1 Analysis of Stress

The concept of stress is the most fundamental concept in the mechanics of solids. The discrete atomistic structure of a material is ignored and the model of a continuum is adopted, according to which the entire space between the boundaries of a considered body is filled with the material. If, at a considered point of a solid body, an infinitesimal surface element $\mathrm{d}S$, with a unit outward normal vector $\mathbf{n}$, transmits a force $\mathrm{d}\mathbf{F}_n$, the traction vector at that point with respect to the considered surface element is defined by the ratio $\mathbf{t}_n = \mathrm{d}\mathbf{F}_n/\mathrm{d}S$. The projection of the traction vector in the direction of the vector orthogonal to the surface element is the normal stress over that surface element, $\sigma_{nn} = \mathbf{t}_n \cdot \mathbf{n}$. The remaining component, tangential to the surface element, is the shear stress $\sigma_{nm}\mathbf{m} = \mathbf{t}_n - \sigma_{nn}\mathbf{n}$, where $\mathbf{m}$ is a unit vector tangential to $\mathrm{d}S$. The first index (n) in the stress component σ_{nm} specifies the orientation of the surface element over which σ_{nm} acts, i.e., the direction of the unit vector orthogonal to $\mathrm{d}S$, while the second index (m) specifies the direction tangential to $\mathrm{d}S$ along which the stress component σ_{nm} physically acts. This chapter is devoted to the analysis of the normal and shear stresses over differently oriented surface elements through a considered material point of a loaded body. The analysis leads to the notion of a stress tensor, originally introduced by the French mathematician, physicist, and engineer Augustin-Louis Cauchy in the nineteenth century. We present the analysis of one-, two-, and three-dimensional states of stress, determine the corresponding principal stresses (maximum and minimum normal stresses) and the maximum shear stress, define the deviatoric and spherical parts of the stress tensor, derive the equations of equilibrium which must be satisfied by the stress field within a loaded body at rest, and formulate the corresponding boundary conditions.

1.1 Traction Vector

At any point of a loaded body, the traction vector $\mathbf{t}_n$, relative to a surface element $\mathrm{d}S$ whose unit normal vector is $\mathbf{n}$ (Fig. 1.1), is defined such that

$$\mathbf{t}_n = \frac{\mathrm{d}\mathbf{f}_n}{\mathrm{d}S}, \tag{1.1}$$

where $\mathrm{d}\mathbf{f}_n$ is the force transmitted by $\mathrm{d}S$. Figure 1.2(a) shows traction vectors acting on four sides of an infinitesimal rectangular material element with sides $\mathrm{d}x$ and $\mathrm{d}y$, having

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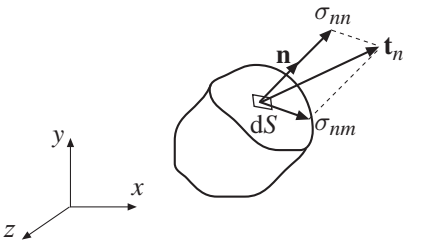


Figure 1.1 Traction vector $\mathbf{t}_n$ over the surface element $\mathrm{d}S$ with unit normal vector $\mathbf{n}$, and its normal and shear stress components (σ_{nn} and σ_{nm}).

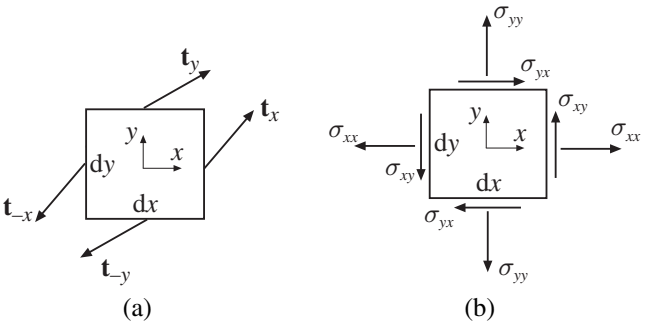


Figure 1.2 (a) Traction vectors $\mathbf{t}_{\pm x}$ and $\mathbf{t}_{\pm y}$ over four sides of a uniformly stressed rectangular material element $(\mathrm{d}x, \mathrm{d}y)$. By the law of action and reaction, $\mathbf{t}_{-x} = -\mathbf{t}_x$ and $\mathbf{t}_{-y} = -\mathbf{t}_y$. (b) The normal and shear stress components of the traction vectors from part (a).

a unit thickness in the z direction. In this two-dimensional case, the traction vectors over the sides whose unit normal vectors are in the positive x and y directions have the normal and shear stress components

$$\mathbf{t}_x = \{\sigma_{xx}, \sigma_{xy}\}, \quad \mathbf{t}_y = \{\sigma_{yx}, \sigma_{yy}\}. \tag{1.2}$$

The normal stresses acting on the two planes are σ_{xx} and σ_{yy}, while the accompanying shear stresses are σ_{xy} and σ_{yx} (Fig. 1.2(b)). The ordering of the indices is such that, for example, σ_{yx} represents a stress component in the x direction, acting over the surface element whose normal vector is in the y direction. In other words, the first index specifies the direction of the surface normal vector, while the second index specifies the direction of the stress component itself. By Newton's third law of action and reaction, or by equilibrium of an infinitesimally thin slice of material whose sides have unit normals in the x and $-x$ directions, it follows that $\mathbf{t}_{-x} = -\mathbf{t}_x$ and $\mathbf{t}_{-y} = -\mathbf{t}_y$. Thus, the directions of the positive stress components are as shown in Fig. 1.2(b), i.e., if the normal vector to the surface element is in the positive coordinate direction, the positive stress components act in the positive coordinate directions, and vice versa.

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The shear stresses σ_{xy} and σ_{yx} obey the conjugacy property

$$\sigma_{xy} = \sigma_{yx}, \tag{1.3}$$

which follows from the moment equilibrium condition (Fig. 1.2(b))

$$\sum M_z = 0: \quad (\sigma_{xy}\mathrm{d}y \cdot 1)\mathrm{d}x - (\sigma_{yx}\mathrm{d}x \cdot 1)\mathrm{d}y = 0. \tag{1.4}$$

The stress matrix (with respect to coordinate axes x and y) is obtained by grouping the two traction vectors, as follows

$$\begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}. \tag{1.5}$$

The three-dimensional generalizations of (1.2) and (1.5) are (Fig. 1.3)

$$\mathbf{t}_x = \{\sigma_{xx}, \sigma_{xy}, \sigma_{xz}\}, \quad \mathbf{t}_y = \{\sigma_{yx}, \sigma_{yy}, \sigma_{yz}\}, \quad \mathbf{t}_z = \{\sigma_{zx}, \sigma_{zy}, \sigma_{zz}\}, \tag{1.6}$$

and

$$\begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \\ \mathbf{t}_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{1.7}$$

The corresponding conjugacy properties of shear stresses are

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{yz} = \sigma_{zy}, \quad \sigma_{zx} = \sigma_{xz}, \tag{1.8}$$

which follow from the moment equilibrium conditions for the z, x, and y axis, respectively, and which make the stress matrix in (1.7) symmetric.

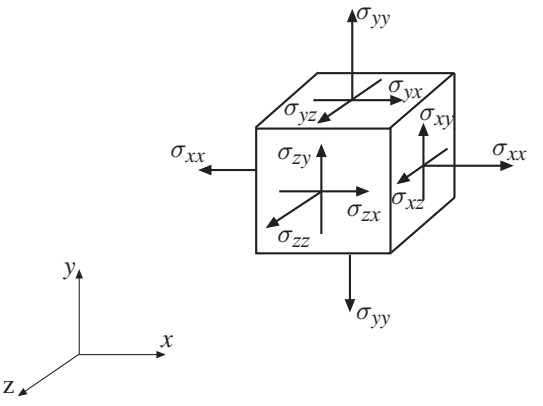


Figure 1.3 Rectangular stress components of a three-dimensional stress.

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1.2 Cauchy Relation for Traction Vectors

If traction vectors in three orthogonal planes through a considered point are known, say $(\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z)$, the traction vector in any other plane through that point is also known and given by the Cauchy relation

$$\mathbf{t}_n = n_x \mathbf{t}_x + n_y \mathbf{t}_y + n_z \mathbf{t}_z, \tag{1.9}$$

where $\mathbf{n} = \{n_x, n_y, n_z\}$ is the unit vector orthogonal to the considered plane (Fig. 1.4).

For simplicity, we prove below the Cauchy relation in the case of two-dimensional state of stress (Fig. 1.5(a)). The free-body diagram of an extracted triangular material element is shown in Fig. 1.5(b). The unit vector orthogonal to the inclined plane is $\mathbf{n} = \{n_x, n_y\}$, where $n_x = \cos\varphi$ and $n_y = \sin\varphi$. For equilibrium, the sum of all forces (per unit length in the z direction) must vanish,

$$\mathbf{t}_{-x}\mathrm{d}y + \mathbf{t}_{-y}(\mathrm{d}y \tan\varphi) + \mathbf{t}_n(\mathrm{d}y/\cos\varphi) + \mathbf{b}(\mathrm{d}y)^2(\tan\varphi)/2 = \mathbf{0}. \tag{1.10}$$

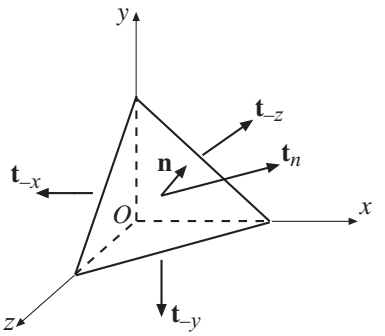


Figure 1.4 A tetrahedron material element with traction vectors on its four sides. The traction vector on the inclined plane with unit normal vector $\mathbf{n}$ is $\mathbf{t}_n$.

