CARTER SUBGROUPS IN CLASSICAL GROUPS

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Dedicated to the memory of Brian Hartley

Introduction

Let F_q be a finite field of characteristic r and order $q = r^a$, $V = F_q^n$ the vector space of finite dimension $n \geqslant 1$ over F_q, and $\mathrm{Mat}(n, q)$ the matrix algebra on V. We denote by $H_n(q)$ (or H_n if q is clear from the context) one of the following classical groups: $\mathrm{Sp}(n, q)$ (with n even), the symplectic group on V; $\mathrm{O}(n, q)$ (with q odd), one of the full orthogonal groups on V in odd characteristic; $\mathrm{U}(n, q)$ (with q a square), the full unitary group on V.

We recall that, if H is any group, a subgroup G of H is called a *Carter subgroup* of H if G is nilpotent and self-normalizing in G. A classical result [**1**] states that any finite solvable group contains exactly one conjugacy class of Carter subgroups. Carter subgroups in finite non-solvable groups have been investigated for their own sake as well as for applications (for example, searching for complete groups, see [**8**]). A non-solvable group may very well contain no Carter subgroups (the smallest example being the alternating group A_5). However, we do not know of any finite group which contains two non-conjugate Carter subgroups. In fact, it has been shown in [**4**] that, if any such group exists, a smallest example would necessarily be almost-simple.

There is strong evidence that in finite groups of Lie type, the existence of Carter subgroups depends on the field characteristic, that is, Carter subgroups exist in general only when the characteristic is odd or the ground field has order 2, in which case they are the normalizers of Sylow 2-subgroups. The only exceptions to this statement appear to be some groups of low rank in characteristic 2 or 3 (see [**7**] and references quoted there). In the context of groups of Lie type, determination of Carter subgroups also seems worthwhile as a piece of the programme for describing 'intermediate' subgroups (see the recent survey [**13**]).

The purpose of this paper is to prove the following.

MAIN THEOREM. *Let G be a Carter subgroup of $H_n = H_n(q)$. Then, either G is the normalizer of a Sylow 2-subgroup of H_n, or H_n is solvable and one of the following holds:*

(i) $H_n = \mathrm{Sp}(2, 3)$ *and G has order* 6*;*

(ii) $H_n = \mathrm{U}(3, 4)$ *and G has order* $3^2 \cdot 2$.

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1. *Preliminary results*

In this section, we recall some basic facts and prove a number of preliminary results to be used in the subsequent parts of the paper.

(A) *Classical groups as norm-groups*

First of all, we recall that, by definition (for example, see [**5**]),

$$H_n = H_n(q) = \{X \in \mathrm{Mat}(n,q) \mid XJ(X^\tau)^t = J\},$$

where τ is an automorphism of the field F_q such that $\tau^2 = \mathrm{id}$, $X \mapsto X^\tau$ is the automorphism of $\mathrm{Mat}(n,q)$ induced by τ, and $J \in \mathrm{Mat}(n,q)$ is non-singular and such that $J = \pm (J^\tau)^t$. Moreover,

(1) $H_n = \mathrm{Sp}(n,q)$ if and only if $\tau = \mathrm{id}$, $J = -J^t$ and J has zero-diagonal if q is even, and up to equivalence (that is, up to conjugation in $\mathrm{GL}(n,q)$) we may assume that $J = \mathrm{antidiag}(I_m, -I_m)$, where $n = 2m$;

(2) $H_n = \mathrm{O}(n,q)$ if and only if $\tau = \mathrm{id}$ and $J = J^t$, and up to equivalence we may assume that

$$J \in \left\{ \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \qquad \text{if } n = 2m+1 \text{ is odd,}$$

$$J \in \left\{ \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}, \quad \mathrm{diag}\left(\begin{pmatrix} 0 & I_{m-1} \\ I_{m-1} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon \end{pmatrix} \right) \right\} \text{ if } n = 2m \text{ is even,}$$

where ε is a non-square in F_q.

(3) $H_n = \mathrm{U}(n,q)$ if and only if $\tau \neq \mathrm{id}$ and $J = (J^\tau)^t$, and up to equivalence we may assume that $J = I_n$.

(Notice that in the orthogonal case, when n is odd there is just one orthogonal group, whereas when n is even there are two isomorphism types of orthogonal groups, according as to whether the Witt index is maximal or not. Also, in the unitary case, $\tau: x \mapsto x^{\sqrt{q}}$ (for $x \in F_q$).)

Now, let $\sigma: \mathrm{Mat}(n,q) \to \mathrm{Mat}(n,q)$ be the map defined by $X \mapsto J(X^\tau)^t J^{-1}$. As $J = \pm (J^\tau)^t$, it follows that σ is an *involution* of $\mathrm{Mat}(n,q)$, by which we mean, as is usual in ring theory, an antiautomorphism of $\mathrm{Mat}(n,q)$ of order 2. It is immediate to check that $H_n = \{X \in \mathrm{Mat}(n,q) \mid \sigma(X)\, X = I_n\}$, the so-called *norm-group* with respect to σ. Conversely, any norm-group is identifiable, up to an automorphism of $\mathrm{Mat}(n,q)$, with one of the classical groups listed above, unless $r = 2$. This is a result of A. O. Wagner [**14**], which we quote, with a complement, in the following.

PROPOSITION 1. *Let σ be an involution of* $\mathrm{Mat}(n,q)$, $n = 2m+s$, $0 \leqslant s \leqslant 1$, *and let* $U^\sigma(n,q) = \{X \in \mathrm{Mat}(n,q) \mid \sigma(X)\, X = I_n\}$. *Then, either*

(i) *up to conjugation in* $\mathrm{Mat}(n,q)$, *we have* $U^\sigma(n,q) = H_n(q)$, *and σ is the map* $X \mapsto J(X^\tau)^t J^{-1}$, *or*

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(ii) *q is even, $n > 1$, and one of the following holds:*
(a) *n is odd and $U^\sigma(n,q) \simeq \mathrm{Sp}(2m,q)$;*
(b) *n is even and greater than 2, and $U^\sigma(n,q)$ is a split extension $N \cdot \mathrm{Sp}(2m-2,q)$, where N is a normal 2-subgroup;*
(c) *$n = 2$, and $U^\sigma(2,q)$ is an elementary 2-group.*

