Finding sums of powers using physical arguments

DAVID TREEBY

1. *Introduction*

In an earlier issue of the *Mathematical Gazette*, Nick Lord established the familiar formula for the sum of the first n squares using a physical argument based on the centroid of a configuration of masses in the plane [1]. In [2] we demonstrate an alternative configuration that gives the same result. This article is a follow-up to these papers, in which we find physical derivations of the formula for

$$S_k(n) = 1^k + 2^k + \ldots + n^k$$

for each $k \in \{1, 2, 3, 4, 5\}$. Let us first summarise the required theory. Take any region $X \subseteq \mathbb{R}^2$ with uniform density and total area A. The *centroid* of X is the arithmetic mean position of all of the points in X. If a region has a line of symmetry then the centroid will be located on that line. If X is decomposed into disjoint parts $X_1, \ldots, X_n$ then the centroid can be found by first finding the centroid $(\bar{x}_j, \bar{y}_j)$ and area A_j of each part and then computing the weighted averages:

$$\bar{x} = \frac{\sum_{j=1}^{n} A_j \bar{x}_j}{A} \quad \text{and} \quad \bar{y} = \frac{\sum_{j=1}^{n} A_j \bar{y}_j}{A}. \tag{1}$$

We will primarily be interested in the x-component of the centroid, which we call the *x-mean*.

2. *The formula for* $S_1(n)$

We first establish the formula for $S_1(n)$, the nth triangular number. The following argument is not new but is given to demonstrate the method that we will deploy throughout this article. Let us begin by arranging unit squares as shown in Figure 1. We find the x-mean of this configuration in two different ways.

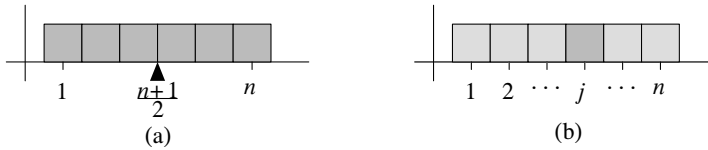


FIGURE 1

Firstly, we see in Figure 1(a) that the rectangle is symmetric. Therefore its x-mean is located at its midpoint:

$$\bar{x} = \frac{n+1}{2}. \tag{2}$$

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On the other hand, in Figure 1(b) we see that block j has area 1 and is located j units from the origin. Therefore,

$$\bar{x} = \frac{1 \times 1 + 1 \times 2 + \ldots + 1 \times n}{n} = \frac{1 + 2 + \ldots + n}{n}. \quad (3)$$

Equating these two results yields

$$\frac{1 + 2 + \ldots + n}{n} = \frac{n + 1}{2},$$

from which we obtain the familiar identity

$$S_1(n) = \frac{n(n + 1)}{2}. \quad (4)$$

3. *The formula for* $S_2(n)$

We now progress to the formula for $S_2(n)$, which gives us the sum of the first n squares. The approaches taken in both [1] and [2] are more elegant than our approach here. Our purpose is to demonstrate a trick that also works when finding the formula for $S_4(n)$, so consider this a warm-up for that result.

We now arrange unit squares as shown in Figure 2. Again we calculate the x-mean of this configuration in two different ways.

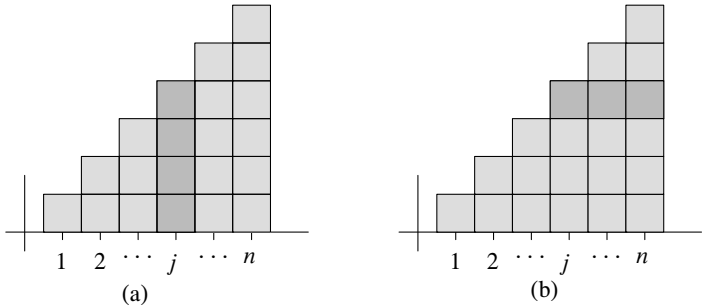


FIGURE 2

In Figure 2(a) the jth column has area j and is centred at position $x = j$. Therefore

$$\bar{x} = \frac{\sum_{j=1}^{n} j^2}{S_1(n)}. \quad (5)$$

On the other hand, the jth row in Figure 2(b) has area $n - j + 1$ and its balancing point is located at its midpoint: $x = \frac{1}{2}(j + n)$. It follows that

$$\bar{x} = \frac{\frac{1}{2}\sum_{j=1}^{n} (n - j + 1)(n + j)}{S_1(n)}. \quad (6)$$

