_{X_t}\mu$ to depend smoothly on t. Then define

$$\mathcal{H}(\{X_t\}) = \int_0^1 \int_M \beta_t \wedge d\beta_t\, dt.$$

This number is again well-defined, and it coincides with the previous definition if X is autonomous. We can also define the helicity when only the time average of $\iota_{X_t}\mu$ is exact but the forms $\iota_{X_t}\mu$ are not necessarily exact for all times. This definition also extends the definition for autonomous X, but the two definitions for time-dependent vector fields may not coincide if $\iota_{X_t}\mu$ happens to be exact for all t. We remark that in the second situation, the flow $\{\phi_X^t\}$ of X_t is isotopic rel endpoints to an exact isotopy $\{\phi_Y^t\}$ [**Ban97**]. However, the helicity does in general depend on the homotopy class (rel endpoints) of the isotopy; see §6.

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The classical Hopf invariant of the homotopy class of a map $p : S^3 \to S^2$ is defined as follows. Choose an area form ω of total area 1 on S^2 and a primitive β of the (closed and hence exact) two-form $p^*\omega$ on S^3. Then define the Hopf invariant to be the integral $\int \beta \wedge d\beta$ over S^3. This is an integer which is also equal to the linking number of the preimages under p of two regular points in S^2. By the non-degeneracy of μ, every closed (exact) two-form on an oriented three-manifold can be written as $\iota_X \mu$ for some divergence-free (exact) vector field. The generalized Hopf invariant is defined even if the two-form $\iota_X \mu$ is not the pull-back of a closed form on S^2, and it can take any real value. In order to prove Theorem 1.2, we will consider the projection $p : M \to B$ of the *Boothby–Wang* (or *prequantization*) bundle over an integral symplectic surface, and relate the form $\iota_X \mu$ to the pull-back of an exact form on the base B. This set-up will be explained in the next two sections, and a similar strategy will be applied in §7 to prove Theorems 1.3 and 1.4.

3. *Regular contact manifolds*

Let M be a closed smooth manifold of dimension $2n + 1$, equipped with a coorientable nowhere-integrable field of hyperplanes (a *contact distribution* or *contact structure*) $\xi \subset TM$. This means that we suppose ξ is given (globally) by the kernel $\xi = \ker \alpha$ of a differential one-form α, and that $\mu = \alpha \wedge (d\alpha)^n$ is a volume form on M. We call μ the *canonical volume form* induced by the *contact form* α. For readers not familiar with contact (and symplectic) geometry, we recommend the monographs [**Gei08, MS98**]. For simplicity, we assume throughout this article that M is connected.

A vector field X on M is said to be *contact* (with respect to ξ) if $\mathcal{L}_X \alpha = h_X \alpha$ for a smooth function h_X on M, and *strictly contact* (with respect to α) if $h_X = 0$. Hence, X is contact if and only if its flow ϕ_X^t preserves the contact structure ξ, and strictly contact if and only if its flow preserves the contact form α. Note that the former concept depends only on the contact structure ξ, whereas the latter depends on the actual choice of contact form α. A vector field on M is divergence-free and contact if and only if it is strictly contact. For any $f \in C^\infty(M)$, the one-form $e^f \alpha$ defines another contact form giving rise to the same coorientation of ξ and orientation of M, and all contact forms representing the cooriented contact structure ξ can be written in this way.

We denote by R_α the *Reeb vector field* of the contact form α, i.e. the unique smooth vector field defined by the equations $\iota_{R_\alpha} d\alpha = 0$ and $\iota_{R_\alpha} \alpha = 1$, and we call its flow the Reeb flow on (M, α). More generally, given a contact vector field X, we call the smooth function $H = \iota_X \alpha$ its *contact Hamiltonian*. Conversely, given a smooth function H on M, there is a unique contact vector field X with contact Hamiltonian H which satisfies the equation $\iota_X d\alpha = (R_\alpha \cdot H)\alpha - dH$. Here we write $X \cdot f = df(X)$ for the derivative of a smooth function f in the direction of a vector field X on M. Indeed, $d\alpha$ restricted to the subbundle $\xi \subset TM$ is non-degenerate (i.e. $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle over M), so that the two equations together uniquely define the contact vector field X. We write X_H for the contact vector field with contact Hamiltonian H, and denote its flow by $\Phi_H = \{\phi_H^t\}$. Observe that $h_X = R_\alpha \cdot H$, so that X is strictly contact if and only if $R_\alpha \cdot H = 0$ or, equivalently, H is preserved under the flow of R_α. Such functions are called *basic* functions.

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The contact form α is said to be *regular* if R_α generates a free S^1-action on M; in particular, all Reeb orbits are closed and of period 1, and M is the total space of a principal S^1-bundle

$$S^1 \xrightarrow{i} M \xrightarrow{p} B \tag{6}$$

over a closed and connected integral symplectic manifold (B, ω) such that $p^*\omega = d\alpha$, known as the *Boothby–Wang* bundle [**BW58**]. Recall that *symplectic* means that the two-form ω on B is closed and non-degenerate, i.e. its top power ω^n defines a volume form on B, and integral means that the cohomology class $[\omega]$ belongs to $H^2(M, \mathbb{Z})$. Here i is the S^1-action of the Reeb vector field, and $p : M \to B \cong M/S^1$ is the projection to the quotient. The projection induces an (algebra) isomorphism $p^* : C^\infty(B) \to C^\infty_b(M)$ between smooth functions on the base B and smooth basic functions on M, as well as a surjective homomorphism $p_*(X_H) = -X_F$ between strictly contact vector fields X_H on (M, α) and Hamiltonian vector fields X_F on (B, ω) (with kernel generated over $\mathbb{R}$ by R_α). Here $H = p^*F = F \circ p$, and the vector field X_F is uniquely defined by the equation $\iota_{X_F}\omega = dF$.

If M has dimension three, then $B = \Sigma_g$ is an oriented closed and connected surface of genus g with integral total area. The most interesting case occurs when the genus g is zero; then (6) is the Hopf bundle $S^1 \to S^3 \to S^2$ (see §5). We would like to point out that every closed orientable three-manifold admits a contact structure [**Mar71**], but none of its contact forms need be regular. For example, the three-torus T^3 does not admit a regular contact form (and, in fact, no torus T^{2n+1} does) [**Bla10**]. This case is discussed separately in Appendix B.

Banyaga [**Ban78a**] has shown that the Boothby–Wang bundle (6) gives rise to a short exact sequence

$$1 \longrightarrow S^1 \xrightarrow{i_*} \mathrm{Diff}_0(M, \alpha) \xrightarrow{p_*} \mathrm{Ham}(B, \omega) \longrightarrow 1 \tag{7}$$

with S^1 in the center of $\mathrm{Diff}_0(M, \alpha)$. Here $\mathrm{Diff}_0(M, \alpha)$ denotes the identity component of the group of *strictly contact diffeomorphisms*, i.e. all those diffeomorphisms that preserve the contact form α and are isotopic to the identity through an isotopy of diffeomorphisms preserving α; $\mathrm{Ham}(B, \omega)$ denotes the group of *Hamiltonian diffeomorphisms* of (B, ω), that is, time-one maps of (the isotopies generated by time-dependent) Hamiltonian vector fields; and i_* is again the S^1-action of the Reeb vector field.

Note that $\iota_{R_\alpha}(\alpha \wedge d\alpha) = d\alpha$, so that $\mathcal{H}(R_\alpha) = \int_M \alpha \wedge d\alpha = \mathrm{vol}(M)$. Theorem 1.2 generalizes this computation to all strictly contact vector fields on a regular contact manifold M.

4. *Helicity of strictly contact vector fields*

Let M be a smooth manifold with a volume form μ, and define a (group) homomorphism $c : C^0(M) \to \mathbb{R}$ by

$$H \mapsto c(H) = c_H = \frac{1}{\mathrm{vol}(M, \mu)} \int_M H\mu, \tag{8}$$

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where $\mathrm{vol}(M,\mu)=\int_M \mu$ is the total volume of M with respect to μ. The next result shows that if $S^1 \to M \to B$ is the Boothby–Wang bundle (6) over an integral symplectic manifold (B^{2n},ω), then the projection p preserves the homomorphism c. In other words, $c_B = c_M \circ p^*$. Here c_M denotes the average value (8) with respect to the canonical volume form $\alpha\wedge(d\alpha)^n$ on the total space M, and similarly c_B denotes the average value (8) with respect to the canonical volume form ω^n on the base B.

PROPOSITION 4.1. *Let (M^{2n+1},α) be a regular contact manifold, and write $H=p^*F=F\circ p$ for $F\in C^0(B)$ where p is the projection map of the Boothby–Wang bundle (6). Then*

$$\int_M H\,\alpha\wedge(d\alpha)^n=\int_B F\omega^n.$$

In particular, we have $c_M(H)=c_B(F)$ with respect to the canonical volume forms $\alpha\wedge(d\alpha)^n$ and ω^n.

Proof. Choose an open cover $\{U_i\}$ of B with the property that the bundle is trivial over each U_i, and let $\{\lambda_i\}$ be a partition of unity subordinate to $\{U_i\}$. Let $V_i=p^{-1}(U_i)\cong U_i\times S^1$, and let $\mu_i=p^*\lambda_i=\lambda_i\circ p$ be the partition of unity subordinate to the open cover $\{V_i\}$ of M. We compute that

$$\begin{aligned}\int_M H\,\alpha\wedge(d\alpha)^n &= \sum_i \int_{U_i\times S^1}(\mu_i H)\,\alpha\wedge(d\alpha)^n\\ &= \sum_i \int_{U_i\times\{\mathrm{pt}\}}(\mu_i H)(d\alpha)^n\\ &= \sum_i \int_{U_i}(\lambda_i F)\omega^n\\ &= \int_B F\omega^n,\end{aligned}$$

which is what we set out to prove. The second equality follows from the fact that μ_i and H are constant on the Reeb orbits S^1. Applying the above formula to the constant function 1 proves the last part of the proposition. □

First proof of Theorem 1.2. The arguments in the proof are valid in any (odd) dimension, except that the definition of helicity makes sense only when $\dim M=3$. Therefore, consider the Boothby–Wang bundle $S^1\to M^{2n+1}\to B^{2n}$ over an integral (B,ω). Denote by $F\in C^\infty(B)$ the unique function such that $H=p^*F$, and write

$$c_F=\frac{1}{\mathrm{vol}(B,\omega^n)}\int_B F\omega^n$$

for the average value of F with respect to the canonical volume form ω^n on the base B. Then $\int_B(F-c_F)\omega^n=0$, and the $2n$-form $(F-c_F)\omega^n$ on B is exact. Choose a primitive γ, i.e. a $(2n-1)$-form such that $d\gamma=(F-c_F)\omega^n$. By Proposition 4.1, we have $c_F=c_H$. Define a $(2n-1)$-form β on M by

$$\beta=(n+1)p^*\gamma+((n+1)c_H-nH)\alpha\wedge(d\alpha)^{n-1}. \tag{9}$$

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By construction,

$$\begin{aligned} d\beta &= (n+1)(p^*d\gamma + c_H(d\alpha)^n) - n(dH\wedge\alpha + Hd\alpha)\wedge(d\alpha)^{n-1} \\ &= (n+1)((H-c_H)(d\alpha)^n + c_H(d\alpha)^n) - n(dH\wedge\alpha + Hd\alpha)\wedge(d\alpha)^{n-1} \\ &= H(d\alpha)^n + n\alpha\wedge dH\wedge(d\alpha)^{n-1} \\ &= H(d\alpha)^n - \alpha\wedge n(\iota_{X_H}d\alpha)\wedge(d\alpha)^{n-1} \\ &= \iota_{X_H}(\alpha\wedge(d\alpha)^n). \end{aligned}$$

In the $n=1$ case, this becomes

$$\beta = 2p^*\gamma + (2c_H - H)\alpha \quad \text{and} \quad d\beta = 2H\,d\alpha - d(H\alpha) = \iota_{X_H}(\alpha\wedge d\alpha).$$

We obtain

$$\begin{aligned} \beta\wedge d\beta &= (2p^*\gamma + (2c_H - H)\alpha)\wedge(2H\,d\alpha - d(H\alpha)) \\ &= 4Hp^*\gamma\wedge d\alpha - 2p^*\gamma\wedge d(H\alpha) + (2c_H - H)2H\alpha\wedge d\alpha \\ &\quad - (2c_H - H)\alpha\wedge d(H\alpha) \\ &= 4Hp^*(\gamma\wedge\omega) - 2Hp^*\,d\gamma\wedge\alpha + \text{exact terms} \\ &\quad + (2c_H - H)H\alpha\wedge d\alpha \\ &= -2H(H - c_H)\,d\alpha\wedge\alpha + (2c_HH - H^2)\alpha\wedge d\alpha + \text{exact terms} \\ &= (4c_HH - 3H^2)\alpha\wedge d\alpha + \text{exact terms}, \end{aligned}$$

where we have used the fact that the wedge product is graded commutative, together with Lemma 2.1 for the third equality. Therefore

$$\mathcal{H}(X_H) = \int_M \beta\wedge d\beta = (4c^2(H) - 3c(H^2))\cdot\mathrm{vol}(M, \alpha\wedge d\alpha). \qquad \square$$

