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Quantum automorphism groups of hypergraphs

Nicolas Faross

Abstract. We introduce a quantum automorphism group for hypergraphs, which turns out to generalize the quantum automorphism group of Bichon for classical graphs. Further, we show that our quantum automorphism group acts on hypergraph C^*-algebras as recently defined. In particular, this action generalizes the one on graph C^*-algebras by Schmidt–Weber in 2018.

1 Introduction

In [34], Wang introduced quantum automorphism groups of finite space. These are compact quantum groups in the sense of Woronowicz [36, 38] and can be used to describe symmetries in the setting of C^*-algebras. A particular example is the quantum permutation group S_n^+, which generalizes the classical symmetry group of n points S_n. It can be defined by the universal unital C^*-algebra

$$C(S_n^+) := C^*(u_{ij} \mid (u_{ij}) \text{ is a magic unitary}),$$

where an $n \times n$ matrix $u := (u_{ij})$ is called a magic unitary if

$$u_{ij}^2 = u_{ij}^* = u_{ij}, \quad \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{kj} = 1 \qquad (1 \le i, j \le n).$$

Magic unitaries are also called quantum permutations since magic unitaries with entries in $\mathbb{C}$ are exactly classical permutation matrices.

In the same setting, Bichon [4] and Banica [2] introduced two different versions of quantum automorphism groups of finite graphs. Both quantum groups generalize the classical automorphism group of a graph by imposing the additional relation $A_\Gamma u = u A_\Gamma$ on a magic unitary u, while Bichon's quantum automorphism group includes some additional relations. Here, A_Γ denotes the adjacency matrix of a graph Γ, which requires the magic unitary u to respect the graph structure. Quantum automorphism groups of graphs provide a large class of examples of compact quantum groups and have, for example, been studied in [9, 13, 20, 29]. In particular, these quantum automorphism groups have been further generalized to different structures,

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such as multigraphs [16], Hadamard matrices [17], matroids [10, 11], and quantum graphs [7, 8].

1.1 Quantum automorphism groups of hypergraphs

In this article, we present a definition of a quantum automorphism group for hypergraphs. Hypergraphs generalize classical graphs by allowing an edge to connect not only two but an arbitrary number of vertices. This makes hypergraphs quite general and gives them many applications in discrete mathematics and computer science [1, 15]. In particular, it is possible to form the dual of a hypergraph by interchanging its vertices and edges, which is, in general, not possible for classical graphs. See [3] for further information on hypergraphs.

In the following, a hypergraph $\Gamma := (V, E)$ is given by a finite set of vertices V, a finite set of edges E, and two maps $s\colon E \to \mathcal{P}(V)$ and $r\colon E \to \mathcal{P}(V)$. An edge $e \in E$ can be depicted by an arrow from the set of source vertices $s(e)$ to the set of range vertices $r(e)$. Thus, we can view classical directed edges as hyperedges with $|s(e)| = |r(e)| = 1$. Note that we consider directed hypergraphs which can also have empty edges and multi-edges.

In this setting, our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ of a hypergraph Γ is given by the following compact matrix quantum group.

Definition 1 (Definition 3.4) Let $\Gamma := (V, E)$ be a hypergraph and $\mathcal{A}$ the universal unital C^*-algebra with generators u_{vw} for all $v, w \in V$ and u_{ef} for all $e, f \in E$ such that

(1) $u_V := (u_{vw})_{v,w\in V}$ and $u_E := (u_{ef})_{e,f\in E}$ are magic unitaries;

(2) $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$, where $A_r, A_s \in \mathbb{C}^{V\times E}$ are defined by

$$(A_s)_{ve} = \begin{cases} 1 & \text{if } v \in s(e), \\ 0 & \text{otherwise,} \end{cases} \qquad (A_r)_{ve} = \begin{cases} 1 & \text{if } v \in r(e), \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v \in V,\ e \in E.$$

Then $\mathrm{Aut}^+(\Gamma) := (\mathcal{A}, u_V \oplus u_E)$ is the *quantum automorphism group of the hypergraph* Γ.

Intuitively, $\mathrm{Aut}^+(\Gamma)$ is given by a quantum permutation u_V on the vertices and a quantum permutation u_E on the edges, which are compatible by intertwining the incidence matrices A_s and A_r. If u_V and u_E are classical permutation matrices, then this definition gives exactly the classical automorphism group of a hypergraph (see Section 3.1).

Note that in contrast to the quantum automorphism groups of Bichon and Banica, we include a second magic unitary for the edges. This is necessary to capture quantum symmetries between multi-edges, which are allowed in our definition of hypergraph. See, for example, Section 3.3 for the quantum symmetries of a concrete family of hypergraphs with multi-edges. However, if a hypergraph Γ or its dual Γ^* have no multi-edges, then our definition reduces to only one magic unitary.

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Theorem 1 (Corollaries 3.17 and 3.18) *Let* $\Gamma := (V, E)$ *be a hypergraph.*

(1) *If* Γ *has no multi-edges, then* $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+$.
(2) *If* Γ^* *has no multi-edges, then* $\mathrm{Aut}^+(\Gamma) \subseteq S_E^+$.

Since hypergraphs are a generalization of classical graphs and multigraphs, it is natural to ask how our quantum automorphism group relates to the quantum automorphism groups of Bichon [4], Banica [2], and Goswami–Hossain [16]. Denote by $\mathrm{Aut}_{\mathrm{Bic}}^+(\Gamma)$ the quantum automorphism group of Bichon and by $\mathrm{Aut}_{\mathrm{GH,Bic}}^+(\Gamma)$ the quantum automorphism group of Goswami–Hossain in the sense of Bichon. Then the following theorem shows that we obtain both quantum groups as a special case when encoding classical graphs and multigraphs as hypergraphs.

Theorem 2 (Theorems 4.3, 4.7, and 4.11)

(1) *Let* $\Gamma := (V, E)$ *be a directed graph as in Definition 2.2. Define the source and range maps* $s(v, w) = \{v\}$ *and* $r(v, w) = \{w\}$ *for all edges* $(v, w) \in E$. *Then* Γ *is a hypergraph with* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}_{\mathrm{Bic}}^+(\Gamma)$.
(2) *Let* $\Gamma := (V, E)$ *be a simple graph as in Definition 2.1. Define the source and range maps* $s(\{v, w\}) = \{v, w\}$ *and* $r(\{v, w\}) = \{v, w\}$ *for all edges* $\{v, w\} \in E$. *Then* Γ *is a hypergraph with* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}_{\mathrm{Bic}}^+(\Gamma)$.
(3) *Let* $\Gamma := (V, E)$ *be a multigraph as in Definition 2.4 with source map* s' *and range map* r'. *Define the new source and range maps* $s(e) = \{s'(e)\}$ *and* $r(e) = \{r'(e)\}$ *for all* $e \in E$. *Then* Γ *is a hypergraph with* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}_{\mathrm{GH,Bic}}^+(\Gamma)$.

Note that the previous theorem can also be used to construct many concrete examples of quantum automorphism groups of hypergraphs.

1.2 Quantum symmetries of hypergraph C^*-algebras

Related to quantum automorphism groups of classical graphs is the study of quantum symmetries of graph C^*-algebras in [19, 30]. These C^*-algebras are defined in terms of an underlying graph and have been studied since the 1980s. They include many examples like matrix algebras, continuous functions on the circle, or the Cuntz algebra (see [24] for more details). Recently, Trieb–Weber–Zenner [32] introduced hypergraph C^*-algebras, which generalize graph C^*-algebras to the setting of hypergraphs. This new class includes all graph C^*-algebras but also new examples of non-nuclear C^*-algebras. See also the recent work by Schäfer–Weber [26] which characterizes the nuclearity of hypergraph C^*-algebras in terms of minors of the underlying hypergraph.

In [30], Schmidt–Weber showed that Banica's quantum automorphism group acts maximally on the corresponding graph C^*-algebra. We generalize this result to hypergraphs by showing that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ acts on the corresponding hypergraph C^*-algebra $C^*(\Gamma)$. As in the case of graph C^*-algebras, hypergraph C^*-algebras are generated by a family of projections $\{p_v\}_{v \in V}$ and a family of partial isometries $\{s_e\}_{e \in E}$ (see Definition 2.33). Our action is then given by permuting these generators using the magic unitaries u_V and u_E, that is,

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$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_{wv}, \qquad \alpha(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \qquad \forall v \in V,\ e \in E.$$

It turns out that this action equals the one of Schmidt–Weber in the case of classical graphs, but our quantum automorphism group is no longer maximal with respect to it. However, we obtain maximality under the additional assumption that $\mathrm{Aut}^+(\Gamma)$ also acts on $C^*(\Gamma')$, where Γ' is obtained by inverting the direction of all edges and by interchanging the vertices and edges of Γ.

Theorem 3 (Theorems 5.2 and 5.7) *Let $\Gamma := (V, E)$ be a hypergraph and $\Gamma' := (\Gamma^*)^{\mathrm{op}}$. Then $\mathrm{Aut}^+(\Gamma)$ is the largest compact matrix quantum group that acts faithfully on both $C^*(\Gamma)$ and $C^*(\Gamma')$ via*

$$\alpha_1 \colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+(\Gamma)), \quad \alpha_2 \colon C^*(\Gamma') \to C^*(\Gamma') \otimes C(\mathrm{Aut}^+(\Gamma))$$

with

$$\alpha_1(p_v) = \sum_{w \in V} p_w \otimes u_{wv} \quad \forall v \in V, \qquad \alpha_1(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \quad \forall e \in E,$$

$$\alpha_2(p_e) = \sum_{f \in E} p_f \otimes u_{fe} \quad \forall e \in E, \qquad \alpha_2(s_v) = \sum_{w \in V} s_w \otimes u_{wv} \quad \forall v \in V.$$

Overview of the article: We begin in Section 2 with some preliminaries on graphs, hypergraphs, quantum groups, as well as graph and hypergraph C^*-algebras. Then we introduce our quantum automorphism groups of hypergraphs in Section 3 and explore some first properties. Further, we give examples of hypergraphs with maximal quantum symmetries and consider the case of hypergraphs without multi-edges. In Section 4, we compute the quantum automorphism groups of hypergraphs that are constructed from classical graphs and multigraphs. In particular, we show that these agree with the quantum automorphism group of Bichon or its multigraph version in the sense of Goswami–Hossain. Finally, we construct the action on hypergraph C^*-algebras in Section 5 and present some remaining open questions in Section 6.

2 Preliminaries

2.1 Notations

We begin by introducing some notations and conventions that will be used throughout the rest of the article. In the following, many results will be formulated in the language C^*-algebras. In particular, we will use universal C^*-algebras and tensor products of C^*-algebras. For more information on these topics, we refer to [6].

Let X and Y be sets. Then we denote by $\mathcal{P}(X)$ the power set of X and by $X \sqcup Y$ the disjoint union of X and Y. Assume X and Y are finite. Then $\mathbb{C}^X$ is the C^*-algebra of all complex-valued functions on X with a basis $\{e_i\}_{i \in X}$ given by indicator functions. Similarly, $\mathbb{C}^{X \times Y}$ is the C^*-algebra of all complex-valued matrices with rows indexed by X and columns indexed by Y. Here, the standard matrix units $e_{ij} \in \mathbb{C}^{X \times Y}$ are given by $(e_{ij})_{k\ell} := \delta_{ik}\delta_{j\ell}$ for all $i, k \in X$ and $j, \ell \in Y$.

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If $\mathcal{A}$ and $\mathcal{B}$ are arbitrary C^*-algebras, then we denote by $\mathcal{A} \otimes \mathcal{B}$ their minimal tensor product. In addition, we identify elements $u \in \mathcal{A} \otimes \mathbb{C}^{X \times Y}$ with $\mathcal{A}$-valued matrices $(u_{ij})_{i \in X, j \in Y}$ via $u = \sum_{i \in X} \sum_{j \in Y} u_{ij} \otimes e_{ij}$.

If $u \in \mathcal{A} \otimes \mathbb{C}^{X \times Y}$ and $v \in \mathcal{A} \otimes \mathbb{C}^{Y \times Y}$, then their direct sum $u \oplus v \in \mathcal{A} \otimes \mathbb{C}^{(X \sqcup Y) \times (X \sqcup Y)}$ is given by

$$(u \oplus v)_{ij} := \begin{cases} u_{ij} & \text{if } i, j \in X, \\ v_{ij} & \text{if } i, j \in Y, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i, j \in X \sqcup Y.$$

Finally, assume that $\mathcal{A}$ is unital. Then we identify scalar matrices $u \in \mathbb{C}^{X \times Y}$ with the $\mathcal{A}$-valued matrices $1 \otimes u \in \mathcal{A} \otimes \mathbb{C}^{X \times Y}$.

2.2 Graphs and hypergraphs

Graphs are combinatorial objects consisting of a set of vertices that are connected by edges. In the following, we begin with the definition of simple and directed graphs before we come to multigraphs and directed hypergraphs as in [15].

Definition 2.1 A *simple graph* $\Gamma := (V, E)$ is given by a finite set of vertices V and a set of edges $E \subseteq \mathcal{P}(V)$ with $|e| = 2$ for all $e \in E$.

Definition 2.2 A *directed graph* $\Gamma := (V, E)$ is given by a finite set of vertices V and a set of edges $E \subseteq V \times V$.

An edge $\{v, w\}$ in a simple graph can be visualized by a line from v to w, whereas an edge (v, w) in a directed graph can be visualized by an arrow from v to w. Further, directed graphs can have self-loops (v, v), which are excluded in our definition of simple graphs. Although edges are modeled differently in both definitions, we can describe them uniformly by an adjacency matrix.

Definition 2.3 Let $\Gamma := (V, E)$ be a simple or directed graph. Then two vertices $v, w \in V$ are *adjacent*, and we write $v \sim w$, if $\{v, w\} \in E$ or $(v, w) \in E$, respectively. Further, we define the *adjacency matrix* $A_\Gamma \in \mathbb{C}^{V \times V}$ by

$$(A_\Gamma)_{vw} := \begin{cases} 1 & \text{if } v \sim w, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v, w \in V.$$

By allowing multiple edges between each pair of vertices, we can generalize directed graphs to multigraphs.

Definition 2.4 A (directed) *multigraph* $\Gamma := (V, E)$ is given by a finite set of vertices V, a finite set of edges E, and two maps $s\colon E \to V$ and $r\colon E \to V$.

In a multigraph, an edge $e \in E$ can be depicted by a directed arrow from the source vertex $s(e)$ to the range vertex $r(e)$ as in the case of directed graphs. Further, in

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the setting of multigraphs, we will be interested in vertices with only incoming or outgoing edges.

Definition 2.5 Let $\Gamma := (V, E)$ be a multigraph and $v \in V$. Then

(1) v is a *source*, if $v \neq r(e)$ for all $e \in E$.
(2) v is a *sink*, if $v \neq s(e)$ for all $e \in E$.
(3) v is *isolated*, if v is a source and a sink.

By replacing the vertices $s(e)$ and $r(e)$ in a multigraph with arbitrary subsets of vertices, we finally arrive at the definition of a hypergraph.

Definition 2.6 A (directed) *hypergraph* $\Gamma := (V, E)$ is given by a finite set of vertices V, a finite set of edges E, and source and range maps $s: E \to \mathcal{P}(V)$ and $r: E \to \mathcal{P}(V)$.

In a hypergraph, an edge $e \in E$ can be depicted by an arrow from the set of source vertices $s(e)$ to the set of range vertices $r(e)$. Thus, classical directed edges correspond to hyperedge with $|s(e)| = |r(e)| = 1$, see also Definitions 2.11 and 2.12 below. Note that we allow empty sets in both the source and range map.

Similarly to the adjacency matrix of classical graphs, it is also possible to describe the edge structure of a hypergraph using an incidence matrix.