Proof. It is shown in [**14**] that either (i) holds, or q is even and the norm-group $U^\sigma(n,q)$ is conjugate to the group $X_D = \{g \in \mathrm{GL}(n,q) \mid gDg^t = D\}$, where D is a non-singular diagonal matrix of size n over F_q. In the latter case, as q is even, $F_q = (F_q)^2$. So let $D = \mathrm{diag}(d_1^2, \ldots, d_n^2)$ for $d_1, \ldots, d_n \in F_q$, and $Y = \mathrm{diag}(d_1, \ldots, d_n)$. Then $Y^{-1}X_D Y$ preserves the matrix $Y^{-1}D(Y^t)^{-1} = I_n$, and therefore $U^\sigma(n,q)$ is indeed conjugate to the group $X = \{g \in \mathrm{Mat}(n,q) \mid gg^t = I_n\}$. Moreover, X is precisely the group preserving the bilinear form $f(x,y) = \sum x_i y_i$ for each $x, y \in V = F_q^n$. Notice that $g^t g = I_n$ implies that $\sum_i g_{ij}^2 = 1 = (\sum_i g_{ij})^2$, hence $\sum_i g_{ij} = 1$ for every j. Also, as $g^t g = I_n$ is equivalent to $gg^t = I_n$, we have $\sum_j g_{ij} = 1$ for every i. Thus, denoting by $\{v_i, \ldots, v_n\}$ the canonical basis of V, it follows that X fixes the vector $u = \sum_i v_i$ (as $ug = \sum_i (\sum_j g_{ij}) v_i = u$ for every $g \in X$), and therefore also preserves the hyperplane $U = u^\perp = \{x \in V \mid \sum x_i = 0\}$.

Let us first consider the case $n = 2m+1$. Then $V = U \perp \langle u \rangle$, an orthogonal X-invariant decomposition. It follows that the restriction f_U of the form f to U is non-degenerate and symplectic (as $f(x,x) = \sum (x_i)^2 = (\sum x_i)^2 = 0$ for every $x \in U$). Therefore the monomorphism $g \mapsto g_U$ embeds X into $\mathrm{Sp}(f_U) = \mathrm{Sp}(2m,q)$. On the other hand, it is clear that any element of $\mathrm{GL}(n,q)$ fixing u and preserving f_U belongs to X. Thus the map $g \mapsto g_U$ is also surjective, and we conclude that $X \simeq \mathrm{Sp}(2m,q)$.

Now assume that $n = 2m > 2$. Then $u \in U$, and X stabilizes the quotient space $U^1 = U/\langle u \rangle$. The map $\phi: g \mapsto g_{U^1}$, where g_{U^1} denotes the automorphism of U^1 induced by g for every $g \in X$, is a homomorphism of X into $\mathrm{GL}(n-2,q)$. The form f_U is degenerate on U, and its radical is precisely $\langle u \rangle$. Accordingly, f_U induces a non-degenerate symplectic form on U^1, which is preserved by $\phi(X)$. In other words, ϕ maps X into $\mathrm{Sp}(n-2,q)$. Now, consider the subgroup S of X consisting of the matrices of shape $\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$, where s of size $n-1$. By the argument above, as $n-1$ is odd, S is isomorphic to $\mathrm{Sp}(n-2,q)$. It is easy to check that $S \cap \mathrm{Ker}(\phi) = 1$, whence $\phi(S) \simeq S \simeq \mathrm{Sp}(n-2,q)$, and, a fortiori, $\phi(X) \simeq \mathrm{Sp}(n-2,q)$. Also, if we take the basis

$$\{w_1 = v_1 + v_2, w_2 = v_2 + v_3, \ldots, w_{n-2} = v_{n-2} + v_{n-1}, w_{n-1} = u, w_n = v_n\}$$

of V, we observe that $w_i g^2 = w_i + \sum_{j<i} \lambda_{ij} w_j$ (with $\lambda_{ij} = 0$ if $i \leqslant n-1$) for each $g \in \mathrm{Ker}(\phi)$ and for each w_i. Hence $\mathrm{Ker}(\phi)$ is a 2-group. Finally, if $n = 2$, then X is conjugate to a group of matrices of shape $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, whence the result.

(B) *The structure of the enveloping algebra of a finite linear group*

It is well known (for example, compare [**3**, Chapter 8]) that any ring A with unity 1, satisfying the minimum condition on left ideals, can be decomposed in a unique way as a direct sum of indecomposable two-sided ideals: $A = \bigoplus A_i$ for $1 \leqslant i \leqslant k$. Such a decomposition is unique in the strong sense that, if B is any two-sided ideal

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of A such that $A = B \oplus B'$ for some two-sided ideal B', then B is the sum of some of the A_i. (The A_i are the so-called *blocks* of A.) Let $1 = \sum e_i$ (for $e_i \in A_i$) be the block-decomposition of the unity of A. Then, e_i is the unity of $A_i = e_i A = A e_i$ for each i, and $e_i e_j = 0$ if $i \neq j$. Thus the e_i form a set of orthogonal central idempotents of A. In fact the following is true: if e is a central idempotent of A, then the two-sided ideal Ae is a block of A if and only if e is *primitive central*, that is, e cannot be written as a sum of two orthogonal central idempotents. Thus the e_i are characterized by the fact that they are the set of primitive central idempotents of A. We shall call the e_i the *minimal central idempotents* of A.

In the particular case when A is a semisimple ring, the blocks A_i are simple rings, that is, full matrix rings over suitable skew-fields, k is the number of representatives, say $L_1, \ldots, L_k$, of isomorphism classes of minimal left ideals of A, and for each i, the block A_i is defined as the sum of all minimal left ideals of A isomorphic to L_i. (What we have just said essentially embodies the so-called Wedderburn–Artin Structure Theorem.)

The instance we are interested in, is when T is a subgroup of $\mathrm{GL}(n, q)$, and $A = F_q T$ is the enveloping algebra (= linear span) of T in $\mathrm{Mat}(n, q)$. Here 1 is the identity matrix I_n, and the minimal central idempotents e_i of A, as they commute pairwise, are simultaneously diagonalizable. As the e_i are also pairwise orthogonal, we may assume, up to an automorphism of $\mathrm{Mat}(n, q)$, that $e_1 = \mathrm{diag}(I_{n_1}, 0_{n-n_1}), \ldots, e_k = \mathrm{diag}(0_{n-n_k}, I_{n_k})$, with $n_1 + \cdots + n_k = n$. Thus, if we set $M_i = e_i \mathrm{Mat}(n, q) e_i$, we see that the block-diagonal sum $\bigoplus M_i$ is a subalgebra of $\mathrm{Mat}(n, q)$, and each M_i may be identified with the full matrix algebra $\mathrm{Mat}(n_i, q)$ with $n_1 + \cdots + n_k = n$. Moreover, if S denotes the centralizer of T in $\mathrm{GL}(n, q)$, then clearly $F_q(TS)$ centralizes each e_i, and therefore is contained in $\bigoplus M_i$.