Second proof of Theorem 1.2. Alternatively, suppose that β is given by (9); then

$$\beta(X_H) = (n+1)(p^*\gamma)(X_H) + ((n+1)c_H - nH)(\alpha\wedge(d\alpha)^{n-1})(X_H).$$

Since $p^*(\iota_{p_*X_H}\omega) = \iota_{X_H}p^*\omega = \iota_{X_H}\,d\alpha = -dH = -p^*(dF) = p^*(\iota_{-X_F}\omega)$ and p^* is an isomorphism on exact one-forms, X_H indeed has a well-defined projection $p_*X_H = -X_F$. We see that $(p^*\gamma)(X_H) = p^*(\gamma(-X_F))$ (compare to (5)), and by the same argument as in the proof of Proposition 4.1 we get

$$\int_M (p^*\gamma)(X_H)\wedge\alpha\wedge d\alpha = -\int_B \gamma(X_F)\wedge\omega = -\int_B \gamma\wedge(\iota_{X_F}\omega).$$

The latter coincides with

$$-\int_B \gamma\wedge dF = -\int_B F\,d\gamma = -\int_B F(F - c_F)\omega^n.$$

Recalling that p^* preserves average values by Proposition 4.1, we obtain that

$$\int_M (p^*\gamma)(X_H)\wedge\alpha\wedge d\alpha = (c^2(H) - c(H^2))\cdot\mathrm{vol}(M, \alpha\wedge(d\alpha)^n).$$

Integrating the above expression for $\beta(X_H)$ over M in the case where $n=1$ completes the proof. $\square$

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By Theorem 1.2, for volume-preserving contact isotopies on regular contact manifolds, the helicity is an invariant of the generating Hamiltonian function rather than the corresponding vector field or isotopy. By Theorems 1.3 and 1.4, the same holds for suspensions of surface isotopies on the solid two-torus or the three-sphere.

We would like to alert the reader to the fact that this formula is valid only for the canonical volume form $\alpha \wedge d\alpha$. If $\mu = c\,\alpha \wedge d\alpha$, then $\mathcal{H}(X;\mu) = c^2\mathcal{H}(X;\alpha\wedge d\alpha)$. If M admits a regular α, then, unless explicitly stated otherwise, we will always assume that the volume form is the canonical one induced by the contact form α. If α is not regular, then X_H need not be exact [**Mül11**]; see also Appendix B.

The same argument proves a relative version of Theorem 1.2.

THEOREM 4.2. *Let M be a closed three-manifold equipped with a contact form α as in Theorem 1.2, and let X_H and X_K be strictly contact vector fields with contact Hamiltonian functions H and $K \in C_b^\infty(M)$. Then X_H and X_K are exact divergence-free, and*

$$\mathcal{R}(X_H, X_K) = (4c(H)c(K) - 3c(H\cdot K))\cdot \mathrm{vol}(M, \alpha\wedge d\alpha).$$

The map $\mathcal{R}(H,K) = (4c(H)c(K) - 3c(H\cdot K))\cdot \mathrm{vol}(M,\alpha\wedge d\alpha)$ is symmetric and $\mathbb{R}$-bilinear, and defines a quadratic form on $C_b^\infty(M)$ given by $\mathcal{H}(H) = \mathcal{R}(H,H) = (4c^2(H) - 3c(H^2))\cdot \mathrm{vol}(M,\alpha\wedge d\alpha)$. By definition, $\mathcal{R}(X_H, X_K) = \mathcal{R}(H,K)$ and $\mathcal{H}(X_H) = \mathcal{H}(H)$. We again see that $\mathcal{H}(H\pm K) = \mathcal{H}(H) \pm 2\mathcal{R}(H,K) + \mathcal{H}(K)$, and thus the helicity of a C^0-generic strictly contact vector field is non-zero.

Alternatively, the average value $c_H = c_F$ can be computed as follows.

LEMMA 4.3. *Let $S \subset M$ be a Reeb circle, i.e. the preimage $p^{-1}(b)$ of a point $b \in B$, and let $D \subset M$ be any filling disk, i.e. $\partial D = S$. Then*

$$c_H = \int_D H\, d\alpha$$

independently of the point $b \in B$ and the disk $D \subset M$ with boundary $S = p^{-1}(b)$.

Proof. One way to see the above identity is as follows. First, note that

$$\int_D H\, d\alpha = \int_D d(p^*\gamma + c_F\alpha) = \int_S p^*\gamma + c_F\alpha.$$

Let $t \mapsto x(t)$ be a parameterization of the Reeb circle by arc length. Then the last integral is equal to

$$\int_0^1 \iota_{R_\alpha}(p^*\gamma + c_F\alpha)(x(t))\, dt = \int_0^1 \iota_{p_*R_\alpha}\gamma(b)\, dt + c_F = c_F,$$

since $p_*R_\alpha = 0$. The lemma now follows from Proposition 4.1. □

Example 4.4. We can decompose any strictly contact vector field X_H into its horizontal and vertical parts as $(X_H - HR_\alpha) + HR_\alpha$, with respect to the projection p_*. Note that $X_{H+c} = X_H + cR_\alpha$, so the kernel of the (surjective) homomorphism p_* is indeed

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generated over the reals by R_α. For a smooth function $F \in C^\infty(B)$, the horizontal lift $Y_F = HR_\alpha - X_H$ of the Hamiltonian vector field X_F is exact; and by Hölder's inequality,

$$\mathcal{H}(Y_F) = (c^2(H) - c(H^2)) \cdot \mathrm{vol}(M) = (c^2(F) - c(F^2)) \cdot \mathrm{vol}(B) \leq 0,$$

with equality if and only if F is constant or, equivalently, $X_F = 0$.

PROPOSITION 4.5. *The absolute value of the helicity $\mathcal{H}(X_H)$ is bounded by a constant times the square of the L^2-norm of H on $(M, \alpha \wedge d\alpha)$. In fact,*

$$-3\|H\|_{L^2}^2 \leq \mathcal{H}(X_H) \leq \|H\|_{L^2}^2,$$

with equality on the left or right if and only if H has mean value zero or is constant, respectively. Moreover, the restriction of the helicity to strictly contact vector fields can take any real value.

Compare this to Arnold's inequality (2).

Proof. Consider a basic function $H \in C_b^\infty(M)$, and denote by $\|\cdot\|_{L^2}$ the L^2-norm on $(M, \alpha \wedge d\alpha)$, i.e. $\|H\|_{L^2}^2 = \int_M H^2 \alpha \wedge d\alpha$. Again, Hölder's inequality implies that $c^2(H) \leq c(H^2)$, with equality if and only if H is a constant function. Thus

$$\begin{aligned} -3\|H\|_{L^2}^2 &= -3c(H^2) \cdot \mathrm{vol}(M) \\ &\leq \mathcal{H}(X_H) \\ &= (4c^2(H) - 3c(H^2)) \cdot \mathrm{vol}(M) \\ &\leq c(H^2) \cdot \mathrm{vol}(M) \\ &= \|H\|_{L^2}^2. \end{aligned}$$

A straightforward calculation shows that

$$\mathcal{H}(X_{H-c}) = \mathcal{H}(X_H) - 2cc_H \cdot \mathrm{vol}(M) + c^2 \cdot \mathrm{vol}(M),$$

which is a quadratic function of $c \in \mathbb{R}$ with global minimum at $c = c_H$. Thus, if H is non-zero, the helicity takes any real value on vector fields of the form $c_1 X_{H-c_2}$ with $c_1, c_2 \in \mathbb{R}$. □

5. *The Hopf bundle*

On the unit three-sphere $S^3 \subset \mathbb{C}^2$, write

$$\begin{aligned} z_1 = x_1 + iy_1 &= e^{i\xi_1} \sin\eta = \cos\xi_1 \sin\eta + i \sin\xi_1 \sin\eta, \\ z_2 = x_2 + iy_2 &= e^{i\xi_2} \cos\eta = \cos\xi_2 \cos\eta + i \sin\xi_2 \cos\eta \end{aligned}$$

for $(z_1, z_2) \in S^3$, where $0 \leq \eta \leq \pi/2$ and $0 \leq \xi_1, \xi_2 < 2\pi$ are *Hopf coordinates*. The standard regular contact form on S^3 is

$$\alpha = \frac{1}{2\pi}(x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2) = \frac{1}{2\pi}(\sin^2\eta\, d\xi_1 + \cos^2\eta\, d\xi_2),$$

with

$$d\alpha = \frac{1}{\pi}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) = \frac{1}{2\pi} \sin(2\eta)\, d\eta \wedge (d\xi_1 - d\xi_2),$$

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so that the Reeb vector field is equal to the Hopf vector field $R_\alpha = 2\pi(\partial/\partial\xi_1 + \partial/\partial\xi_2)$, which generates the (one-periodic) Reeb or Hopf flow on S^3. The corresponding volume form is

$$\alpha \wedge d\alpha = \frac{1}{(2\pi)^2} \sin(2\eta)\, d\eta \wedge d\xi_1 \wedge d\xi_2 = \frac{1}{2\pi^2}\, dV,$$

where dV is the standard volume form on the unit three-sphere. The total volume of S^3 with respect to $\alpha \wedge d\alpha$ equals 1.

On the unit two-sphere $S^2 \subset \mathbb{R}^3$, consider the *spherical coordinates*

$$x = \cos\varphi, \quad y = \sin\varphi\cos\psi, \quad z = \sin\varphi\sin\psi,$$

where $0 \le \varphi \le \pi$ and $0 \le \psi < 2\pi$. The standard area (or symplectic) form is (up to scaling)

$$\omega = \frac{1}{4\pi}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) = \frac{1}{4\pi} \sin\varphi\, d\varphi \wedge d\psi = \frac{1}{4\pi}\, dA,$$

where dA denotes the standard area form on the unit two-sphere. This gives S^2 a total area of 1 with respect to ω.

Recall the Hopf bundle $S^1 \xrightarrow{i} S^3 \xrightarrow{p} S^2$. In the above coordinates, the projection becomes $p(\eta, \xi_1, \xi_2) = (2\eta, \xi_1 - \xi_2) = (\varphi, \psi)$. We have

$$p^*\omega = \frac{1}{4\pi} \sin(2\eta)\, d(2\eta) \wedge d(\xi_1 - \xi_2) = \frac{1}{2\pi} \sin(2\eta)\, d\eta \wedge (d\xi_1 - d\xi_2) = d\alpha,$$

so that $p : (S^3, \alpha) \to (S^2, \omega)$ is the prequantization bundle (6) over the integral symplectic surface (S^2, ω). By Theorem 1.2 and Proposition 4.1,

$$\mathcal{H}(X_H) = 4c_H^2 - 3c_{H^2} = 4c_F^2 - 3c_{F^2},$$

where $H = F \circ p$ and the average values are computed with respect to the volume form $\alpha \wedge d\alpha$ on S^3 and the area form ω on S^2.

Example 5.1. The Reeb vector field $R_\alpha = X_H$ with $H = 1$ generates the Reeb flow on S^3, and $\mathcal{H}(R_\alpha) = 4c(1)^2 - 3c(1^2) = 1$. This vector field (as well as its negative) is an eigenvector of the curl with eigenvalue 1 and is an energy minimizer on its adjoint orbit with respect to an associated Riemannian metric $g = \alpha \otimes \alpha + d\alpha(\cdot\,, J\cdot)$ (see [**Bla10**]).

Let $X_H = 2\pi(\partial/\partial\xi_2 - \partial/\partial\xi_1)$; then $H = \cos(2\eta)$, or $F = \cos\varphi$ with $H = F \circ p$. We have

$$c_F = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \cos\varphi \sin\varphi\, d\varphi\, d\psi = 0$$

and

$$c_{F^2} = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \cos^2\varphi \sin\varphi\, d\varphi\, d\psi = \frac{1}{3},$$

and therefore $\mathcal{H}(X_H) = -1$.

Consider $H = \cos^2\eta = (1 + \cos(2\eta))/2$ and $\sin^2\eta = (1 - \cos(2\eta))/2$, corresponding to the strictly contact vector fields $X_H = 2\pi\ \partial/\partial\xi_2$ and $2\pi\ \partial/\partial\xi_1$. Then $F = (1 + \cos\varphi)/2$ and $(1 - \cos\varphi)/2$, respectively, and we compute, as above, that $\mathcal{H}(X_H) = 0$ in both cases.