Definition 2.7 Let $\Gamma := (V, E)$ be a hypergraph. Then its *incidence matrices* $A_s, A_r \in \mathbb{C}^{V \times E}$ are given by

$$(A_s)_{ve} = \begin{cases} 1 & \text{if } v \in s(e), \\ 0 & \text{otherwise,} \end{cases} \quad (A_r)_{ve} = \begin{cases} 1 & \text{if } v \in r(e), \\ 0 & \text{otherwise,} \end{cases} \qquad \forall v \in V,\ e \in E.$$

It is also possible to generalize the notation of sources and sinks from multigraphs to hypergraphs.

Definition 2.8 Let $\Gamma := (V, E)$ be a hypergraph and $v \in V$. Then

(1) v is a *source*, if $v \notin r(e)$ for all $e \in E$.
(2) v is a *sink*, if $v \notin s(e)$ for all $e \in E$.
(3) v is *isolated*, if v is a source and a sink.

Further, the following properties will be used to describe hypergraphs with a special structure.

Definition 2.9 Let $\Gamma := (V, E)$ be a hypergraph. Then

(1) Γ has *no multi-edges*, if $s(e_1) = s(e_2)$ and $r(e_1) = r(e_2)$ implies $e_1 = e_2$ for all $e_1, e_2 \in E$.
(2) Γ is *undirected*, if $s(e) = r(e)$ for all $e \in E$.
(3) Γ is *k-uniform*, if $|s(e)| = |r(e)| = k$ for all $e \in E$.

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Given a simple graph, a directed graph, or a multi-graph, it is possible to regard it as a hypergraph in a natural way. In the following, we define the corresponding source and range maps and show how the resulting hypergraphs can be characterized using the properties defined previously.

Definition 2.10 Let $\Gamma := (V, E)$ be a simple graph. Then we can regard Γ as a hypergraph with the source and range map given by $s(\{v, w\}) := \{v, w\}$ and $r(\{v, w\}) := \{v, w\}$ for all $\{v, w\} \in E$. Conversely, 2-uniform undirected hypergraphs without multi-edges correspond exactly to simple graphs in this way.

Definition 2.11 Let $\Gamma := (V, E)$ be a directed graph. Then we can regard Γ as a hypergraph by defining the source and range map given by $s(v, w) := \{v\}$ and $r(v, w) := \{w\}$ for all $(v, w) \in E$. Conversely, 1-uniform hypergraphs without multi-edges correspond exactly to directed graphs in this way.

Definition 2.12 Let $\Gamma := (V, E)$ be a multigraph with source map $s' : E \to V$ and range map $r' : E \to V$. Then we can regard Γ as a hypergraph with the new source and range map given by $s(e) := \{s'(e)\}$ and $r(e) := \{r'(e)\}$ for all $e \in E$. Conversely, 1-uniform hypergraphs correspond exactly to multigraphs in this way.

Given any hypergraph, it is always possible to obtain a new hypergraph by interchanging the source and range maps or by interchanging the vertices and edges.

Definition 2.13 Let $\Gamma = (V, E)$ be a hypergraph. Then its *opposite hypergraph* is given by $\Gamma^{\mathrm{op}} := (V, E)$ with $s^{\mathrm{op}}(e) := r(e)$ and $r^{\mathrm{op}}(e) := s(e)$ for all $e \in E$.

Definition 2.14 Let $\Gamma = (V, E)$ be a hypergraph. Then its *dual hypergraph* is given by $\Gamma^* := (E, V)$ with source and range maps s^* and r^* defined by

$$s^*(v) := \{e \in E \mid v \in s(e)\}, \quad r^*(v) := \{e \in E \mid v \in r(e)\}.$$

Note that we can use the dual source and range maps s^* and r^* to rewrite the source and range maps s and r as

$$s(e) = \{v \in V \mid e \in s^*(v)\}, \quad r(e) = \{v \in V \mid e \in r^*(v)\} \quad \forall e \in E.$$

In particular, we have $(\Gamma^*)^* = \Gamma$.

2.3 Compact matrix quantum groups

Compact quantum groups were first introduced by Woronowicz in [36, 38] and are a generalization of classical compact groups to describe symmetries in the setting of C^*-algebras. In the following, we will focus only on compact matrix quantum groups, which form a subclass of compact quantum groups analogous to classical matrix groups. We begin with some basic definitions before we come to actions on C^*-algebras and the quantum permutation group. For more information on compact quantum groups, we refer to [23, 31].

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Definition 2.15 Let I be a finite index set. A *compact matrix quantum group* $G := (\mathcal{A}, u)$ consists of a unital C^*-algebra $\mathcal{A}$ and a matrix $u := (u_{ij})_{i,j\in I} \in \mathcal{A} \otimes \mathbb{C}^{I\times I}$, such that

(1) $\mathcal{A}$ is generated by the matrix entries u_{ij} for all $i, j \in I$;
(2) u is unitary and $\overline{u} := (u_{ij}^*)_{i,j\in I}$ is invertible;
(3) there exists a unital $*$-homomorphism $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ given by $\Delta(u_{ij}) = \sum_{k\in I} u_{ik} \otimes u_{kj}$ for all $i, j \in I$.

If G is a compact matrix quantum group, then the C^*-algebra $\mathcal{A}$ will be denoted by $C(G)$ and the $*$-algebra generated by the entries u_{ij} will be denoted by $\mathcal{O}(G)$. Further, the matrix u is called the fundamental representation of G.

As for classical groups, it is possible to define subgroups, quotients, and isomorphisms for compact matrix quantum groups. However, in order to formulate these, it is convenient to first introduce morphisms of compact quantum groups.

Definition 2.16 Let G and H be two compact matrix quantum groups. A *morphism of compact quantum groups* is a unital $*$-homomorphism $\varphi: C(H) \to C(G)$, such that $(\varphi \otimes \varphi) \circ \Delta_H = \Delta_G \circ \varphi$.

Note that there also exists a stronger notion of morphism between compact matrix quantum groups that respects their fundamental representations. However, we are interested in morphisms of compact quantum groups since we want to be able to compare compact matrix quantum groups of different sizes.

Definition 2.17 Let G and H be two compact matrix quantum groups.

(1) H is a *subgroup* of G and we write $H \subseteq G$ if there exists a surjective morphism of compact quantum groups $\varphi: C(G) \to C(H)$.
(2) H is a *quotient* of G if there exists an injective morphism of compact quantum groups $\varphi: C(H) \to C(G)$. In this case, we identify $C(H)$ with a C^*-subalgebra of $C(G)$.
(3) G and H are *isomorphic* and we write $G = H$ if there exists a bijective morphism of compact quantum groups $\varphi: C(G) \to C(H)$. Note that in this case, φ^{-1} is again a morphism of compact quantum groups.

While the comultiplication Δ of a quantum group generalizes the multiplication of a classical group, there exists further structure on the $*$-algebra $\mathcal{O}(G)$ that corresponds to the neutral element and the inverse map in the classical case.

Remark 2.18 Let G be a compact matrix quantum group with fundamental representation u. Then there exists a unital $*$-homomorphism $\varepsilon: \mathcal{O}(G) \to \mathbb{C}$ called the *counit* and a unital anti-homomorphism $S: \mathcal{O}(G) \to \mathcal{O}(G)$ called the *antipode*, which are given by $\varepsilon(u_{ij}) = \delta_{ij}$, and $S(u_{ij}) = u_{ji}^*$ for all $i, j \in I$. These turn $\mathcal{O}(G)$ into a Hopf $*$-algebra. For more details, see [31].

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Using the Hopf $*$-algebra structure of $\mathcal{O}(G)$, we can now define actions of compact matrix quantum groups as in [4, 34].

Definition 2.19 Let G be a compact matrix quantum group and $\mathcal{A}$ a unital C^*-algebra. An *action* of G on $\mathcal{A}$ is a unital $*$-homomorphism $\alpha: \mathcal{A} \to \mathcal{A} \otimes C(G)$, such that

(1) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$;
(2) there exists a dense $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ with $\alpha(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{O}(G)$;
(3) $(\mathrm{id} \otimes \varepsilon) \circ \alpha|_{\mathcal{B}} = \mathrm{id}$.

Alternatively, the second and third conditions can be replaced by the more analytical condition that $(1 \otimes C(G))\alpha(\mathcal{A})$ is linearly dense in $\mathcal{A} \otimes C(G)$ (see, e.g., [30]). However, Definition 2.19 will be easier to check in our case.

Definition 2.20 Let α be an action of a compact matrix quantum group G on a C^*-algebra $\mathcal{A}$. Then α is *faithful* if for any quotient H of G, such that $\alpha|_{C(H)}$ is an action on $\mathcal{A}$, we have $C(H) = C(G)$.

Next, we define magic unitaries and the quantum permutation group S_n^+, which was first introduced by Wang [34] as the quantum automorphism group of the finite set $X := \{1, \dots, n\}$. However, we will consider the case of arbitrary finite sets X and denote the corresponding quantum permutation group by S_X^+.

Definition 2.21 Let X be a finite set and $\mathcal{A}$ be a unital C^*-algebra. An element $u := (u_{ij}) \in \mathcal{A} \otimes \mathbb{C}^{X \times X}$ is called *magic unitary*, if

$$u_{ij}^2 = u_{ij}^* = u_{ij} \quad \forall i, j \in X, \qquad \sum_{j \in I} u_{ij} = \sum_{j \in I} u_{ji} = 1 \quad \forall i \in X.$$

Note that magic unitaries with entries in $\mathbb{C}$ are exactly classical permutation matrices, which gives magic unitaries also the name quantum permutation matrices. These matrices have applications in quantum information via non-local games [21] and concrete magic unitaries have, for example, been recently studied in [14, 22]. For more information and open problems related to magic unitaries, we refer to [35].

Now, the quantum permutation group S_X^+ is the compact matrix quantum group with a universal magic unitary as its fundamental representation.

Definition 2.22 Let X be a finite set and $\mathcal{A}$ be the universal unital C^*-algebra with generators u_{ij} for all $i, j \in X$ such that $u := (u_{ij})_{i,j \in X}$ is a magic unitary. Then $S_X^+ := (\mathcal{A}, u)$ is the *quantum permutation group* on X.

In [34], Wang showed that the quantum permutation group S_X^+ is the largest compact matrix quantum group that acts on $\mathbb{C}^X$ in the following sense. Thus, we can think of S_X^+ as the quantum automorphism group of a finite set X.

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Proposition 2.23 [34] *Let X be a finite set. Then S_X^+ acts faithfully on $\mathbb{C}^X$ via the map $\alpha: \mathbb{C}^X \to \mathbb{C}^X \otimes C(S_X^+)$ given by*

$$\alpha(e_i) = \sum_{j \in X} e_j \otimes u_{ji} \quad \forall i \in X,$$

where u denotes the fundamental representation of S_X^+ and $\{e_i\}_{i \in X}$ the standard basis of $\mathbb{C}^X$. Further, if G is any compact matrix quantum group that acts faithfully on $\mathbb{C}^X$ via the action α, then $G \subseteq S_X^+$.

When defining quantum automorphism groups of graphs or hypergraphs, we will consider intertwiner relations of the form $Au = vA$, where u and v are unitary matrices over a C^*-algebra and A is a scalar matrix. The following proposition can be verified by a simple computation and gives a useful reformulation of these relations.

Proposition 2.24 *Let $\mathcal{A}$ be a unital C^*-algebra. Further, let $u \in \mathcal{A} \otimes \mathbb{C}^{I \times I}$, $v \in \mathcal{A} \otimes \mathbb{C}^{J \times J}$ be unitaries, and $T \in \mathbb{C}^{J \times I}$. Then $Tu = vT$ if and only if $T^* v = u T^*$.*

Further, we will use the free product of compact matrix quantum groups, which was introduced in [33].

Definition 2.25 Let $G := (\mathcal{A}, u)$ and $H := (\mathcal{B}, v)$ be two compact matrix quantum groups. Then the *free product* $G * H$ is the compact matrix quantum group

$$G * H := (\mathcal{A} * \mathcal{B}, u \oplus v),$$

where $\mathcal{A} * \mathcal{B}$ denotes the universal unital free product of C^*-algebras and we identify u and v with $\mathcal{A} * \mathcal{B}$-valued matrices via the canonical inclusions $\mathcal{A}, \mathcal{B} \hookrightarrow \mathcal{A} * \mathcal{B}$.

Example 2.26 Consider two finite sets X and Y with corresponding quantum permutation groups S_X^+ and S_Y^+. Then their free product $S_X^+ * S_Y^+$ is given by the universal unital C^*-algebra

$$C(S_X^+ * S_Y^+) = C^*(u_{ij}, v_{k\ell} \mid (u_{ij}) \text{ and } (v_{k\ell}) \text{are magic unitaries}),$$

where the elements u_{ij} are indexed over X and the elements $v_{k\ell}$ are indexed over Y. Further, the comultiplication is defined by

$$\Delta(u_{ij}) = \sum_{s \in X} u_{is} \otimes u_{sj}, \quad \Delta(v_{k\ell}) = \sum_{t \in Y} v_{it} \otimes v_{tj}.$$

2.4 Quantum automorphism groups of graphs

In the case of graphs, there exist two different versions of quantum automorphism groups, which have been introduced by Bichon [4] and Banica [2]. These quantum groups have been further studied, for example, in [9, 13, 20, 25, 29] and have been recently generalized to multigraphs in [16]. In the following, we begin with the definition of the quantum automorphism groups of Bichon, before we come to the version of Banica and the recent version for multigraphs by Goswami–Hossain. For more information on quantum automorphism groups of graphs, we refer to [29].

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Definition 2.27 Let $\Gamma := (V, E)$ be a simple or directed graph and denote by $\mathcal{A}$ the universal unital C^*-algebra with generators u_{ij} for all $i, j \in V$ and the following relations:

(1) $u := (u_{ij})_{i,j\in V}$ is a magic unitary;
(2) $A_\Gamma u = uA_\Gamma$, where A_Γ is the adjacency matrix of Γ;
(3) $u_{ik}u_{j\ell} = u_{j\ell}u_{ik}$ for all $i \sim j$ and $k \sim \ell$.

Then $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma) := (\mathcal{A}, u)$ is *Bichon's quantum automorphism group* of Γ.

Note that the second relation $A_\Gamma u = uA_\Gamma$ was originally formulated using a different set of relations. However, the following proposition shows that both versions are equivalent.

Proposition 2.28 [29, Proposition 2.1.3] *Let $\Gamma := (V, E)$ be a simple or directed graph, $\mathcal{A}$ a unital C^*-algebra, and $u \in \mathcal{A} \otimes \mathbb{C}^{V\times V}$ a magic unitary. Then $A_\Gamma u = uA_\Gamma$ is equivalent to*

$$u_{ik}u_{j\ell} = u_{j\ell}u_{ik} = 0 \quad \forall i \sim j,\ k \not\sim \ell,$$
$$u_{ik}u_{j\ell} = u_{j\ell}u_{ik} = 0 \quad \forall i \not\sim j,\ k \sim \ell.$$

Using this reformulation of the intertwiner relations $A_\Gamma u = uA_\Gamma$, one can also prove the following proposition, which will be useful later. See [30] for further details.

Proposition 2.29 *Let $\Gamma := (V, E)$ be a simple or directed graph and denote by u the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Then*

$$\sum_{\substack{k,\ell\in V\\ k\sim\ell}} u_{ik}u_{j\ell} = \sum_{\substack{k,\ell\in V\\ k\sim\ell}} u_{ki}u_{\ell j} = 1 \quad \forall i \sim j.$$

By dropping Relation 3 in Definition 2.27, we obtain the quantum automorphism group of Banica, which is often studied instead of Bichon's version. However, we will mainly be interested in Bichon's version and its following generalization to multigraphs as recently defined by Goswami–Hossain [16].