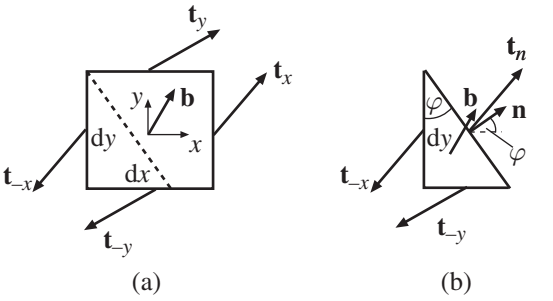


Figure 1.5 (a) Traction vectors $\mathbf{t}_{\pm x}$ and $\mathbf{t}_{\pm y}$ over four sides of the rectangular material element $(\mathrm{d}x, \mathrm{d}y)$. The body force per unit volume is $\mathbf{b}$. (b) The free-body diagram of a triangular element extracted from the rectangular element of part (a) along an inclined plane whose normal vector is $\mathbf{n} = \{n_x, n_y\} = \{\cos\varphi, \sin\varphi\}$.

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The body force (per unit volume) is $\mathbf{b}$, and the area of the triangular element is equal to $(\mathrm{d}y)^2(\tan\varphi)/2$. Upon dividing (1.10) by $\mathrm{d}y$ and performing the limit $\mathrm{d}y \to 0$, it follows that

$$\mathbf{t}_n = (\cos\varphi)\mathbf{t}_{-x} + (\sin\varphi)\mathbf{t}_{-y}. \tag{1.11}$$

Since $\mathbf{t}_{-x} = -\mathbf{t}_x$, $\mathbf{t}_{-y} = -\mathbf{t}_y$, $n_x = \cos\varphi$, and $n_y = \sin\varphi$, (1.11) becomes

$$\mathbf{t}_n = n_x\mathbf{t}_x + n_y\mathbf{t}_y, \tag{1.12}$$

which is a two-dimensional version of the Cauchy relation (1.9).

The matrix representation of the Cauchy relation (1.9) is $\mathbf{t}_n = [\sigma]\cdot\mathbf{n}$, i.e., in expanded form,

$$\begin{bmatrix} t_{nx} \\ t_{ny} \\ t_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \tag{1.13}$$

Thus, the (x, y, z) components of the traction vector $\mathbf{t}_n$ are

$$\begin{aligned} t_{nx} &= \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z, \\ t_{ny} &= \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z, \\ t_{nz} &= \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z. \end{aligned} \tag{1.14}$$

Note that in the matrix representation (1.13), the vectors $\mathbf{t}_n$ and $\mathbf{n}$ are considered to be column vectors of dimension (3×1).

Example 1.1 Prove the conjugacy relation for traction vectors $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{n}\cdot\mathbf{t}_m$, where $\mathbf{m}$ and $\mathbf{n}$ are two, not necessarily orthogonal, unit vectors.

Solution

By the matrix form of the Cauchy relation (1.13), we can write $\mathbf{t}_n = [\sigma]\cdot\mathbf{n}$ and $\mathbf{t}_m = [\sigma]\cdot\mathbf{m}$. Thus, $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{m}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{n}$ and $\mathbf{n}\cdot\mathbf{t}_m = \mathbf{n}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{m}$, where $()^{\mathrm{T}}$ denotes the transpose, so that $\mathbf{m}^{\mathrm{T}}$ and $\mathbf{n}^{\mathrm{T}}$ are row vectors. Since $[\sigma]$ is a symmetric matrix, it follows (by the definition of symmetric matrices) that $\mathbf{m}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{n} = \mathbf{n}^{\mathrm{T}}\cdot[\sigma]\cdot\mathbf{m}$, for any two vectors $\mathbf{m}$ and $\mathbf{n}$. Thus, $\mathbf{m}\cdot\mathbf{t}_n = \mathbf{n}\cdot\mathbf{t}_m$.

1.3 Normal and Shear Stresses over an Inclined Plane

The normal stress over an inclined plane is obtained by projecting the traction vector $\mathbf{t}_n$ onto the normal vector $\mathbf{n}$ (Fig. 1.4),

$$\sigma_{nn} = \mathbf{n}\cdot\mathbf{t}_n = n_x t_{nx} + n_y t_{ny} + n_z t_{nz}. \tag{1.15}$$

Substituting (1.14) into (1.15) gives

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2(\sigma_{xy}n_x n_y + \sigma_{yz}n_y n_z + \sigma_{zx}n_z n_x). \tag{1.16}$$

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The matrix representation of (1.15) is

$$\sigma_{nn} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \tag{1.17}$$

The traction vector in an inclined plane can be decomposed into its normal and shear stress components, such that

$$\mathbf{t}_n = \sigma_{nn}\mathbf{n} + \sigma_{nm}\mathbf{m}, \tag{1.18}$$

where $\mathbf{m}$ is a unit vector within the plane under consideration, and σ_{nm} is the corresponding shear stress in that plane. The magnitude of σ_{nm} can be obtained from Pythagoras' theorem as

$$\sigma_{nm} = (t_n^2 - \sigma_{nn}^2)^{1/2} = (t_{nx}^2 + t_{ny}^2 + t_{nz}^2 - \sigma_{nn}^2)^{1/2}, \tag{1.19}$$

where (t_{nx}, t_{ny}, t_{nz}) are given by (1.14), and σ_{nn} by (1.16).

1.3.1 Two-Dimensional State of Stress

In the two-dimensional case, $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$ and the expression (1.16) reduces to

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + 2\sigma_{xy}n_x n_y. \tag{1.20}$$

This can also be obtained directly by projecting the traction vector $\mathbf{t}_n$, as given by (1.12), in the direction $\mathbf{n} = \{n_x, n_y\}$ (Fig. 1.6).

Since $n_x = \cos\varphi$ and $n_y = \sin\varphi$, (1.20) can be rewritten as

$$\sigma_{nn} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\varphi + \sigma_{xy}\sin 2\varphi, \tag{1.21}$$

with the trigonometric identities

$$\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi), \quad \sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi) \tag{1.22}$$

conveniently used.

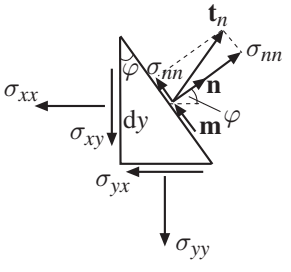


Figure 1.6 The traction vector $\mathbf{t}_n$ acting over an inclined plane whose unit normal vector is $\mathbf{n}$, with its normal and shear stress components (σ_{nn} and σ_{nm}).

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The shear stress σ_{nm} is obtained by projecting the traction vector $\mathbf{t}_n$ from (1.12) in the direction $\mathbf{m} = \{m_x, m_y\}$ (Fig. 1.6), i.e., $\sigma_{nm} = \mathbf{m} \cdot \mathbf{t}_n$. This gives

$$\sigma_{nm} = \sigma_{xx} n_x m_x + \sigma_{yy} n_y m_y + \sigma_{xy} n_x m_y + \sigma_{yx} n_y m_x. \tag{1.23}$$

Since $m_x = -n_y$ and $m_y = n_x$, the above can be rewritten as

$$\sigma_{nm} = -(\sigma_{xx} - \sigma_{yy}) n_x n_y + \sigma_{xy}(n_x^2 - n_y^2), \tag{1.24}$$

or

$$\sigma_{nm} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\varphi + \sigma_{xy} \cos 2\varphi. \tag{1.25}$$

1.4 Tensorial Nature of Stress

By (1.20) and (1.23), we have in the two-dimensional case

$$\begin{aligned}
\sigma_{nn} &= \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2\sigma_{xy} n_x n_y, \\
\sigma_{mm} &= \sigma_{xx} m_x^2 + \sigma_{yy} m_y^2 + 2\sigma_{xy} m_x m_y, \\
\sigma_{nm} &= \sigma_{xx} n_x m_x + \sigma_{yy} n_y m_y + \sigma_{xy} n_x m_y + \sigma_{yx} n_y m_x, \\
\sigma_{mn} &= \sigma_{xx} m_x n_x + \sigma_{yy} m_y n_y + \sigma_{xy} m_x n_y + \sigma_{yx} m_y n_x.
\end{aligned} \tag{1.26}$$

This can be cast in the matrix form

$$\begin{bmatrix} \sigma_{nn} & \sigma_{nm} \\ \sigma_{mn} & \sigma_{mm} \end{bmatrix} = \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \cdot \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix}, \tag{1.27}$$

where T denotes the transpose, and

$$\begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{1.28}$$

is the rotation matrix that transforms the (x, y) components of any vector $\mathbf{r}$ into its (n, m) components

$$\begin{bmatrix} r_n \\ r_m \end{bmatrix} = \begin{bmatrix} n_x & m_x \\ n_y & m_y \end{bmatrix} \cdot \begin{bmatrix} r_x \\ r_y \end{bmatrix}. \tag{1.29}$$

When the components of a 2×2 matrix transform under the rotation of the coordinate system according to (1.27), the matrix is said to be a second-order tensor.