By equation (4), we can compute the relative helicity of these vector fields. For example, $4\pi\ \partial/\partial\xi_2 = R_\alpha + 2\pi(\partial/\partial\xi_2 - \partial/\partial\xi_1)$, and thus $\mathcal{R}(R_\alpha, 2\pi(\partial/\partial\xi_2 - \partial/\partial\xi_1)) = 0$.

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6. *Homotopies rel endpoints*

We begin by recalling the following proposition, which is essentially contained in [**Ban78a**]. For the reader's convenience, a complete proof is given in Appendix A.

PROPOSITION 6.1. *Let (M, α) be a closed and connected regular contact three-manifold, and let $S^1 \to M \to B$ be the associated Boothby–Wang bundle (6) over the closed and connected integral symplectic surface (B, ω). If the base B has positive genus, then the inclusion $S^1 \hookrightarrow \mathrm{Diff}_0(M, \alpha)$ into the identity component of the group of strictly contact diffeomorphisms is a homotopy equivalence. In particular, the fundamental group of $\mathrm{Diff}(M, \alpha)$ is $\mathbb{Z}$, with generator being the homotopy class of the one-periodic Reeb flow, and for $k > 1$, $\pi_k(\mathrm{Diff}(M, \alpha))$ is trivial. If the base B is S^2, i.e. if the Boothby–Wang bundle is the Hopf fibration, then the one-periodic Reeb flow represents twice the generator of $\pi_1(\mathrm{Diff}(S^3, \alpha)) = \mathbb{Z}$, and the fundamental group is generated by the homotopy class of the flow of the vector field $2\pi\ \partial/\partial\xi_1$, which coincides with the homotopy class of the flow of the vector field $2\pi\ \partial/\partial\xi_2$. Moreover, we have $\pi_k(\mathrm{Diff}(S^3, \alpha)) \cong \pi_k(S^3)$ for $k > 1$.*

COROLLARY 6.2. *If $M = S^3$, the helicity of a strictly contact vector field depends on the homotopy class rel endpoints of the isotopy it generates. This holds true whether we consider homotopies in $\mathrm{Diff}(S^3, \alpha)$ or in $\mathrm{Diff}(S^3, \alpha \wedge d\alpha)$.*

Proof. By Example 5.1, we have $\mathcal{H}(4\pi\ \partial/\partial\xi_i) = 0 \neq 1 = \mathcal{H}(R_\alpha)$ for $i = 1, 2$, or $\mathcal{H}(2\pi(\partial/\partial\xi_2 - \partial/\partial\xi_1)) = -1 \neq 0 = \mathcal{H}(0)$. Thus, by Proposition 6.1, the helicity depends on the homotopy class rel endpoints. The last statement follows from the induced homomorphism on fundamental groups. □

In other words, the helicity is not an invariant on the universal covering space of $\mathrm{Diff}_0(M, \alpha)$, i.e. of the homotopy class rel endpoints of an isotopy. We write $H \sim K$ if the isotopies Φ_H and Φ_K are homotopic rel endpoints through a homotopy of strictly contact isotopies, that is, a two-parameter family $\phi_{s,t}$ of strictly contact diffeomorphisms with $\phi_{s,0} = \mathrm{id}$ and $\phi_{s,1} = \phi \in \mathrm{Diff}(M, \alpha)$ for all $0 \leq s \leq 1$. Denote by $X_{s,t}$ and $Y_{s,t}$ the vector fields defined by

$$\frac{d}{dt}\phi_{s,t} = X_{s,t} \circ \phi_{s,t}, \quad \frac{d}{ds}\phi_{s,t} = Y_{s,t} \circ \phi_{s,t}.$$

In particular, $Y_{s,0} = 0 = Y_{s,1}$. Since the diffeomorphisms $\phi_{s,t}$ preserve α, the vector fields $X_{s,t}$ and $Y_{s,t}$ are strictly contact.

LEMMA 6.3. [**Ban78a**] *If $H \sim K$, then $c(H) = c(K)$.*

Proof. It is well-known [**Ban78b**] that

$$\frac{d}{ds}X_{s,t} = \frac{d}{dt}Y_{s,t} + [X_{s,t}, Y_{s,t}].$$

Contracting α with this equation and integrating over $[0, 1] \times [0, 1] \times M$ (the bracket of two basic functions has vanishing average value) proves the lemma. □

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COROLLARY 6.4. *Suppose that $H \sim K$ for two basic functions H and K. Then*

$$\mathcal{H}(X_H) - \mathcal{H}(X_K) = 3(\|K\|_{L^2}^2 - \|H\|_{L^2}^2).$$

Thus the helicities of X_H and X_K are equal if and only if the L^2-norms of H and K coincide. Furthermore, if $\mathcal{H}(X_H) \geq \mathcal{H}(X_K)$, then $\|H\|_{L^2} \leq \|K\|_{L^2}$; and if $\mathcal{H}(X_H) > \mathcal{H}(X_F)$, then $\|H\|_{L^2} < \|F\|_{L^2}$. Conversely, suppose that two basic functions H and K generate contact isotopies with the same endpoint. If either $\mathcal{H}(X_H) \geq \mathcal{H}(X_K)$ and $\|H\|_{L^2} > \|K\|_{L^2}$, or $\mathcal{H}(X_H) > \mathcal{H}(X_K)$ and $\|H\|_{L^2} \geq \|K\|_{L^2}$, then $H \not\sim K$.

For example, suppose $H \sim K$. Then the helicity of the strictly contact vector field generating the composed isotopy $\Phi_H^{-1} \circ \Phi_K$ is $-3\|H - K\|_{L^2}^2 \leq 0$. If X_H generates a loop and $\mathcal{H}(X_H) > 0$, the loop is not contractible.

7. *Suspensions of surface isotopies*

In this section we improve, using different methods, results due to Gambaudo and Ghys [**GG97**], which relate the helicity to the Calabi invariant of surface isotopies. Denote by $D^2 \subset \mathbb{R}^2$ the unit disk in the plane with polar coordinates (r, θ), where $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$, and standard area (or symplectic) form $\omega = r\, dr \wedge d\theta$. Consider also the cylinder $D^2 \times \mathbb{R}$ with volume form $\omega \wedge dt$, where t is the coordinate on the real line. Let $\phi \in \mathrm{Diff}(D^2, \partial D^2, \omega)$ be an area-preserving diffeomorphism that is the identity near the boundary ∂D^2 of the disk. Consider the solid torus

$$T_\phi = D^2 \times \mathbb{R} \,/\, \{(\phi^n(x), t) \sim (x, t + n) \mid n \in \mathbb{Z}\}$$

with the induced volume form $\omega \wedge dt$, and denote by $p : D^2 \times \mathbb{R} \to T_\phi$ the canonical projection to the *mapping torus* T_ϕ. The divergence-free vector field $\partial/\partial t$ projects to a divergence-free vector field $p_*(\partial/\partial t) = X(\phi)$, called the *suspension* of the *surface diffeomorphism* ϕ [**GG97**]. Clearly, $\iota(X(\phi))(\omega \wedge dt) = \omega = d\lambda$ for a one-form λ on D^2, and the three-form $\lambda \wedge \omega$ vanishes for dimension reasons. The situation becomes more interesting after embedding T_ϕ into standard $\mathbb{R}^3$.

Let $\{\phi_t\}$ be an isotopy generated by a one-periodic smooth Hamiltonian function $F : D^2 \times \mathbb{R} \to \mathbb{R}$ that is compactly supported in the interior, with $\phi_0 = \mathrm{id}$ and time-one map $\phi_1 = \phi$. This isotopy gives rise to a volume-preserving embedding $T_\phi \to D^2 \times \mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{R}^3$ by composition of the map $S(\{\phi_t\}) : T_\phi \to D^2 \times \mathbb{R}/\mathbb{Z}$ given by $(x, t) \mapsto (\phi_t(x), t)$ with an embedding $D^2 \times \mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{R}^3$ of the solid torus that is volume-preserving with respect to the standard volume form dV on $\mathbb{R}^3$. An explicit (orientation-preserving) embedding of $D^2 \times \mathbb{R}/\mathbb{Z}$ into $\mathbb{R}^3$ is given by

$$((r, \theta), t) \mapsto ((A + Br\cos\theta)\cos(2\pi t),\, Br\sin\theta,\, (A + Br\cos\theta)\sin(2\pi t)),$$

which preserves total volume for an appropriate choice of constants $A > B > 0$. By Moser's argument, it can be deformed to a volume-preserving embedding τ (that preserves the boundary). The vector field $X(\phi)$ defines a divergence-free vector field $S(\{\phi_t\})_*(X(\phi)) = X(\{\phi_t\}) = X + \partial/\partial t$ on the solid torus $D^2 \times \mathbb{R}/\mathbb{Z}$, where $X(x, t) = X_F^t(x)$ is the Hamiltonian vector field generating the isotopy $\{\phi_t\}$. We call $X(\{\phi_t\})$

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the *suspension* of the *surface isotopy* $\{\phi_t\}$. By identifying $D^2 \times \mathbb{R}/\mathbb{Z}$ with its image $\tau(D^2 \times \mathbb{R}/\mathbb{Z})$ in $\mathbb{R}^3$, we can identity the vector fields $X(\{\phi_t\})$ and $\tau_* X(\{\phi_t\})$, and we refer to the latter also as the suspension of the isotopy $\{\phi_t\}$.

Proof of Theorem 1.3. By restricting to differential forms on the torus $D^2 \times \mathbb{R}/\mathbb{Z}$ which are pull-backs of differential forms on the image of τ that extend to global differential forms on $\mathbb{R}^3$, the same argument as in the proof of Lemma 2.2 gives $\mathcal{H}(\tau_* X(\{\phi_t\}), dV) = \mathcal{H}(X(\{\phi_t\}), \omega \wedge dt)$. On the other hand,

$$\iota_{X(\{\phi_t\})}(\omega \wedge dt) = \iota_X \omega \wedge dt + \omega = dF_t \wedge dt + d\lambda = d(F_t\, dt + \lambda),$$

where λ is a one-form on D^2 (that extends to $\mathbb{R}^2$) with $d\lambda = \omega$. The primitive $F_t\, dt + \lambda$ extends to a global one-form on $\mathbb{R}^3$. Moreover,

$$(F_t\, dt + \lambda) \wedge (dF_t \wedge dt + \omega) = F_t \omega \wedge dt - dF_t \wedge \lambda \wedge dt.$$

Since F vanishes near the boundary, Stokes' theorem implies that

$$0 = \int_{D^2 \times \mathbb{R}/\mathbb{Z}} d(F_t\, \lambda \wedge dt) = \int_{D^2 \times \mathbb{R}/\mathbb{Z}} dF_t \wedge \lambda \wedge dt + F_t\, \omega \wedge dt,$$

and thus

$$\mathcal{H}(X(\{\phi_t\}), \omega \wedge dt) = 2 \int_{D^2 \times \mathbb{R}/\mathbb{Z}} F_t \omega \wedge dt = 2 \int_0^1 \int_{D^2} F_t \omega\, dt = 2\,\mathrm{Cal}(\phi). \qquad \square$$

Example 7.1. Consider the solid torus $D^2 \times \mathbb{R}/\mathbb{Z}$ with coordinates $((r, \theta), t)$, where $0 \le r \le 1$, $0 \le \theta < 2\pi$ and $0 \le t < 1$ (considering a disk of arbitrary radius corresponds to rescaling the area form). Let $\rho : [0, 1] \to \mathbb{R}$ be a smooth function that is identically zero near $r = 1$, and consider the area-preserving diffeomorphism $\phi_\rho : D^2 \to D^2$ defined by $(r, \theta) \mapsto (r, \theta + \rho(r))$ for $r > 0$ and $\phi_\rho(0) = 0$, where 0 denotes the origin in $\mathbb{R}^2$. The suspension of the isotopy $\{\phi_\rho^t\} = \{\phi_{t\rho}\}$ is the vector field

$$X(\{\phi_{t\rho}\}) = \rho(r) \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}$$

(cf. [**GG01**, §1.4]), and its generating Hamiltonian function is

$$F(r, \theta) = \int_r^1 s\rho(s)\, ds$$

(see [**MO07**, Example 4.2]). An easy computation shows that

$$\iota_{X(\{\phi_{t\rho}\})}(r\, dr \wedge d\theta \wedge dt) = d\left(\frac{1}{2} r^2\, d\theta + \left(\int_r^1 s\rho(s)\, ds\right) dt\right) = d\beta_\rho.$$