Definition 2.30 Let $\Gamma := (V, E)$ be a multigraph without isolated vertices. Denote by $\mathcal{A}$ the universal unital C^*-algebra with generators u_{ef} for all $e, f \in E$ and the following relations:

(1) The matrix $u := (u_{ef})_{e,f\in E}$ is a magic unitary.
(2) Let $v \in V$ and $e_1, e_2 \in E$.
   - If $s(e_1) = s(e_2)$, then $\sum_{\substack{f\in E\\ s(f)=v}} u_{e_1 f} = \sum_{\substack{f\in E\\ s(f)=v}} u_{e_2 f}$.
   - If $r(e_1) = r(e_2)$, then $\sum_{\substack{f\in E\\ r(f)=v}} u_{e_1 f} = \sum_{\substack{f\in E\\ r(f)=v}} u_{e_2 f}$.
(3) Let $e, f \in E$. Then $u_{ef} = 0$ if
   - $s(e)$ is neither a source nor a sink and $s(f)$ is a source;
   - $r(e)$ is neither a source nor a sink and $r(f)$ is a sink.

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(4) Let $v \in V$ be neither a source nor a sink and $e_1, e_2 \in E$ such that $s(e_1) = r(e_2)$ is neither a source nor a sink. Then $\sum_{\substack{f \in E \\ s(f)=v}} u_{e_1 f} = \sum_{\substack{f \in E \\ r(f)=v}} u_{e_2 f}$.

Then $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma) := (\mathcal{A}, u)$ is the *quantum automorphism group of* Γ *by Goswami–Hossain in Bichon's sense.*

Note that we swapped the conditions in Relation 3 in contrast to the original definition in [16, Definition 3.16].

2.5 Graph and hypergraph C^*-algebras

Graph C^*-algebras are a well-studied class of C^*-algebras that are defined in terms of an underlying graph and generalize Cuntz–Krieger algebras [12]. They include many concrete examples like matrix algebras, continuous functions on the circle, or the Cuntz algebra. In the following, we define graph C^*-algebras only for finite directed graphs. See [24] for more information and the general case of infinite graphs.

Definition 2.31 Let $\Gamma := (V, E)$ be a directed graph. The *graph* C^**-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e with orthogonal ranges for all $e \in E$ such that

(1) $s^*_{(v,w)} s_{(v,w)} = p_w$ for all $(v, w) \in E$;
(2) $s_{(v,w)} s^*_{(v,w)} \leq p_v$ for all $(v, w) \in E$;
(3) $p_v \leq \sum_{\substack{e \in E \\ v=s(e)}} s_e s_e^*$ for all $v \in V$ that are not a sink.

In [30], Schmidt and Weber then showed that Banica's quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ of a finite graph Γ acts maximally on the corresponding graph C^*-algebra $C^*(\Gamma)$.

Proposition 2.32 [30] *Let* $\Gamma := (V, E)$ *be a directed graph. Then* $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ *acts maximally on* $C^*(\Gamma)$ *via* $\alpha \colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma))$ *given by*

$$\alpha(p_v) = \sum_{w \in V} u_{vw} \otimes p_w, \quad \alpha(s_{(v_1,w_1)}) = \sum_{(v_2,w_2) \in E} u_{v_1 v_2} u_{w_1 w_2} \otimes s_{(v_2,w_2)}$$

for all $v \in V$ *and* $(v_1, w_1) \in E$.

See also [19] for more information on actions of quantum groups on graph C^*-algebras. Recently, Trieb–Weber–Zenner [32] introduced hypergraph C^*-algebras by replacing the graph in the definition of graph C^*-algebras with a hypergraph.

Definition 2.33 Let $\Gamma := (V, E)$ be a hypergraph. Then the *hypergraph* C^**-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that

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(1) $s_e^* s_f = \delta_{ef} \sum_{v \in r(e)} p_v$ for all $e, f \in E$;
(2) $s_e s_e^* \leq \sum_{v \in s(e)} p_v$ for all $e \in E$;
(3) $p_v \leq \sum_{e \in s(e)} s_e s_e^*$ for all $v \in V$ that are not a sink.

If a directed graph is encoded as a hypergraph as in Definition 2.11, then the corresponding hypergraph C^*-algebra agrees with the classical graph C^*-algebra. However, hypergraph C^*-algebras also include new examples of non-nuclear C^*-algebras. For more on the nuclearity of hypergraph C^*-algebras, we refer to the recent work of Schäfer and Weber [26], where the nuclearity of hypergraph C^*-algebras is characterized in terms of minors of the underlying hypergraph.

3 Quantum symmetries of hypergraphs

3.1 Automorphism groups of hypergraphs

We first recall the definition of the classical automorphism group of a hypergraph and characterize it in terms of permutation matrices, before we introduce the quantum automorphism group of a hypergraph.

Definition 3.1 Let $\Gamma := (V, E)$ be a hypergraph. Then its *automorphism group* is

$$\mathrm{Aut}(\Gamma) := \left\{ (\sigma, \tau) \in S_V \times S_E \mid \forall e \in E \colon \sigma(s(e)) = s(\tau(e)) \wedge \sigma(r(e)) = r(\tau(e)) \right\},$$

where $\sigma \in S_V$ acts on $\mathcal{P}(V)$ by $\sigma(\{v_1, \dots v_k\}) := \{\sigma(v_1), \dots, \sigma(v_k)\}$.

Thus, a hypergraph automorphism consists of a permutation of the vertices and a permutation of the edges that are compatible by respecting the source and range maps. Next, we show how these compatibility conditions can be reformulated when both permutations are given as permutation matrices.

Definition 3.2 Let X be a finite set. The *permutation representation* of S_X is given by $S_X \to \mathbb{C}^{X \times X}$, $\sigma \mapsto P_\sigma$ with $(P_\sigma)_{ij} = \delta_{i\sigma(j)}$ for all $i, j \in X$.

Note that the permutation representation is faithful, so we have an embedding $S_X \hookrightarrow \mathbb{C}^{X \times X}$ given by permutation matrices.

Proposition 3.3 *Let $\Gamma := (V, E)$ be a hypergraph and $(\sigma, \tau) \in S_V \times S_E$. Then*
(1) *$\sigma(s(e)) = s(\tau(e))$ for all $e \in E$ if and only if $A_s P_\tau = P_\sigma A_s$,*
(2) *$\sigma(r(e)) = r(\tau(e))$ for all $e \in E$ if and only if $A_r P_\tau = P_\sigma A_r$,*

where A_s and A_r are the incidence matrices of Γ.

Proof We prove only the first statement about the source map s. The second statement about the range map r follows in a similar way. Using the definitions of

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A_s and permutation matrices, one verifies that

$$A_s P_\tau = P_\sigma A_s \iff (A_s)_{v\tau(e)} = (A_s)_{\sigma^{-1}(v)e} \quad \forall v \in V,\ e \in E,$$

where the right side is equivalent to

$$v \in s(\tau(e)) \iff \sigma^{-1}(v) \in s(e) \qquad \forall v \in V,\ e \in E.$$

However, $\sigma^{-1}(v) \in s(e)$ can be rewritten as $v \in \sigma(s(e))$, so that we obtain

$$v \in s(\tau(e)) \iff v \in \sigma(s(e)) \qquad \forall v \in V,\ e \in E.$$

This means that $s(\tau(e)) = \sigma(s(e))$ for all $e \in E$. ∎

3.2 Quantum automorphism groups of hypergraphs

Using the characterization of classical hypergraph automorphisms in terms of permutation matrices, we can now define the quantum automorphism group of a hypergraph.

Definition 3.4 Let $\Gamma := (V, E)$ be a hypergraph and denote by $\mathcal{A}$ the universal unital C^*-algebra with generators u_{vw} for all $v, w \in V$ and u_{ef} for all $e, f \in E$, such that

(1) $u_V := (u_{vw})_{v,w\in V}$ and $u_E := (u_{ef})_{e,f\in E}$ are magic unitaries;

(2) $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$, where $A_s, A_r \in \mathbb{C}^{V\times E}$ are the incidence matrices of Γ.

Then $\mathrm{Aut}^+(\Gamma) := (\mathcal{A}, u_V \oplus u_E)$ is the *quantum automorphism group of* Γ.

Intuitively, we replace the permutations $\sigma \in S_V$ and $\tau \in S_E$ in the definition of the classical automorphism group by quantum permutation matrices u_V and u_E. The compatibility conditions between these two matrices can then be expressed by the intertwining relations from the previous sections.

Before we show that the previous definition of $\mathrm{Aut}^+(\Gamma)$ indeed generalizes the classical automorphism group $\mathrm{Aut}(\Gamma)$, we first comment on the relations in Definition 3.4.

Remark 3.5 For $\mathrm{Aut}^+(\Gamma)$ to be well-defined as a compact matrix quantum group, the magic unitary relations and intertwiner relations have to be compatible with the comultiplication $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. However, this follows directly from the existence of S_n^+ and the fact that additional intertwiner relations always respect the comultiplication (see [34, 37]).

Remark 3.6 Recall that the magic unitary relations of u_V and u_E are given by

$$u_{vw}^2 = u_{vw}^* = u_{vw} \quad \forall v, w \in V, \qquad u_{ef}^2 = u_{ef}^* = u_{ef} \quad \forall e, f \in E,$$

$$\sum_{w\in V} u_{vw} = \sum_{w\in V} u_{wv} = 1 \quad \forall v \in V, \qquad \sum_{f\in E} u_{ef} = \sum_{f\in E} u_{fe} = 1 \quad \forall e \in E,$$

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and the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ can be written as

$$\sum_{\substack{f\in E\\ v\in s(f)}} u_{fe} = \sum_{\substack{w\in V\\ w\in s(e)}} u_{vw}, \qquad \sum_{\substack{f\in E\\ v\in r(f)}} u_{fe} = \sum_{\substack{w\in V\\ w\in r(e)}} u_{vw} \qquad \forall v\in V, e\in E.$$

Note that A_s^* and A_r^* are also intertwiners by Proposition 2.24 since u_V and u_E are unitaries. Thus, we have the additional relations $A_s^* u_V = u_E A_s^*$ and $A_r^* u_V = u_E A_r^*$, which can be written as

$$\sum_{\substack{w\in V\\ w\in s(e)}} u_{wv} = \sum_{\substack{f\in E\\ v\in s(f)}} u_{ef}, \qquad \sum_{\substack{w\in V\\ w\in r(e)}} u_{wv} = \sum_{\substack{f\in E\\ v\in r(f)}} u_{ef} \qquad \forall v\in V, e\in E.$$

Remark 3.7 Let $u := u_V \oplus u_E$ and define the block matrix

$$A := \begin{pmatrix} 0 & A_s \\ A_r^* & 0 \end{pmatrix} \in \mathbb{C}^{(V\sqcup E)\times(V\sqcup E)}.$$

Then $Au = uA$ is equivalent to $A_s u_E = u_V A_s$ and $A_r^* u_V = u_E A_r^*$, where the second equation is again equivalent to $A_r u_E = u_V A_r$ by Proposition 2.24. Therefore, the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in Definition 3.4 can be formulated as the single intertwiner relation $Au = uA$.

The following proposition shows that the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ indeed generalizes the classical automorphism group $\mathrm{Aut}(\Gamma)$ in the sense of compact matrix quantum groups.

Proposition 3.8 *Let Γ be a hypergraph. Then* $\operatorname{Spec} C(\mathrm{Aut}^+(\Gamma)) \cong \mathrm{Aut}(\Gamma)$ *as finite groups.*

Proof Let $\Gamma := (V, E)$ and denote by $\mathcal{A}$ the C^*-algebra $C(\mathrm{Aut}^+(\Gamma))$. Then $\operatorname{Spec}\mathcal{A}$ is a group with multiplication given by $\varphi * \psi := (\varphi\otimes\psi)\circ\Delta$ and is isomorphic to a subgroup of unitary matrices $G \subseteq \mathbb{C}^{(V\sqcup E)\times(V\sqcup E)}$ via

$$\varphi \in \operatorname{Spec}\mathcal{A} \;\longleftrightarrow\; \varphi(u) := (\varphi(u_{ij})) \in \mathbb{C}^{(V\sqcup E)\times(V\sqcup E)},$$

see, for example, [31, Proposition 6.1.11]. Further, we have the decomposition $u = u_E \oplus u_V$, so that $\varphi(u) = \varphi(u_E)\oplus\varphi(u_V)$ is given by a pair of matrices $\varphi(u_E)$ and $\varphi(u_V)$. Since u_V and u_E are magic unitaries, $\varphi(u_E)$ and $\varphi(u_V)$ are exactly permutation matrices, which correspond to a pair of permutations $(\sigma,\tau)\in S_V\times S_E$ via the permutation representation from Definition 3.2. Proposition 3.3 then implies that (σ,τ) are exactly automorphisms of Γ. ∎

3.3 Examples of hypergraphs with maximal quantum symmetry

Before we come to examples of quantum automorphism groups of hypergraphs, we first show that these quantum groups are always subgroups of $S_V^+ * S_E^+$.

Proposition 3.9 *Let $\Gamma := (V, E)$ be a hypergraph. Then* $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+ * S_E^+$.

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Proof Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $S_V^+ * S_E^+$. Since u_V and u_E are magic unitaries, the universal property of $C(S_V^+ * S_E^+)$ implies the existence of a unital $*$-homomorphism $\varphi\colon C(S_V^+ * S_E^+) \to C(\mathrm{Aut}^+(\Gamma))$ with $\varphi(\widehat{u}_{vw}) = u_{vw}$ and $\varphi(\widehat{u}_{ef}) = u_{ef}$. This $*$-homomorphism is surjective since $C(\mathrm{Aut}^+(\Gamma))$ is generated by the entries of u_V and u_E. Further, one verifies directly that it is a morphism of compact quantum groups. Thus, $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+ * S_E^+$. ∎

Next, we construct a concrete family of hypergraphs $\Gamma_{n,m}$ for which equality in the previous proposition holds. Thus, these hypergraphs have maximal possible quantum symmetries in the sense of Definition 3.4.

Definition 3.10 Let $n, m \in \mathbb{N}$. Define the hypergraph $\Gamma_{n,m} := (V, E)$ with vertices $V = \{1, \dots, n\}$, edges $E = \{1, \dots, m\}$, and source and range maps $s(e) := V$ and $r(e) := V$ for all $e \in E$.

Proposition 3.11 *Let $n, m \in \mathbb{N}$ and $\Gamma_{n,m} := (V, E)$ be the hypergraph of Definition 3.10. Then $\mathrm{Aut}^+(\Gamma_{n,m}) = S_V^+ * S_E^+$.*

Proof Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma_{n,m})$ and by $\widehat{u}$ the fundamental representation of $S_V^+ * S_E^+$. By the proof of Proposition 3.9, we have $\mathrm{Aut}^+(\Gamma_{n,m}) \subseteq S_V^+ * S_E^+$ via a unital $*$-homomorphism $\varphi\colon C(S_V^+ * S_E^+) \to C(\mathrm{Aut}^+(\Gamma_{n,m}))$ with $\varphi(\widehat{u}_{vw}) = u_{vw}$ and $\varphi(\widehat{u}_{ef}) = u_{ef}$. To show the other inclusion, we construct the inverse $*$-homomorphism using the universal property of $C(\mathrm{Aut}^+(\Gamma_{n,m}))$. Thus, we have to show that $\widehat{u}_V$ and $\widehat{u}_E$ satisfy the relations from Definition 3.4. However, $\widehat{u}_V$ and $\widehat{u}_E$ are magic unitaries by definition, and we compute

$$\sum_{\substack{f \in E \\ v \in s(f)}} \widehat{u}_{fe} = \sum_{f \in E} \widehat{u}_{fe} = 1 = \sum_{w \in V} \widehat{u}_{vw} = \sum_{\substack{w \in V \\ w \in s(e)}} \widehat{u}_{vw} \quad \forall v \in V, e \in E,$$

so that $A_s \widehat{u}_E = \widehat{u}_V A_s$ by Remark 3.6. Similarly, one shows that $A_r \widehat{u}_E = \widehat{u}_V A_r$ by replacing the source map s with the range map r. Thus, the $*$-homomorphism from Proposition 3.9 is invertible, which shows $\mathrm{Aut}^+(\Gamma_{n,m}) = S_V^+ * S_E^+$. ∎

3.4 Opposite and dual hypergraphs

Next, we compute the quantum automorphism groups of the opposite and dual of a hypergraph. Recall from Definition 2.14 that the opposite hypergraph Γ^{op} is obtained by interchanging the source map and the range map of a hypergraph Γ. One can directly verify that in the classical case, both $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma^{\mathrm{op}})$ coincide. In the following, we show that this fact can be generalized to the quantum setting.