A three-dimensional stress, represented by a 3×3 matrix appearing in (1.7), is a second-order tensor, because it obeys the transformation rule of the type (1.27), with the rotation tensor

$$[Q] = \begin{bmatrix} n_x & m_x & s_x \\ n_y & m_y & s_y \\ n_z & m_z & s_z \end{bmatrix}, \tag{1.30}$$

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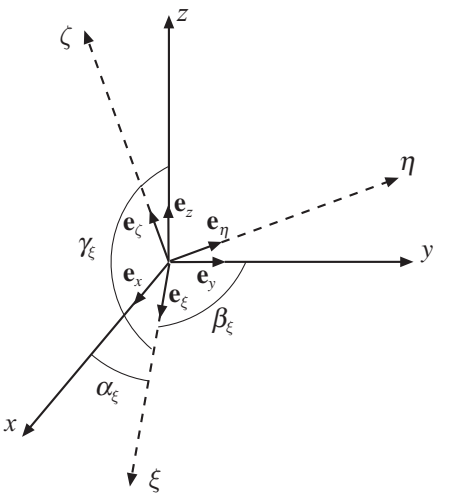


Figure 1.7 The coordinate system (ξ, η, ζ) obtained by the rotation from the coordinate system (x, y, z). The unit vectors of the two sets of coordinate axes are $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and $(\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta)$.

where (**n**, **m**, **s**) are the unit vectors of the rotated coordinate system. Thus, if the new coordinate axes are (ξ, η, ζ), we have

$$[\sigma]_{(\xi,\eta,\zeta)} = [Q]^{\mathrm{T}} \cdot [\sigma]_{(x,y,z)} \cdot [Q]. \tag{1.31}$$

Geometrically, if the ξ axis makes the angles $(\alpha_\xi, \beta_\xi, \gamma_\xi)$ with the positive x axis (Fig. 1.7), then the unit vector **n** along the positive ξ axis is

$$\mathbf{n} = \mathbf{e}_\xi = \{\cos\alpha_\xi, \cos\beta_\xi, \cos\gamma_\xi\}. \tag{1.32}$$

Similarly, for the unit vectors **m** and **s** along the positive η and ζ axes,

$$\mathbf{m} = \mathbf{e}_\eta = \{\cos\alpha_\eta, \cos\beta_\eta, \cos\gamma_\eta\}, \quad \mathbf{s} = \mathbf{e}_\zeta = \{\cos\alpha_\zeta, \cos\beta_\zeta, \cos\gamma_\zeta\}. \tag{1.33}$$

The rotation matrix in (1.30) is thus

$$[Q] = \begin{bmatrix} \cos\alpha_\xi & \cos\alpha_\eta & \cos\alpha_\zeta \\ \cos\beta_\xi & \cos\beta_\eta & \cos\beta_\zeta \\ \cos\gamma_\xi & \cos\gamma_\eta & \cos\gamma_\zeta \end{bmatrix}. \tag{1.34}$$

1.5 Principal Stresses: 2D State of Stress

For design purposes it is of fundamental importance to determine the maximum normal stress at a considered point of a loaded body. Considering first a two-dimensional state of stress, the normal stress on an inclined plane whose unit normal vector is **n** is given by (1.20), i.e.,

$$\sigma_{nn} = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2\sigma_{xy} n_x n_y. \tag{1.35}$$

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The objective is to find the plane (specified by the components n_x and n_y of its normal vector), over which σ_{nn} is maximum (or minimum). Since n_x and n_y are the components of a unit vector, we search for an extreme value of the function $\sigma_{nn}(n_x, n_y)$ subject to the constraint $n_x^2 + n_y^2 = 1$. Thus, we introduce the Lagrangian multiplier σ and search for an unconstrained extremum of the function $f = \sigma_{nn} - \sigma(n_x^2 + n_y^2 - 1)$, i.e.,

$$f = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2\sigma_{xy} n_x n_y - \sigma(n_x^2 + n_y^2 - 1). \tag{1.36}$$

The stationarity conditions for f are

$$\frac{\partial f}{\partial n_x} = 0, \quad \frac{\partial f}{\partial n_y} = 0, \tag{1.37}$$

which give

$$\begin{aligned} (\sigma_{xx} - \sigma) n_x + \sigma_{xy} n_y &= 0, \\ \sigma_{xy} n_x + (\sigma_{yy} - \sigma) n_y &= 0. \end{aligned} \tag{1.38}$$

We next prove that σ is in fact the maximum or minimum normal stress. By multiplying the first equation in (1.38) by $\mathbf{e}_x$ and the second by $\mathbf{e}_y$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors along the x and y axes, by adding up the resulting two expressions, and by using (1.2), the traction vector in the principal plane is found to be

$$\mathbf{t}_n = n_x \mathbf{t}_x + n_y \mathbf{t}_y = \sigma \mathbf{n}. \tag{1.39}$$

Thus, $\mathbf{t}_n$ in the principal plane is entirely in the $\mathbf{n}$ direction, having no shear component at all, and σ represents the (extreme) normal stress in that plane.

Since $\mathbf{t}_n = [\sigma] \cdot \mathbf{n}$, (1.39) can be rewritten as

$$[\sigma] \cdot \mathbf{n} = \sigma \mathbf{n}, \tag{1.40}$$

which means that σ and $\mathbf{n}$ satisfying (1.40) are the eigenvalues and eigenvectors of the stress tensor $[\sigma]$.

To determine σ, we use the fact that (1.38) is a homogeneous system of two linear algebraic equations for n_x and n_y, which has a solution if and only if the determinant of the system vanishes,

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = 0. \tag{1.41}$$

Upon expansion, (1.41) yields a quadratic equation for σ,

$$\sigma^2 - I_1 \sigma - I_2 = 0, \tag{1.42}$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy}, \quad I_2 = -\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 \tag{1.43}$$

are two invariants of the 2×2 stress tensor $[\sigma]$. These are called invariants because they have the same values regardless of the coordinate system used to express the stress tensor, i.e., the (x, y) coordinate system or any other coordinate system obtained from the (x, y) system by rotation. Physically, the principal stress σ cannot depend on the

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coordinate system used to express the stress tensor, and thus the coefficients I_1 and I_2 in equation (1.42) cannot depend on the coordinate system either.

The solution of the quadratic equation (1.42) is

$$\sigma_{1,2} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2}. \tag{1.44}$$

These are the so-called principal stresses; $\sigma_1 = \sigma_{\max}$ is the maximum normal stress, and $\sigma_2 = \sigma_{\min}$ is the minimum normal stress. The corresponding directions are the principal directions (1) and (2). They are specified by either the first or second equation in (1.38). For example, from the first equation it follows that

$$\tan\varphi_{1,2} = \left(\frac{n_y}{n_x}\right)_{1,2} = \frac{\sigma_{1,2} - \sigma_{xx}}{\sigma_{xy}}. \tag{1.45}$$

Upon the substitution of (1.44), this becomes

$$\tan\varphi_{1,2} = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}} \pm \frac{\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2}}{2\sigma_{xy}}. \tag{1.46}$$

It can be readily verified that

$$\tan\varphi_1 \cdot \tan\varphi_2 = -1, \tag{1.47}$$

which implies that the angles φ_1 and φ_2 are 90° apart. Also, since $\sigma_1 > \sigma_{xx}$, it follows from (1.45) that $\tan\varphi_1 > 0$ (i.e., $0^\circ < \varphi_1 < 90^\circ$) if $\sigma_{xy} > 0$. If $\sigma_{xy} < 0$, then $-90^\circ < \varphi_1 < 0^\circ$.

Alternatively, by multiplying the first equation of (1.38) by n_y and the second by n_x, and subtracting the resulting two expressions, we obtain

$$\tan 2\varphi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \tag{1.48}$$

whose two solutions give the angles $\varphi_{1,2}$ in accord with (1.46). A rectangular material element with the sides along the principal directions is shown in Fig. 1.8(a).

Exercise 1.1 Derive (1.48) directly from (1.21) by finding its extremum from the condition $\partial\sigma_{nn}/\partial\varphi = 0$.

Exercise 1.2 The shear stress in the principal plane is always zero. Prove that $\sigma_{nm} = 0$ in the principal plane by substituting (1.48) into (1.25).

1.6 Maximum Shear Stress: 2D Case

For ductile materials it is of particular importance to determine the maximum shear stress at any point of a loaded body, because large shear stresses can cause plastic deformation. The shear stress on an inclined plane whose unit normal vector is $\mathbf{n}$ is given by (1.24), i.e.,

$$\sigma_{nm} = -(\sigma_{xx} - \sigma_{yy})n_x n_y + \sigma_{xy}(n_x^2 - n_y^2). \tag{1.49}$$

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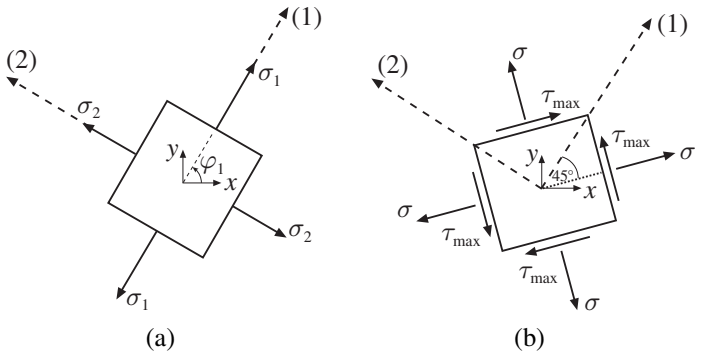


Figure 1.8 (a) Principal stresses $\sigma_1 = \sigma_{\max}$ and $\sigma_2 = \sigma_{\min}$ along the principal directions (1) and (2). Direction (1) is specified by the angle φ_1 relative to the positive x axis. (b) The planes of maximum shear stress $\tau_{\max} = (\sigma_1 - \sigma_2)/2$ are at 45° relative to the principal planes. The normal stress in these planes is the average normal stress $\sigma = (\sigma_1 + \sigma_2)/2$.