Then

$$\beta_\rho \wedge d\beta_\rho = \left(\frac{1}{2} r^2 \rho(r) + \int_r^1 s\rho(s)\, ds\right) r\, dr \wedge d\theta \wedge dt.$$

Using integration by parts for the second summand, we find that

$$\mathcal{H}(X(\{\phi_{t\rho}\})) = \int_0^1 \int_0^{2\pi} \int_0^1 \left(\frac{1}{2} r^3 \rho(r) + r \int_r^1 s\rho(s)\, ds\right) dr\, d\theta\, dt = 2\pi \int_0^1 r^3 \rho(r)\, dr.$$

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Let $\phi_1, \phi_2 \in \mathrm{Diff}(D^2, \partial D^2, \omega)$ be two area-preserving diffeomorphisms that are the identity near the boundary, and consider the corresponding solid tori T_{ϕ_1} and T_{ϕ_2}. Let $\{\phi_1^t\}$ and $\{\phi_2^t\}$ be Hamiltonian isotopies as above with $\phi_1^0 = \phi_2^0 = \mathrm{id}$ and with time-one maps $\phi_1^1 = \phi_1$ and $\phi_2^1 = \phi_2$. Again, consider $X(\{\phi_1^t\}) = X_1 + \partial/\partial t$ and $X(\{\phi_2^t\}) = X_2 + \partial/\partial t$ as vector fields on $D^2 \times \mathbb{R}/\mathbb{Z}$, and embed the solid torus into the three-sphere by

$$\tau^1 : ((r, \theta), t) \mapsto (\tfrac{1}{2} \sin^{-1} r, \theta + 2\pi t, 2\pi t) = (\eta, \xi_1, \xi_2) \in S^3$$

(rather than the standard $((r, \theta), t) \mapsto (\frac{1}{2} \sin^{-1} r, \theta, 2\pi t)$) and

$$\tau^2 : ((r, \theta), t) \mapsto \left(\tfrac{1}{2}(\pi - \sin^{-1} r), 2\pi t, \theta + 2\pi t\right) = (\eta, \xi_1, \xi_2) \in S^3,$$

so that the vector fields $\tau^1_*(X(\{\phi_1^t\}))$ and $\tau^2_*(X(\{\phi_2^t\}))$ coincide near the torus $\{\eta = \pi/4\}$. Here the image of τ^1 is the solid torus $\{\eta \le \pi/4\}$ in S^3, and the image of τ^2 is the solid torus $\{\eta \ge \pi/4\}$. Denote by $X(\{\phi_1^t\}, \{\phi_2^t\})$ the smooth vector field that coincides with $\tau^1_*(X(\{\phi_1^t\}))$ on the set $\{\eta \le \pi/4\}$, and with $\tau^2_*(X(\{\phi_2^t\}))$ on the set $\{\eta \ge \pi/4\}$. This divergence-free vector field on the three-sphere S^3 is called the *double suspension* of $\{\phi_1^t\}$ and $\{\phi_2^t\}$. Up to scaling, the volume form μ on S^3 obtained from gluing together the two copies of the solid torus is the standard one, and it has total volume equal to 2π.

Proof of Theorem 1.4. By Lemma 2.2,

$$\mathcal{H}(X(\{\phi_1^t\}, \{\phi_2^t\}); dV) = \mathcal{H}(X(\{\phi_1^t\}, \{\phi_2^t\}); \pi\mu) = \pi^2 \cdot \mathcal{H}(X(\{\phi_1^t\}, \{\phi_2^t\}); \mu).$$

By construction, $X(\{\phi_1^t\}, \{\phi_2^t\}) = \tau^1_*(X_1) + \tau^2_*(X_2) + R_\alpha$, so that

$$\mathcal{H}(X(\{\phi_1^t\}, \{\phi_2^t\})) = \mathcal{H}(\tau^1_*(X_1) + \tau^2_*(X_2)) + 2\mathcal{R}(\tau^1_*(X_1) + \tau^2_*(X_2), R_\alpha) + \mathcal{H}(R_\alpha).$$

For the first term, we compute that

$$\begin{aligned}\mathcal{H}(\tau^1_*(X_1) + \tau^2_*(X_2); \mu) &= \mathcal{H}(\tau^1_*(X_1); \mu) + 2\mathcal{R}(\tau^1_*(X_1), \tau^2_*(X_2); \mu) + \mathcal{H}(\tau^2_*(X_2); \mu) \\ &= \mathcal{H}(X_1, \omega \wedge dt) + \mathcal{H}(X_2, \omega \wedge dt) = 0,\end{aligned}$$

by using the fact that $\tau^1_*(X_1)$ and $\tau^2_*(X_2)$ have disjoint supports on S^3 and the last statement of Theorem 1.3. Moreover,

$$\begin{aligned}\mathcal{R}(\tau^1_*(X_1) + \tau^2_*(X_2), R_\alpha; \mu) &= \mathcal{R}(\tau^1_*(X_1), R_\alpha; \mu) + \mathcal{R}(\tau^2_*(X_2), R_\alpha; \mu) \\ &= \mathcal{R}\left(X_1, \frac{\partial}{\partial t}; \omega \wedge dt\right) + \mathcal{R}\left(X_2, \frac{\partial}{\partial t}; \omega \wedge dt\right) \\ &= \mathrm{Cal}(\phi_1) + \mathrm{Cal}(\phi_2).\end{aligned}$$

Since $\mathcal{H}(R_\alpha) = (\mathrm{vol}(S^3))^2$, combining all the above statements proves the claim. □

8. *Continuous contact isotopies*

Let (M, ξ) be a closed contact manifold equipped with a contact form α. A continuous isotopy $\Phi = \{\phi_t\}$ in the group $\mathrm{Homeo}(M)$ of homeomorphisms is a *continuous strictly contact isotopy* if there exists a Cauchy sequence of smooth basic contact Hamiltonian functions $H_i : [0, 1] \times M \to \mathbb{R}$ such that the sequence Φ_{H_i} of strictly contact isotopies

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converges uniformly to Φ. Here the norm [**BS12**] used to define the metric on the space of contact Hamiltonian functions is

$$\|H\| = \max_{0\le t\le 1}\Big(\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x) + |c(H_t)|\Big), \tag{10}$$

which means that the Cauchy sequence H_i converges uniformly. It is also possible to replace the maximum over $0 \le t \le 1$ by the integral over the interval $[0, 1]$, but we will restrict to the former in this article. For a detailed study of continuous contact isotopies and related notions, see [**BS12, MS11**].

Suppose now that (M, α) is regular. Then the *continuous contact Hamiltonian function* $H = \lim_i H_i$ associated to the continuous strictly contact isotopy Φ is unique [**BS12**]. In other words, if H_i and K_i are two Cauchy sequences of smooth basic contact Hamiltonian functions with $\lim_i \Phi_{H_i} = \Phi = \lim_i \Phi_{K_i}$, then we must have $\lim_i H_i = \lim_i K_i$. Thus the limit $\mathcal{H}(\Phi)$ of the sequence $\mathcal{H}(X_{H_i})$ exists and does not depend on the sequence H_i but only on the continuous strictly contact isotopy Φ. This proves Corollary 1.5 and verifies that Definition 1.6 is well-defined.

We point out that the contact Hamiltonian functions H_i are time-dependent in general, even if the limit H is autonomous. However, by our earlier remark, the helicity of a time-dependent divergence-free vector field that is exact at all times t is well-defined, and Definition 1.6 makes sense for any continuous strictly contact isotopy.

Conversely, a continuous strictly contact isotopy is uniquely determined by its continuous contact Hamiltonian function [**BS12**]. To be more precise, suppose that two sequences of smooth basic contact Hamiltonian functions H_i and K_i generate two sequences Φ_{H_i} and Φ_{K_i} of uniformly convergent strictly contact isotopies. If $\lim_i H_i = \lim_i K_i$, then $\lim_i \Phi_{H_i} = \lim_i \Phi_{K_i}$. Denote the common limit by Φ. In the terminology of topological (strictly) contact dynamics, the continuous Hamiltonian function $H = \lim_i H_i$ 'generates' the isotopy Φ, and we write $\Phi_H = \Phi$.

By the above uniqueness theorems, the continuous strictly contact isotopy Φ_H is a one-parameter subgroup if and only if its continuous contact Hamiltonian function is autonomous. Furthermore, a continuous contact Hamiltonian function H is invariant under the Reeb flow, and we call H a *continuous basic function* on M (see [**MS11**]). In particular, there exists a unique function F on B such that $H = p^*F = F \circ p$. This function F is a continuous Hamiltonian function in the sense explained in the next section. Moreover, the S^1-extension (7) extends to so-called strictly contact homeomorphisms of M, i.e. time-one maps of continuous strictly contact isotopies, and Hamiltonian homeomorphisms of B; see [**BS12**]. The latter were defined and studied in [**Mül08a, Mül08b, MO07**]. See the next section for a brief summary. Example 4.4 concerning horizontal lifts can be generalized verbatim to continuous Hamiltonian isotopies.

Suppose that $\phi \in \mathrm{Homeo}(M)$ is the uniform limit $\phi = \lim_i \phi_i$ of a sequence of strictly contact diffeomorphisms, i.e. $\phi_i^*\alpha = \alpha$ for all i. We denote the group of strictly contact diffeomorphisms by $\mathrm{Diff}(M, \alpha)$ and the group of limit homeomorphisms by $\overline{\mathrm{Diff}}(M, \alpha)$. By rigidity, if a homeomorphism $\phi \in \overline{\mathrm{Diff}}(M, \alpha)$ is smooth, then $\phi \in \mathrm{Diff}(M, \alpha)$ (see [**MS11**]), justifying our notation.

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Proof of Theorem 1.7. Let Φ be a continuous strictly contact isotopy with continuous contact Hamiltonian function H, and let H_i be a Cauchy sequence with limit H and such that $\Phi_{H_i} \to \Phi$ uniformly, whose existence is guaranteed by the definition of a continuous strictly contact isotopy. Then the conjugated smooth isotopy $\phi_i^{-1} \circ \Phi_{H_i} \circ \phi_i$ has the smooth contact Hamiltonian function $H_i \circ \phi_i$ and, moreover, converges to $\phi^{-1} \circ \Phi \circ \phi$ uniformly, while $H_i \circ \phi_i$ converges to $H \circ \phi$ in the metric defined by (10) [**BS12**]. This extension of the usual transformation law provides further justification for our notation. Since ϕ preserves the (measure induced by the) volume form $\alpha \wedge d\alpha$ on M, the change-of-variables formula shows that ϕ preserves the average value c of a function on M. Thus the following identities hold:

$$\begin{aligned}\mathcal{H}(\phi^{-1} \circ \Phi \circ \phi) &= \left(4c^2(H \circ \phi) - 3c((H \circ \phi)^2)\right) \cdot \mathrm{vol}(M) \\ &= \left(4c^2(H) - 3c(H^2)\right) \cdot \mathrm{vol}(M) \\ &= \mathcal{H}(\Phi).\end{aligned}$$

□

9. *Continuous Hamiltonian isotopies*

We briefly recall the definition of a continuous Hamiltonian isotopy, which is similar to the case of a continuous contact isotopy discussed in the previous section. See [**Mül08a, Mül08b, MO07**] for details. Let (B^{2n}, ω) be a closed and connected symplectic manifold. Recall from §3 the one-to-one correspondence between Hamiltonian vector fields $X = X_F$ and smooth mean-value-zero normalized functions F on M, given by the relation $\iota_X \omega = dF$. In this context, a smooth function on M is generally referred to as a *Hamiltonian* function. Suppose that $F_i : [0, 1] \times B \to \mathbb{R}$ is a sequence of smooth mean-value-zero normalized time-dependent Hamiltonian functions, and denote by $\Phi_{F_i} = \{\phi_{F_i}^t\}$ the sequence of smooth Hamiltonian isotopies corresponding to the Hamiltonian functions F_i, i.e. the isotopies generated by the vector fields X_{F_i}. If F_i is a Cauchy sequence with respect to the metric induced by the norm (10) and Φ_{F_i} converges uniformly to a continuous isotopy $\Phi = \{\phi_t\}$ of homeomorphisms, then Φ is called a *continuous Hamiltonian isotopy*, and the limit $F = \lim F_i$ is called a *continuous Hamiltonian function*. The group of time-one maps of continuous Hamiltonian isotopies is denoted by $\mathrm{Hameo}(B, \omega)$. Note that we assume the Hamiltonian functions have mean value zero (with respect to the canonical volume form ω^n), so that the term $c(F(t, \cdot))$ in (10) vanishes in the present situation. It is again possible to work with the (weaker) norm obtained by replacing the maximum with the time average over $0 \le t \le 1$. These definitions make sense for non-compact manifolds and manifolds with non-empty boundary, provided one considers only Hamiltonian functions that are compactly supported in the interior of B and adjusts the definition accordingly. In this case, the mean value of a Hamiltonian function need no longer vanish identically; it is instead normalized by the requirement of having compact support in the interior.