Proposition 3.12 *Let Γ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^{\mathrm{op}})$.*

Proof Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma^{\mathrm{op}})$. By definition, we have $A_{s^{\mathrm{op}}} = A_r$ and $A_{r^{\mathrm{op}}} = A_s$, so that the entries of u and $\widehat{u}$ satisfy exactly the same

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relations. Hence, by the universal properties of $C(\mathrm{Aut}^+(\Gamma))$ and $C(\mathrm{Aut}^+(\Gamma^{\mathrm{op}}))$, there exists a $*$-isomorphism $\varphi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma^{\mathrm{op}}))$ with $\varphi(u_{vw}) = \widehat{u}_{vw}$ and $\varphi(u_{ef}) = \widehat{u}_{ef}$. Further, a direct computation shows that this $*$-isomorphism is indeed a morphism of compact quantum groups. ∎

Next, we consider dual hypergraphs. Recall from Definition 2.14 that the dual Γ^* of a hypergraph Γ is obtained by interchanging the vertices and edges. As in the case of the opposite hypergraph, one verifies that a hypergraph and its dual have isomorphic classical automorphism groups via the map $(\sigma, \tau) \mapsto (\tau, \sigma)$. This again generalized to the quantum setting as follows.

Proposition 3.13 *Let Γ be a hypergraph. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^*)$.

Proof Let $\Gamma := (V, E)$ and denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma^*)$. We begin by constructing a $*$-isomorphism $\varphi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma^*))$ with $\varphi(u_{vw}) = \widehat{u}_{vw}$ and $\varphi(u_{ef}) = \widehat{u}_{ef}$ using the universal properties of $C(\mathrm{Aut}^+(\Gamma))$ and $C(\mathrm{Aut}^+(\Gamma^*))$. Hence, we have to show that the entries of u and $\widehat{u}$ satisfy the same relations. Since u_V, u_E, $\widehat{u}_E$, and $\widehat{u}_V$ are all magic unitaries, it only remains to show that

$$A_s u^{(1)} = u^{(2)} A_s \iff A_{s^*} u^{(2)} = u^{(1)} A_{s^*},$$

$$A_r u^{(1)} = u^{(2)} A_r \iff A_{r^*} u^{(2)} = u^{(1)} A_{r^*}$$

for two arbitrary magic unitaries $u^{(1)}$ and $u^{(2)}$ indexed by V and E, respectively. However, this follows directly from Proposition 2.24, since

$$(A_{s^*})_{ev} = \begin{cases} 1 & \text{if } e \in s^*(v), \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } v \in s(e), \\ 0 & \text{otherwise,} \end{cases} = (A_s)_{ve} = (A_s^T)_{ev}$$

for all $e \in E$ and $v \in V$, so that $A_{s^*} = A_s^T = A_s^*$ and similarly $A_{r^*} = A_r^*$. Thus, the $*$-isomorphism φ exists and one verifies directly that it is a morphism of compact quantum groups. ∎

3.5 Hypergraphs without multi-edges

In contrast to the quantum automorphism group of graphs by Bichon and Banica, our quantum automorphism group includes an additional magic unitary u_E for the edges. This magic unitary is necessary to capture quantum symmetries between multi-edges, see, for example, the family of hypergraphs in Section 3.3. However, it turns out that if a hypergraph has no multi-edges, then the magic unitary u_E is redundant and we can express the entries of u_E in terms of the entries of u_V.

We begin with the following lemma, which relates the entries of u_E to the entries of u_V.

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Lemma 3.14 *Let $\Gamma := (V, E)$ be a hypergraph and $X \subseteq V$. Then*

$$\sum_{\substack{f \in E \\ X \subseteq s(f)}} u_{fe} = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_{vw} \quad \forall e \in E.$$

In particular, the product commutes. The same statement also holds for the range map r.

Proof Let $e \in E$ and $X \subseteq V$. Fix an arbitrary ordering $X = \{v_1, \dots, v_k\}$, where all v_i are distinct. Then

$$\prod_{v \in X} (A_s u_E)_{ve} = \prod_{v \in X} \sum_{\substack{f \in E \\ v \in s(f)}} u_{fe} = \sum_{f_1 \in E, v_1 \in s(f_1)} \cdots \sum_{f_k \in E, v_k \in s(f_k)} u_{f_1 e} \dots u_{f_k e}.$$

Since $u_{f_1 e} \dots u_{f_k e} = \delta_{f_1 f_2} \dots \delta_{f_1 f_k} u_{f_1 e}$, it follows that

$$\prod_{v \in X} (A_s u_E)_{ve} = \sum_{f \in E, v_1 \in s(f), \dots, v_k \in s(f)} u_{fe} = \sum_{\substack{f \in E \\ X \subseteq s(f)}} u_{fe}.$$

On the other hand, we can apply $A_s u_E = u_V A_s$ to the original expression to obtain

$$\prod_{v \in X} (A_s u_E)_{ve} = \prod_{v \in X} (u_V A_s)_{ve} = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_{vw}.$$

Thus,

$$\sum_{\substack{f \in E \\ X \subseteq s(f)}} u_{fe} = \prod_{v \in X} \sum_{\substack{w \in V \\ w \in s(e)}} u_{vw}.$$

Since the left side does not depend on the order of X, the product on the right side commutes. Further, we can replace s with r to obtain a proof of the corresponding statement for the range map r. ∎

Using an inclusion–exclusion argument, we can further improve the previous lemma to obtain the equality $X = s(f)$ instead of $X \subseteq s(f)$ on the left side.

Lemma 3.15 *Let $\Gamma := (V, E)$ be a hypergraph and $X \subseteq V$. Then*

$$\sum_{\substack{f \in E \\ s(f) = X}} u_{fe} = \sum_{X \subseteq Y \subseteq V} (-1)^{|Y| - |X|} \prod_{v \in Y} \sum_{\substack{w \in V \\ w \in s(e)}} u_{vw} \quad \forall e \in E.$$

The same statement also holds for the range map r.

Proof By rewriting the statement using Lemma 3.14, we have to show that

$$\sum_{\substack{f \in E \\ s(f) = X}} u_{fe} = \sum_{X \subseteq Y \subseteq V} (-1)^{|Y| - |X|} \sum_{\substack{f \in E \\ Y \subseteq s(f)}} u_{fe} \quad \forall e \in E.$$

Now, consider an element u_{fe} with $X \subseteq s(f)$ and define $k := |s(f)| - |X|$. Then there are exactly $\binom{k}{\ell}$ subsets Y with $X \subseteq Y \subseteq s(f)$ and $|X| + \ell$ elements. Further, each is

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weighted with a factor of

$$(-1)^{|Y|-|X|} = (-1)^{(|X|+\ell)-|X|} = (-1)^{\ell}$$

on the right side of the equation. Thus, by the binomial theorem, the total contribution of u_{ef} on the right side is

$$\sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} = ((-1)+1)^{k} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

Therefore, the right side of the equation contains exactly each u_{fe} with $k = 0$, which is equivalent to $s(f) = X$. The corresponding statement for the range map r can be proven in the same way. ∎

The previous lemma can now be used to show that the elements of u_E can be expressed in terms of the elements of u_V if the underlying hypergraph has no multi-edges.

Theorem 3.16 *Let $\Gamma := (V, E)$ be a hypergraph without multi-edges. Denote by $C^*(u_V)$ the C^*-algebra generated by u_{vw} for all $v, w \in V$. Then $u_{ef} \in C^*(u_V)$ for all $e, f \in E$.*

Proof Let $e, f \in E$. Then

$$\sum_{\substack{g \in E \\ s(g)=s(e)}} u_{gf}, \quad \sum_{\substack{g \in E \\ r(g)=r(e)}} u_{gf} \in C^*(u_V)$$

by choosing $X = s(e)$ and $X = r(e)$ in Lemma 3.15. This implies

$$\left(\sum_{\substack{g \in E \\ s(g)=s(e)}} u_{gf} \right) \left(\sum_{\substack{g \in E \\ r(g)=r(e)}} u_{gf} \right) = \sum_{\substack{g_1, g_2 \in E \\ s(g_1)=s(e) \\ r(g_2)=r(e)}} \underbrace{u_{g_1 f} u_{g_2 f}}_{\delta_{g_1 g_2} u_{g_1 f}} = \sum_{\substack{g \in E \\ s(g)=s(e) \\ r(g)=r(e)}} u_{gf} \in C^*(u_V).$$

However, Γ has no multi-edges, so that

$$u_{ef} = \sum_{\substack{g \in E \\ s(g)=s(e) \\ r(g)=r(e)}} u_{gf} \in C^*(u_V).$$

∎

Translating the previous theorem to the setting of quantum groups yields the following two corollaries.

Corollary 3.17 *Let $\Gamma := (V, E)$ be a hypergraph without multi-edges. Then* $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+$.

Proof Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of S_V^+. As in the proof of Proposition 3.9, there exists

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a morphism of compact quantum groups $\varphi: C(S_V^+) \to \mathrm{Aut}^+(\Gamma)$ with $\varphi(\widehat{u}_{vw}) = u_{vw}$. By Theorem 3.16, this morphism is surjective, so that $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+$. ∎

Corollary 3.18 *Let $\Gamma := (V, E)$ be a hypergraph such that Γ^* has no multi-edges. Then $\mathrm{Aut}^+(\Gamma) \subseteq S_E^+$.*

Proof By combining Proposition 3.4 with Corollary 3.17, we obtain $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+(\Gamma^*) \subseteq S_E^+$. ∎

4 Link to quantum symmetries of classical graphs

In this section, we study the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ for hypergraphs which come from a classical directed, simple, or multigraph as defined in Section 2.2. In particular, we show that, in this case, our quantum automorphism group for hypergraphs agrees with the quantum automorphism group of Bichon for classical graphs or its multigraph version by Goswami–Hossain. Thus, we can view our quantum automorphism group for hypergraphs as a generalization of the quantum automorphism group of Bichon for graphs.

4.1 Directed graphs

We begin with directed graphs as in Definition 2.2. Recall from Definition 2.11 that we can identify a directed graph $\Gamma := (V, E)$ with a 1-uniform hypergraph without multi-edges by defining the source and range maps $s(v, w) := \{v\}$ and $r(v, w) = \{w\}$ for all $(v, w) \in E$. In this way, we can apply Definition 3.4 to obtain a hypergraph quantum automorphism group $\mathrm{Aut}^+(\Gamma)$. In the following, we show that $\mathrm{Aut}^+(\Gamma)$ agrees with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ of Bichon. We begin by reformulating the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the setting of directed graphs.

Lemma 4.1 *Let $\Gamma := (V, E)$ be a directed graph, $\mathcal{A}$ a unital C^*-algebra, $u_V \in \mathcal{A} \otimes \mathbb{C}^{V \times V}$, and $u_E \in \mathcal{A} \otimes \mathbb{C}^{E \times E}$. Then the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to*

$$\sum_{\substack{(v_2, w_2) \in E \\ v_0 = v_2}} u_{(v_2, w_2)(v_1, w_1)} = u_{v_0 v_1}, \qquad \sum_{\substack{(v_2, w_2) \in E \\ v_0 = w_2}} u_{(v_2, w_2)(v_1, w_1)} = u_{v_0 w_1}$$

for all $v_0 \in V$ and $(v_1, w_1) \in E$.

Proof Let $v_0 \in V$, $e := (v_1, w_1) \in E$. Then

$$(A_s u_E)_{v_0 e} = \sum_{\substack{f \in E \\ v_0 \in s(f)}} u_{fe} = \sum_{\substack{(v_2, w_2) \in E \\ v_0 = v_2}} u_{(v_2, w_2)(v_1, w_1)}.$$

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On the other hand, the image of s always contains one element and there are no multi-edges so that

$$\left(u_V A_s\right)_{v_0 e} = \sum_{\substack{w \in V \\ w \in s(e)}} u_{v_0 w} = u_{v_0 v_1}.$$

Thus, $A_s u_E = u_V A_s$ is equivalent to

$$\sum_{\substack{(v_2,w_2)\in E \\ v_0 = v_2}} u_{(v_2,w_2)(v_1,w_1)} = u_{v_0 v_1} \qquad \forall v_0 \in V,\ (v_1,w_1) \in E.$$

The equivalence between $A_r u_E = u_V A_r$ and the second equation in the statement follows similarly. ∎

Using the previous lemma, we can now express the entries of u_E in terms of the entries of u_V.

Lemma 4.2 *Let $\Gamma := (V, E)$ be a directed graph and denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then*

$$u_{(v_1,w_1)(v_2,w_2)} = u_{v_1 v_2} u_{w_1 w_2} = u_{w_1 w_2} u_{v_1 v_2} \quad \forall (v_1,w_1),(v_2,w_2) \in E.$$

Proof Let $(v_1,w_1),(v_2,w_2) \in E$. By Lemma 4.1, we have

$$u_{v_1 v_2} = \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3}} u_{(v_3,w_3)(v_2,w_2)}, \qquad u_{w_1 w_2} = \sum_{\substack{(v_4,w_4)\in E \\ w_1 = w_4}} u_{(v_4,w_4)(v_2,w_2)},$$

which yields

$$u_{v_1 v_2} u_{w_1 w_2} = \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3}} \sum_{\substack{(v_4,w_4)\in E \\ w_1 = w_4}} \underbrace{u_{(v_3,w_3)(v_2,w_2)} u_{(v_4,w_4)(v_2,w_2)}}_{\delta_{(v_3,w_3)(v_4,w_4)} u_{(v_3,w_3)(v_2,w_2)}} = \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3 \\ w_1 = w_3}} u_{(v_3,w_3)(v_2,w_2)},$$

$$u_{w_1 w_2} u_{v_1 v_2} = \sum_{\substack{(v_4,w_4)\in E \\ w_1 = w_4}} \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3}} \underbrace{u_{(v_4,w_4)(v_2,w_2)} u_{(v_3,w_3)(v_2,w_2)}}_{\delta_{(v_4,w_4)(v_3,w_3)} u_{(v_3,w_3)(v_2,w_2)}} = \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3 \\ w_1 = w_3}} u_{(v_3,w_3)(v_2,w_2)}.$$

Since Γ has no multi-edges, we have

$$u_{(v_1,w_1)(v_2,w_2)} = \sum_{\substack{(v_3,w_3)\in E \\ v_1 = v_3 \\ w_1 = w_3}} u_{(v_3,w_3)(v_2,w_2)}.$$

Thus,

$$u_{(v_1,w_1)(v_2,w_2)} = u_{v_1 v_2} u_{w_1 w_2} = u_{w_1 w_2} u_{v_1 v_2}.$$

∎

Now, we can show that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ agrees with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ when we identify the magic unitary u_V with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Theorem 4.3 *Let Γ be a directed graph. Then $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.*

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Proof Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Further, define the elements $\widehat{u}_{(v_1,w_1)(v_2,w_2)} := \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2}$ for all $(v_1, w_1), (v_2, w_2) \in E$, and the matrices $\widehat{u}_V := (\widehat{u}_{vw})_{v,w\in V}$ and $\widehat{u}_E := (\widehat{u}_{ef})_{e,f\in E}$.