To determine the extreme values of σ_{nm}, we introduce the Lagrangian multiplier τ and search for the free extremum of the auxiliary function $g = \sigma_{nm}(n_x, n_y) - \tau(n_x^2 + n_y^2 - 1)$, i.e.,

$$g = -(\sigma_{xx} - \sigma_{yy})n_x n_y + \sigma_{xy}(n_x^2 - n_y^2) - \tau(n_x^2 + n_y^2 - 1). \tag{1.50}$$

The stationarity conditions

$$\frac{\partial g}{\partial n_x} = 0, \quad \frac{\partial g}{\partial n_y} = 0 \tag{1.51}$$

then give

$$\begin{aligned} (\sigma_{xy} - \tau)n_x - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})n_y &= 0, \\ \frac{1}{2}(\sigma_{xx} - \sigma_{yy})n_x + (\sigma_{xy} + \tau)n_y &= 0. \end{aligned} \tag{1.52}$$

We now prove that τ represents the maximum or minimum shear stress. By multiplying the first equation of (1.52) by $\mathbf{e}_y$ and the second by $\mathbf{e}_x$, and by adding up the resulting two expressions, it follows that the traction vector in the plane of extreme shear stress is

$$\mathbf{t}_n = n_x \mathbf{t}_x + n_y \mathbf{t}_y = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\mathbf{n} + \tau \mathbf{m}. \tag{1.53}$$

Thus, τ is the shear stress, while the corresponding normal stress in the planes of extreme shear stress is $\sigma = (\sigma_{xx} + \sigma_{yy})/2 \equiv (\sigma_1 + \sigma_2)/2$, which is the average normal stress acting on the material element at a considered point. (Recall that $I_1 = \sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2$ is the first stress invariant of a two-dimensional state of stress.)

To determine τ, we use the fact that (1.52) is a homogeneous system of two linear algebraic equations for n_x and n_y. This has a solution if and only if the determinant of the system vanishes,

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$$\begin{vmatrix} \sigma_{xy} - \tau & -(\sigma_{xx} - \sigma_{yy})/2 \\ (\sigma_{xx} - \sigma_{yy})/2 & \sigma_{xy} + \tau \end{vmatrix} = 0, \tag{1.54}$$

which, upon expansion, yields

$$\sigma_{xy}^2 - \tau^2 + \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 = 0. \tag{1.55}$$

The solution of the quadratic equation (1.55) is

$$\tau_{1,2} = \pm\frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2} \equiv \pm\frac{1}{2}(\sigma_1 - \sigma_2). \tag{1.56}$$

The corresponding directions follow from (1.52),

$$\tan\varphi_{1,2} = \left(\frac{n_y}{n_x}\right)_{1,2} = \frac{2(\sigma_{xy} - \tau_{1,2})}{\sigma_{xx} - \sigma_{yy}}. \tag{1.57}$$

It can be readily verified that

$$\tan\varphi_1 \cdot \tan\varphi_2 = -1, \tag{1.58}$$

i.e., the angles φ_1 and φ_2 are 90° apart.

Alternatively, by multiplying the first equation of (1.52) by n_y and the second by n_x, and subtracting the resulting two expressions, we obtain

$$\tan 2\varphi = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}}, \tag{1.59}$$

whose two solutions give the angles $\varphi_{1,2}$ in accord with (1.57). Furthermore, by comparing (1.48) and (1.59), we find that

$$\tan 2\varphi^\sigma \cdot \tan 2\varphi^\tau = -1, \tag{1.60}$$

which implies that the planes of extreme shear stress are at 45° relative to the principal planes of extreme normal stress (Fig. 1.8(b)).

Exercise 1.3 Derive (1.59) directly from (1.25) by searching its extremum with respect to φ, i.e., from $\partial\sigma_{nm}/\partial\varphi = 0$.

Exercise 1.4 Prove that in the plane of extreme shear stress the normal stress is $\sigma_{nn} = (\sigma_{xx} + \sigma_{yy})/2$ by substituting (1.59) into (1.21).

1.7 Mohr's Circle for 2D State of Stress

From (1.21) and (1.25), using the notation $\sigma_n = \sigma_{nn}$ and $\tau_n = -\sigma_{nm}$, we can write

$$\begin{aligned} \sigma_n - \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) &= \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\varphi + \sigma_{xy}\sin 2\varphi, \\ \tau_n &= \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\varphi - \sigma_{xy}\cos 2\varphi. \end{aligned} \tag{1.61}$$

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By squaring and adding up the above two expressions, it follows that

$$\left(\sigma_n - \frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 + \tau_n^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2. \tag{1.62}$$

Therefore, for the two-dimensional state of stress $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$, the normal and shear stress (σ_n, τ_n) at an arbitrary plane through a considered material point are coordinates of a point on the circle defined by (1.62), which is known as Mohr's circle. The center of the circle and its radius are

$$C\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right), \quad R = \tau_{\max} = \frac{1}{2}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\sigma_{xy}^2} = \frac{1}{2}(\sigma_1 - \sigma_2). \tag{1.63}$$

The circle is easily constructed by locating the points C and $A(\sigma_{xx}, -\sigma_{xy})$, and drawing a circle of radius $R = CA$, centered at C. The principal stresses (σ_1, σ_2), the maximum shear stress $(\tau_{\max})$, and the angles that define the corresponding directions of vectors orthogonal to material planes over which these stresses act, are shown in Fig. 1.9. They are

$$\sigma_{1,2} = \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) \pm R, \quad \tan 2\varphi_{1,2} = \frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}}, \quad \tan\varphi_1 = \frac{\sigma_{xy}}{\sigma_{xx}-\sigma_2}. \tag{1.64}$$

Exercise 1.5 Show that the radius of Mohr's circle in (1.63) can be expressed as $R = (1/2)(I_1^2 + 4I_2)^{1/2}$, where I_1 and I_2 are the stress invariants defined in (1.43).

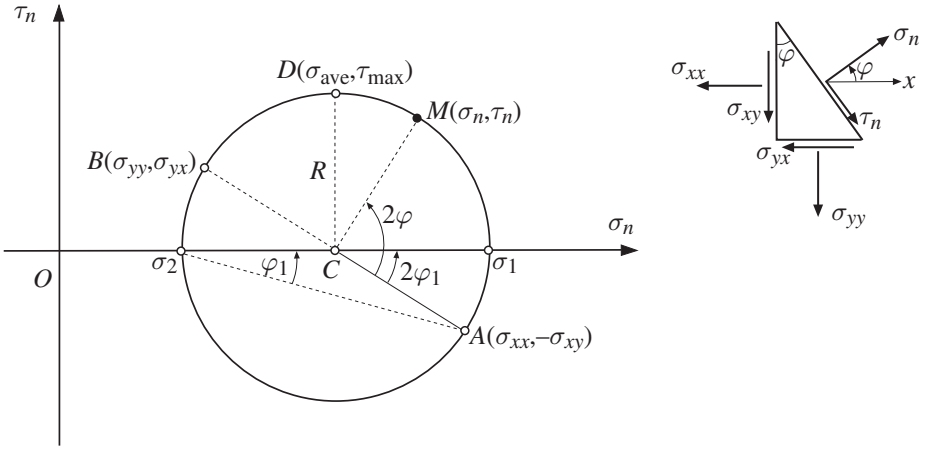


Figure 1.9 Mohr's circle for two-dimensional state of stress $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$. The principal stresses are σ_1 and σ_2, in the directions specified by the angles φ_1 and $\varphi_2 = \varphi_1 + 90^\circ$. The point $M(\sigma_n, \tau_n)$ on Mohr's circle corresponds to the material plane whose normal makes an angle φ with the x axis. Point $D(\sigma_{ave}, \tau_{max})$ corresponds to the plane of the maximum shear stress $\tau_{max} = R = (\sigma_1 - \sigma_2)/2$, where R is the radius of Mohr's circle. The normal stress in that plane is the average normal stress $\sigma_{ave} = (\sigma_1 + \sigma_2)/2$.