Moreover, $A = F_q T$ is semisimple precisely when $(r, |T|) = 1$. In investigating this case, and particularly when $A = F_q T$ happens to be simple, we shall find it convenient to make use of a very concrete version of Wedderburn's Theorem. Recall that a *primitive idempotent* of a semisimple ring R is a generator of a minimal left ideal of R. The results we shall need are summarized in the following proposition.

PROPOSITION 2 (Wedderburn). *Let R be a simple subalgebra of $\mathrm{Mat}(n, q)$ containing the identity I_n, and let k be the rank of a primitive idempotent of R. Then,*

(i) *the centre $Z(R)$ of R is a field extension of F_q of degree (say) h;*

(ii) *$n = km$, where $k = ht$ (for $t, m \in \mathbb{N}$).*

Up to an automorphism of $\mathrm{Mat}(n, q)$, we may assume that,

(iii) *$R = \mathrm{Mat}(m, D)$, where $D = S \otimes I_t$, S the subfield of $\mathrm{Mat}(h, q)$ generated by the companion matrix of a suitable polynomial of degree h irreducible over F_q (here we mean that R may be viewed as block-matrices with blocks of size k, each block being a tensor product $s \otimes I_t$ (for $s \in S$);*

(iv) *$C = C_{\mathrm{Mat}(n, q)}(R) = \mathrm{Mat}(t, S) \otimes I_m$; in particular C is also simple, and $Z(R) = Z(C)$.*

(Of course, an early standard reference such as [**11**] contains all the ingredients that are needed to deduce the above statement. However, we could not find in the literature an explicit and detailed proof. For such a proof we refer to [**9**].)

An easy consequence of Proposition 2 is the following.

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COROLLARY. *Let F_{q^h} be a field extension of F_q of degree h, let ϕ be an embedding of* $\mathrm{Mat}(m, F_{q^h})$ *into* $\mathrm{Mat}(n,q)$ *as F_q-algebras, and let $\varphi: F_{q^h} \to S$ be a field isomorphism, where S is defined as in Proposition* 2(iii). *Then, up to conjugation, we may identify ϕ with the monomorphism ρ defined by $\alpha e_{ij} \mapsto (\varphi(\alpha) \otimes I_t) \otimes e_{ij}$, where $\alpha \in F_{q^h}$, e_{ij} (for $1 \leqslant i,j \leqslant m$) denotes the $m \times m$ matrix having entry* 1 *in position (i,j) and all other entries* 0, *and $ht = k$ is the rank of a primitive idempotent of* $\mathrm{Im}(\phi)$.

Proof. By the Skolem–Noether Theorem, every automorphism of a full matrix algebra over a field is inner. Thus, by Proposition 2, we may assume up to conjugation that $\mathrm{Im}(\phi) = \mathrm{Mat}(m, D)$. Now $\rho\phi^{-1}$ is an automorphism of $\mathrm{Mat}(m, D)$, hence is inner. This gives the result.

(C) *Preliminary results*

LEMMA 1. *Let G be a Carter subgroup of a finite group H, and let $f: H \to S$ be an epimorphism. Then, if* $\mathrm{Ker}(f)$ *is soluble, $f(G)$ is a Carter subgroup of S.*

Proof. Let $n \in N_S(f(G))$, and let $h \in H$ be a preimage of n under f. Then G and G^h are both Carter subgroups of the soluble group $\mathrm{Ker}(f)\,G$. Since a finite soluble group contains exactly one conjugacy class of Carter subgroups [**1**], there exists $k \in \mathrm{Ker}(f)$ such that $G^h = G^k$. It follows $hk^{-1} \in G$, hence $n = f(hk^{-1}) \in f(G)$. Therefore, $f(G)$ is self-normalizing in S. As $f(G)$ is clearly nilpotent, we are done.

LEMMA 2. *Let $H_n = H_n(q)$ be as in* (A). *Assume that*

$$n = n_1 + \cdots + n_k, \quad J = \mathrm{diag}(J_1, \ldots, J_k)$$

with $J_i \in \mathrm{Mat}(n_i, q)$ (for $i = 1, \ldots, k$), and let $H_{n_i} = \{X \in \mathrm{Mat}(n_i, q) \mid XJ_i(X^\tau)^t = J_i\}$. Then,

(i) *$H_{n_i} = \mathrm{Sp}(n_i, q)$, $\mathrm{O}(n_i, q)$ or $\mathrm{U}(n_i, q)$, according as to whether H_n is $\mathrm{Sp}(n,q)$, $\mathrm{O}(n,q)$ or $\mathrm{U}(n,q)$, respectively;*

(ii) *whenever $n_i = n_j$, we may assume that $H_{n_i} = H_{n_j}$, except possibly when $H_n = \mathrm{O}(n,q)$;*

(iii) *assume that G is a Carter subgroup of H_n contained in the block-diagonal sum*

$$\mathrm{Mat}(n_1, q) \oplus \cdots \oplus \mathrm{Mat}(n_k, q),$$

and let G_i denote the projection of G into $\mathrm{Mat}(n_i, q)$ for $i \in \{1, \ldots, k\}$, then G_i is a Carter subgroup of H_{n_i} and $G = G_1 \times \cdots \times G_k$, where each G_i is identified with $\mathrm{diag}(I_{n_1}, \ldots, G_i, \ldots, I_{n_k})$.

Proof. (i) This is obvious.

(ii) This is also clear, by the remarks above concerning the possible choices of J, up to equivalence.

(iii) Clearly G_i is a subgroup of H_{n_i} for each i, hence

$$G \leqslant G_1 \times \cdots \times G_k \leqslant H_{n_1} \times \cdots \times H_{n_k} \leqslant H_n$$

(here each G_i is identified with $\mathrm{diag}(I_{n_1}, \ldots, G_i, \ldots, I_{n_k})$, and each H_{n_i} is identified with $\mathrm{diag}(I_{n_1}, \ldots, H_{n_i}, \ldots, I_{n_k})$). As $G_1 \times \cdots \times G_k$ is nilpotent, the assertion follows.

WARNING. As we have done in the previous lemma, we shall find it convenient in the sequel to use the same notation G_i, H_{n_i} (for $i \in \{1, \ldots, k\}$) to indicate either the

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relevant subgroups of $\mathrm{GL}(n_i, q)$, or their natural embeddings $\mathrm{diag}(I_{n_1}, \ldots, G_i, \ldots, I_{n_k})$, $\mathrm{diag}(I_{n_1}, \ldots, H_{n_i}, \ldots, I_{n_k})$ into $\mathrm{GL}(n, q)$. It will be clear from the context which is meant.