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Equating these results gives

$$\sum_{j=1}^{n} j^2 = \frac{1}{2}\sum_{j=1}^{n}(n - j + 1)(n + j)$$

$$\Rightarrow \quad S_2(n) = \frac{1}{2}\sum_{j=1}^{n}\left(n^2 - j^2 + n + j\right)$$

$$\Rightarrow \quad \frac{3}{2}S_2(n) = \frac{1}{2}n^3 + \frac{1}{2}n^2 + \frac{1}{2}S_1(n).$$

This implies that

$$S_2(n) = \frac{1}{3}n^3 + \frac{1}{3}n^2 + \frac{1}{3}S_1(n) = \frac{n(n + 1)(2n + 1)}{6}.$$

4. *The formula for $S_3(n)$*

The physical derivation of the formula for $S_3(n)$ is more pleasant. We begin by arranging unit squares as shown in Figure 3.

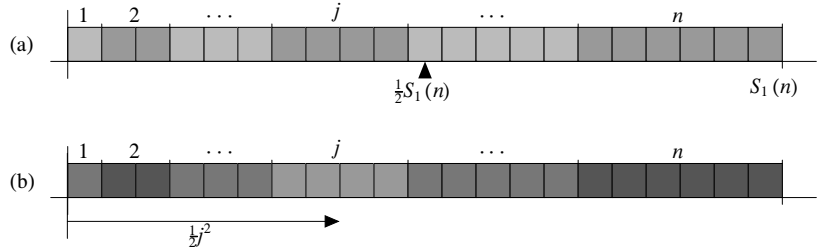


FIGURE 3

Firstly, the rectangle in Figure 3(a) has area $1 + 2 + \ldots + n = S_1(n)$ and its balancing point is

$$\bar{x} = \tfrac{1}{2}S_1(n). \tag{8}$$

On the other hand, the jth rectangle in Figure 3(b) comprises j blocks whose balancing point is located at position

$$1 + 2 + \ldots + (j - 1) + \tfrac{1}{2}j = \tfrac{1}{2}j^2.$$

It follows that the x-mean of this configuration is also given by

$$\bar{x} = \frac{\sum_{j=1}^{n} j \cdot \frac{j^2}{2}}{S_1(n)} = \frac{\sum_{j=1}^{n} j^3}{2S_1(n)}. \tag{9}$$

Equating (8) and (9) gives

$$\sum_{j=1}^{n} j^3 = S_1^2(n),$$

that is,

$$S_3(n) = S_1^2(n) = \frac{n^2(n + 1)^2}{4}.$$

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5. *The formula for* $S_4(n)$

The derivation here borrows ideas from the derivations above for $S_2(n)$ and $S_3(n)$. We begin by arranging unit squares as shown in Figure 4 below.