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1.8 Principal Stresses: 3D State of Stress

The normal stress on an inclined plane with a unit normal vector $\mathbf{n} = \{n_x, n_y, n_z\}$ is given by (1.16), i.e.,

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2(\sigma_{xy}n_xn_y + \sigma_{yz}n_yn_z + \sigma_{zx}n_zn_x). \tag{1.65}$$

To find the extreme values of the function $\sigma_{nn}(n_x, n_y, n_z)$, subject to the constraint $n_x^2 + n_y^2 + n_z^2 = 1$, we introduce the Lagrangian multiplier σ and search for an unconstrained extremum of the function

$$\begin{aligned} f = {} & \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2(\sigma_{xy}n_xn_y + \sigma_{yz}n_yn_z + \sigma_{zx}n_zn_x) \\ & - \sigma(n_x^2 + n_y^2 + n_z^2 - 1). \end{aligned} \tag{1.66}$$

The stationarity conditions for f are

$$\frac{\partial f}{\partial n_x} = 0, \quad \frac{\partial f}{\partial n_y} = 0, \quad \frac{\partial f}{\partial n_z} = 0, \tag{1.67}$$

which give

$$\begin{aligned} (\sigma_{xx} - \sigma)n_x + \sigma_{xy}n_y + \sigma_{xz}n_z &= 0, \\ \sigma_{yx}n_x + (\sigma_{yy} - \sigma)n_y + \sigma_{yz}n_z &= 0, \\ \sigma_{zx}n_x + \sigma_{zy}n_y + (\sigma_{zz} - \sigma)n_z &= 0. \end{aligned} \tag{1.68}$$

The Lagrangian multiplier σ physically represents the maximum or minimum normal stress. To show this, we multiply the first equation in (1.68) by $\mathbf{e}_x$, the second by $\mathbf{e}_y$, and the third by $\mathbf{e}_z$, and add up the resulting three expressions, to obtain

$$n_x\mathbf{t}_x + n_y\mathbf{t}_y + n_z\mathbf{t}_z = \sigma\mathbf{n}, \tag{1.69}$$

where (1.6) has been used. Thus, in view of (1.9), we have

$$\mathbf{t}_n = \sigma\mathbf{n}, \tag{1.70}$$

i.e., the traction $\mathbf{t}_n$ in the principal plane is entirely in the $\mathbf{n}$ direction, and σ represents the corresponding (extreme) normal stress in that plane. There is no shear stress in the principal plane.

Since, by (1.13), $\mathbf{t}_n = [\sigma] \cdot \mathbf{n}$, expression (1.70) can be rewritten as

$$[\sigma] \cdot \mathbf{n} = \sigma\mathbf{n}, \tag{1.71}$$

which can also be recognized directly from (1.68). This means that σ and $\mathbf{n}$ satisfying (1.71) are the eigenvalues and eigenvectors of the stress tensor $[\sigma]$.

To solve the eigenvalue problem analytically, we use the fact that (1.68) is a homogeneous system of three linear algebraic equations for n_x, n_y, and n_z, which has a solution if and only if its determinant vanishes,

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma \end{vmatrix} = 0. \tag{1.72}$$

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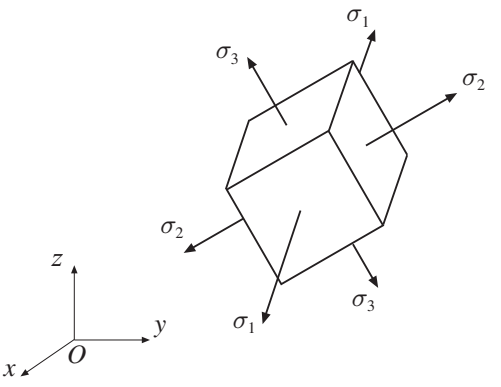


Figure 1.10 A material element with its sides along the principal directions. The principal stresses are $\sigma_1 = \sigma_{\max}$, σ_2, and $\sigma_3 = \sigma_{\min}$.

Upon expansion, (1.72) yields a cubic equation for σ,

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0, \tag{1.73}$$

where

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz}, \\ I_2 &= -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2, \\ I_3 &= \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \end{aligned} \tag{1.74}$$

are three invariants of the 3×3 symmetric stress tensor $[\sigma]$. The three roots of (1.73) are the principal stresses σ_1, σ_2, and σ_3. They are labeled such that $\sigma_1 = \sigma_{\max}$ is the maximum normal stress, σ_2 is the intermediate, and $\sigma_3 = \sigma_{\min}$ is the minimum normal stress. The corresponding directions are the principal directions (1), (2), and (3).

If all three principal stresses are different, than the three principal directions are orthogonal to each other (Fig. 1.10). If $\sigma_1 = \sigma_2 \neq \sigma_3$, then the directions (1) and (2) can be any two directions in the plane orthogonal to the principal direction (3). If $\sigma_1 = \sigma_2 = \sigma_3$, then the state of stress is purely spherical ("hydrostatic," without shear stress in any plane), and any direction is a principal direction.

If the stress tensor is expressed in the coordinate system of principal directions, the stress matrix reads

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{1.75}$$

REMARK The eigenvalue problem given by (1.71) can be easily solved computationally by passing the stress matrix $[\sigma]$ as the input argument to the MATLAB function *eig*, which returns the corresponding three eigenvalues and three eigenvectors.

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1.9 Maximum Shear Stress: 3D Case

This will be determined conveniently from the stress expressions written with respect to principal directions as the coordinate axes (Fig. 1.11(a)). The traction vector in an arbitrary inclined plane with a unit normal vector $\{n_1, n_2, n_3\}$ is, by the Cauchy relation (1.9),

$$\mathbf{t}_n = n_1\mathbf{t}_1 + n_2\mathbf{t}_2 + n_3\mathbf{t}_3 = \{\sigma_1 n_1, \sigma_2 n_2, \sigma_3 n_3\}, \tag{1.76}$$

because $\mathbf{t}_1 = \{\sigma_1, 0, 0\}$, $\mathbf{t}_2 = \{0, \sigma_2, 0\}$, and $\mathbf{t}_3 = \{0, 0, \sigma_3\}$. The normal stress component of $\mathbf{t}_n$ in (1.76) is, by (1.16),

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2, \tag{1.77}$$

where we have used a condensed notation $\sigma_n = \sigma_{nn}$. The magnitude τ_n of the total shear stress in the considered inclined plane can then be determined by Pythagoras' theorem from

$$\tau_n^2 = t_n^2 - \sigma_n^2, \quad t_n^2 = \mathbf{t}_n \cdot \mathbf{t}_n = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2, \tag{1.78}$$

which gives

$$\tau_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2. \tag{1.79}$$

To determine the extreme values of τ_n, we search for unconstrained extreme values of the auxiliary function $g = \tau_n^2 - \tau^2(n_1^2 + n_2^2 + n_3^2 - 1)$, i.e.,

$$g = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 - \lambda(n_1^2 + n_2^2 + n_3^2 - 1), \tag{1.80}$$

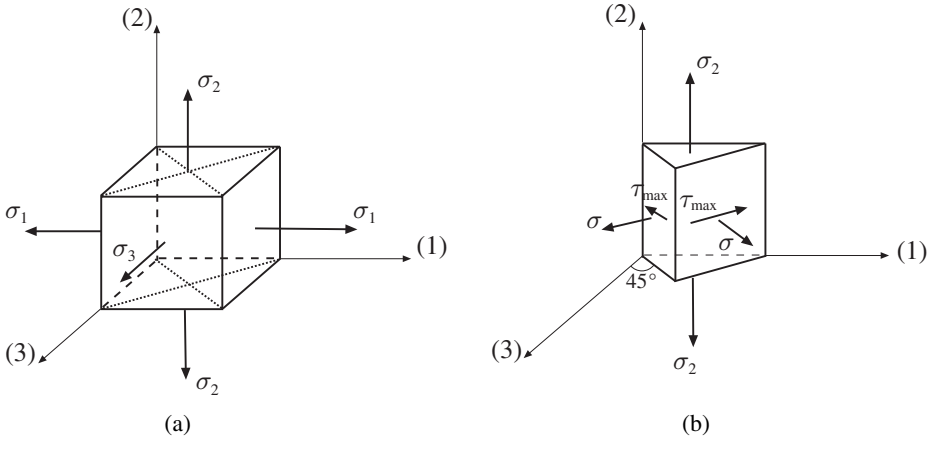


Figure 1.11 (a) A material element with its sides along the principal directions. Principal stresses are σ_1, σ_2, and σ_3. Two planes at 45° to principal directions (1) and (3) are shown by dotted lines. (b) An extracted triangular prism with the maximum shear stress $\tau_{\max} = (\sigma_1 - \sigma_3)/2$ along the planes at 45° relative to the principal planes. The normal stress in these planes is $\sigma = (\sigma_1 + \sigma_3)/2$.

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where λ is the corresponding Lagrangian multiplier. The stationarity conditions for $g = g(n_1, n_2, n_3)$ are

$$\frac{\partial g}{\partial n_1} = 0, \quad \frac{\partial g}{\partial n_2} = 0, \quad \frac{\partial g}{\partial n_3} = 0, \tag{1.81}$$

which lead to

$$\begin{aligned} n_1 \left[\sigma_1^2 - 2\sigma_1(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2) - \lambda\right] &= 0, \\ n_2 \left[\sigma_2^2 - 2\sigma_2(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2) - \lambda\right] &= 0, \\ n_3 \left[\sigma_3^2 - 2\sigma_3(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2) - \lambda\right] &= 0. \end{aligned} \tag{1.82}$$

These equations are satisfied in six distinct cases. The first three cases correspond to principal planes whose unit normals **n** are

$$\{\pm 1, 0, 0\}, \quad \{0, \pm 1, 0\}, \quad \{0, 0, \pm 1\}. \tag{1.83}$$

The corresponding values of the Lagrangian multiplier are

$$\lambda = -\sigma_1^2, \quad \lambda = -\sigma_2^2, \quad \lambda = -\sigma_3^2, \tag{1.84}$$

with the shear stresses all being equal to zero ($\tau_n = 0$). These are the minimum values of the magnitude of the shear stress.