Now, let us consider V as a natural H_n-module. For later use, we need to recall a few basic facts (see [**12**, Lemma 4.1.12 and Corollary 4.1.15]). Namely, let W be a maximal totally singular subspace of V, and let $N = N_{H_n}(W)$ be the stabilizer of W in H_n. The following hold.

(1) There exists another maximal totally singular subspace U, such that $V = (W \oplus U) \perp X$, where $W \oplus U$ is non-degenerate and $X = (W \oplus U)^{\perp}$.

(2) Let L denote the subgroup of N fixing each of the subspaces W, U, X. Then $N = [O_r(N)]\, L$ (split extension), and

(i) if n is even and $H_n \neq O^-(n, q)$, then $X = \{0\}$ and $L \simeq \mathrm{GL}(W)$;

(ii) if n is odd, then $\dim(X) = 1$ and $L \simeq \mathrm{GL}(W) \times \mathrm{I}(X)$, where $\mathrm{I}(X)$ denotes the isometry group of X, a subgroup of F_q^*;

(iii) if $H_n = O^-(n, q)$, then $\dim(X) = 2$ and $L \simeq \mathrm{GL}(W) \times O^-(2, q)$. (In Lie theory, $O_r(N)$, and L are called the 'unipotent radical' and a 'Levi complement' of N, respectively, and N is a 'maximal parabolic' subgroup of H_n.)

(3) The group N contains a Sylow r-subgroup of H_n.

In particular, let us assume that the case $X = \{0\}$ holds, in other words $n = 2m$, and V admits a hyperbolic splitting $V = W \oplus U$, where U and W both have dimension m. Then the following lemma holds.

LEMMA 3. (i) *The group L defined above does not contain any Carter subgroup of H_n.*

(ii) *Let G be a Carter subgroup of H_n contained in N.*

(a) *If $r \neq 2$, then $G = R \times M$, where $R = G \cap O_r(N)$ is an elementary abelian r-group, $(r, |M|) = 1$, and M is isomorphic to the normalizer of a Sylow 2-subgroup of* $\mathrm{GL}(m, q)$.

(b) *If $r = 2$, either $m = 1$, or $q = 2$ and G is a 2-group.*

Proof. (i) For a suitable choice of basis in V, we may assume that $J = \mathrm{antidiag}(J_1, J_2)$, where J_1 and J_2 both have size m and, since $J = \pm(J^{\tau})^t$, it follows that $J_2 = \pm(J_1^{\tau})^t$. It is readily checked that $O_r(N)$ consists of matrices of shape $\begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}$, where $Z \in \mathrm{Mat}(m, q)$, and

$$L = \{\mathrm{diag}(A, J_2((A^{\tau})^t)^{-1} J_2^{-1}) \mid A \in \mathrm{GL}(m, q)\} \simeq \mathrm{GL}(m, q).$$

Assume that G is a Carter subgroup of H_n contained in L. Then it follows from [**6**] that G is the normalizer of a Sylow 2-subgroup of L. Also, it is straightforward to see that $h = \mathrm{antidiag}(J_1 J_2^{-1}, I)$ belongs to H_n and normalizes L. Therefore, $G^h = G^x$ for some $x \in L$. It follows $hx^{-1} \in G$, hence $h \in L$, a contradiction.

(ii) We consider the projection π of N onto L. We have that $\mathrm{Ker}(\pi) = O_r(N)$ is an elementary abelian r-group. Thus $\pi(G)$ is a Carter subgroup of L, by Lemma 1. If $r \neq 2$, it follows from [**6**] that $G/(G \cap O_r(N)) = M$, where $(r, |M|) = 1$, and M is isomorphic to a 2-Sylow normalizer of $\mathrm{GL}(m, q)$. The statement for $r = 2$ is an immediate consequence of [**6**].

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THEOREM 1. *Let G be a Carter subgroup of H_n, T any subgroup of G such that $G \leqslant TC_G(T)$, $\{e_1, \dots, e_k\}$ the set of minimal central idempotents of $A = F_q T$, and n_i the rank of e_i (for $i = 1, \dots, k$). Let σ be the involution of $\mathrm{Mat}(n, q)$ defined by $X \mapsto J(X^\tau)^t J^{-1}$. Then σ fixes each e_i, and the situation of Lemma* 2(iii) *occurs, namely*

$$G = G_1 \times \cdots \times G_k \leqslant H_{n_1} \times \cdots \times H_{n_k} \leqslant H_n,$$

where G_i is a Carter subgroup of the classical group H_{n_i} (for $i = 1, \dots, k$).

Proof. As, by definition, $H_n = \{X \in \mathrm{Mat}(n, q) \mid \sigma(X)\, X = I_n\}$, it follows that σ fixes A, and therefore preserves the set $\{e_1, \dots, e_k\}$. Assume that σ acts non-trivially on the e_i, hence σ interchanges, say, e_{2j+1} with e_{2j+2}, for $0 \leqslant j \leqslant h-1$, and fixes e_i for $i > 2h$. As observed in (B), we may choose a basis of V such that $e_1 = \mathrm{diag}(I_{n_1}, 0_{n-n_1})$, $\dots, e_k = \mathrm{diag}(0_{n-n_k}, I_{n_k})$. Then clearly $Je_{2j+1} = e_{2j+2} J$ for $0 \leqslant j \leqslant h-1$, and $Je_i = e_i J$ for $i > 2h$. As $I_n = e_1 + \cdots + e_k$, it readily follows that J has shape

$$\mathrm{diag}(B_0, \dots, B_{h-1}, J_{2h+1}, \dots, J_k),$$

where each block B_j is anti-diagonal. In particular, B_0 has shape $\mathrm{antidiag}(J_1, J_2)$, where J_1 and J_2 are square blocks of size n_1, the common rank of e_1 and e_2. By Lemma 2(iii), the projection of G into $\mathrm{Mat}(2n_1, q)$ is a Carter subgroup of H_{2n_1}. On the other hand, by our assumptions on T, observe that G is contained in the block-diagonal sum $\mathrm{Mat}(n_1, q) \oplus \mathrm{Mat}(n_2, q) \oplus \cdots \oplus \mathrm{Mat}(n_k, q)$ (compare Section (B)). Therefore this projection consists of matrices of shape $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ (for $X, Y \in \mathrm{GL}(n_1, q)$), contradicting Lemma 3(i). We conclude that σ must act trivially on the e_i. The statement of the Theorem 1 follows by Lemma 2(iii).

2. *Reduction to the indecomposable case*

Let G be a Carter subgroup of H_n. We aim to show that we can reduce to the case when the algebra $F_q G$ is indecomposable. We need two more lemmas.