To prove the statement, we begin by constructing a unital $*$-homomorphism $\varphi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$ defined by $\varphi(u_{vw}) = \widehat{u}_{vw}$ and $\varphi(u_{(v_1,w_1)(v_2,w_2)}) = \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2}$ using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Thus, we have to show that the matrices $\widehat{u}_V$ and $\widehat{u}_E$ satisfy the relations from Definition 3.4. By Definition 2.27, $\widehat{u}_V$ is a magic unitary, and we have

$$\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = \widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2} \quad \forall (v_1, w_1), (v_2, w_2) \in E,$$

which implies

$$\begin{aligned}(\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2})^* &= (\widehat{u}_{w_1w_2})^*(\widehat{u}_{v_1v_2})^* = \widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2} = \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_1},\\ (\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2})^2 &= \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = (\widehat{u}_{v_1v_2})^2(\widehat{u}_{w_1w_2})^2 = \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2}.\end{aligned}$$

Further, the additional relations from Proposition 2.29 yield

$$\begin{aligned}\sum_{(v_2,w_2)\in E} \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} &= \sum_{\substack{v_2,w_2\in V\\(v_2,w_2)\in E}} \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = 1,\\ \sum_{(v_2,w_2)\in E} \widehat{u}_{v_2v_1}\widehat{u}_{w_2w_1} &= \sum_{\substack{v_2,w_2\in V\\(v_2,w_2)\in E}} \widehat{u}_{v_2v_1}\widehat{u}_{w_2w_1} = 1\end{aligned}$$

for all $(v_1, w_1) \in E$. Hence, $\widehat{u}_E$ is a magic unitary. Next, we check the intertwiner relation $A_s\widehat{u}_E = \widehat{u}_V A_s$. Let $v_0 \in V$ and $(v_1, w_1) \in E$. Then

$$\sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}_{v_2v_1}\widehat{u}_{w_2w_1} = \widehat{u}_{v_0v_1} \sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}_{w_2w_1} = \widehat{u}_{v_0v_1} \sum_{w_2\in V} (A_\Gamma)_{v_0w_2}\widehat{u}_{w_2w_1}.$$

By Definition 2.27, we have $A_\Gamma\widehat{u}_V = \widehat{u}_V A_\Gamma$, so that

$$\widehat{u}_{v_0v_1} \sum_{w_2\in V} (A_\Gamma)_{v_0w_2}\widehat{u}_{w_2w_1} = \widehat{u}_{v_0v_1} \sum_{w_2\in V} \widehat{u}_{v_0w_2}(A_\Gamma)_{w_2w_1}.$$

Since $\widehat{u}_{v_0v_1}\widehat{u}_{v_0w_2} = \delta_{v_1w_2}\widehat{u}_{v_0v_1}$, it follows that

$$\widehat{u}_{v_0v_1} \sum_{w_2\in V} \widehat{u}_{v_0w_2}(A_\Gamma)_{w_2w_1} = \sum_{w_2\in V} \delta_{v_1w_2}\widehat{u}_{v_0v_1}(A_\Gamma)_{w_2w_1} = \widehat{u}_{v_0v_1}(A_\Gamma)_{v_1w_1} = \widehat{u}_{v_0v_1}.$$

Thus, we have in total

$$\sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}_{(v_2,w_2)(v_1,w_1)} = \sum_{\substack{(v_2,w_2)\in E\\v_0=v_2}} \widehat{u}_{v_2v_1}\widehat{u}_{w_2w_1} = \widehat{u}_{v_0v_1},$$

which is equivalent to $A_s\widehat{u}_E = \widehat{u}_V A_s$ by Lemma 4.1. Similarly, one shows $A_r\widehat{u}_E = \widehat{u}_V A_r$ so that the map φ exists.

Next, we construct the inverse map $\psi\colon C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma))$ given by $\psi(\widehat{u}_{vw}) = u_{vw}$ using the universal property of $C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$. Thus, we have to show that the matrix u_V satisfies the relations from Definition 2.27. First, note that u_V

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is a magic unitary by Definition 3.4. Second, we have to show that $A_\Gamma u_V = u_V A_\Gamma$. Observe that $A_\Gamma = A_s A_r^*$, since

$$(A_s A_r^*)_{vw} = \sum_{e \in E} (A_s)_{ve} (A_r)_{we} = \sum_{e \in E} \delta_{(v,w)e} = \begin{cases} 1 & \text{if } (v,w) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

for all $v, w \in V$. Hence,

$$A_\Gamma u_V = A_s A_r^* u_V = A_s u_E A_r^* = u_V A_s A_r^* = u_V A_\Gamma,$$

because A_r^* also intertwines u_V and u_E by Remark 3.6. Finally, we have to check that $u_{v_1 v_2} u_{w_1 w_2} = u_{w_1 w_2} u_{v_1 v_2}$ for all edges $(v_1, w_1), (v_2, w_2) \in E$. But this follows directly from Lemma 4.2, since

$$u_{v_1 v_2} u_{w_1 w_2} = u_{(v_1, w_1)(v_2, w_2)} = u_{w_1 w_2} u_{v_1 v_2}.$$

Thus, the $*$-homomorphism ψ exists. Note that the maps φ and ψ are indeed inverse, because

$$\widehat{u}_{vw} \longleftrightarrow u_{vw}, \qquad \widehat{u}_{v_1 v_2} \widehat{u}_{w_1 w_2} \longleftrightarrow u_{v_1 v_2} u_{w_1 w_2} = u_{(v_1, w_1)(v_2, w_2)}$$

by Lemma 4.2. Further, a direct computation shows that they are morphisms of compact quantum groups. ∎

4.2 Simple graphs

Next, we come to simple graphs as in Definition 2.1. Recall from Definition 2.10 that we can regard simple graphs as 2-uniform undirected hypergraphs without multi-edges by defining the source and range maps $s(\{v, w\}) := \{v, w\}$ and $r(\{v, w\}) := \{v, w\}$ for all $\{v, w\} \in E$. In the following, we show that, for a simple graph Γ, our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ agrees again with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ by Bichon. We begin with a reformulation of the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the setting of simple graphs.

Lemma 4.4 *Let $\Gamma := (V, E)$ be a simple graph, $\mathcal{A}$ a unital C^*-algebra, and $u_V \in \mathcal{A} \otimes \mathbb{C}^{V \times V}$ and $u_E \in \mathcal{A} \otimes \mathbb{C}^{E \times E}$ be two unitaries. Then the following are equivalent:*

(1) *$A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$;*
(2) *$\sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_0, v_2\}\{v_1, w_1\}} = u_{v_0 v_1} + u_{v_0 w_1}$ for all $v_0 \in V$ and $\{v_1, w_1\} \in E$;*
(3) *$\sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_1, w_1\}\{v_0, v_2\}} = u_{v_1 v_0} + u_{w_1 v_0}$ for all $v_0 \in V$ and $\{v_1, w_1\} \in E$.*

Proof Since $A_s = A_r$, we only have to consider $A_s u_E = u_V A_s$. Let $v_0 \in V$ and $e := \{v_1, w_1\} \in E$. Then

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$$(A_s u_E)_{v_0 e} = \sum_{f \in E} (A_s)_{v_0 f} u_{fe} = \sum_{\substack{f \in E \\ v_0 \in f}} u_{fe} = \sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_0, v_2\}\{v_1, w_1\}},$$
$$(u_V A_s)_{v_0 e} = \sum_{w \in V} u_{v_0 w} (A_s)_{we} = \sum_{\substack{w \in V \\ w \in e}} u_{v_0 w} = u_{v_0 v_1} + u_{v_0 w_1}.$$

Hence, $A_s u_E = u_V A_s$ is equivalent to

$$\sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_0, v_2\}\{v_1, w_1\}} = u_{v_0 v_1} + u_{v_0 w_1} \quad \forall v_0 \in V,\ \{v_1, w_1\} \in E.$$

Similarly, we compute

$$(u_E A_s^*)_{e v_0} = \sum_{f \in E} u_{ef} (A_s^*)_{f v_0} = \sum_{\substack{f \in E \\ v_0 \in f}} u_{ef} = \sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_1, w_1\}\{v_0, v_2\}},$$
$$(A_s^* u_V)_{e v_0} = \sum_{w \in V} (A_s)_{ew} u_{w v_0} = \sum_{\substack{w \in V \\ w \in e}} u_{w v_0} = u_{v_1 v_0} + u_{w_1 v_0}.$$

Thus, $u_E A_s^* = A_s^* u_V$ is equivalent to

$$\sum_{\substack{v_2 \in V \\ \{v_0, v_2\} \in E}} u_{\{v_1, w_1\}\{v_0, v_2\}} = u_{v_1 v_0} + u_{w_1 v_0} \quad \forall v_0 \in V,\ \{v_1, w_1\} \in E,$$

which is again equivalent to $A_s u_E = u_V A_s$ by Proposition 2.24. ∎

As in the case of directed graphs, we can now use the previous lemma to express the entries of the magic unitary u_E in terms of the entries of u_V.

Lemma 4.5 *Let $\Gamma := (V, E)$ be a simple graph, denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and let $\{v_1, w_1\}, \{v_2, w_2\} \in E$. Then*

$$u_{\{v_1, w_1\}\{v_2, w_2\}} = u_{v_1 v_2} u_{w_1 w_2} + u_{v_1 w_2} u_{w_1 v_2} = u_{v_1 v_2} u_{w_1 w_2} + u_{w_1 v_2} u_{v_1 w_2}.$$

Proof Since $v_1 \neq w_1$, we have

$$\begin{aligned}(u_{v_1 v_2} + u_{v_1 w_2})(u_{w_1 v_2} + u_{w_1 w_2}) &= \underbrace{u_{v_1 v_2} u_{w_1 v_2}}_{0} + u_{v_1 v_2} u_{w_1 w_2} + u_{v_1 w_2} u_{w_1 v_2} + \underbrace{u_{v_1 w_2} u_{w_1 w_2}}_{0} \\ &= u_{v_1 v_2} u_{w_1 w_2} + u_{v_1 w_2} u_{w_1 v_2}.\end{aligned}$$

On the other hand, Lemma 4.4 yields

$$\begin{aligned}(u_{v_1 v_2} + u_{v_1 w_2})(u_{w_1 v_2} + u_{w_1 w_2}) &= \sum_{\substack{v_3 \in V \\ \{v_1, v_3\} \in E}} u_{\{v_1, v_3\}\{v_2, w_2\}} \sum_{\substack{v_4 \in V \\ \{w_1, v_4\} \in E}} u_{\{w_1, v_4\}\{v_2, w_2\}} \\ &= \sum_{\substack{v_3, v_4 \in V \\ \{v_1, v_3\}, \{w_1, v_4\} \in E}} \underbrace{u_{\{v_1, v_3\}\{v_2, w_2\}} u_{\{w_1, v_4\}\{v_2, w_2\}}}_{\delta_{\{v_1, v_3\}\{w_1, v_4\}} u_{\{v_1, v_3\}\{v_2, w_2\}}} \\ &= u_{\{v_1, w_1\}\{v_2, w_2\}},\end{aligned}$$

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where $\{v_1, v_3\} = \{w_1, v_4\}$ and $v_1 \neq w_1$ implies $v_3 = w_1$ in the last step. Thus,

$$u_{\{v_1,w_1\}\{v_2,w_2\}} = (u_{v_1v_2} + u_{v_1w_2})(u_{w_1v_2} + u_{w_1w_2}) = u_{v_1v_2}u_{w_1w_2} + u_{v_1w_2}u_{w_1v_2}.$$

Similarly, we compute

$$(u_{v_1v_2} + u_{w_1v_2})(u_{v_1w_2} + u_{w_1w_2}) = u_{v_1v_2}u_{w_1w_2} + u_{w_1v_2}u_{v_1w_2}$$

and

$$\begin{aligned}(u_{v_1v_2} + u_{w_1v_2})(u_{v_1w_2} + u_{w_1w_2}) &= \sum_{\substack{v_3,v_4\in V\\ \{v_1,v_3\},\{v_2,v_4\}\in E}} \underbrace{u_{\{v_1,w_1\}\{v_2,v_3\}}u_{\{v_1,w_1\}\{w_2,v_4\}}}_{\delta_{\{v_2,v_3\}\{w_2,v_4\}}u_{\{v_1,w_1\}\{v_2,v_3\}}}\\ &= u_{\{v_1,w_1\}\{v_2,w_2\}}\end{aligned}$$

using $v_2 \neq w_2$ and the other part of Lemma 4.4. Hence,

$$u_{\{v_1,w_1\}\{v_2,w_2\}} = u_{v_1v_2}u_{w_1w_2} + u_{w_1v_2}u_{v_1w_2} \quad \forall \{v_1, w_1\}, \{v_2, w_2\} \in E.$$

■

In addition to the previous lemmas, we need the following proposition for general hypergraphs, which states that $u_{vw} = 0$ if the vertices v and w are contained in a different number of edges. This proposition will also be used in Section 4.3 when computing the quantum automorphism groups of multigraphs.

Note that this type of relation seems to be useful when computing quantum symmetries of concrete hypergraphs. For example, a similar relation $u_{ij}u_{kl} = 0$ for $d(i,k) \neq d(j,l)$ was used in [27, 28] to compute the quantum symmetries of classical graphs, where d denotes the distance between two vertices.

Proposition 4.6 *Let $\Gamma := (V, E)$ be a hypergraph and denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Define*

$$N_s(v) := |\{e \in E \mid v \in s(e)\}| \quad \forall v \in V.$$

Then $N_s(v) \cdot u_{vw} = u_{vw} \cdot N_s(w)$ for all $v, w \in V$. In particular, $N_s(v) \neq N_s(w)$ implies $u_{vw} = 0$. The same statement also holds for the range map r.

Proof Let $v \in V$. By summing both sides of $(A_s u_E)_{ve} = (u_V A_s)_{ve}$ over all $e \in E$, we obtain

$$\sum_{e\in E}\sum_{f\in E}(A_s)_{vf}u_{fe} = \sum_{f\in E}(A_s)_{vf}\underbrace{\left(\sum_{e\in E}u_{fe}\right)}_{1} = \sum_{f\in E}(A_s)_{vf} = N_s(v),$$

$$\sum_{e\in E}\sum_{w\in V}u_{vw}(A_s)_{we} = \sum_{w\in V}u_{vw}\left(\sum_{e\in E}(A_s)_{we}\right) = \sum_{w\in V}u_{vw}\cdot N_s(w).$$

Thus, $N_s(v) = \sum_{w\in V} u_{vw} \cdot N_s(w)$, which implies

$$N_s(v)\cdot u_{vw} = u_{vw}\cdot N_s(v) = \sum_{x\in V}\underbrace{u_{vw}u_{vx}}_{\delta_{wx}u_{vw}}\cdot N_s(x) = u_{vw}\cdot N_s(w) \quad \forall v, w \in V.$$

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In particular, if $N_s(v) \neq N_s(w)$, then

$$N_s(v) \cdot u_{vw} = u_{vw} \cdot N_s(w) \iff (N_s(v) - N_s(w)) \cdot u_{vw} = 0 \iff u_{vw} = 0.$$

The corresponding statement and proof for the range map can be obtained by replacing s with r. ∎

We can now show that, for simple graphs, our quantum automorphism group agrees with the quantum automorphism group by Bichon when we identify the magic unitary u_V with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Theorem 4.7 *Let Γ be a simple graph. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$.