The remaining three cases of the extreme values of the shear stress are in the planes whose unit normals **n** are

$$\{\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0\}, \quad \{\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2}\}, \quad \{0, \pm 1/\sqrt{2}, \pm 1/\sqrt{2}\}. \tag{1.85}$$

The corresponding Lagrangian multipliers are

$$\lambda = -\sigma_1\sigma_2, \quad \lambda = -\sigma_1\sigma_3, \quad \lambda = -\sigma_2\sigma_3, \tag{1.86}$$

with the magnitudes of the extreme values of the shear stress

$$\tau_n = \frac{\sigma_1 - \sigma_2}{2}, \quad \tau_n = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_n = \frac{\sigma_2 - \sigma_3}{2}. \tag{1.87}$$

These extreme shear stresses occur in the planes that are parallel to one of the principal directions and are at $\pm 45^\circ$ with respect to other two principal directions (Fig. 1.11(b)). The normal stresses in these planes are, from (1.77),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma_n = \frac{\sigma_1 + \sigma_3}{2}, \quad \sigma_n = \frac{\sigma_2 + \sigma_3}{2}. \tag{1.88}$$

The absolute maximum of the shear stress and the corresponding normal stress are

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}, \quad \sigma = \frac{\sigma_1 + \sigma_3}{2}. \tag{1.89}$$

The direction of $\tau_{\max}$, shown in Fig. 1.11(b), does not have a component in the (2) direction. This is because the principal plane (1, 3) is free of shear stress, and by conjugacy of shear stresses there cannot be a vertical component of shear stress in any plane orthogonal to the (1, 3) plane.

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1.10 Mohr's Circles for 3D State of Stress

By using the principal stress directions, we can write, from (1.65) and (1.78),

$$\begin{aligned} \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 &= \sigma_n, \\ \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 &= \sigma_n^2 + \tau_n^2, \\ n_1^2 + n_2^2 + n_3^2 &= 1. \end{aligned} \tag{1.90}$$

If σ_n and τ_n are given, (1.90) is a system of three linear algebraic equations for the corresponding values of n_1^2, n_2^2, and n_3^2. Its solution is

$$\begin{aligned} n_1^2 &= \frac{\tau_n^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}, \\ n_2^2 &= \frac{\tau_n^2 + (\sigma_n - \sigma_3)(\sigma_n - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}, \\ n_3^2 &= \frac{\tau_n^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}, \end{aligned} \tag{1.91}$$

where it is assumed that $\sigma_1 > \sigma_2 > \sigma_3$. Since $n_i^2 \geq 0$ for each i, (1.91) impose the conditions

$$\begin{aligned} \tau_n^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3) &\geq 0, \\ \tau_n^2 + (\sigma_n - \sigma_3)(\sigma_n - \sigma_1) &\leq 0, \\ \tau_n^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2) &\geq 0. \end{aligned} \tag{1.92}$$

These inequalities can be rewritten as

$$\begin{aligned} \left(\sigma_n - \frac{\sigma_2 + \sigma_3}{2}\right)^2 + \tau_n^2 &\geq \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2, \\ \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau_n^2 &\leq \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2, \\ \left(\sigma_n - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau_n^2 &\geq \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2. \end{aligned} \tag{1.93}$$

The inequalities in (1.93) define the domain in the (σ_n, τ_n) plane between the three circles, i.e., within the outer (largest) circle and outside the inner (smaller) two circles (Fig. 1.12). The centers of these circles and their radii are

$$\begin{aligned} C_1 &\left(\frac{\sigma_2 + \sigma_3}{2}, 0\right), \quad R_1 = \frac{\sigma_2 - \sigma_3}{2}, \\ C_2 &\left(\frac{\sigma_1 + \sigma_3}{2}, 0\right), \quad R_2 = \frac{\sigma_1 - \sigma_3}{2}, \\ C_3 &\left(\frac{\sigma_1 + \sigma_2}{2}, 0\right), \quad R_3 = \frac{\sigma_1 - \sigma_2}{2}. \end{aligned} \tag{1.94}$$

The three circles are Mohr's circles for a three-dimensional state of stress. If the principal stresses at a considered material point are $(\sigma_1, \sigma_2, \sigma_3)$, the coordinates of a point

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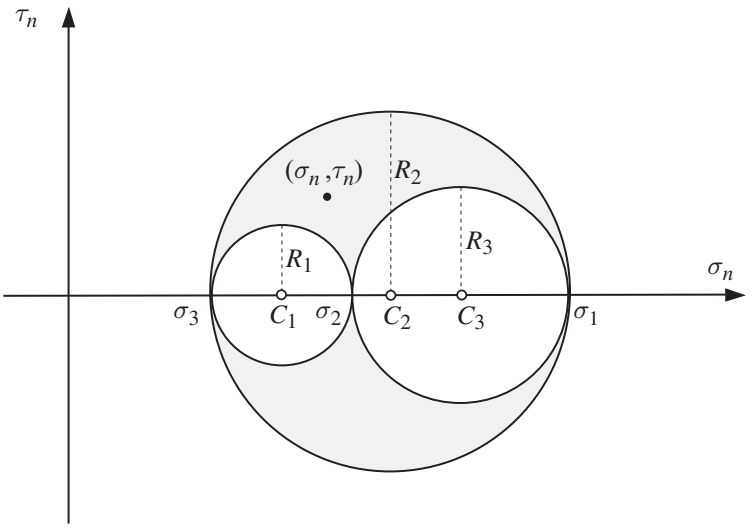


Figure 1.12 Mohr's circles for three-dimensional state of stress characterized by the principal stresses σ_1, σ_2, and σ_3. The normal and shear stress over an arbitrary plane through a considered material point are the coordinates of the corresponding point (σ_n, τ_n) within the domain bounded by three circles. The radii of the circles, R_1, R_2, and R_3, are the maximum shear stresses for the considered state of stress.

(σ_n, τ_n) within the domain bounded by three Mohr's circles represent the normal and shear stresses in the material plane whose normal vector is specified by (1.91). The radii R_1, R_2, and R_3 represent the local maxima of the shear stress. The σ_n coordinates of the centers C_1, C_2, and C_3 are the normal stresses in the planes of the local maximum of the shear stress.

If $\sigma_1 = \sigma_2 = \sigma_3$, Mohr's circles degenerate into a point, because there is no shear stress in any material plane under hydrostatic state of stress. If $\sigma_2 = \sigma_3 \neq \sigma_1$, two Mohr's circles overlap and the third degenerates to a point $(\sigma_2, 0)$. An arbitrary state of stress in this case corresponds to a point (σ_n, τ_n) on the circle

$$\left(\sigma_n - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau_n^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2, \tag{1.95}$$

as recognized from (1.93). The corresponding n_1 is obtained from the first expression in (1.90) by using $\sigma_3 = \sigma_2$ and $n_2^2 + n_3^2 = 1 - n_1^2$,

$$n_1^2 = \frac{\sigma_n - \sigma_2}{\sigma_1 - \sigma_2}. \tag{1.96}$$

The values of n_2 and n_3 satisfy the condition $n_2^2 + n_3^2 = 1 - n_1^2$, but are otherwise undetermined. Indeed, if $\sigma_2 = \sigma_3 \neq \sigma_1$, any two directions in the plane orthogonal to σ_1 can be taken as principal directions, which means that n_2 and n_3 are not unique.

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1.11 Deviatoric and Spherical Parts of Stress

The deviatoric and spherical parts of the stress tensor are important for the study of inelastic deformation and failure criteria of materials and structural elements. The spherical part of stress is defined in terms of the average normal stress at a considered material point, which is the arithmetic mean of the normal stress components in any three mutually orthogonal planes through that point,

$$\sigma^{\mathrm{s}} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3). \tag{1.97}$$

Thus, $\sigma^{\mathrm{s}} = (1/3)I_1$, where I_1 is the first stress invariant, appearing in (1.74). The part of the stress tensor $[\sigma]$ defined by

$$[\sigma]^{\mathrm{s}} = \begin{bmatrix} \sigma^{\mathrm{s}} & 0 & 0 \\ 0 & \sigma^{\mathrm{s}} & 0 \\ 0 & 0 & \sigma^{\mathrm{s}} \end{bmatrix} = \sigma^{\mathrm{s}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.98}$$

is referred to as the spherical part of the stress tensor. Shortly written, $[\sigma]^{\mathrm{s}} = \sigma^{\mathrm{s}}[I]$, where $[I]$ is the unit or identity matrix, whose diagonal entries are equal to 1 and the off-diagonal entries are equal to 0.