A continuous Hamiltonian isotopy is uniquely determined by its generating Hamiltonian function [**MO07**]. That is, suppose that a sequence Φ_{F_i} of smooth Hamiltonian isotopies generated by normalized smooth Hamiltonian functions F_i converges uniformly

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to an isotopy Φ of homeomorphisms, and that the Hamiltonian functions converge to a continuous function F uniformly. If another such sequence Φ_{G_i} satisfies $\lim_i F_i = \lim_i G_i$, then $\lim_i \Phi_{F_i} = \lim_i \Phi_{G_i}$ and we write $\Phi_F = \Phi$.

Conversely, Buhovsky and Seyfaddini [**BS11**] generalized (and simplified the proof of) a previous result by Viterbo [**Vit06**] on the uniqueness of the 'generating Hamiltonian' F. That is, if $F_i \to F$ and $G_i \to G$, and if the isotopies Φ_{F_i} and Φ_{G_i} have the same uniform limit Φ, then $F = G$. When $B = D^2$, the real number

$$\int_0^1 \int_{D^2} F_t \, \omega \, dt \tag{11}$$

is well-defined and depends only on the continuous Hamiltonian isotopy Φ. If Φ is a smooth Hamiltonian isotopy, it equals the Calabi invariant of the time-one map of Φ. We thus call (11) the *Calabi invariant* $\mathrm{Cal}(\Phi)$ of the isotopy Φ, and note that $\mathrm{Cal}(\Phi) = \lim_i \mathrm{Cal}(\Phi_{F_i})$ for a (and thus any) sequence of Hamiltonian isotopies converging to Φ in the sense of the definition of a continuous Hamiltonian isotopy. Note that every smooth area-preserving isotopy of $(D^2, \partial D^2, \omega)$ is Hamiltonian.

Suppose that $\Phi = \{\phi_t\}$ is a continuous Hamiltonian isotopy, F_i is a Cauchy sequence in the sense explained above, and the smooth Hamiltonian isotopies Φ_{F_i} converge uniformly to Φ. Consider the suspensions $X(\Phi_{F_i})$ defined in §7. These do not necessarily converge to a (continuous) vector field on $D^2 \times \mathbb{R}/\mathbb{Z}$ as $i \to \infty$. However, the flows (with time-s maps) $(x, t) \mapsto (\phi_{F_i}^{t+s} \circ (\phi_{F_i}^t)^{-1}(x), t + s)$ converge uniformly to the flow $(x, t) \mapsto (\phi_{t+s} \circ \phi_t^{-1}(x), t + s)$, which we call the *suspension* of Φ. As remarked above, $\mathrm{Cal}(\Phi_{F_i}) \to \mathrm{Cal}(\Phi)$, independently of the choice of the sequence F_i in Definition 1.9. This proves Corollary 1.8, shows that Definition 1.9 is well-defined, and extends the definition of the helicity of the suspension of a smooth Hamiltonian isotopy. However, recall again that the helicity is in general not C^0-continuous with respect to the isotopy Φ; if a sequence of isotopies Φ_i converges only uniformly to Φ, their helicities need not converge.

The group $\mathrm{Sympeo}(M, \omega)$ of symplectic homeomorphisms is by definition the C^0-closure of the group $\mathrm{Symp}(M, \omega) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$ of symplectic diffeomorphisms in $\mathrm{Homeo}(M)$ [**MO07**]. The usual transformation law continues to hold for homeomorphisms, i.e. $\phi^{-1} \circ \Phi \circ \phi = \{\phi^{-1} \circ \phi_t \circ \phi\}$ has continuous Hamiltonian function $F \circ \phi$ for any continuous Hamiltonian isotopy $\Phi = \{\phi_t\}$ with continuous Hamiltonian function F and symplectic homeomorphism ϕ. Again by rigidity, an element of $\mathrm{Sympeo}(M, \omega)$ that is smooth belongs to $\mathrm{Symp}(M, \omega)$, which, together with the transformation law, justifies our notation.

The Calabi invariant of Φ is conjugation-invariant by area-preserving diffeomorphisms of the two-disk. Any area-preserving homeomorphism can be approximated uniformly by diffeomorphisms [**Hir63, Mun59, Mun60, Mun65**] and thus by area-preserving (or symplectic) diffeomorphisms [**Oh06, Sik07**]. Therefore the Calabi invariant (11) of a (smooth or continuous) Hamiltonian isotopy is invariant under conjugation by any area-preserving homeomorphism of the two-disk. For smooth isotopies, this was known to Gambaudo and Ghys; see [**GG97**] for a different proof.

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Proof of Theorem 1.10. Denote the suspensions by $\tilde{\phi}^s(x,t) = (\phi_{t+s} \circ \phi_t^{-1}(x), t+s)$ and $\tilde{\psi}^s(x,t) = (\psi_{t+s} \circ \psi_t^{-1}(x), t+s)$, and write $\tilde{\varphi}(x,t) = (\varphi(x), t)$. By hypothesis,

$$\tilde{\psi}^s(x,t) = \tilde{\varphi} \circ \tilde{\phi}^s \circ \tilde{\varphi}^{-1}(x,t) = (\varphi \circ (\phi_{t+s} \circ \phi_t^{-1}) \circ \varphi^{-1}(x), t+s).$$

The right-hand side is the suspension of the isotopy $\varphi \circ \Phi \circ \varphi^{-1} = \{\varphi \circ \phi_t \circ \varphi^{-1}\}$ and thus, by conjugation invariance, has helicity $2\,\mathrm{Cal}(\varphi \circ \Phi \circ \varphi^{-1}) = 2\,\mathrm{Cal}(\Phi)$, which in turn equals the helicity of the suspension of Φ. □

10. *Topologically conjugate diffeomorphisms*

Recall Arnold's first question presented in the introduction. Suppose that (the volume-preserving isotopies generated by) two smooth exact divergence-free vector fields X and Y are topologically conjugate; this means that there exists a (volume-preserving) homeomorphism ψ such that $\{\psi \circ \phi_X^t \circ \psi^{-1}\} = \{\phi_Y^t\}$. If ψ is a C^1-diffeomorphism, this is equivalent to $\psi_* X = Y$, and it is easy to see that the helicities of X and Y coincide (Lemma 2.2). Does this identity hold in general, even when ψ is not a C^1-diffeomorphism and, thus, $\psi_* X$ is not well-defined? Theorems 1.7 and 1.10 provide positive answers in two particular cases coming from the contact geometry of regular contact three-manifolds and the symplectic geometry of surfaces. The discussion in this section and the next is intended to illustrate these results.

The proof of the following algebraic lemma is trivial. For a group G, we denote by $Z_g = \{c \in G \mid gc = cg\}$ the centralizer of $g \in G$.

LEMMA 10.1. *If for elements a, b, c of some group G we have $cac^{-1} = b$, then $dad^{-1} = b$ if and only if $d \in c \cdot Z_a = Z_b \cdot c$.*

The lemma applied to the group $G = \mathrm{Homeo}(M)$ says that if $\psi \circ \phi \circ \psi^{-1} = \varphi$, then in general ψ is not unique. Note that it is trivial to find examples of two diffeomorphisms or isotopies of diffeomorphisms that are topologically conjugate. For example, if a diffeomorphism ϕ equals the identity on some open set U, and ψ is a diffeomorphism outside an open set $V \subset U$ but non-smooth inside V, then $\psi \circ \phi \circ \psi^{-1}$ is a diffeomorphism. Similar examples can be constructed for ϕ being the identity on some factor of a product manifold. The actual problem is to find examples of topologically conjugate diffeomorphisms or isotopies of diffeomorphisms that are not conjugated by a diffeomorphism. Indeed, it seems quite a rare situation that two diffeomorphisms are conjugated by a homeomorphism but not by a C^1-diffeomorphism.

The following construction is due to Furstenberg [**Fur61, Rou90**]. Let θ be an irrational number, d an integer, and f a smooth function on S^1. A *Furstenberg transformation* $\phi_{\theta,d,f}$ is a diffeomorphism of T^2 defined by

$$\phi_{\theta,d,f}(x,y) = (xe^{2\pi i\theta}, x^d y e^{2\pi i f(x)}).$$

Furstenberg transformations are always area-preserving (with respect to the standard area form $dx \wedge dy$) and *minimal* (that is, every orbit is dense in the torus T^2) provided d is non-zero [**Fur61, Rou90**]. The following lemma is essentially contained in [**Kod95a**].

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LEMMA 10.2. *Let θ be an irrational number, d a non-zero integer, and f a smooth function on S^1. Consider the Furstenberg transformations $\phi_{\theta,d,f}$ and $\phi_{\theta,d,0}$ of T^2. There exists a continuous map $\psi : T^2 \to T^2$ that satisfies the identity $\psi \circ \phi_{\theta,d,f} = \phi_{\theta,d,0} \circ \psi$ if and only if f can be split with respect to the circle action $x \mapsto e^{2\pi i\theta}x$ on S^1, i.e. there exists a continuous function g on S^1 that satisfies the equation $g(x) - g(e^{2\pi i\theta}x) = f(x) - \eta$, where η denotes the average value of f (with respect to the measure induced by dx). In that case,*

$$\psi(x, y) = (xe^{2\pi i(m\theta+\eta+k)/d}, x^m ye^{2\pi ig(x)}),$$

where m and k are integers. In particular, ψ is (a posteriori) an area-preserving homeomorphism, and $\phi_{\theta,d,f}$ is topologically conjugate to $\phi_{\theta,d,0}$. Moreover, the function g is unique up to (adding) a real constant. Thus, if f is smooth and g is not C^1, then $\phi_{\theta,d,f}$ is topologically conjugate but not C^1-conjugate to $\phi_{\theta,d,0}$.

Proof. The 'if' part is a straightforward computation. For the 'only if' part, suppose that g_1 and g_2 are continuous functions with $g_i(x) - g_i(e^{2\pi i\theta}x) = f(x) - \eta$ for $i = 1, 2$. Their difference then satisfies $(g_1 - g_2)(x) = (g_1 - g_2)(e^{2\pi i\theta}x)$ with θ irrational, and by continuity $g_1 - g_2$ is constant. By the homotopy lifting theorem, we may write

$$\psi(x, y) = (x^{m_1}y^{n_1}e^{2\pi iF_1(x,y)}, x^{m_2}y^{n_2}e^{2\pi iF_2(x,y)})$$

for integers m_1, n_1, m_2, n_2 and smooth functions F_1, F_2 on the torus. Calculating explicitly $\psi \circ \phi_{\theta,d,f} = \phi_{\theta,d,0} \circ \psi$ and using that θ is irrational and ϕ_f is area-preserving and minimal yields

$$\psi(x, y) = (xe^{2\pi i(m\theta+\eta+k)/d}, x^m ye^{2\pi iF(x,y)}),$$

where $F(x, y) - F(\phi_f(x, y)) = f(x) - \eta$ and $F : T^2 \to \mathbb{R}$ is a continuous function [**Kod95a**]. Kodaka gave the proof for the $d = 1$ case, and the general case can be proved verbatim. Kodaka moreover assumed *a priori* that ψ is a homeomorphism; however, this assumption is not necessary. The function G defined by $G(x, y) = g(x)$ also satisfies the equality $G(x, y) - G(\phi_f(x, y)) = f(x) - \eta$. Subtracting shows that the continuous function $F - G$ is constant since $\phi_{\theta,d,f}$ is minimal. Then $F(x, y) = g(x) + c$ as claimed. This shows that the map ψ has the required form, and it is easy to see that it is injective and surjective and thus a homeomorphism (since T^2 is compact and Hausdorff). □

Example 10.3. [**Fur61**] For an irrational number θ, choose a sequence of integers $n_k \geq 2^k$, $k \geq 1$, such that $0 < n_k\theta - [n_k\theta] \leq 2^{-n_k}$, where $[x]$ denotes, as usual, the greatest integer less than or equal to x. Furstenberg originally constructed only one such number θ, but by Poincaré's recurrence theorem this can be done for any irrational number θ. Define $n_k = -n_{-k}$ for $k < 0$. Then the real function $f : S^1 \to \mathbb{R}$ defined by

$$f(e^{2\pi it}) = \sum_{k\neq 0} \frac{1}{k^2}(1 - e^{2\pi in_k\theta})e^{2\pi in_k t}$$

is a smooth function on S^1 with mean value zero. Now define the real function $g : S^1 \to \mathbb{R}$ by

$$g(e^{2\pi it}) = \sum_{k\neq 0} \frac{1}{k^2}e^{2\pi in_k t},$$

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which is continuous in $x \in S^1$ but not C^1-smooth. It is immediate to check that $g(x) - g(e^{2\pi i\theta}x) = f(x)$.