Proof Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Further, define the elements

$$\widehat{u}_{\{v_1,w_1\}\{v_2,w_2\}} := \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2} \quad \forall \{v_1,w_1\},\{v_2,w_2\} \in E,$$

and the matrices $\widehat{u}_V := (\widehat{u}_{vw})_{v,w\in V}$ and $\widehat{u}_E := (\widehat{u}_{ef})_{e,f\in E}$. Note that $\widehat{u}_{\{v_1,w_1\}\{v_2,w_2\}}$ is well-defined, since $\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = \widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2}$ by Definition 2.27.

First, we construct the unital $*$-homomorphism $\varphi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$ given by $\varphi(u_{vw}) = \widehat{u}_{vw}$ and $\varphi(u_{ef}) = \widehat{u}_{ef}$ using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Thus, we have to show that $\widehat{u}_V$ and $\widehat{u}_E$ are magic unitaries that are intertwined by $A_s = A_r$. By Definition 2.27, the matrix $\widehat{u}_V$ is a magic unitary and we can compute that

$$\begin{aligned}\widehat{u}^*_{\{v_1,w_1\},\{v_2,w_2\}} &= (\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2})^* + (\widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2})^* = \widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2} + \widehat{u}_{w_1v_2}\widehat{u}_{v_1w_2}\\ &= \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2} = \widehat{u}_{\{v_1,w_1\},\{v_2,w_2\}}\end{aligned}$$

for all $\{v_1,w_1\},\{v_2,w_2\} \in E$. Similarly, we compute

$$\begin{aligned}\widehat{u}^2_{\{v_1,w_1\}\{v_2,w_2\}} &= (\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2})(\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2})\\ &= \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2}\widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2}\widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2}\\ &= \widehat{u}^2_{v_1v_2}\widehat{u}^2_{w_1w_2} + \widehat{u}^2_{v_1w_2}\widehat{u}^2_{w_1v_2}\\ &= \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2}\\ &= \widehat{u}_{\{v_1,w_1\}\{v_2,w_2\}}\end{aligned}$$

for all $\{v_1,w_1\},\{v_2,w_2\} \in E$, where we additionally used the fact that $\widehat{u}_{w_1w_2}\widehat{u}_{v_1w_2} = 0$ and $\widehat{u}_{w_1v_2}\widehat{u}_{v_1v_2} = 0$, since $v_1 \neq w_1$. Next, we want to show that the rows and columns of $\widehat{u}_E$ sum up to 1. Observe that

$$\sum_{\{v_1,w_1\}\in E} \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = \frac{1}{2} \sum_{\substack{v_1,w_1\in V\\ \{v_1,w_1\}\in V}} \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} = \frac{1}{2} \quad \forall \{v_2,w_2\} \in E$$

by Proposition 2.29. Therefore,

$$\sum_{\{v_1,w_1\}\in E} \widehat{u}_{\{v_1,w_1\}\{v_2,w_2\}} = \sum_{\{v_1,w_1\}\in E} \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \sum_{\{v_1,w_1\}\in E} \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2} = \frac{1}{2} + \frac{1}{2} = 1$$

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for all $\{v_2, w_2\} \in E$. Similarly, we have

$$\sum_{\{v_2,w_2\}\in E} \widehat{u}_{\{v_1,w_1\}\{v_2,w_2\}} = \frac{1}{2} + \frac{1}{2} = 1$$

for all $\{v_1, w_1\} \in E$. Hence, $\widehat{u}_E$ is a magic unitary. It remains to show that $A_s\widehat{u}_E = \widehat{u}_V A_s$. But this follows from Lemma 4.4 and $A_\Gamma \widehat{u}_V = \widehat{u}_V A_\Gamma$, because

$$\begin{aligned}
\sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}_{\{v_0,v_2\}\{v_1,w_1\}} &= \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} (\widehat{u}_{v_0v_1}\widehat{u}_{v_2w_1} + \widehat{u}_{v_0w_1}\widehat{u}_{v_2v_1}) \\
&= \widehat{u}_{v_0v_1} \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}_{v_2w_1} + \widehat{u}_{v_0w_1} \sum_{\substack{v_2\in V\\ \{v_0,v_2\}\in E}} \widehat{u}_{v_2v_1} \\
&= \widehat{u}_{v_0v_1} \sum_{v_2\in V} (A_\Gamma)_{v_0v_2}\widehat{u}_{v_2w_1} + \widehat{u}_{v_0w_1} \sum_{v_2\in V} (A_\Gamma)_{v_0v_2}\widehat{u}_{v_2v_1} \\
&= \widehat{u}_{v_0v_1} \sum_{v_2\in V} \widehat{u}_{v_0v_2}(A_\Gamma)_{v_2w_1} + \widehat{u}_{v_0w_1} \sum_{v_2\in V} \widehat{u}_{v_0v_2}(A_\Gamma)_{v_2v_1} \\
&= \sum_{v_2\in V} \underbrace{\widehat{u}_{v_0v_1}\widehat{u}_{v_0v_2}}_{\delta_{v_1v_2}\widehat{u}_{v_0v_1}} (A_\Gamma)_{v_2w_1} + \sum_{v_2\in V} \underbrace{\widehat{u}_{v_0w_1}\widehat{u}_{v_0v_2}}_{\delta_{w_1v_2}\widehat{u}_{v_0w_1}} (A_\Gamma)_{v_2v_1} \\
&= \widehat{u}_{v_0v_1} + \widehat{u}_{v_0w_1}
\end{aligned}$$

for all $v_0 \in V$ and $\{v_1, w_1\} \in E$. Hence, the map φ exists.

Next, we construct the inverse map $\psi\colon C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma))$ given by $\psi(\widehat{u}_{vw}) = u_{vw}$ using the universal property of $C(\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma))$. Hence, we have to show that u_V satisfies the relations from Definition 2.27. By Definition 3.4, u_V is a magic unitary. To show that $A_\Gamma u_V = u_V A_\Gamma$, observe that

$$(A_sA_s^*)_{vw} = \sum_{e\in E} (A_s)_{ve}(A_s)_{we} = |\{e \in E \mid v \in s(e) \ \wedge \ w \in s(e)\}| \quad \forall v, w \in V.$$

Since Γ is a simple graph, each $e \in E$ contains exactly two elements so that

$$(A_sA_s^*)_{vw} = \begin{cases} 0 & \text{if } v \neq w, \{v, w\} \notin E, \\ 1 & \text{if } v \neq w, \{v, w\} \in E, \\ N_s(v) & \text{if } v = w, \end{cases}$$

where $N_s(v) := |\{e \in E \mid v \in s(e)\}|$ as in Proposition 4.6. Hence, we can write $A_sA_s^* = A_\Gamma + T$ with $T \in \mathbb{C}^{V\times V}$ defined by

$$T_{vw} := \begin{cases} 0 & \text{if } v \neq w, \\ N_s(v) & \text{if } v = w, \end{cases} \quad \forall v, w \in V.$$

Proposition 4.6 shows that $Tu_V = u_V T$, which implies

$$A_\Gamma u_V = A_sA_s^*u_V - Tu_V = A_su_EA_s^* - u_VT = u_VA_sA_s^* - u_VT = u_VA_\Gamma.$$

Finally, we have to show that $u_{v_1v_2}u_{w_1w_2} = u_{w_1w_2}u_{v_1v_2}$ for all $\{v_1, w_1\}, \{v_2, w_2\} \in E$. But this follows directly from Lemma 4.5, since

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$$u_{v_1v_2}u_{w_1w_2} = u_{\{v_1,w_1\}\{v_2,w_2\}} - u_{v_1w_2}u_{w_1v_2} = u_{\{w_1,v_1\}\{w_2,v_2\}} - u_{v_1w_2}u_{w_1v_2}$$
$$= u_{w_1w_2}u_{v_1v_2} + u_{v_1w_2}u_{w_1v_2} - u_{v_1w_2}u_{w_1v_2} = u_{w_1w_2}u_{v_1v_2}.$$

Thus, the $*$-homomorphism ψ exists. By the definitions of φ and ψ, we have

$$u_{vw} \longleftrightarrow \widehat{u}_{vw}, \qquad u_{\{v_1,w_1\}\{v_2,w_2\}} \longleftrightarrow \widehat{u}_{v_1v_2}\widehat{u}_{w_1w_2} + \widehat{u}_{v_1w_2}\widehat{u}_{w_1v_2}$$

for all $v, w \in V$ and $\{v_1, w_1\}, \{v_2, w_2\} \in E$, which shows that both maps are indeed inverse. Further, one directly verifies that ψ is a morphism of compact quantum groups. Hence, ψ is an isomorphism of compact quantum groups, which shows $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. ∎

4.3 Multigraphs

Finally, we consider the case of multigraphs as in Definition 2.4. Recall from Definition 2.12 that we can identify a multigraph $\Gamma := (V, E)$ with the source map $s' \colon E \to V$ and the range map $r' \colon E \to V$ with a 1-uniform hypergraphs by defining new source and range maps $s \colon E \to \mathcal{P}(V)$ and $r \colon E \to \mathcal{P}(V)$ given by $s(e) := \{s'(e)\}$ and $r(e) := \{r'(e)\}$ for all $e \in E$. Throughout the rest of this section, we will identify the maps s' and r' with the maps s and r. In particular, we will write $u_{s(e)v}$ instead of $u_{s'(e)v}$ for an edge $e \in E$ and a vertex $v \in V$.

The goal of this section is to show that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ agrees with the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$ of Goswami–Hossain for multigraphs without isolated vertices. We begin by reformulating the intertwiner relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ in the setting of multigraphs.

Lemma 4.8 *Let $\Gamma := (V, E)$ be a multigraph and denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then the relations $A_s u_E = u_V A_s$ and $A_r u_E = u_V A_r$ are equivalent to*

$$\sum_{\substack{f \in E \\ s(f)=v}} u_{ef} = u_{s(e)v}, \qquad \sum_{\substack{f \in E \\ r(f)=v}} u_{ef} = u_{r(e)v} \qquad \forall v \in V,\ e \in E.$$

Proof First, consider the equation $A_s u_E = u_V A_s$, which is equivalent to $A_s^* u_V = u_E A_s^*$ by Proposition 2.24. A direct computation yields

$$(u_E A_s^*)_{ev} = \sum_{\substack{f \in E \\ s(f)=v}} u_{ef}, \quad (A_s^* u_V)_{ev} = \sum_{\substack{w \in V \\ s(e)=w}} u_{wv} = u_{s(e)v} \qquad \forall v \in V,\ e \in E,$$

such that $A_s^* u_V = u_E A_s^*$ is equivalent to $\sum_{\substack{f \in E \\ s(f)=v}} u_{ef} = u_{s(e)v}$. The statement for the range map r can be obtained by replacing s with r in the previous computation. ∎

To show that our quantum automorphism group agrees with $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$, we will identify the magic unitary u_E with the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. However, we need additional elements in $C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma))$ that correspond to the magic unitary u_V.

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Definition 4.9 Let $\Gamma := (V, E)$ be a multigraph without isolated vertices and denote by u the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Define the elements

$$u_{vw} := \begin{cases} \sum\limits_{\substack{f\in E \\ s(f)=w}} u_{ef} & \text{if there exists } e \in E \text{ with } s(e) = v, \\ \sum\limits_{\substack{f\in E \\ r(f)=w}} u_{ef} & \text{if there exists } e \in E \text{ with } r(e) = v, \end{cases}$$

for all $v, w \in V$.

Next, we show that these elements u_{vw} are well-defined and form a quantum permutation on the vertices.

Lemma 4.10 *Let $\Gamma := (V, E)$ be a multigraph without isolated vertices. Then the elements u_{vw} in Definition 4.9 are well-defined. Further, the matrix $u_V := (u_{vw})_{v,w\in V}$ is a magic unitary.*

Proof First, note that at least one case in Definition 4.9 applies to each vertex v, since Γ has no isolated vertices. Further, each case does not depend on the edge e by Relation 2 in Definition 2.30. To show that overlapping cases are well-defined, assume $v, w \in V$ such that v is neither a source nor a sink. If w is neither a source nor a sink, then both cases agree by Relation 4. On the other hand, if w is a source or a sink, then both cases yield $u_{vw} = 0$, since the sum is empty in one case, while each $u_{ef} = 0$ by Relation 3 in the other case. Thus, the elements u_{vw} are well-defined.

Next, we show that the matrix $u_V := (u_{vw})_{v,w\in V}$ is a magic unitary. Let $v, w \in V$ and assume without loss of generality that there exists an edge $e \in E$ with $v = s(e)$. Using Lemma 4.8 and the fact that $(u_{ef})_{e,f\in E}$ is a magic unitary, we compute

$$u^*_{vw} = \sum_{\substack{f\in E \\ s(f)=w}} u^*_{ef} = \sum_{\substack{f\in E \\ s(f)=w}} u_{ef} = u_{vw},$$

$$u^2_{vw} = \sum_{\substack{f_1\in E \\ s(f_1)=w}} \sum_{\substack{f_2\in E \\ s(f_2)=w}} \underbrace{u_{ef_1}u_{ef_2}}_{\delta_{f_1f_2}u_{ef_1}} = \sum_{\substack{f\in E \\ s(f)=w}} u_{ef} = u_{vw}$$

and

$$\sum_{w\in V} u_{vw} = \sum_{w\in V} \sum_{\substack{f\in E \\ s(f)=w}} u_{ef} = \sum_{f\in E} \underbrace{\sum_{\substack{w\in V \\ s(f)=w}} u_{ef}}_{u_{ef}} = \sum_{f\in E} u_{ef} = 1.$$

To show that the rows of u_V sum to 1, we use an argument from the proof of [34, Theorem 3.1]. By our previous computation, we have

$$(u_V u^*_V)_{vw} = \sum_{x\in V} \underbrace{u_{vx}u^*_{wx}}_{\delta_{vw}u_{vx}} = \delta_{vw} \sum_{x\in V} u_{vx} = \delta_{vw} \quad \forall v, w \in V.$$

Hence, u_V is right-invertible with $u_V u^*_V = 1$. If we show that u_V is a representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$, that is, $\Delta(u_{vw}) = \sum_{x\in V} u_{vx} \otimes u_{xw}$, then [36, Proposition 3.2] implies

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that u_V is also left-invertibe, that is, $u_V^* u_V = 1$. Thus,

$$1 = (u_V^* u_V)_{vv} = \sum_{w \in V} u_{wv}^* u_{wv} = \sum_{w \in V} u_{wv} \quad \forall v \in V.$$

Therefore, it remains to show that u_V is a representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Again, let $v, w \in V$ and assume without loss of generality that there exists an edge $e \in E$ with $v = s(e)$. Then

$$\begin{aligned}\Delta(u_{vw}) &= \sum_{\substack{f \in E \\ s(f)=w}} \Delta(u_{ef}) = \sum_{\substack{f \in E \\ s(f)=w}} \sum_{g \in E} u_{eg} \otimes u_{gf} = \sum_{g \in E} u_{eg} \otimes \left(\sum_{\substack{f \in E \\ s(f)=w}} u_{gf} \right) \\ &= \sum_{g \in E} u_{eg} \otimes u_{s(g)w},\end{aligned}$$

and on the other hand,

$$\sum_{x \in V} u_{vx} \otimes u_{xw} = \sum_{x \in V} \sum_{\substack{f \in E \\ s(f)=x}} u_{ef} \otimes u_{xw} = \sum_{f \in E} \sum_{\substack{x \in V \\ s(f)=x}} u_{ef} \otimes u_{xw} = \sum_{f \in E} u_{ef} \otimes u_{s(f)w}.$$

Thus, $\Delta(u_{vw}) = \sum_{x \in V} u_{vx} \otimes u_{xw}$ for all $v, w \in V$. ∎

Using the previous lemmas, we can now show that our quantum automorphism group agrees with the quantum automorphism group of Goswami–Hossain for multigraphs without isolated vertices.