LEMMA 4. (i) *Let T be a subgroup of G such that $G = T \times U$. Then, if $F_q G$ is indecomposable, $F_q T$ is also indecomposable.*

(ii) *Let r be odd, and P be a Sylow 2-subgroup of G. Then the minimal central idempotents of $F_q G$ are the same as those of $F_q P$.*

Proof. (i) Let $e_1, \dots, e_k$ be the minimal central idempotents of $F_q T$. As $G = T \times U$, the e_i are central in $F_q G$. Thus $F_q G = e_1 F_q G e_1 \oplus \cdots \oplus e_k F_q G e_k$ is a decomposition as a direct sum of 2-sided ideals. This forces $k = 1$.

(ii) Let $e_1, \dots, e_k$ be the minimal central idempotents of $F_q G$. With the notation of Theorem 1, setting $T = G$, we have $G = G_1 \times \cdots \times G_k \leqslant H_{n_1} \times \cdots \times H_{n_k} \leqslant H_n$. Hence, for each $i \in \{1, \dots, k\}$, the matrix $D_i = e_1 + \cdots + (-e_i) + \cdots + e_k$ belongs to $F_q P$. Thus $I_n - D_i = 2e_i \in F_q P$, whence, as r is odd, $e_i \in F_q P$, and therefore

$$F_q P = e_1 F_q P e_1 \oplus \cdots \oplus e_k F_q P e_k.$$

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We claim that this is the block-decomposition of $F_q P$. If not, for some i we have $e_i = b_1 + b_2$, where b_1 and b_2 are orthogonal central idempotents of $F_q P$. It follows that $e_i F_q G e_i = b_1 F_q G b_1 \oplus b_2 F_q G b_2$. But this is a decomposition of $e_i F_q G e_i$ as a direct sum of proper two-sided ideals of $F_q G$, a contradiction.

LEMMA 5. *For a prime p, let Syl_p denote a non-trivial Sylow p-subgroup of H_n for $n > 1$. Then the following holds:*

(a) *if r is odd, $N_{H_n}(\mathrm{Syl}_r)$ is nilpotent if and only if $r = 3$ and $H_n = \mathrm{Sp}(2,3)$;*

(b) *$N_{H_n}(\mathrm{Syl}_2)$ is nilpotent if and only if*

(i) $H_n = \mathrm{Sp}(n,q)$, $q \equiv \pm 1 (\mathrm{mod}\, 8)$,

(ii) $H_n = \mathrm{O}(n,q)$,

(iii) $H_n = \mathrm{U}(n,q)$, *q odd*,

(iv) $H_n = \mathrm{U}(2,4)$,

(v) $q = 2$.

In particular, $N_{H_n}(\mathrm{Syl}_2) = \mathrm{Syl}_2$ if and only if H_n is as in (i), (ii), *or* (v).

Proof. As, by assumption, $\mathrm{Syl}_r \neq 1$, and $O^-(2,q) \simeq D_{2(q+1)}$, it follows that H_n is not $O^-(2,q)$. Thus we may assume that Syl_r is contained in the stabilizer

$$N = N_{H_n}(W) = [O_r(N)]\, L$$

of a non-zero maximal totally singular subspace W of V. Assume $N_{H_n}(\mathrm{Syl}_r)$ to be a Carter subgroup of H_n. Then, a fortiori, $N_N(\mathrm{Syl}_r)$ is a Carter subgroup of N. Now, $N_N(\mathrm{Syl}_r)$ maps via the natural homomorphism onto $N_L(\mathrm{Syl}'_r)$ with Syl'_r a Sylow r-subgroup of L; hence, by Lemma 1, $N_L(\mathrm{Syl}'_r)$ is a Carter subgroup of L. As $\mathrm{Syl}'_r \leqslant \mathrm{GL}(W)$, if $\dim(W) > 1$ it follows from [**6**] that $q = 2$, and therefore $H_n = \mathrm{Sp}(n,2)$. If $\dim(W) = 1$, $\mathrm{Syl}_r = O_r(N)$ and $n \leqslant 4$. A direct inspection shows that L does not centralize $O_r(N)$, unless $H_n = \mathrm{Sp}(2,2)$, $\mathrm{U}(2,4)$ or $\mathrm{Sp}(2,3)$. In all these cases $N_{H_n}(\mathrm{Syl}_r)$ is indeed a Carter subgroup of H_n (being a Sylow 2-subgroup for $H_n = \mathrm{Sp}(n,2)$, and a cyclic group of order 6 for $H_n = \mathrm{Sp}(2,3)$ or $\mathrm{U}(2,4)$). The above arguments prove (a) and parts of (b). The remaining parts of (b) follow from [**2**]. (Actually, if n is odd, [**2**] deals with $\mathrm{SO}(n,q)$, not the full $\mathrm{O}(n,q)$. However, from the fact that Sylow 2-subgroup of $\mathrm{SO}(n,q)$ is self-normalizing in $\mathrm{SO}(n,q)$, it follows immediately that Syl_2 is self-normalizing in $\mathrm{O}(n,q)$.)

THEOREM 2. *Let H_n and G be counterexamples to the Main Theorem, with n minimal. Then $F_q G$ is indecomposable.*

Proof. Clearly we may assume that $n > 1$. Let $e_1, \ldots, e_k$ be the minimal central idempotents of $F_q G$, and assume that $k > 1$. By Theorem 1,

$$G = G_1 \times \cdots \times G_k \leqslant H_{n_1} \times \cdots \times H_{n_k} \leqslant H_n,$$

where each G_i is a Carter subgroup of H_{n_i} (for $i = 1, \ldots, k$).

CLAIM 1: *H_n is not an orthogonal group.* For assume the contrary. Then q is odd, H_{n_i} is an orthogonal group (by Lemma 2(i)), and the Main Theorem applies as $n_i < n$. Thus, by Lemma 5, G_i is a 2-group (this is also true if $n_i = 1$). It follows that G is a (self-normalizing) Sylow 2-subgroup of H_n, a contradiction.

CLAIM 2: *the minimal central idempotents of $F_q G$ have pairwise different ranks.* For let (say) e_1 and e_2 have the same rank. Then, by Lemma 2(ii) we may assume that

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$J_1 = J_2$ and $H_{n_1} = H_{n_2}$. By the minimality of n, there exists $h_1 \in H_{n_1}$ such that $G_2 = h_1 G_1 h_1^{-1}$. Conjugating G by $h = \mathrm{diag}(I_{n_1}, h_1, I_{n_3}, \ldots, I_{n_k})$, we may assume that $G_1 = G_2$ (while J is left unchanged, as $h \in H_n$). But now we see that the matrix

$$m = \mathrm{diag}\left(\begin{pmatrix} 0 & I_{n_1} \\ I_{n_1} & 0 \end{pmatrix}, \quad I_{n-2n_1}\right)$$

fixes J, that is, belongs to H_n, and normalizes G. Hence $m \in G$, which is clearly impossible as $m \notin H_{n_1} \times \cdots \times H_{n_k}$.