A necessary and sufficient criterion for when a function can be split (with respect to some minimal homeomorphism) was proved in [**GH55**, p. 135]; see also [**Rou90, Kod95a**]. Examples of such functions are most easily constructed as above by using Fourier series [**Kod95b**, Lemma 2.1] and Plancherel's theorem, where the number η is the coefficient of the constant term.

The following proposition generalizes the preceding example (in which M is void) to trivial T^2-bundles of any dimension.

PROPOSITION 10.4. *For a smooth manifold M, there exist pairs of diffeomorphisms of $M \times T^2$ that are conjugated by a homeomorphism but not by any C^1-diffeomorphism. If, moreover, M supports a volume form μ and $M \times T^2$ is equipped with the product volume form $\mu \wedge dx \wedge dy$, then there are pairs of diffeomorphisms as above which, in addition, are volume-preserving, and the conjugating homeomorphism may also be chosen to preserve volume.*

Proof. A straightforward computation shows that

$$(\mathrm{id} \times \psi) \circ (\mathrm{id} \times \phi_{\theta,d,f}) = (\mathrm{id} \times \phi_{\theta,d,0}) \circ (\mathrm{id} \times \psi),$$

where $\phi_{\theta,d,f}$, $\phi_{\theta,d,0}$ and ψ are as in the example above. We can, in fact, replace the second and third identity maps by any homeomorphism ϕ of M here and in the argument below, and also choose $g(p, x) = g(x) + c(p)$ for a continuous function c on M. This proves the existence part of the proposition.

Let $\psi = \psi_1 \times \psi_2 : M \times T^2 \to M \times T^2$ be a C^1-diffeomorphism, where the factors $\psi_1 : M \times T^2 \to M$ and $\psi_2 : M \times T^2 \to T^2$ are both C^1-smooth maps, and assume that $\psi \circ (\mathrm{id} \times \phi_{\theta,d,f}) = (\mathrm{id} \times \phi_{\theta,d,0}) \circ \psi$. For fixed $p \in M$, consider the restriction $\psi_2^p = \psi_2(p, \cdot)$ of ψ_2 to the fiber over p. By a routine computation, we obtain $\psi_2^p \circ \phi_{\theta,d,f} = \phi_{\theta,d,0} \circ \psi_2^p$. But, by Lemma 10.2, ψ_2^p cannot be C^1-smooth, and this is a contradiction. □

Alternatively, the identity $\psi \circ (\mathrm{id} \times \phi_{\theta,d,f}) = (\mathrm{id} \times \phi_{\theta,d,0}) \circ \psi$ also implies $\psi_1^p = \psi_1^p \circ \phi_{\theta,d,f}$, where $\psi_1^p : T^2 \to M$ is the restriction of ψ_1 to the fiber over p. Since $\phi_{\theta,d,f}$ is minimal ($d \neq 0$), this implies that ψ_1 depends only on p but not on $(x, y) \in T^2$. Thus, for fixed $p \in M$, the map ψ_2^p is a local diffeomorphism. Its image is open and closed, and therefore ψ_2^p is surjective. Since it is also injective, it is a diffeomorphism of T^2, and we may proceed as above to derive a contradiction. However, the proof given above shows that there does not even exist a C^1-smooth *map* ψ such that $\psi \circ (\mathrm{id} \times \phi_{\theta,d,f}) = (\mathrm{id} \times \phi_{\theta,d,0}) \circ \psi$. The following result can be proved similarly to Lemma 10.2.

LEMMA 10.5. *If M is connected and simply connected or if $M = T^n$ in Proposition 10.4, then the homeomorphism ψ conjugating* $\mathrm{id} \times \phi_{\theta,d,f}$ *and* $\mathrm{id} \times \phi_{\theta,d,0}$ *is of the form*

$$\psi(p, (x, y)) = (\psi_1(p), (xe^{2\pi i(m\theta+\eta+k)/d}, x^m y e^{2\pi i(g(x)+c(p))}))$$

or

$$\psi(p, (x, y)) = (\psi_1(p), (xe^{2\pi i(m\theta+\eta+k)/d}, x^m y e^{2\pi i(g(x)+c(p))} p_1^{q_1} \cdots p_n^{q_n})),$$

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respectively, for a homeomorphism ψ_1 of M, integers m, k and $q_1, \dots, q_n$, and a continuous function c on M. The homeomorphism ψ is volume-preserving (with respect to a product volume form on $M \times T^2$) if and only if ψ_1 is.

Conversely, we have the following example.

Example 10.6. Suppose that $f = \eta$ is constant and $\psi \circ \phi_{\theta,d,\eta} = \phi_{\theta,d,0} \circ \psi$. Then, again, ψ has the above form for a function g that satisfies $g(x) - g(e^{2\pi i\theta}x) = f(x) - \eta = 0$. Assuming ψ is continuous, g must be continuous and thus constant. Therefore

$$\psi(x, y) = (xe^{2\pi i(m\theta+\eta+k)/d}, x^m y e^{2\pi ic})$$

for some $c \in \mathbb{R}$ and is, in particular, an area-preserving diffeomorphism. This provides an example of two area-preserving diffeomorphisms that are conjugated only by (area-preserving) diffeomorphisms.

A thorough study of topological and smooth conjugacy of diffeomorphisms of the circle S^1 is carried out in the book [**KH95**] by Katok and Hasselblatt. In particular, for any integer $r \geq 0$, the authors construct examples of C^{r+1}-smooth diffeomorphisms of S^1 that are conjugated only by diffeomorphisms (or homeomorphisms when $r = 0$) of class C^r.

11. *Topologically conjugate smooth dynamical systems*

We would like to point out that none of the diffeomorphisms in §10 that are conjugated by homeomorphisms (and only homeomorphisms) are isotopic to the identity, since the integer d in the definition of the Furstenberg transformation is non-zero. We now construct examples of smooth Hamiltonian and strictly contact *isotopies* that are conjugated by a homeomorphism but not by symplectic or contact C^1-diffeomorphisms, respectively. We begin by recalling some facts previously used in this work in the form of a well-known and easy-to-verify lemma. We state it for autonomous vector fields, but the conclusions of the lemma are equally valid for time-dependent vector fields.

LEMMA 11.1. *Let X and Y be smooth vector fields on M, and let ϕ be a diffeomorphism of M. Then $\phi \circ \phi_X^t \circ \phi^{-1} = \phi_Y^t$ for all t if and only if $\phi_* X = Y$. If $X = X_H$ and $Y = X_F$ are Hamiltonian (with respect to some symplectic form if M has even dimension) or strictly contact (with respect to some contact form if M is odd-dimensional) and ϕ is symplectic or contact, respectively, then $\phi \circ \phi_H^t \circ \phi^{-1} = \phi_F^t$ for all t if and only if $e^h H = \phi^* F = F \circ \phi$. Here $\phi^*\alpha = e^h\alpha$ if ϕ is a contact diffeomorphism, and $h = 0$ if ϕ is symplectic.*

The proposition that we state next bypasses reference to the vector fields and directly relates the isotopies to their Hamiltonian functions.

PROPOSITION 11.2. [**BS12, MO07**] *Suppose that Φ_H and Φ_F are continuous or smooth Hamiltonian isotopies (of a symplectic manifold) or strictly contact isotopies (of a regular contact manifold) and that ϕ is a symplectic homeomorphism or the uniform limit of strictly contact diffeomorphisms. Then $\phi \circ \phi_H^t \circ \phi^{-1} = \phi_F^t$ for all t if and only if $H = F \circ \phi$.*

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Example 11.3. Let (M^2, ω) be a symplectic surface, and let F be a smooth function on M which in local (Darboux) coordinates near some point in M has the form $F(r, \theta) = e^{-f(r,\theta)}$, where

$$f(r, \theta) = \frac{4}{r^2(1 + 15\cos^2\theta)}$$

is the composition of the map $(r, \theta) \mapsto 1/r^2$ with the area-preserving change of coordinates $(x, y) \mapsto (2x, y/2)$. Here $r \in \mathbb{R}_{\geq 0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ denote polar coordinates, and $x = r\cos\theta$ and $y = r\sin\theta$ denote rectangular coordinates in the plane. By cutting off the Hamiltonian F outside a neighborhood of the origin, we may assume it is compactly supported in the domain of the Darboux chart.

For a disk $D^2 \subset \mathbb{R}^2$ contained in the domain of the Darboux chart and centered at the origin, let $\phi_\rho : D^2 \to D^2$ be an area-preserving homeomorphism compactly supported in the interior of D^2, defined by $(r, \theta) \mapsto (r, \theta + \rho(r))$ for $r > 0$ and $\phi_\rho(0) = 0$ at the origin, where $\rho : (0, 1] \to \mathbb{R}$ is a smooth function with $\rho(r) = 0$ near $r = 1$; cf. [**MO07**, Example 4.2] or [**Mül08b**, Example 2.6.5]. This extends to an area-preserving homeomorphism of M by the identity outside $D^2 \subset M$, which is smooth everywhere except at the origin with an appropriate choice of ρ. Indeed, by imposing the conditions that $\rho > 0$ and $\rho'(r) \to -\infty$ sufficiently fast, ϕ_ρ is not even Lipschitz. Consider the function $H(r, \theta) = F \circ \phi_\rho(r, \theta)$, which is obviously smooth away from the origin. Since F decays exponentially as $r \to 0^+$, $F \circ \phi_\rho$ converges to zero as $r \to 0^+$. Similarly, one sees that all partial derivatives at the origin exist and vanish, and thus H is a smooth function on M. By Proposition 11.2, we have $\phi_\rho \circ \phi_H^t \circ \phi_\rho^{-1} = \phi_F^t$, that is, the smooth Hamiltonian vector fields X_H and X_F are topologically conjugate. If $\rho(r)$ grows like r^{-a} as $r \to 0^+$, where $0 < a < 2$, then ϕ_ρ becomes a Hamiltonian homeomorphism that is not Lipschitz.

Recall that on a smooth manifold M of dimension at most three, every homeomorphism can be approximated uniformly by diffeomorphisms; and if a volume-preserving homeomorphism can be approximated uniformly by diffeomorphisms, it can also be approximated uniformly by volume-preserving diffeomorphisms. This, in particular, means that every area-preserving homeomorphism is a symplectic homeomorphism.

LEMMA 11.4. *Suppose that $\rho(r)$ grows like r^{-a-1} as $r \to 0^+$, where $a > 0$. If ψ is an area-preserving homeomorphism of M that conjugates the flows of H and F, then ψ is not Lipschitz; consequently, no C^1-symplectic diffeomorphism conjugates the Hamiltonian vector fields X_H and X_F.*

Proof. Because ψ is a symplectic homeomorphism, $F \circ \phi_\rho = H = F \circ \psi$ by Proposition 11.2, and thus $F \circ (\psi \circ \phi_\rho^{-1}) = F$. That is, the homeomorphism $\psi \circ \phi_\rho^{-1}$ preserves the level sets of F, which near the origin are concentric ellipses centered at the origin. By hypothesis, there exist two sequences of positive numbers $r_n > r_n' \to 0^+$ such that $r_n - r_n' < r_n^2$, $\rho(r_n) = \pi/2 \bmod 2\pi$, and $\rho(r_n') = \pi \bmod 2\pi$. Assuming ψ is Lipschitz, its Lipschitz constant $L > 0$ must obey the inequalities

$$L \geq \frac{|\psi(\phi_\rho^{-1}(r_n, \pi/2)) - \psi(\phi_\rho^{-1}(r_n', \pi))|}{|\phi_\rho^{-1}(r_n, \pi/2) - \phi_\rho^{-1}(r_n', \pi)|} \geq \frac{r_n - r_n'/4}{r_n - r_n'} > \frac{1}{4}\left(\frac{3}{r_n} + 1\right) \to +\infty.$$

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The middle inequality holds because $\phi_\rho^{-1} = \phi_{-\rho}$ and $\psi \circ \phi_\rho^{-1}$ preserves the concentric ellipses. Thus ψ cannot be Lipschitz continuous, and there is no symplectic C^1-diffeomorphism conjugating the two Hamiltonian vector fields. □

In fact, the lemma still holds if $\rho(r)$ grows like r^{-a} for $a > 0$. Then there exists $0 < \epsilon < a$ and sequences r_n and r_n' as above, except that now $r_n - r_n' < r_n^{a-\epsilon+1}$. In this case, one obtains the inequality $L > (1/4)(3r_n^{-a+\epsilon} + 1) \to +\infty$. It is possible to embed countably many disjoint disks (of shrinking radii) into any surface (M, ω), producing examples where ψ fails to be C^1 at at least countably many points.