The remaining part of the stress tensor in the additive decomposition

$$[\sigma] = [\sigma]^{\mathrm{s}} + [\sigma]^{\mathrm{d}} \tag{1.99}$$

is referred to as a deviatoric part of the stress tensor. Its components are

$$[\sigma]^{\mathrm{d}} = \begin{bmatrix} \sigma_{xx} - \sigma^{\mathrm{s}} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma^{\mathrm{s}} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma^{\mathrm{s}} \end{bmatrix}. \tag{1.100}$$

Since any direction is a principal direction of the spherical part of stress tensor, the principal directions of the deviatoric part of the stress tensor coincide with the principal directions of the total stress tensor, while their principal values are related by

$$\sigma_1^{\mathrm{d}} = \sigma_1 - \sigma^{\mathrm{s}}, \quad \sigma_2^{\mathrm{d}} = \sigma_2 - \sigma^{\mathrm{s}}, \quad \sigma_3^{\mathrm{d}} = \sigma_3 - \sigma^{\mathrm{s}}. \tag{1.101}$$

The deviatoric part of the stress tensor is a traceless tensor, i.e.,

$$J_1 = \sigma_{xx}^{\mathrm{d}} + \sigma_{yy}^{\mathrm{d}} + \sigma_{zz}^{\mathrm{d}} = 0. \tag{1.102}$$

Thus, in general, there are only two independent nonvanishing deviatoric stress invariants, J_2 and J_3. Expressed with respect to the principal directions, they are, from (1.74),

$$J_2 = -(\sigma_1^{\mathrm{d}}\sigma_2^{\mathrm{d}} + \sigma_2^{\mathrm{d}}\sigma_3^{\mathrm{d}} + \sigma_3^{\mathrm{d}}\sigma_1^{\mathrm{d}}), \quad J_3 = \sigma_1^{\mathrm{d}}\sigma_2^{\mathrm{d}}\sigma_3^{\mathrm{d}}. \tag{1.103}$$

Of particular interest for the analysis of plastic deformation is the second (quadratic) invariant J_2. By using (1.97) and (1.101), J_2 in (1.103) can be expressed in terms of the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ as

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \tag{1.104}$$

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If the stress tensor is expressed with respect to the axes (x, y, z), the expression for J_2 is

$$J_2 = \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2. \quad (1.105)$$

Exercise 1.6 By substituting (1.101) in the expression for J_2 in (1.103), derive the expression (1.104). Also, by using (1.102) prove that

$$J_2 = -(\sigma_1^{\mathrm{d}}\sigma_2^{\mathrm{d}} + \sigma_2^{\mathrm{d}}\sigma_3^{\mathrm{d}} + \sigma_3^{\mathrm{d}}\sigma_1^{\mathrm{d}}) = \frac{1}{2}\left[(\sigma_1^{\mathrm{d}})^2 + (\sigma_2^{\mathrm{d}})^2 + (\sigma_3^{\mathrm{d}})^2\right]. \quad (1.106)$$

REMARK The deviatoric stress components are often denoted by the letter S, i.e., $S_{xx} = \sigma_{xx}^{\mathrm{d}}$, etc.

1.12 Octahedral Shear Stress

The octahedral shear stress is of importance for the analysis of inelastic deformation of ductile materials. The octahedral shear stress at a considered material point is the shear stress that acts in the octahedral plane through this point. The octahedral plane is defined as the plane which makes equal angles with the principal directions of stress at a considered point. The unit vector orthogonal to an octahedral plane has the components which satisfy the relation $n_1^2 = n_2^2 = n_3^2 = 1/3$, thus there are eight octahedral planes. One of them is shown in Fig. 1.13. From (1.14), expressed with respect to the principal axes (1), (2), and (3), the components of the traction vector in an octahedral plane are

$$t_{n_1} = \sigma_1 n_1, \quad t_{n_2} = \sigma_2 n_2, \quad t_{n_3} = \sigma_3 n_3, \quad (1.107)$$

hence

$$t_n^2 = \frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^3\right). \quad (1.108)$$

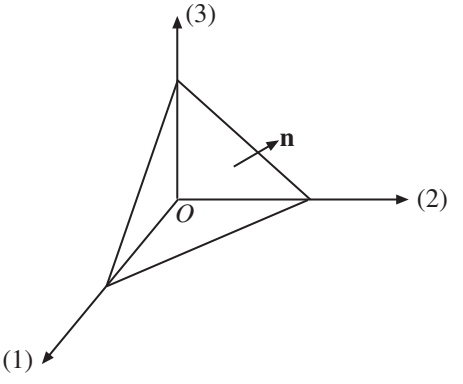


Figure 1.13 The octahedral plane whose unit normal vector is $\mathbf{n} = \{1, 1, 1\}/\sqrt{3}$ with respect to the principal stress axes (1), (2), and (3).

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The octahedral normal stress is, from (1.15), given by $\sigma_{\text{oct}} = n_1 t_{n_1} + n_2 t_{n_2} + n_3 t_{n_3}$, which gives, by using (1.107),

$$\sigma_{\text{oct}} = \sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3). \tag{1.109}$$

The square of the octahedral shear stress can then be determined from

$$\tau_{\text{oct}}^2 = t_n^2 - \sigma_{\text{oct}}^2 = \frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^3\right) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2. \tag{1.110}$$

This gives

$$\tau_{\text{oct}}^2 = \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \equiv \frac{2}{3} J_2. \tag{1.111}$$

It can be shown that the magnitude of the octahedral shear stress represents the average of the shear stresses over all planes through a considered material point.

1.13 Differential Equations of Equilibrium

In previous sections we analyzed the state of stress at a point. In general, the state of stress within a body changes from point to point, and we need to formulate the governing differential equations whose solution specifies the variation of the stress state within the considered body. Figure 1.14 shows a material body under a surface traction $\mathbf{t}_n$ specified over a part S_t of the total bounding surface S of the body. A displacement vector $\mathbf{u}$ is prescribed over the remaining part S_u of S. Often, displacements may be required to be zero over certain portions of S_u, or tractions may be zero over certain portions of S_t. Mixed boundary conditions could also be specified in which certain components of the traction and the displacements are prescribed at a point. For example, the tangential component of the traction and the normal component of the displacement may be prescribed to be zero. In general, one can prescribe either a component of the traction or a component of the displacement in a given direction, but not both in the

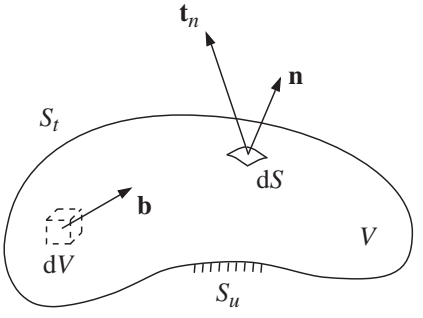


Figure 1.14 A material body of volume V bounded by surface $S = S_t + S_u$. The external loading consists of a surface traction $\mathbf{t}_n$ acting over S_t and a body force $\mathbf{b}$ (per unit volume) acting within V. Displacement vector $\mathbf{u}$ is prescribed over S_u.

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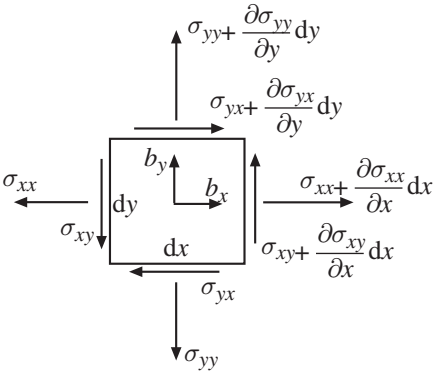


Figure 1.15 An infinitesimal rectangular element of unit thickness in the z direction, whose sides in the x and y directions are $\mathrm{d}x$ and $\mathrm{d}y$. If the stress components on the left and lower sides of the element are $(\sigma_{xx}, \sigma_{xy})$ and $(\sigma_{yy}, \sigma_{yx})$, the stress components on the opposite, right, and upper sides of the element are as shown. The body force components are b_x and b_y.

same direction. In addition to the surface loads, the body may be under prescribed body forces **b** within the volume V, usually from gravity.

In the equilibrium configuration of the body, the sum of all forces and the sum of all moments for any reference point must vanish. If the entire body is in equilibrium, every portion of it must be in equilibrium as well (and vice versa). To derive the differential equations of equilibrium, we shall apply the equilibrium conditions to an infinitesimal material element imagined to be extracted from the body. For simplicity, we shall consider first the two-dimensional case, i.e., an infinitesimal rectangular element of unit thickness in the z direction, whose sides in the x and y directions are $\mathrm{d}x$ and $\mathrm{d}y$. Its free-body diagram is shown in Fig. 1.15. If the average traction components on the left side of the block are σ_{xx} and σ_{xy}, the average traction components on the right side of the block are

$$\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\,\mathrm{d}x, \quad \sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}\,\mathrm{d}x. \tag{1.112}$$

Likewise, if the average traction components on the lower side of the block are σ_{yy} and σ_{yx}, the average traction components on the upper side of the block are

$$\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\,\mathrm{d}y, \quad \sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\,\mathrm{d}y. \tag{1.113}$$

The material element may also be under a body force (per unit volume), whose in-plane components are b_x and b_y. For equilibrium, the sum of all forces in the x and y directions must vanish. The condition $\sum F_x = 0$ is

$$-\sigma_{xx}\mathrm{d}y + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y - \sigma_{yx}\mathrm{d}x + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial yx}\,\mathrm{d}y\right)\mathrm{d}x + b_x\mathrm{d}x\mathrm{d}y = 0. \tag{1.114}$$

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Upon cancelation of opposite terms, the use of $\sigma_{yx} = \sigma_{xy}$, and division by $\mathrm{d}x\mathrm{d}y$, (1.114) reduces to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0. \tag{1.115}$$

Similarly, from the equilibrium condition $\sum F_y = 0$ we obtain

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0. \tag{1.116}$$

Partial differential equations (1.115) and (1.116) are the differential equations of equilibrium for two-dimensional problems of mechanics of deformable bodies. Their three-dimensional generalization is obtained in a straightforward manner by the consideration of the free-body diagram of a three-dimensional material element whose sides are $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. By making zero the sums of all forces in the x, y, and z directions, we obtain

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x &= 0, \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y &= 0, \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0.
\end{aligned} \tag{1.117}$$

These are the differential equations of equilibrium for three-dimensional problems of the mechanics of deformable bodies.