Assume first that $H_n \notin \{\mathrm{Sp}(n,3), \mathrm{U}(n,4)\}$. By the minimality of n, we have $G_i = N_{H_{n_i}}(P_i)$, where P_i is a Sylow 2-subgroup of H_{n_i} (for $i = 1, \ldots, k$), and $P = P_1 \times \cdots \times P_k$ is clearly a Sylow 2-subgroup of G. If q is even, then $q = 2$ and G_i is a 2-group, for every i, by Lemma 5. Thus, as above, G is a (self-normalizing) Sylow 2-subgroup of H_n, a contradiction. So, assume that q is odd, let $m \in N_{H_n}(P)$, and consider the map $\tau: X \mapsto m^{-1}Xm$ for $X \in F_q P$. Then τ is an automorphism of $F_q P$, and hence permutes its minimal central idempotents, which are the same as those of $F_q G$, by Lemma 4(ii). As the e_i have pairwise different ranks by the argument above, m must centralize each of them. It follows that $m \in \mathrm{Mat}(n_1, q) \oplus \cdots \oplus \mathrm{Mat}(n_k, q)$, hence $m = m_1 \cdots m_k$, where $m_i \in H_{n_i}$ (for $i = 1, \ldots, k$). Now, as $P_i = P \cap H_{n_i}$, we get

$$m_i^{-1} P_i m_i = m^{-1} P_i m = m^{-1} Pm \cap m^{-1} H_{n_i} m = P \cap H_{n_i} = P_i.$$

In other words, $m_i \in N_{H_{n_i}}(P_i) = G_i$ (for $i = 1, \ldots, k$), that is, $m \in G$. We conclude that $G = N_{H_n}(P)$, where P is a Sylow 2-subgroup of H_n, a contradiction.

Now let $H_n = \mathrm{Sp}(n,3)$. As n is minimal, Lemma 5 tells us, for each i, that H_{n_i} has Carter subgroups only if $n_i = 2$ (notice that $n_i = 1$ cannot occur here). Since the minimal central idempotents of $F_q G$ have pairwise different ranks, this forces $k = 1$, against our assumptions.

Finally, let $H_n = \mathrm{U}(n,4)$. Again Lemma 5 and the minimality of n tell us, for each i, that H_{n_i} has Carter subgroups only if $n_i \leqslant 3$. Now we notice that if $H_n = \mathrm{U}(3,4)$, up to conjugacy we may assume that

$$G = \left\langle \mathrm{diag}(a,a,b), \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in (F_4)^* \right\rangle,$$

and therefore $F_4 G$ is decomposable with $k = 2$, $n_1 = 2$, $n_2 = 1$. This information rules out the case $n_i = 3$, thus forcing $k \leqslant 2$, and so $n \leqslant 3$. But for $n \leqslant 3$, H_n is not a counterexample to the Main Theorem, and we are done.

3. *Proof of the Main Theorem*

In this section we restate and prove our Main Theorem.

MAIN THEOREM. *Let G be a Carter subgroup of $H_n = H_n(q)$. Then, either G is the normalizer of a Sylow 2-subgroup of H_n, or H_n is solvable and one of the following holds:*

(i) $H_n = \mathrm{Sp}(2,3)$ *and G has order 6;*

(ii) $H_n = \mathrm{U}(3,4)$ *and G has order of* $3^2 \cdot 2$.

We require some further lemmas.

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LEMMA 6. *For a prime $p \notin \{2, r\}$, let Syl_p be an absolutely irreducible Sylow p-subgroup of H_n for $n > 1$. Then $N_{H_n}(\mathrm{Syl}_p)$ is non-nilpotent.*

Proof. Assume that $N_{H_n}(\mathrm{Syl}_p)$ is nilpotent. As Syl_p is absolutely irreducible, n is a power of p and $C_{H_n}(\mathrm{Syl}_p)$ is scalar; thus, in particular, $p \mid (q-1)$. As $p \neq 2$, and $Z(H_n)$ has order at most 2 for $H_n = \mathrm{Sp}(n, q)$, $\mathrm{O}(n, q)$, we are left with the case $H_n = \mathrm{U}(n, q)$ and $p \mid (1 + \sqrt{q})$. Now, let $J = I_n$, p^a be the highest power of p dividing $1 + \sqrt{q}$, D be the subgroup of all diagonal matrices in $\mathrm{GL}(n, q)$ of order dividing p^a, S be a Sylow p-subgroup of the group of permutation matrices. Then the subgroup $[D]S$ is unitary, and comparison with the order of $\mathrm{U}(n, q)$ shows that it is indeed a Sylow p-subgroup of $\mathrm{U}(n, q)$ (see [**15**]). As S is not self-normalizing as a subgroup of the symmetric group S_n (compare [**6**]), there exists $m \in H_n$ normalizing S, and hence also normalizing $N_{H_n}(\mathrm{Syl}_p) = (\mathrm{Syl}_p) Z(H_n)$, but not belonging to $N_{H_n}(\mathrm{Syl}_p)$, a contradiction.

LEMMA 7. *Let σ be an involution of $\mathrm{Mat}(n, q)$, C be an irreducible subfield of $\mathrm{Mat}(n, q)$ containing F_q, and assume that σ preserves and acts non-trivially on C. Then $N = N_{\mathrm{GL}(n, q)}(C) = (N \cap U^\sigma(n, q)) \cdot C$. In particular, every Galois automorphism of C over F_q is realized via conjugation by an element of $U^\sigma(n, q)$.*

Proof. Let x be a generator of C^*. We note that, as C is a field, the minimum polynomial of x is irreducible, and, as C is irreducible, x has only one invariant factor over F_q. This means that the characteristic and the minimum polynomial of x coincide. Therefore $C = F_q(x)$ has degree n over F_q, and $\langle x \rangle$ is a so-called Singer-cycle of $\mathrm{GL}(n, q)$. In particular [**10**, II.7.3], $C_{\mathrm{GL}(n, q)}(x) = \langle x \rangle$, and $N_{\mathrm{GL}(n, q)}(\langle x \rangle)$ acts on $\langle x \rangle$ as a cyclic group of order n, realizing the Galois group of C over F_q (generated by the automorphism $a \mapsto a^q$).