Example 11.5. Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$. In local Darboux coordinates $(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n)$, consider the autonomous Hamiltonian

$$G(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n) = \int_r^1 s\rho(s)\, ds,$$

where $r = \sqrt{r_1^2 + \cdots + r_n^2}$. For an appropriate choice of ρ as above, this Hamiltonian generates the Hamiltonian homeomorphism $\phi_\rho : M \to M$ given by $\phi_\rho(0) = 0$ and

$$\phi_\rho(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n) = (r_1, \ldots, r_n, \theta_1 + \rho(r), \ldots, \theta_n + \rho(r));$$

see [**BS11, Mül08a, MO07**]. Define $\overline{F} = e^{-f(r_1,\theta_1)}$, where the function f is as in Example 11.3, and let $\overline{H} = \overline{F} \circ \phi_\rho$. Arguing as above, we obtain two smooth Hamiltonian (and, in particular, exact divergence-free) vector fields $X_{\overline{F}}$ and $X_{\overline{H}}$ on M whose Hamiltonian isotopies are conjugated by a symplectic homeomorphism that is not even Lipschitz. Moreover, there does not exist a C^1-symplectic diffeomorphism or a Lipschitz symplectic homeomorphism conjugating the two isotopies.

If M is non-compact and ψ is a conformally symplectic C^1-diffeomorphism, i.e. $\psi^*\omega = c\omega$, then $cF = F \circ (\psi \circ \phi_\rho^{-1})$, and the same argument as above can be used to show that ψ is not Lipschitz.

Example 11.6. Let $S^1 \to M^{2n+1} \to B^{2n}$ be the prequantization bundle over a closed symplectic manifold (B^{2n}, ω) with projection $p : M \to B$. Let F and $H = F \circ \phi_\rho$ be smooth functions on B as in Example 11.5. Then $\tilde{F} = F \circ p$ and $\tilde{H} = H \circ p = F \circ (\phi_\rho \circ p)$ are basic functions on M, and thus generate strictly contact isotopies of M. Since ϕ_ρ is a Hamiltonian homeomorphism, it induces a well-defined strictly contact homeomorphism $\tilde{\phi}_\rho$ of M with $\phi_\rho \circ p = p \circ \tilde{\phi}_\rho$ [**BS12**]. Then $\tilde{H} = F \circ (\phi_\rho \circ p) = F \circ (p \circ \tilde{\phi}_\rho) = \tilde{F} \circ \tilde{\phi}_\rho$, and the strictly contact isotopies of $\tilde{F}$ and $\tilde{H}$ are topologically conjugate by Proposition 11.2. If $\tilde{\psi}$ is any other contact diffeomorphism on M (or is the uniform limit of strictly contact diffeomorphisms) conjugating the two isotopies, then $e^h \tilde{F} = \tilde{F} \circ (\tilde{\psi} \circ \tilde{\phi}_\rho^{-1})$ for a smooth and thus bounded function h on M. By the same argument as above, any such $\tilde{\psi}$ has regularity less than Lipschitz.

If we allow the Hamiltonian vector fields to be time-dependent, we can produce examples of vector fields that are not conjugated by *any* C^1-diffeomorphism.

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Example 11.7. Let $f_t(r, \theta)$ be a smooth function on $[0, 1] \times \mathbb{R}^2$ that near the origin is given by composition of the function $(r, \theta) \mapsto 1/r^2$ with a time-dependent area-preserving change of coordinates equal to $(x, y) \mapsto (\sigma(t)x, y/\sigma(t))$, where $\sigma(t) = 2$ near $t = 0$ and $\sigma(t) = 1/2$ near $t = 1$, and define a smooth function by $F_t(r, \theta) = e^{-f_t(r,\theta)}$. Let ϕ_ρ be as above, and let $H_t = F_t \circ \phi_\rho^{-1}$. By the same argument as before, this gives rise to smooth Hamiltonian functions on the surface (M, ω), and their Hamiltonian isotopies are topologically conjugate, i.e. $\phi_\rho \circ \phi_F^t = \phi_H^t \circ \phi_\rho$.

LEMMA 11.8. *The Hamiltonian isotopies $\{\phi_F^t\}$ and $\{\phi_H^t\}$ of M are not C^1-conjugate, i.e. there exists no C^1-diffeomorphism ψ of M such that $\psi \circ \phi_F^t = \phi_H^t \circ \psi$.*

Proof. Arguing by contradiction, suppose there exists a C^1-diffeomorphism ψ such that $\psi \circ \phi_F^t = \phi_H^t \circ \psi$. Then, in local coordinates, the diffeomorphism ϕ_F^t preserves the area form $\psi^*\omega = \det d\psi(x)\omega$. Thus $\det d\psi(x) = \det d\psi(\phi_F^t(x))$ for all x near zero. Near $t = 0$ and $t = 1$, the flow of F follows concentric ellipses centered at the origin but with major axis at $t = 0$ perpendicular to the major axis at $t = 1$. This implies that $\det d\psi(x) = c$ is independent of x, or ψ is conformally symplectic, at least near the origin. Since the transformation law is a local statement, we have $cF_t = H_t \circ \psi$, or $cF_t \circ (\psi^{-1} \circ \phi_\rho) = F_t$ near the origin. By essentially the same argument as above, the local inverse ψ^{-1} is not Lipschitz near the origin, which is a contradiction. □

Note that the examples can be modified so that F_t is C^∞-close to an autonomous Hamiltonian. The statement that there exists no C^1-map ψ such that $\psi \circ \phi_F^t = \phi_H^t \circ \psi$ is false: if ψ_2 is the constant map $\psi_2(p, x) = x_0$, where x_0 is any point in M at which X_H^t vanishes for all t (e.g. the origin or a point on the corresponding Reeb circle in the examples above), the above identity holds.

As mentioned in the introduction, one can also define the helicity as follows. For two distinct points $x, y \in M$ and two times t_1 and t_2, consider the pieces of trajectories $\phi_X^t(x)$, $0 \le t \le t_1$, and $\phi_X^t(y)$, $0 \le t \le t_2$, and close them up into loops using a 'system of short paths' in M. The asymptotic linking number of these two loops is defined, and the helicity equals the average of these asymptotic linking numbers over $M \times M$ minus the diagonal. See, for example, [**Ghy07**] for details. This alternate definition suggests that the helicity could be invariant under topological conjugation. However, the system of short paths considered above to close up the pieces of trajectories may become tangled when conjugating with a homeomorphism, so invariance of the helicity is not obvious. We now observe that the problem is indeed a local one.

LEMMA 11.9. *Let $\{U_i\}$ be an open cover of a closed smooth three-manifold M with volume form μ. The helicity is invariant under conjugation by volume-preserving homeomorphisms if and only if it is invariant under conjugation by volume-preserving homeomorphisms that are isotopic to the identity through isotopies of volume-preserving homeomorphisms that are supported in an open set U_i (and have vanishing mass flow).*

Proof. As noted before, in dimension less than or equal to three, every homeomorphism can be approximated uniformly by diffeomorphisms, and a homeomorphism ϕ that

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preserves the measure induced by μ can be approximated uniformly by volume-preserving diffeomorphisms. The group $\mathrm{Homeo}(M, \mu)$ of volume-preserving homeomorphisms of M is locally path-connected in the compact-open topology [**Fat80**]. Thus there exists a volume-preserving diffeomorphism ψ sufficiently close to ϕ that they are isotopic inside $\mathrm{Homeo}(M, \mu)$ or, equivalently, such that the volume-preserving homeomorphism $\phi \circ \psi^{-1}$ is isotopic to the identity in $\mathrm{Homeo}(M, \mu)$. The helicity is invariant under conjugation by $\phi = (\phi \circ \psi^{-1}) \circ \psi$ if and only if it is invariant under conjugation by $\phi \circ \psi^{-1}$ (Lemma 2.2). Thus, without loss of generality, we may assume that ϕ is isotopic to the identity through an isotopy of volume-preserving homeomorphisms. If that is the case, its mass flow is well-defined [**Fat80**]. There is a dual homomorphism, namely the flux mentioned in §2, for isotopies of volume-preserving diffeomorphisms [**Ban97**]. By surjectivity of the flux, there exists an isotopy of volume-preserving diffeomorphisms with the same mass flow as the isotopy connecting ϕ to the identity. By the same argument as above, the general case reduces to considering volume-preserving homeomorphisms ϕ with vanishing mass flow. Such homeomorphisms can be fragmented into a finite composition $\phi = \phi_m \circ \cdots \circ \phi_1$ of volume-preserving homeomorphisms so that each ϕ_k is supported (and isotopic to the identity with vanishing mass flow) inside an element of an open cover of M [**Fat80**]. □

For example, one may choose as subsets the domains of a Darboux atlas with respect to a contact form on M. We note that an isotopy of volume-preserving diffeomorphisms with vanishing flux or mass flow may also be fragmented into isotopies with 'small' support; however, the helicity is not a homomorphism.

12. *Higher-dimensional helicities*

There are several generalizations of helicity to higher dimensions, which are studied, for example, in [**Khe03, KV03, Riv02**]; see also [**AK98**, Ch. III, §7.B]. In [**KV03**], the authors consider the linking number of a divergence-free vector field on a manifold of arbitrary dimension with a codimension-two foliation endowed with an invariant transverse measure. In this short section, we compute this linking number for a strictly contact vector field on a regular contact manifold, simultaneously generalizing [**KV03**, Example 3.8] (Hamiltonian vector fields on closed symplectic manifolds) and [**KV03**, Example 3.9] (Reeb vector fields on closed contact manifolds).

PROPOSITION 12.1. *Let (M^{2n+1}, α) be a closed manifold with a regular contact form α, and let (B^{2n}, ω) be the base of the corresponding Boothby–Wang bundle. Suppose that $A \subset B$ is a closed, oriented and null-homologous codimension-two submanifold, and write $N = p^{-1}(A) \subset M$ where $p : M \to B$ is the projection. Suppose, further, that X_H is a strictly contact vector field on (M, α), and write F for the unique smooth function on B satisfying $p^*F = H$. Then*

$$\mathcal{H}(X_H, N) = -n \int_A F\, \omega^{n-1}.$$

This number obviously extends to an invariant of continuous strictly contact isotopies on (M, α), and it is invariant under conjugation by uniform limits of strictly contact diffeomorphisms, provided the limit preserves N.

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Proof. By definition (see [**KV03**]),

$$\mathcal{H}(X_H, N) = \int_N \beta,$$

where β is a primitive of $\iota_X \mu$ and μ is the canonical volume form on M induced by α. Since $\int_N \tau = \int_A p_* \tau = 0$ for any closed $(2n-1)$-form τ on M (A is null-homologous), N is null-homologous as well, so that the above integral is well-defined. Here $p_* : H^{2n-1}(M) \to H^{2n-2}(B)$ is the induced map in the Gysin sequence of the S^1-bundle $S^1 \to M \to B$. Recall from (9) that

$$\beta = (n+1)p^*\gamma + ((n+1)c_H - nH)\alpha \wedge (d\alpha)^{n-1},$$

so that by the same partition-of-unity argument as above, and since $p_* R_\alpha = 0$,

$$\int_N \beta = \int_A ((n+1)c_H - nF)\omega^{n-1} = -n \int_A F\omega^{n-1}$$

as claimed. □

See [**KV03**] for further replacing the submanifold A by an oriented (possibly singular) codimension-two foliation $\mathcal{F}$ with a holonomy-invariant transverse measure.

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A. *Appendix. Proof of Proposition 6.1*

The short exact sequence (7) is a Serre fibration. Indeed, a homotopy $\Phi_t : D^k \to \mathrm{Ham}(M, \omega)$, $0 \le t \le 1$, with respect to the C^∞-topology on $\mathrm{Ham}(B, \omega)$ is a smooth map $D^k \times [0, 1] \to \mathrm{Ham}(B, \omega)$, $(s, t) \mapsto \phi_{s,t} \in \mathrm{Ham}(B, \omega)$. There exists a unique family $F_{s,t} = F_s(t, \cdot)$ of (normalized) smooth Hamiltonian functions such that $\phi_{s,t} = \phi^t_{F_s} \circ \phi_{s,0}$, where $t \mapsto \phi^t_{F_s}$ is the Hamiltonian flow of F_s starting at the identity. Define $H_{s,t} = p^* F_{s,t} = F_{s,t} \circ p : M \to \mathbb{R}$ for $0 \le t \le 1$ and $s \in D^k$, and denote by $t \mapsto \psi^t_{H_s}$ the strictly contact flow of H_s starting at the identity. Given a lift $\Psi_0 : D^k \to \mathrm{Diff}_0(M, \alpha)$, say $s \mapsto \psi_{s,0}$, of Φ_0, the homotopy $\Psi_t : D^k \to \mathrm{Diff}_0(M, \alpha)$ defined by $s \mapsto \psi_{s,t} = \psi^t_{H_s} \circ \psi_{s,0} \in \mathrm{Diff}(M, \alpha)$ lifts the homotopy Φ_t. Thus (7) has the homotopy lifting property with respect to all disks.