In dynamics (vibration and wave propagation) problems, one needs to include the inertia–acceleration terms and use Newton's second law of motion $\mathbf{F} = m\mathbf{a}$, where m is the mass and $\mathbf{a} = \partial^2 \mathbf{u}/\partial t^2$ is the acceleration (t being the time). In this case, we obtain

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x &= \rho \frac{\partial^2 u_x}{\partial t^2}, \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y &= \rho \frac{\partial^2 u_y}{\partial t^2}, \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= \rho \frac{\partial^2 u_z}{\partial t^2},
\end{aligned} \tag{1.118}$$

where ρ denotes the mass density ($\rho = \mathrm{d}m/\mathrm{d}V$). The partial differential equations (1.118) are known as the Cauchy equations of motion. They apply to both solids and fluids, as long as they are considered to be continuums for which the concept of stress can be used.

REMARK If the coordinate axes are labeled as (x_1, x_2, x_3), rather than (x, y, z), then the equations of motion (1.118) can be cast in a compact form as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (i = 1, 2, 3). \tag{1.119}$$

The so-called summation convention over the repeated index is adopted in (1.119), according to which

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$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} \equiv \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{(for each } i = 1, 2, 3\text{)}. \tag{1.120}$$

1.13.1 Boundary Conditions

Returning to the equilibrium equations (1.117), these partial differential equations must be accompanied by boundary conditions, which correspond to a given loading at the points of the external boundary of the body S_t, where traction is prescribed. If the traction vector $\mathbf{t}_n$ is prescribed to be equal to $\mathbf{t}_n^0$ at the points of S_t, then, according to the Cauchy relation (1.13), the components of the stress tensor $[\sigma]$ must satisfy, at each point of S_t,

$$[\sigma] \cdot \mathbf{n} = \mathbf{t}_n^0, \tag{1.121}$$

where $\mathbf{n} = \{n_x, n_y, n_z\}$ is a unit vector orthogonal to S_t at a considered point. The three scalar expressions associated with the matrix expression (1.121) are

$$\begin{aligned} \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z &= t_{nx}^0, \\ \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z &= t_{ny}^0, \\ \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z &= t_{nz}^0. \end{aligned} \tag{1.122}$$

These are the traction (stress) boundary conditions accompanying the equations of equilibrium (1.117). Displacement boundary conditions will be discussed in subsequent chapters.

1.13.2 Statical Indeterminacy

Equilibrium equations (1.117) are three linear first-order partial differential equations for six unknown stress components. In general, we cannot determine six unknowns from only three equations and, thus, the problem of the determination of stress components is, in general, statically indeterminate. We need to introduce additional equations based on additional physical considerations, and for that we need to consider the deformation caused by the applied loading. Because the deformation depends on material properties, we shall need also to introduce the so-called constitutive equations of the material, which relate the stress and the strain components. The definitions of the strain components, which represent the measures of the relative deformation of material at a considered point, is the subject of Chapter 2. The constitutive equations are considered in Chapter 3.

Problems

Problem 1.1 A rectangular material element is in the state of stress as shown. Determine: (a) the traction vector in the plane whose normal vector is at 60° with respect to the x axis; (b) the normal and shear stresses in that plane; (c) the principal stresses and their directions; (d) the maximum shear stress and the corresponding normal stress.

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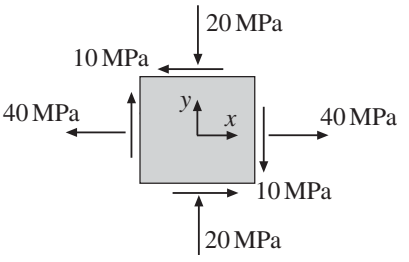


Figure P1.1

Problem 1.2 (a) For the stress state from Problem 1.1, construct the Mohr circle. (b) Determine in which plane the normal stress is 20 MPa, and (c) in which planes the shear stress is 15 MPa. What are the values of the corresponding shear and normal stresses in these planes?

Problem 1.3 The traction vectors along the sides of a trapezoidal material element are as shown in Fig. P1.3. Determine: (a) the stress components σ_{xx}, σ_{yy}, and σ_{xy}; (b) the maximum shear stress $\tau_{\max}$ and the planes over which $\tau_{\max}$ acts.

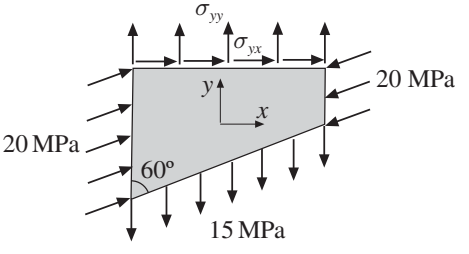


Figure P1.3

Problem 1.4 The normal and shear stresses over the sides of an equilateral triangular element are as shown in Fig. P1.4. Determine the unknown quantities τ, σ_{xx}, and σ_{xy}. Determine also the normal stress σ_{yy}.

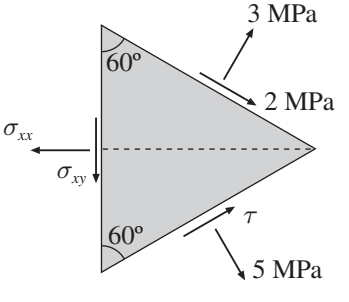


Figure P1.4

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Problem 1.5 A three-dimensional rectangular block of material of dimensions $a \times b \times c$ is under uniform uniaxial compression of magnitude 50 MPa, as shown in Fig. P1.5. (a) By performing the cross product of vectors **AB** and **AC**, show that the components of the unit vector orthogonal to the plane ABC are

$$n_x = \frac{bc}{d^2}, \quad n_y = \frac{ca}{d^2}, \quad n_z = \frac{ab}{d^2}, \quad d^2 = (a^2b^2 + b^2c^2 + c^2a^2)^{1/2}.$$

(b) Determine the direction and the magnitude of the normal and shear stresses in the plane ABC.

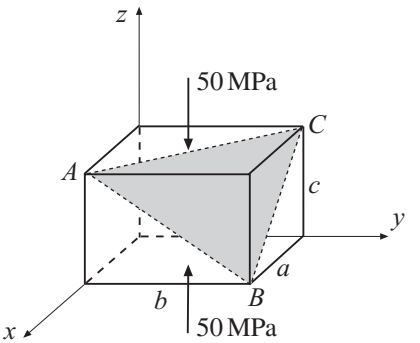


Figure P1.5

Problem 1.6 The stress state at a material point with respect to the (x, y, z) coordinate system is

$$[\sigma] = \begin{bmatrix} 20 & 10 & 0 \\ 10 & -10 & 5 \\ 0 & 5 & 30 \end{bmatrix} \text{ (MPa)}.$$

Determine the components of the stress tensor relative to the coordinate system (ξ, η, ζ), obtained from the (x, y, z) system by a 30° counterclockwise rotation about the z axis.

Problem 1.7 The three-dimensional stress state is

$$[\sigma] = \begin{bmatrix} 10 & -10 & 5 \\ -10 & 0 & 3 \\ 5 & 3 & -15 \end{bmatrix} \text{ (MPa)}.$$

Determine: (a) the traction vector in the plane whose normal is $\mathbf{n} = \{1, 1, 4\}\sqrt{2}/6$; (b) the normal and shear stresses in that plane; (c) the principal stresses and their directions; (d) the maximum shear stress and the corresponding normal stress; (e) the spherical and deviatoric parts of the stress tensor; and (f) the octahedral shear stress and the octahedral normal stress.

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Problem 1.8 By using the expression for the second invariant of the deviatoric stress tensor

$$J_2 = -(\sigma^{\mathrm{d}}_{xx}\sigma^{\mathrm{d}}_{yy} + \sigma^{\mathrm{d}}_{yy}\sigma^{\mathrm{d}}_{zz} + \sigma^{\mathrm{d}}_{zz}\sigma^{\mathrm{d}}_{xx}) + \sigma^2_{xy} + \sigma^2_{yz} + \sigma^2_{zx},$$

which follows from (1.74), when applied to the deviatoric stress tensor, derive expression (1.105).

Problem 1.9 Consider the stress field from elementary beam bending analysis,

$$\sigma_{zy} = \frac{3F}{8ab^3}(b^2 - y^2), \quad \sigma_{zz} = -\frac{3F(L-z)}{4ab^3}\,y,$$

where F is the applied force at the right end of the cantilever beam of length L and cross-sectional dimensions $2a \times 2b$ (Fig. P1.9). Verify that all equilibrium equations in (1.117) are satisfied by this approximate stress field. Assume that body forces are absent and that other stress components are equal to zero. (The stress field is approximate because one of the compatibility conditions, to be discussed in Chapters 2 and 3, is not satisfied by this stress field; see Problem 4.5 of Chapter 4).

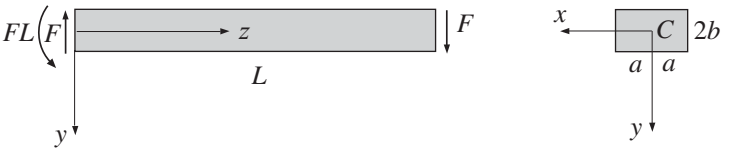


Figure P1.9

Problem 1.10 A rectangular block (Fig. P1.10) whose sides are (a, b, c) is placed between two smooth wide rigid plates at $x = \pm a/2$ which prevent displacement in the x direction. The block is loaded on its upper side by a uniform pressure p, while it is supported at its bottom by a rigid smooth surface which prevents displacement in the y direction. The block is free to expand in the out-of-plane z direction. Write down the boundary conditions for all six sides of the block.

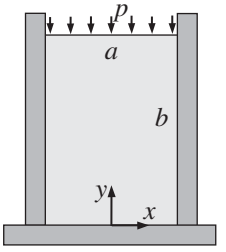


Figure P1.10

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