Now, as σ acts on C as an automorphism of order 2, the order of C is a square, say s^2, and $\sigma(c) = c^s$ for every $c \in C$. Thus, for every $a \in N$ we have $\sigma(aca^{-1}) = \sigma(a^{-1}) c^s \sigma(a)$. On the other hand, we also have $\sigma(aca^{-1}) = (aca^{-1})^s = ac^s a^{-1}$. Whence $c^s \sigma(a) a = \sigma(a) a c^s$, and therefore $\sigma(a) a \in C_{\mathrm{GL}(n, q)}(c^s)$. If we take $c = x^s$, we see that $\sigma(a) a \in C_{\mathrm{GL}(n, q)}(x) = \langle x \rangle$, by our assumptions. Write $\sigma(a) a = c_0 \in C$. Then $\sigma(c_0) = \sigma(\sigma(a) a) = \sigma(a) a = c_0$, hence c_0 belongs to the subfield C_0 of order s, consisting of the elements of C fixed by σ. The map $\rho: c \mapsto \sigma(c) c$ (for $c \in C$) is a homomorphism from C^* to C_0^*, whose kernel is the subgroup of C^* of order $s+1$. Thus ρ is surjective, and therefore there exists $c \in C$ such that $\sigma(c) c = c_0^{-1}$. It follows that

$$\sigma(ac)\, ac = \sigma(c)\, \sigma(a)\, ac = 1,$$

that is, $ac \in U^\sigma(n, q)$. Whence Lemma 7.

Finally, in order to deal with the critical case of a symplectic group over the field of three elements, we also need the following easy lemma.

LEMMA 8. *Let $H_n = \mathrm{Sp}(2m, 3)$ with m even, and assume that J has shape $\mathrm{diag}(J_1, \ldots, J_m)$, where each J_i has size 2. Then*

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes I_m \in H_n,$$

and x is conjugate to its inverse in H_n.

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Proof. Clearly we may assume that $m = 2$ and $J_i = \begin{pmatrix} 0 & \varepsilon_i \\ -\varepsilon_i & 0 \end{pmatrix}$ (with $\varepsilon_i = \pm 1$ for $i = 1, 2$). Let

$$y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{if } \varepsilon_1 = \varepsilon_2, \qquad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{if } \varepsilon_1 = -\varepsilon_2.$$

Then $y \in H_4$, and $y^{-1}xy = x^{-1}$.

Proof of the Main Theorem. Let H_n and G be counterexamples to our theorem with n minimal. Clearly we may assume that $n > 1$, and, by Theorem 2, that $F_q G$ is indecomposable. Let R be the Sylow r-subgroup of $G = R \times T$. It follows from Lemma 4 that $A = F_q T$ is indecomposable, hence simple. Therefore, by Proposition 2, $C = C_{\mathrm{Mat}(n,q)}(A)$ is also simple, and

$$(*) \qquad A \simeq \mathrm{Mat}(m, q^h), \qquad C \simeq \mathrm{Mat}(t, q^h) \quad \text{with } n = mth.$$

Moreover, if σ denotes the involution of $\mathrm{Mat}(n, q)$ defining H_n as a norm-group, then clearly $\sigma(A) = A$ and $\sigma(C) = C$.

Step 1. Let us first assume that T is abelian. Then $m = 1$, $A \simeq F_{q^h}$ is a field extension of F_q, and G is contained in C. If $h = 1$, the equality $t = n$ forces T to be scalar. Then $G = N_{H_n}(R)$, where R is a Sylow r-subgroup of H_n. If $R \neq 1$, by Lemma 5 either $r = 2$, or $H_n = \mathrm{Sp}(2, 3)$, $|G| = 6$, and G is not a counterexample. If $R = 1$, then $G = T$ is scalar, a contradiction as $n > 1$. Thus, we may assume that $h > 1$. Now, G is a Carter subgroup of $H_n \cap C = K_t$, a group satisfying the conditions of Proposition 1. First let $t = 1$. In this case, as $m = 1$, we have $n = h$ and $A = C \simeq \mathrm{Mat}(1, q^n)$ is a subfield of $\mathrm{Mat}(n, q)$ containing F_q. As C is self-centralizing, by Schur's Lemma C is an irreducible subfield of $\mathrm{Mat}(n, q)$. Also $G = H_n \cap C = H_1$, a cyclic subgroup of C^*. Now $|G| > 2$, otherwise G would be scalar, a contradiction. Thus σ acts non-trivially on C, and we may apply Lemma 7. We get that

$$N = N_{\mathrm{GL}(n,q)}(C) = (N \cap H_n) \cdot C$$

has order $n|C|$. Now G is normalized by N, and in particular by the element of H_n realizing $\mathrm{Gal}(C : F_q)$, a contradiction. Therefore we may assume that $1 < t < n$. Now, T lies in the centre of C, by its very definition; hence T, as a subgroup of $H_n \cap C = K_t$, is scalar. As above, this implies that G is the normalizer of a non-trivial Sylow r-subgroup of K_t. If $K_t = H_t(q^h)$, by Lemma 5, either $q^h \in \{2, 3\}$, or $q^h = 4$, $t = 2$, and $K_2 = \mathrm{U}(2, 4)$. As $h > 1$, the first two cases are ruled out. In the last case, $q = 2$, $n = 4$ and $|G| = 6$. But this gives $H_n = \mathrm{Sp}(4, 2) \simeq S_6$, which forces G to be a Sylow 2-subgroup of $\mathrm{Sp}(4, 2)$, a contradiction. According to Proposition 1, we are left with the case of q even and $K_t = N \cdot \mathrm{Sp}(t-s, q^h)$, where $s \in \{1, 2\}$ and N is a (possibly trivial) normal 2-subgroup (and $\mathrm{Sp}(0, q^h)$ denotes 1). Then

$$R = N(R \cap \mathrm{Sp}(t-s, q^h)).$$

If $t > 2$, it follows that $R \cap \mathrm{Sp}(t-s, q^h)$ is a Sylow 2-subgroup of $\mathrm{Sp}(t-s, q^h)$ with nilpotent normalizer. This contradicts Lemma 5, as $h > 1$. Finally, if $t = 2$, $K_2 = N$ is a 2-group. This forces $G = N$, a self-normalizing Sylow 2-subgroup of H_n, and G is not a counterexample.

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Step 2. Having reduced the problem to the case when T is non-abelian, we may assume that $G = P \times S$, where P is a non-abelian Sylow p-subgroup of G with $p \neq r$. Again, by Lemma 4 (and using the same notation as in $(*)$, with P replacing T), $A = F_q P$ is simple, hence $C = C_{\mathrm{Mat}(n,q)}(A)$ is also simple, and $A \simeq \mathrm{Mat}(m, q^h)$, $C \simeq \mathrm{Mat}(t, q^h)$ with $n = mth$.