Theorem 4.11 *Let Γ be a multigraph without isolated vertices. Then* $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$.

Proof Let $\Gamma := (V, E)$. Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. Further, define the elements

$$\widehat{u}_{vw} := \begin{cases} \sum_{\substack{f \in E \\ s(f)=w}} \widehat{u}_{ef} & \text{if there exists } e \in E \text{ with } s(e) = v, \\ \sum_{\substack{f \in E \\ r(f)=w}} \widehat{u}_{ef} & \text{if there exists } e \in E \text{ with } r(e) = v, \end{cases}$$

for all $v, w \in V$ as in Definition 4.9.

We begin by constructing the unital $*$-homomorphism $\varphi\colon C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma))$ given by $\varphi(u_{vw}) = \widehat{u}_{vw}$ and $\varphi(u_{ef}) = \widehat{u}_{ef}$ using the universal property of $C(\mathrm{Aut}^+(\Gamma))$. Hence, we have to show that the matrices $\widehat{u}_V := (\widehat{u}_{vw})_{v,w \in V}$ and $\widehat{u}_E := (\widehat{u}_{ef})_{e,f \in E}$ satisfy the relations of Definition 3.4. The matrix $\widehat{u}_E$ is a magic unitary by Relation 1 of Definition 2.30 and $\widehat{u}_V$ is a magic unitary by Lemma 4.10. Next, consider the relation $A_s \widehat{u}_E = \widehat{u}_V A_s$, which is equivalent to $A_s^* \widehat{u}_V = \widehat{u}_E A_s^*$ by Proposition 2.24. However, this relation follows directly, since

$$(A_s^* \widehat{u}_V)_{ev} = \sum_{w \in V} (A_s^*)_{ew} \widehat{u}_{wv} = \widehat{u}_{s(e)v} = \sum_{\substack{f \in E \\ s(f)=v}} \widehat{u}_{ef} = \sum_{f \in E} \widehat{u}_{ef} (A_s^*)_{fv} = (\widehat{u}_E A_s^*)_{ev}$$

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for all $e \in E$ and $v \in V$. Similarly, one shows $A_r \widehat{u}_E = \widehat{u}_V A_r$, such that the $*$-homomorphism φ exists.

Next, we construct the inverse $*$-homomorphism $\psi\colon C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)) \to C(\mathrm{Aut}^+(\Gamma))$ given by $\psi(\widehat{u}_{ef}) = u_{ef}$ using the universal property of $C(\mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma))$. Thus, we have to show that the entries of u_E satisfy the relations in Definition 2.30. By Definition 3.4, u_E is a magic unitary. Further, we can use Lemma 4.8 to compute

$$\sum_{\substack{f\in E\\ s(f)=v}} u_{e_1 f} = u_{s(e_1)v} = u_{s(e_2)v} = \sum_{\substack{f\in E\\ s(f)=v}} u_{e_2 f},$$
$$\sum_{\substack{f\in E\\ r(f)=v}} u_{e_1 f} = u_{r(e_1)v} = u_{r(e_2)v} = \sum_{\substack{f\in E\\ r(f)=v}} u_{e_2 f}$$

for all $v \in V$ and $e_1, e_2 \in E$ with $s(e_1) = s(e_2)$ or $r(e_1) = r(e_2)$, respectively. Thus, Relation 2 is satisfied. Next, consider Relation 3 and let $e, f \in E$. If $s(e)$ is neither a source or sink and $s(f)$ is a source, then $N_r(s(e)) > 0$ and $N_r(s(f)) = 0$ in the notation of Proposition 4.6. Therefore, Proposition 4.6 implies that $u_{s(e)s(f)} = 0$, such that

$$0 = u_{s(e)s(f)} u_{ef} = \sum_{\substack{g\in E\\ s(g)=s(f)}} \underbrace{u_{eg} u_{ef}}_{\delta_{gf} u_{ef}} = u_{ef}.$$

Similarly, one shows $u_{s(e)s(f)} = 0$ if $r(f)$ is a sink. Hence, Relation 3 is satisfied. Finally, Lemma 4.8 implies

$$\sum_{\substack{f\in E\\ s(f)=v}} u_{e_1 f} = u_{s(e_1)v} = u_{r(e_2)v} = \sum_{\substack{f\in E\\ r(f)=v}} u_{e_2 f}$$

for all $v \in V$ and $e_1, e_2 \in E$ with $s(e_1) = r(e_2)$. Thus, Relation 4 holds and the $*$-homomorphism φ exist.

The $*$-homomorphisms φ and ϕ are indeed inverse, since

$$u_{ef} \longleftrightarrow \widehat{u}_{ef} \quad \forall e, f \in E, \qquad u_{vw} \longleftrightarrow \widehat{u}_{vw} \quad \forall v, w \in V$$

by Lemma 4.8 and Definition 4.9. Further, a direct computation shows that ψ is a morphism of compact quantum groups. Hence, $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{GH,Bic}}(\Gamma)$. ∎

5 Action on hypergraph C^*-algebras

In [30], Schmidt–Weber showed that the quantum automorphism group $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ of a directed graph Γ acts maximally on the graph C^*-algebra $C^*(\Gamma)$. In our notation, this action is given by

$$\alpha(p_v) = \sum_{w\in V} p_w \otimes u_{wv}, \quad \alpha(s_{(v_1,w_1)}) = \sum_{(v_2,w_2)\in E} s_{(v_2,w_2)} \otimes u_{v_2 v_1} u_{w_2 w_1}$$

for all $v \in V$ and $(v_1, w_1) \in E$, where $\Gamma := (V, E)$ is a directed graph and we flipped both tensor legs to agree with the notation in [34].

In the following, we generalize this result to hypergraphs by showing that our quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ of a hypergraph Γ acts faithfully on the

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hypergraph C^*-algebra $C^*(\Gamma)$ from Definition 2.33. Here, the action is given by

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_{wv} \quad \forall v \in V, \qquad \alpha(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \quad \forall e \in E.$$

Further, we show that this action is maximal when we also consider the dual action

$$\alpha'(p_e) = \sum_{f \in E} p_f \otimes u_{fe} \quad \forall e \in E, \qquad \alpha'(s_v) = \sum_{w \in V} s_w \otimes u_{wv} \quad \forall v \in V$$

on $C^*(\Gamma')$, where $\Gamma' := (\Gamma^*)^{\mathrm{op}}$ is obtained by flipping the source and range maps in the dual hypergraph Γ^*.

5.1 Existence of the action

We begin by showing the existence of the action of $\mathrm{Aut}^+(\Gamma)$ on the hypergraph C^*-algebra $C^*(\Gamma)$, where the main step will be the construction of the underlying $*$-homomorphism in the following lemma.

Lemma 5.1 *Let $\Gamma := (V, E)$ be a hypergraph and denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then there exists a unital $*$-homomorphism $\alpha\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(\mathrm{Aut}^+(\Gamma))$ defined by*

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_{wv}, \quad \alpha(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \qquad \forall v \in V, \forall e \in E.$$

Proof We use the universal property of $C^*(\Gamma)$ to construct the map α from the statement. Thus, we have to show that $\alpha(p_v)$ are orthogonal projections, $\alpha(s_e)$ are partial isometries, and both satisfy the relations from Definition 2.33. Recall the magic unitary relations of u_V and u_E and the intertwiner relations of A_s and A_r from Remark 3.6. Then, we can show that $\alpha(p_v)$ are orthogonal projections by computing

$$\begin{aligned}\alpha(p_{v_1})\alpha(p_{v_2}) &= \sum_{w_1, w_2 \in V} \underbrace{p_{w_1} p_{w_2}}_{\delta_{w_1 w_2} p_{w_1}} \otimes u_{w_1 v_1} u_{w_2 v_2} \\ &= \sum_{w \in V} p_w \otimes \underbrace{u_{w v_1} u_{w v_2}}_{\delta_{v_1 v_2} u_{w v_1}} = \delta_{v_1 v_2} \sum_{w \in V} p_w \otimes u_{w v_1} = \delta_{v_1 v_2}\, \alpha(p_{v_1})\end{aligned}$$

for all $v_1, v_2 \in V$ and

$$\alpha(p_v)^* = \sum_{w \in V} p_w^* \otimes u_{wv}^* = \sum_{w \in V} p_w \otimes u_{wv} = \alpha(p_v)$$

for all $v \in V$. Similarly, we show that $\alpha(s_e)$ are partial isometries by computing

$$\begin{aligned}\alpha(s_e)\alpha(s_e)^*\alpha(s_e) &= \sum_{f_1, f_2, f_3 \in E} s_{f_1} s_{f_2} s_{f_3} \otimes \underbrace{u_{f_1 e} u_{f_2 e}^* u_{f_3 e}}_{\delta_{f_1 f_2} \delta_{f_1 f_3} u_{f_1 e}} \\ &= \sum_{f \in E} \underbrace{s_f s_f^* s_f}_{s_f} \otimes u_{fe} = \sum_{f \in E} s_f \otimes u_{fe} = \alpha(s_e)\end{aligned}$$

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for all $e \in E$. Next, consider Relation 1 from Definition 2.33, which states that $s_e^* s_f = \delta_{ef} \sum_{\substack{v \in V \\ v \in r(e)}} p_v$ for all $e, f \in E$. When applying α to the left side, we obtain

$$\begin{aligned}\alpha(s_e)^* \alpha(s_f) &= \sum_{g_1, g_2 \in E} \underbrace{s_{g_1}^* s_{g_2}}_{\text{Rel. 1}} \otimes u_{g_1 e}^* u_{g_2 f} \\ &= \sum_{g \in E} \sum_{\substack{v \in V \\ v \in r(g)}} p_v \otimes \underbrace{u_{ge} u_{gf}}_{\delta_{ef} u_{ge}} = \delta_{ef} \sum_{v \in V} \sum_{\substack{g \in E \\ v \in r(g)}} p_v \otimes u_{ge}.\end{aligned}$$

Using Remark 3.6, we have $\sum_{\substack{g \in E \\ v \in r(g)}} u_{ge} = \sum_{\substack{w \in V \\ w \in r(e)}} u_{vw}$ for all $v \in V$ and $e \in E$, which can be used to further rewrite $\alpha(s_e)^* \alpha(s_f)$ to

$$\begin{aligned}\alpha(s_e)^* \alpha(s_f) &= \delta_{ef} \sum_{v \in V} \sum_{\substack{w \in V \\ w \in r(e)}} p_v \otimes u_{vw} = \delta_{ef} \sum_{\substack{w \in V \\ w \in r(e)}} \sum_{v \in V} p_v \otimes u_{vw} \\ &= \delta_{ef} \sum_{\substack{w \in V \\ w \in r(e)}} \alpha(p_w).\end{aligned}$$

Thus, Relation 1 is satisfied. Next, consider Relation 2, which is given by $s_e s_e^* \leq \sum_{\substack{v \in V \\ v \in s(e)}} p_v$ for all $e \in E$. As before, we apply α to the left side and compute

$$\alpha(s_e)\alpha(s_e)^* = \sum_{f_1, f_2 \in E} s_{f_1} s_{f_2}^* \otimes \underbrace{u_{f_1 e} u_{f_2 e}^*}_{\delta_{f_1 f_2} u_{f_1 e}} = \sum_{f \in E} s_f s_f^* \otimes u_{fe}.$$

Since each $u_{fe} \geq 0$, we can use Relation 2 to obtain

$$\alpha(s_e)\alpha(s_e)^* \leq \sum_{f \in E} \left(\sum_{\substack{v \in V \\ v \in s(f)}} p_v \right) \otimes u_{fe} = \sum_{v \in V} \sum_{\substack{f \in E \\ v \in s(f)}} p_v \otimes u_{fe}.$$

By Remark 3.6, we have $\sum_{\substack{f \in E \\ v \in s(f)}} u_{fe} = \sum_{\substack{w \in V \\ w \in s(e)}} u_{wv}$ for all $v \in V$ and $e \in E$, such that

$$\alpha(s_e)\alpha(s_e)^* \leq \sum_{v \in V} \sum_{\substack{w \in V \\ w \in s(e)}} p_v \otimes u_{vw} = \sum_{\substack{w \in V \\ w \in s(e)}} \sum_{v \in V} p_v \otimes u_{vw} = \sum_{\substack{w \in V \\ w \in s(e)}} \alpha(p_w).$$

Hence, Relation 2 is satisfied. Finally, consider Relation 3, which states that $p_v \leq \sum_{\substack{e \in E \\ v \in s(e)}} s_e s_e^*$ for all $v \in V$ that are not a sink. Let $v \in V$ be a vertex that is not a sink. By Proposition 4.6, we have $u_{vw} = 0$ for all $w \in V$ that are a sink, since $N_s(v) > 0$ if v is not a sink and $N_s(w) = 0$ if w is a sink. Therefore,

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_{wv} = \sum_{\substack{w \in V \\ w \neq \text{sink}}} p_w \otimes u_{wv}.$$

Since $u_{vw} \geq 0$, Relation 3 yields

$$\alpha(p_v) \leq \sum_{\substack{w \in V \\ w \neq sink}} \left(\sum_{\substack{e \in E \\ w \in s(e)}} s_e s_e^* \right) \otimes u_{wv} = \sum_{e \in E} \sum_{\substack{w \in V \\ w \in s(e)}} s_e s_e^* \otimes u_{wv},$$

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where we used again the fact that $u_{vw} = 0$ if w is a sink. On the other hand, we have

$$\sum_{\substack{f\in E\\ v\in s(f)}} \alpha(s_f)\alpha(s_f)^* = \sum_{e_1,e_2\in E}\sum_{\substack{f\in E\\ v\in s(f)}} s_{e_1}s_{e_2}^* \otimes \underbrace{u_{e_1 f}u_{e_2 f}^*}_{\delta_{e_1e_2}u_{e_1f}} = \sum_{e\in E}\sum_{\substack{f\in E\\ v\in s(f)}} s_e s_e^* \otimes u_{ef}.$$

By Remark 3.6, we have $\sum_{\substack{w\in V\\ w\in s(e)}} u_{wv} = \sum_{\substack{f\in E\\ v\in s(f)}} u_{ef}$ for all $v \in V$ and $e \in E$, which implies

$$\alpha(p_v) \leq \sum_{e\in E}\sum_{\substack{w\in V\\ w\in s(e)}} s_e s_e^* \otimes u_{wv} = \sum_{e\in E}\sum_{\substack{f\in E\\ v\in s(f)}} s_e s_e^* \otimes u_{ef} = \sum_{\substack{f\in E\\ v\in s(f)}} \alpha(s_f)\alpha(s_f)^*.$$

■

Next, we show that the previous $*$-homomorphism α defines a faithful action in the sense of Definitions 2.19 and 2.20.