Since $\mathrm{Ham}(B, \omega)$ is path-connected, (7) gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_k(S^1) \to \pi_k(\mathrm{Diff}_0(M, \alpha)) \to \pi_k(\mathrm{Ham}(B, \omega)) \to \pi_{k-1}(S^1) \to \cdots . \quad \text{(A.1)}$$

According to [**Pol01**, §7.2], if B is a closed and connected surface, then the fundamental group of $\mathrm{Ham}(B, \omega)$ is $\mathbb{Z}_2$ for $B = S^2$, with generator being the one-turn rotation of the sphere, and is trivial otherwise. Moreover, $\pi_2(\mathrm{Ham}(B, \omega)) = 0$ for all closed surfaces B. To see this, recall that the inclusion $\mathrm{Ham}(B, \omega) \hookrightarrow \mathrm{Symp}_0(B, \omega)$ induces an isomorphism on homotopy groups π_k for $k > 1$; see [**MS98**, §10.2]. Moser's argument shows that for

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closed surfaces B, the inclusion of $\mathrm{Symp}_0(B, \omega)$ into $\mathrm{Diff}_0(B)$ induces isomorphisms on all homotopy groups; see [**Ban97**, §1.5] or, again, [**Pol01**]. Moreover, the latter is contractible for genus at least 2, and has strong deformation retract T^2 or SO(3) for genus 1 or 0, respectively [**EE69**]. Now $\pi_k(T^2)$ obviously vanishes for $k > 1$, and since $\mathrm{SO}(3) \cong \mathbb{R}P^3$ has universal covering space S^3, we get $\pi_k(\mathrm{SO}(3)) \cong \pi_k(S^3)$ for $k > 1$. Upon combining these facts, we see that $\pi_2(\mathrm{Ham}(B, \omega))$ indeed vanishes for all closed surfaces B.

From the long exact sequence (A.1) we obtain the description of the fundamental group of $\mathrm{Diff}(M, \alpha)$. For $k = 2$, recall that $\pi_2(\mathrm{Ham}(B, \omega))$ and $\pi_2(S^1)$ are zero, and therefore $\pi_2(\mathrm{Diff}(M, \alpha)) = 0$. For $k > 2$, the homotopy groups $\pi_k(S^1)$ and $\pi_{k-1}(S^1)$ to the left and right vanish; thus $\pi_k(\mathrm{Diff}(M, \alpha)) \cong \pi_k(\mathrm{Ham}(B, \omega))$, and the claim follows from the same argument as above. □

B. *Appendix. A non-regular contact three-manifold*

By Martinet's theorem, any closed three-manifold admits a contact structure. On the other hand, the torus T^3 does not admit a regular contact form. Consider the contact form $\alpha = \cos z\, dx - \sin z\, dy$ on T^3 with induced volume form $dx \wedge dy \wedge dz$, where $x, y, z \in \mathbb{R}/(2\pi\mathbb{Z})$. A basic function on (T^3, α) depends only on z [**Mül11**] and thus can be written as a Fourier series

$$H(z) = \sum_{n\in\mathbb{Z}} c_n e^{inz} = \sum_{n=0}^{\infty} a_n \cos(nz) + b_n \sin(nz).$$

Its strictly contact vector field X_H has flux

$$[\iota_{X_H}(\alpha \wedge d\alpha)] = a_1[dy \wedge dz] + b_1[dx \wedge dz],$$

and thus X_H is exact if and only if $a_1 = 0 = b_1$ or, equivalently, $c_1 = 0$ [**Mül11**].

PROPOSITION B.1. *The helicity of a strictly contact vector field X_H whose contact Hamiltonian $H : T^3 \to \mathbb{R}$ satisfies $c_1 = 0$ is given by*

$$\mathcal{H}(X_H) = -\sum_{n\in\mathbb{Z}} \left(3 + \frac{4}{n^2-1}\right)|c_n|^2 = c_0^2 - 2\sum_{n>0}\left(3 + \frac{4}{n^2-1}\right)|c_n|^2. \tag{B.1}$$

In particular, the helicity is bounded from above and below by a multiple of $\|H\|_{L^2}^2$.

Proof. If $c_1 = 0$, then also $c_{-1} = \overline{c}_1 = 0$, and we can define *real* functions

$$F(z) = \sum_{n\in\mathbb{Z}} c_n \left(\frac{1}{n+1} e^{i(n+1)z} - \frac{1}{n-1} e^{i(n-1)z}\right),$$
$$G(z) = \sum_{n\in\mathbb{Z}} i c_n \left(\frac{1}{n+1} e^{i(n+1)z} + \frac{1}{n-1} e^{i(n-1)z}\right).$$

Then

$$d(F\, dx + G\, dy - H\alpha) = 2H\, d\alpha - d(H\alpha) = \iota_{X_H}(\alpha \wedge d\alpha),$$

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and a direct computation shows that

$$\begin{aligned}&(F\,dx + G\,dy - H\alpha) \wedge d(F\,dx + G\,dy - H\alpha)\\ &\quad = (2H(F\cos z - G\sin z) - 3H^2)\,dx \wedge dy \wedge dz + \text{exact terms}\end{aligned}$$

and

$$2H(F\cos z - G\sin z) - 3H^2 = \sum_{n,m\in\mathbb{Z}} \left(-\frac{4}{n^2-1} - 3\right) c_n c_m e^{i(n+m)z}.$$

We note that the constant term is $c(H) = c_0 = \int_{T^3} H\,dx \wedge dy \wedge dz$, and also that $\sum_{n\in\mathbb{Z}} |c_n|^2 = \|H\|^2_{L^2}$; thus (B.1) follows. □

References

[AK98] V. I. Arnold and B. A. Khesin. *Topological Methods in Hydrodynamics (Applied Mathematical Sciences, 125)*. Springer, New York, 1998.

[Arn86] V. I. Arnol'd. The asymptotic Hopf invariant and its applications. *Selecta Math. Soviet.* **5**(4) (1986), 327–345, selected translations.

[Ban78a] A. Banyaga. The group of diffeomorphisms preserving a regular contact form. *Topology and Algebra (Proc. Colloq., Eidgenöss. Tech. Hochsch., Zurich, 1977) (Monographies de L'Enseignement Mathématique, 26)*. Université de Genève, Geneva, 1978, pp. 47–53.

[Ban78b] A. Banyaga. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv.* **53**(2) (1978), 174–227.

[Ban97] A. Banyaga. *The Structure of Classical Diffeomorphism Groups (Mathematics and its Applications, 400)*. Kluwer, Dordrecht, 1997.

[Bla10] D. E. Blair. *Riemannian Geometry of Contact and Symplectic Manifolds*, 2nd edn *(Progress in Mathematics, 203)*. Birkhäuser, Boston, MA, 2010.

[BS11] L. Buhovsky and S. Seyfaddini. Uniqueness of generating Hamiltonians for continuous Hamiltonian flows. *J. Symplectic Geom.* to appear, arXiv:1003.2612v2.

[BS12] A. Banyaga and P. Spaeth. On the uniqueness of generating Hamiltonians for topological strictly contact isotopies. *Preprint*, 2012, http://newton.kias.re.kr/∼spaeth.

[BW58] W. M. Boothby and H. C. Wang. On contact manifolds. *Ann. of Math.* (2) **68** (1958), 721–734.

[EE69] C. J. Earle and J. Eells. A fibre bundle description of Teichmüller theory. *J. Differential Geom.* **3** (1969), 19–43.

[EG00] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture. *Nonlinearity* **13**(2) (2000), 441–458.

[Fat80] A. Fathi. Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. Éc. Norm. Supér.* (4) **13**(1) (1980), 45–93.

[Fur61] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.* **83** (1961), 573–601.

[Gei08] H. Geiges. *An introduction to Contact Topology (Cambridge Studies in Advanced Mathematics, 109)*. Cambridge University Press, Cambridge, 2008.

[GG97] J.-M. Gambaudo and É. Ghys. Enlacements asymptotiques. *Topology* **36**(6) (1997), 1355–1379.

[GG01] J.-M. Gambaudo and É. Ghys. Signature asymptotique d'un champ de vecteurs en dimension 3. *Duke Math. J.* **106**(1) (2001), 41–79.

[GH55] W. H. Gottschalk and G. A. Hedlund. *Topological Dynamics (American Mathematical Society Colloquium Publications, 36)*. American Mathematical Society, Providence, RI, 1955.

[Ghy07] É. Ghys. Knots and dynamics. *Proceedings of the International Congress of Mathematicians (Madrid, Spain, 2006)*, I. European Mathematical Society, Zürich, 2007, pp. 247–277.

[Hir63] M. W. Hirsch. Obstruction theories for smoothing manifolds and maps. *Bull. Amer. Math. Soc.* **69** (1963), 352–356.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995.

https://doi.org/10.1017/S0143385712000387 Published online by Cambridge University Press

[Khe03] B. A. Khesin. Geometry of higher helicities. *Mosc. Math. J.* **3**(3) (2003), 989–1011, dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday.

[Kod95a] K. Kodaka. Anzai and Furstenberg transformations on the 2-torus and topologically quasi-discrete spectrum. *Canad. Math. Bull.* **38**(1) (1995), 87–92.

[Kod95b] K. Kodaka. Tracial states on crossed products associated with Furstenberg transformations on the 2-torus. *Studia Math.* **115**(2) (1995), 183–187.

[KV03] D. Kotschick and T. Vogel. Linking numbers of measured foliations. *Ergod. Th. & Dynam. Sys.* **23**(2) (2003), 541–558.

[Mar71] J. Martinet. Formes de contact sur les variétés de dimension 3. *Singularities – Symposium, II (Liverpool, UK 1969/1970) (Lecture Notes in Mathematics, 209)*. Springer, Berlin, 1971, pp. 142–163.

[MO07] S. Müller and Y.-G. Oh. The group of Hamiltonian homeomorphisms and C^0-symplectic topology. *J. Symplectic Geom.* **5**(2) (2007), 167–219.

[MS98] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*, 2nd edn *(Oxford Mathematical Monographs)*. Oxford University Press, New York, 1998.

[MS11] S. Müller and P. Spaeth. Topological contact dynamics I: symplectization and applications of the energy-capacity inequality. *Preprint*, 2011, arXiv:1110.6705v2.

[Mül08a] S. Müller. The group of Hamiltonian homeomorphisms and topological symplectic topology. *PhD Thesis*, Department of Mathematics, University of Wisconsin – Madison, 2008.

[Mül08b] S. Müller. The group of Hamiltonian homeomorphisms in the L^∞-norm. *J. Korean Math. Soc.* **45**(6) (2008), 1769–1784.

[Mül11] S. Müller. A note on the volume flux of smooth and continuous strictly contact isotopies. *Preprint*, 2011, arXiv:1107.4869v1.

[Mun59] J. Munkres. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Bull. Amer. Math. Soc.* **65** (1959), 332–334.

[Mun60] J. Munkres. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Ann. of Math.* (2) **72** (1960), 521–554.

[Mun65] J. R. Munkres. Higher obstructions to smoothing. *Topology* **4** (1965), 27–45.

[Oh06] Y.-G. Oh. C^0-coerciveness of Moser's problem and smoothing area preserving homeomorphisms. *Preprint*, 2006, arXiv:math/0601183v5.

[Pol01] L. Polterovich. *The Geometry of the Group of Symplectic Diffeomorphisms (Lectures in Mathematics ETH Zürich)*. Birkhäuser, Basel, 2001.

[Riv02] T. Rivière. High-dimensional helicities and rigidity of linked foliations. *Asian J. Math.* **6**(3) (2002), 505–533.

[Rou90] H. Rouhani. A Furstenberg transformation of the 2-torus without quasi-discrete spectrum. *Canad. Math. Bull.* **33**(3) (1990), 316–322.

[Sik07] J.-C. Sikorav. Approximation of a volume-preserving homeomorphism by a volume-preserving diffeomorphism. *Preprint*, 2007, http://www.umpa.ens-lyon.fr/∼symplexe/publications.php.

[Vit06] C. Viterbo. On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows. *Int. Math. Res. Not.* (2006), Art. ID 34028; MR 2233715(2007h:37081a). Erratum. *Int. Math. Res. Not.* (2006), Art. ID 38748; MR 2272093 (2007h:37081b).

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