We claim that $P \times (S \cap A)$ and $Z(P) \times S$ are Carter subgroups of $H_n \cap A$ and $H_n \cap C$ respectively. For, let $K_m = H_n \cap A = \{X \in A \mid \sigma(X)\, X = I_n\}$. As P is contained in K_m, it follows that $G \cap K_m = P \times (S \cap A)$. Let $x \in N_{K_m}(P \times (S \cap A))$. Then x normalizes P and centralizes S, hence $x \in N_{H_n}(G) = G$. We conclude that $x \in P \times (S \cap A)$. Similarly, let $K_t = H_n \cap C$. Then, as S is contained in K_t, it follows that $G \cap K_t = Z(P) \times S$. Every $x \in N_{K_t}(Z(P) \times S)$ normalizes S and centralizes P, and hence belongs to G. Therefore $x \in Z(P) \times S$.

Now we notice that, as P is non-abelian, $m > 1$, hence $t < n$. Also the fact that P is non-abelian implies that $K_m \notin \{\mathrm{Sp}(2,3), \mathrm{U}(3,4)\}$. For, otherwise, we would get $|P \times (S \cap A)| = 3 \cdot 2$ or $3^2 \cdot 2$, hence $|P| = 2$ or 3^2 respectively.

Step 3. Assume that $m < n$, $K_m = H_m$, and $K_t = H_t$. By the minimality of n, we have that $P \times (S \cap A)$ is the normalizer of a Sylow 2-subgroup of H_m. Then it follows, using Lemma 5, that P is a 2-group. (This is clear if H_m is symplectic or orthogonal; on the other hand, if H_m is unitary, the $2'$-part of a 2-Sylow normalizer is abelian (compare **[2]**).) Thus we may assume that $G = P \times S$, where P is a (non-abelian) Sylow 2-subgroup of G. Now $t < n$ forces $N_{H_t}(Z(P)) = Z(P) \times S$, or $H_t \in \{\mathrm{Sp}(2,3), \mathrm{U}(3,4)\}$. But the last case is ruled out, as P is a 2-group, and therefore $r \neq 2$.

So, assume first that $H_t \neq \mathrm{Sp}(2,3)$, and let $h \in N_{H_n}(P)$. Conjugating by h leaves invariant A, and hence C; hence also $H_n \cap C = H_t$. In particular, S^h is contained in H_t. But S^h centralizes $P^h = P$, and hence, a fortiori, $Z(P)$. So S^h must be contained in $Z(P) \times S$, whence $S^h = S$. We conclude that $h \in N_{H_n}(G) = G$, and therefore $G = N_{H_n}(P)$, where P is a Sylow 2-subgroup of H_n, a contradiction.

A more detailed analysis is required when $t = 2$, and $H_2 = \mathrm{Sp}(2,3) = \mathrm{SL}(2,3)$. In this case $Z(P) = \langle -I \rangle$, $|S| = 3$, and $|G| = 2^a \cdot 3$. By Proposition 2 and its Corollary, and using the notation introduced there, we may assume that $A = \mathrm{Mat}(m, F_3 \otimes I_2)$, $C = \mathrm{Mat}(2,3) \otimes I_m$, $H_2 = \mathrm{SL}(2,3) \otimes I_m$, $S = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes I_m \right\rangle$ with $n = 2m$. As $Z(A) = Z(C)$, it follows that $S \cap A = 1$, hence P must be a self-normalizing Sylow 2-subgroup of $H_n \cap A = H_m$. As P is non-abelian, m is greater than 1, which, by Lemma 5, forces H_m to be an orthogonal group. Also, as $A = F_q P$ is isomorphic to the full matrix algebra $\mathrm{Mat}(m,3)$, we have that P is absolutely irreducible as a subgroup of $\mathrm{GL}(m,3)$. Thus m is a 2-power. Now, up to conjugations in $A = \mathrm{Mat}(m, F_3 \otimes I_2)$, we may assume that the matrix of the form fixed by $H_m = H_n \cap A$ has shape $\mathrm{diag}(I_2, \ldots, \pm I_2)$, and P contains the elementary 2-subgroup of order 2^m consisting of the matrices of shape $\mathrm{diag}(\pm I_2, \ldots, \pm I_2)$. This in turn implies that the form fixed by H_n has a matrix J of shape $\mathrm{diag}(J_1, \ldots, J_m)$. Observe that the previous conjugations occur inside A, and hence leave C, and therefore also $H_2 = H_n \cap C$, unchanged. Thus J is preserved by H_2 as well, and this forces $J_i = \begin{pmatrix} 0 & \varepsilon_i \\ -\varepsilon_i & 0 \end{pmatrix}$ (with $\varepsilon_i = \pm 1$ for $i = 1, \ldots, m$), whence $H_n = \mathrm{Sp}(2m, 3)$. Since $G = P \times S \leqslant N_{H_n}(S)$, it follows that P has to be a Sylow 2-subgroup of $N_{H_n}(S)$. As $|S| = 3$, we have $|N_{H_n}(S) : C_{H_n}(S)| \leqslant 2$, hence $N_{H_n}(S) = C_{H_n}(S)$. But this contradicts Lemma 8.

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Step 4. Assume that $m < n$, and either $K_m \neq H_m$ or $K_t \neq H_t$. Then q is even, and either K_m or K_t satisfies condition (ii) of Proposition 1. Let $K_m = N \cdot \mathrm{Sp}(m-s, q^h)$, where $s \in \{1, 2\}$, N is a (possibly trivial) normal 2-subgroup (and $\mathrm{Sp}(0, q^h)$ denotes 1). If $m > 2$, by Lemma 1 $(P \times (S \cap A))\, N/N$ is a Carter subgroup of $\mathrm{Sp}(m-s, q^h)$, hence a 2-group by the minimality of n and Lemma 5. This gives a contradiction, as $p \neq 2$. Clearly, we get the same contradiction if $m = 2$, $K_2 = N$. Arguing exactly in the same way, we can rule out the possibility that $K_t = N \cdot \mathrm{Sp}(t-s, q^h)$ (for in this case $(Z(P) \times S)\, N/N$ should be a 2-group).

Step 5. Finally, let $m = n$, whence $t = h = 1$. As $A = F_q P = \mathrm{Mat}(n, q)$, it follows that C is scalar and P is an absolutely irreducible Sylow p-subgroup of H_n. Then $G = P \times S = N_{H_n}(P)$. But, as $p \neq r$, by Lemma 6 $N_{H_n}(P)$ cannot be nilpotent, a contradiction.

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