Theorem 5.2 *Let Γ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma)$ acts faithfully on $C^*(\Gamma)$ via the map α from Lemma 5.1.*

Proof Let $\Gamma := (V, E)$ and define $\mathcal{B} \subseteq C^*(\Gamma)$ as the $*$-subalgebra generated by p_v for all $v \in V$ and p_e for all $e \in E$. Then $\mathcal{B}$ is dense in $C^*(\Gamma)$ and we have $\alpha(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{O}(\mathrm{Aut}^+(\Gamma))$ by the definition of α. Next, let $v \in V$. Then

$$\sum_{w\in V} \alpha(p_w) \otimes u_{wv} = \sum_{w_1,w_2\in V} p_{w_2} \otimes u_{w_2w_1} \otimes u_{w_1v} = \sum_{w_2\in V} p_{w_2} \otimes \Delta(u_{w_2v})$$

implies $(\alpha \otimes \mathrm{id})(\alpha(p_v)) = (\mathrm{id} \otimes \Delta)(\alpha(p_v))$. Further, we compute

$$(\mathrm{id} \otimes \varepsilon)(\alpha(p_v)) = (\mathrm{id} \otimes \varepsilon)\left(\sum_{w\in V} p_w \otimes u_{wv}\right) = \sum_{w\in V} p_w \cdot \delta_{wv} = p_v.$$

The previous computations also show that $(\alpha \otimes \mathrm{id})(\alpha(s_e)) = (\mathrm{id} \otimes \Delta)(\alpha(s_e))$ and $(\mathrm{id} \otimes \varepsilon)(\alpha(s_e)) = s_e$ for all $e \in E$ by replacing p_v with s_e. Hence, $(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}) \circ \alpha$ and $(\varepsilon \otimes \mathrm{id}) \circ \alpha|_{\mathcal{B}} = \mathrm{id}$. Thus, α defines an action of $\mathrm{Aut}^+(\Gamma)$ on $C^*(\Gamma)$. To show that α is faithful, assume that there exists a quotient G of $\mathrm{Aut}^+(\Gamma)$ such that $\alpha|_{C(G)}$ is also an action on $C^*(\Gamma)$. Then

$$\alpha|_{C(G)}(p_v) = \sum_{w\in V} p_w \otimes u_{wv} \in C^*(\Gamma) \otimes C(G) \qquad \forall v \in V,$$
$$\alpha|_{C(G)}(s_e) = \sum_{f\in E} s_f \otimes u_{fe} \in C^*(\Gamma) \otimes C(G) \qquad \forall e \in E.$$

The representation of $\alpha|_{C(G)}(p_v)$ with respect to p_v is unique since the p_v are linearly independent as orthogonal projections. Thus, $u_{wv} \in C(G)$ for all $v, w \in V$. Similarly, $u_{ef} \in C(G)$ for all $e, f \in E$, since the s_e are linearly independent as partial isometries with orthogonal ranges. Hence, $C(G) = C(\mathrm{Aut}^+(\Gamma))$, which shows that α is faithful. ■

Although our action appears to be different from the action of Schmidt–Weber, the following remark shows that it reduces to the action of Schmidt–Weber in the case

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of classical directed graphs. In particular, it justifies the special form of the action retrospectively, since the action does not appear to be canonical from the point of view of classical directed graphs.

Remark 5.3 Let $\Gamma := (V, E)$ be a directed graph. By the proof of Theorem 4.3, we have $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ via $u_{vw} = \widehat{u}_{vw}$ and $u_{(v_1,v_2)(w_1,w_2)} = \widehat{u}_{v_1w_1}\widehat{u}_{v_2w_2}$, where u denotes the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and $\widehat{u}$ denotes the fundamental representation of $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$. Under this isomorphism, the action α of Theorem 5.2 has the form

$$\alpha(p_v) = \sum_{w\in V} p_w \otimes \widehat{u}_{wv}, \quad \alpha(s_{(v_1,w_1)}) = \sum_{(v_2,w_2)\in E} s_{(v_2,w_2)} \otimes \widehat{u}_{v_2v_1}\widehat{u}_{w_2w_1}$$

for all $v \in V$ and $(v_1, w_1) \in E$, which is the same form as in [30]. Hence, we obtain the action of Schmidt–Weber for $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ as a special case. Note that it was already shown in [19] that $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ acts on $C^*(\Gamma)$ via this action.

5.2 Maximality of the action

In [30], Schmidt–Weber showed that their action is maximal in the sense that $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ is the largest quantum group that acts on the graph C^*-algebra $C^*(\Gamma)$ via the map α described before. However, if Γ is a classical directed graph, then our quantum automorphism group agrees with $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma)$ by Theorem 4.3, and our action α in Theorem 5.2 agrees with the action of Schmidt–Weber by Remark 5.3. Since $\mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma) \neq \mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ for some direct graphs, our action on hypergraph C^*-algebras is not maximal, even in the case of directed graphs.

In the following, we show that we can obtain maximality of our action by including an additional assumption. We begin by introducing some notation.

Definition 5.4 Let $\Gamma := (V, E)$ be a hypergraph. Then define $\Gamma' := (E, V)$ with

$$s'(e) := \{v \in V \mid v \in r(e)\}, \quad r'(e) := \{v \in V \mid v \in s(e)\} \quad \forall e \in E.$$

Note that $\Gamma' = (\Gamma^{\mathrm{op}})^* = (\Gamma^*)^{\mathrm{op}}$, where Γ^{op} and Γ^* are the opposite and dual hypergraph from Definitions 2.13 and 2.14.

Using the results in Section 3.4, it follows directly that $\mathrm{Aut}^+(\Gamma)$ not only acts on $C^*(\Gamma)$ but also on $C^*(\Gamma')$ in a natural way.

Proposition 5.5 *Let $\Gamma := (V, E)$ be a hypergraph. Then $\mathrm{Aut}^+(\Gamma)$ acts faithfully on $C^*(\Gamma')$ via $\alpha\colon C^*(\Gamma') \to C^*(\Gamma') \otimes C(\mathrm{Aut}^+(\Gamma))$ defined by*

$$\alpha(p_e) = \sum_{f\in E} p_f \otimes u_{fe}, \quad \alpha(s_v) = \sum_{w\in V} s_w \otimes u_{wv} \qquad \forall e \in E,\ \forall v \in V.$$

Proof Denote by u the fundamental representation of $\mathrm{Aut}^+(\Gamma)$ and by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma')$. By Propositions 3.12 and 3.4, we have

$$\mathrm{Aut}^+(\Gamma') = \mathrm{Aut}^+\left((\Gamma^*)^{\mathrm{op}}\right) = \mathrm{Aut}^+(\Gamma^*) = \mathrm{Aut}^+(\Gamma)$$

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via the mapping $\widehat{u}_V \mapsto u_E$ and $\widehat{u}_E \mapsto u_V$. The statement follows by applying this isomorphism to the action of $\mathrm{Aut}^+(\Gamma')$ on $C^*(\Gamma')$ from Theorem 5.2. ∎

The main observation to prove maximality is the following lemma, which shows that at least some of the relations in Definition 3.4 can be recovered from the action on $C^*(\Gamma)$.

Lemma 5.6 *Let $\Gamma := (V, E)$ be a hypergraph and G be a compact matrix quantum group that acts on $C^*(\Gamma)$ via $\alpha: C^*(\Gamma) \to C^*(\Gamma) \otimes C(G)$ with*

$$\alpha(p_v) = \sum_{w \in V} p_w \otimes u_{wv} \quad \forall v \in V, \qquad \alpha(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \quad \forall e \in E,$$

for some elements $u_{vw} \in C(G)$ for all $v, w \in V$ and $u_{ef} \in C(G)$ for all $e, f \in E$. Then $u_V := (u_{vw})_{v,w \in V}$ is a magic unitary. If additionally $u_E := (u_{ef})_{e,f \in E}$ is a magic unitary, then $A_r u_E = u_V A_r$.

Proof The proof that u_V is a magic unitary is contained in the proof of [34, Theorem 3.1], since the elements p_v are orthogonal projections that sum up to 1. For the second part of the statement, assume that u_E is also a magic unitary and consider the relation $s_e^* s_e = \sum_{\substack{v \in V \\ v \in r(e)}} p_v$ from Definition 2.33. By applying α to the left side and using the relation, we obtain

$$\alpha(s_e^* s_e) = \sum_{f_1, f_2 \in E} s_{f_1}^* s_{f_2} \otimes \underbrace{u_{f_1 e}^* u_{f_2 e}}_{\delta_{f_1 f_2} u_{f_1 e}} = \sum_{f \in E} s_f^* s_f \otimes u_{fe} = \sum_{f \in E} \sum_{\substack{v \in V \\ v \in r(f)}} p_v \otimes u_{fe},$$

which can be rewritten as

$$\alpha(s_e^* s_e) = \sum_{v \in V} \sum_{f \in E} (A_r)_{vf}\, p_v \otimes u_{fe} = \sum_{v \in V} p_v \otimes \left(\sum_{f \in E} (A_r)_{vf}\, u_{fe} \right).$$

Similarly, by applying α to the right side of the original equation, we obtain

$$\begin{aligned} \sum_{\substack{v \in V \\ v \in r(e)}} \alpha(p_v) &= \sum_{\substack{v \in V \\ v \in r(e)}} \sum_{w \in V} p_w \otimes u_{wv} \\ &= \sum_{v, w \in V} (A_r)_{ve}\, p_w \otimes u_{wv} = \sum_{w \in V} p_w \otimes \left(\sum_{v \in V} u_{wv} (A_r)_{ve} \right). \end{aligned}$$

Thus,

$$(A_r u_E)_{ve} = \sum_{f \in E} (A_r)_{vf}\, u_{fe} = \sum_{w \in V} u_{wv} (A_r)_{ve} = (A_r u_V)_{ve} \quad \forall v \in V$$

by using the fact that the elements p_v are linearly independent as orthogonal projections, so that we can compare the terms in the previous sums. This shows $A_r u_E = u_V A_r$. ∎

By combining the previous lemma with the actions on $C^*(\Gamma)$ and $C^*(\Gamma')$, we can now show that $\mathrm{Aut}^+(\Gamma)$ is the largest quantum group that acts both on $C^*(\Gamma)$ and $C^*(\Gamma')$ in the sense of Theorem 5.2 and Lemma 5.5.

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Theorem 5.7 *Let $\Gamma := (V, E)$ be a hypergraph and G be a compact matrix quantum group acting faithfully on $C^*(\Gamma)$ and $C^*(\Gamma')$ via maps*

$$\alpha_1 \colon C^*(\Gamma) \to C^*(\Gamma) \otimes C(G), \quad \alpha_2 \colon C^*(\Gamma') \to C^*(\Gamma') \otimes C(G),$$

which satisfy

$$\alpha_1(p_v) = \sum_{w \in V} p_w \otimes u_{wv}, \qquad \alpha_1(s_e) = \sum_{f \in E} s_f \otimes u_{fe} \qquad \forall v \in V,\ e \in E,$$
$$\alpha_2(p_e) = \sum_{f \in E} p_f \otimes u_{fe}, \qquad \alpha_2(s_v) = \sum_{w \in V} s_w \otimes u_{wv} \qquad \forall e \in E,\ v \in V$$

for some elements $u_{vw} \in C(G)$ and $u_{ef} \in C(G)$. Then $G \subseteq \mathrm{Aut}^+(\Gamma)$.

Proof By applying the first part of Lemma 5.6 to α_1 and α_2, we obtain that the matrices $u_V := (u_{vw})_{v,w \in V}$ and $u_E := (u_{ef})_{e,f \in E}$ are magic unitaries. The second part of Lemma 5.6 then yields $A_r u_E = u_V A_r$ and $A_{r'} u_V = u_E A_{r'}$. But $A_{r'} = A_s^*$, such that $A_s^* u_V = u_E A_s^*$, which implies $A_s u_E = u_V A_s$ by Proposition 2.24. This shows that the elements u_{vw} and u_{ef} satisfy the relations of Definition 3.4. Hence, by the universal property of $C(\mathrm{Aut}^+(\Gamma))$, there exists a unital $*$-homomorphism $\varphi \colon C(\mathrm{Aut}^+(\Gamma)) \to C(G)$ that maps the generators of $C(\mathrm{Aut}^+(\Gamma))$ to the entries of u_V and u_E. Next, we show that φ is a morphism of compact quantum groups. Let $w \in V$ and observe that

$$\begin{aligned}(\alpha_1 \otimes \mathrm{id})(\alpha_1(w)) &= \sum_{x \in V} \alpha_1(p_x) \otimes u_{xw} \\ &= \sum_{x \in V} \sum_{v \in V} p_v \otimes u_{vx} \otimes u_{xw} = \sum_{v \in V} p_v \otimes \left(\sum_{x \in V} u_{vx} \otimes u_{xw} \right),\end{aligned}$$

which implies $(\mathrm{id} \otimes \Delta)(\alpha_1(w)) = \sum_{v \in V} p_v \otimes \Delta(u_{vw})$. Because α_1 is an action, it satisfies $(\alpha_1 \otimes \mathrm{id}) \circ \alpha_1 = (\mathrm{id} \otimes \Delta) \circ \alpha_1$. Further, all p_v are linearly independent as orthogonal projections, which implies $\Delta(u_{vw}) = \sum_{w \in V} u_{vx} \otimes u_{xw}$ for all $v, w \in V$. Denote by $\widehat{u}$ the fundamental representation of $\mathrm{Aut}^+(\Gamma)$. Then the previous equation yields

$$\Delta(\varphi(\widehat{u}_{vw})) = \sum_{x \in V} u_{vx} \otimes u_{xw} = \sum_{x \in V} \varphi(\widehat{u}_{vx}) \otimes \varphi(\widehat{u}_{xw}), = (\varphi \otimes \varphi)(\Delta(\widehat{u}_{vw}))$$

for all $v, w \in V$. Similarly, one shows that $\Delta(\varphi(\widehat{u}_{ef})) = (\varphi \otimes \varphi)(\Delta(\widehat{u}_{ef}))$ for all $e, f \in E$ by using the action α_2. Thus, φ is a morphism of compact quantum groups. Further, φ is surjective because α_1 and α_2 are faithful. Otherwise, the image of φ would define a proper quotient quantum group of G acting on $C^*(\Gamma)$ and $C^*(\Gamma')$ in the same way, which is impossible. Therefore, $G \subseteq \mathrm{Aut}^+(\Gamma)$. ∎

Note that if Γ is an undirected hypergraph, then $\Gamma' = \Gamma^*$ in Definition 5.4. In this case, $\mathrm{Aut}^+(\Gamma)$ is the maximal quantum group that acts faithfully on both $C^*(\Gamma)$ and

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$C^*(\Gamma^*)$ in a compatible way. Further, since $\mathrm{Aut}^+(\Gamma)$ acts maximally, we can view $\mathrm{Aut}^+(\Gamma)$ as the quantum symmetry group of $C^*(\Gamma)$ in the sense of Theorem 5.7.

6 Open questions

In this last section, we present some remaining open questions. First, it would be interesting to compute quantum automorphism groups of concrete hypergraphs such as the following example.

Question 6.1 What is the quantum automorphism group of the complete hypergraph $\Gamma := (V, \mathcal{P}(V) \times \mathcal{P}(V))$ with source map $s(X, Y) = X$ and range map $r(X, Y) = Y$.

Since Γ has no multi-edges, we know that $\mathrm{Aut}^+(\Gamma) \subseteq S_V^+$ by Corollary 3.17. However, it remains open if this inclusion is proper or if $\mathrm{Aut}^+(\Gamma) = S_V^+$ holds.

Additionally, it remains open if we obtain a new class of quantum groups or if each quantum automorphism group of a hypergraph can be realized as the quantum automorphism group of a possibly larger classical graph.

Question 6.2 Let Γ be a hypergraph. Can one construct a classical graph Γ' such that $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Bic}}(\Gamma')$ or $\mathrm{Aut}^+(\Gamma) = \mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma')$?

It also remains open if there exists a Banica version for the quantum automorphism group of hypergraphs in the following sense.

Question 6.3 Is there an alternative definition of quantum automorphism groups of hypergraphs that reduces to $\mathrm{Aut}^+_{\mathrm{Ban}}(\Gamma)$ in the case of classical directed graphs and that acts on $C^*(\Gamma)$ in a natural way?

Finally, Hahn [18] characterized hypergraphs for which the classical automorphism group of their product is given by the wreath product of their automorphism groups.

Question 6.4 Can the results of Hahn be generalized to the quantum setting using the free wreath product of Bichon [5]?

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