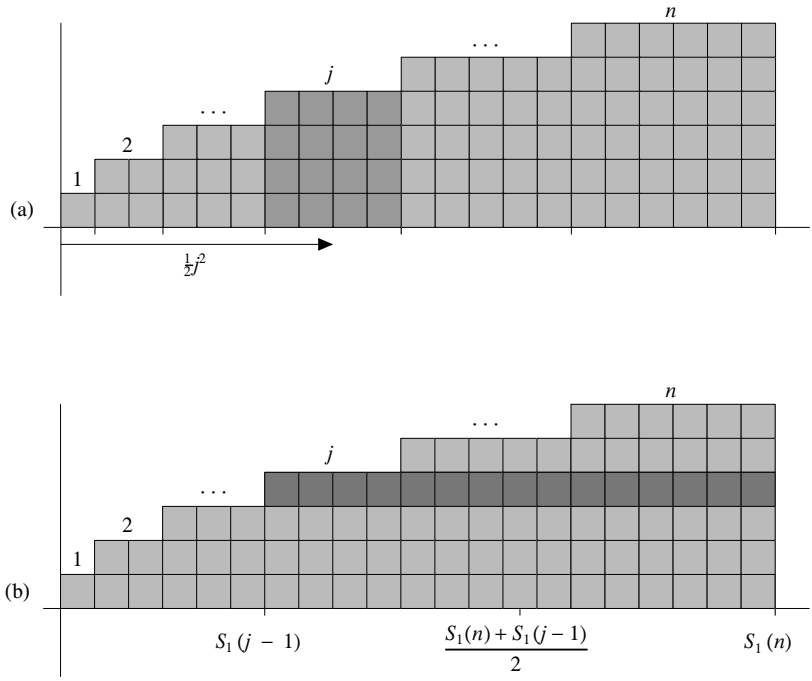


FIGURE 4

The jth square in Figure 4(a) has area j^2 and is centred at $x = \frac{1}{2}j^2$. Therefore the x-mean of this configuration is located at

$$\bar{x} = \frac{\sum_{j=1}^{n} j^2 \cdot \frac{j^2}{2}}{S_2(n)} = \frac{S_4(n)}{2S_2(n)}. \tag{10}$$

On the other hand, the jth row in Figure 4(b) has area $S_1(n) - S_1(j-1)$ and is balanced at its midpoint, $\frac{1}{2}\big(S_1(n) + S_1(j-1)\big)$. Therefore the x-mean of this configuration is also given by

$$\begin{aligned}\bar{x} &= \frac{\sum_{j=1}^{n} \frac{1}{2}\left(S_1(n) - S_1(j-1)\right)\left(S_1(n) + S_1(j-1)\right)}{S_2(n)} \\ &= \frac{\sum_{j=1}^{n} \left(S_1^2(n) - S_1^2(j-1)\right)}{2S_2(n)}.\end{aligned} \tag{11}$$

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Equating (10) and (11) gives

$$\begin{aligned} S_4(n) &= \sum_{j=1}^{n} \left(S_1^2(n) - S_1^2(j-1)\right) \\ &= nS_1^2(n) - \sum_{j=1}^{n} S_1^2(j-1) \\ &= nS_1^2(n) - \sum_{j=1}^{n} \frac{j^2(j-1)^2}{4} \\ &= nS_1^2(n) - \frac{1}{4}\sum_{j=1}^{n} \left(j^4 - 2j^3 + j^2\right) \\ &= nS_1^2(n) - \frac{1}{4}S_4(n) + \frac{1}{2}S_3(n) - \frac{1}{4}S_2(n). \end{aligned}$$

Therefore

$$\frac{5}{4}S_4(n) = nS_1^2(n) + \frac{1}{2}S_3(n) - \frac{1}{4}S_2(n).$$

This implies that

$$\begin{aligned} S_4(n) &= \frac{4}{5}nS_1^2(n) + \frac{2}{5}S_3(n) - \frac{1}{5}S_2(n) \\ &= \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right). \end{aligned}$$

6. *The formula for* $S_5(n)$

A physical derivation of the formula for $S_5(n)$ can be found by dissecting a right-angled triangle into trapezia. We will see that the same dissection also gives us another derivation of the formula for $S_3(n)$. However, before considering this, we need a little preliminary theory.

Proposition 1: The x-mean of the right-angled triangle with vertices $(0,\ 0)$, $(a,\ 0)$, $(0,\ b)$ is $\bar{x} = \dfrac{a}{3}$.

This is shown in Figure 5(a). While this can be most efficiently established using calculus, it is traditionally seen to follow from the fact that the centroid of any triangle divides its medians in the ratio 2 : 1 (see for example [3] and [4]). In Section 7 we give an alternative proof.

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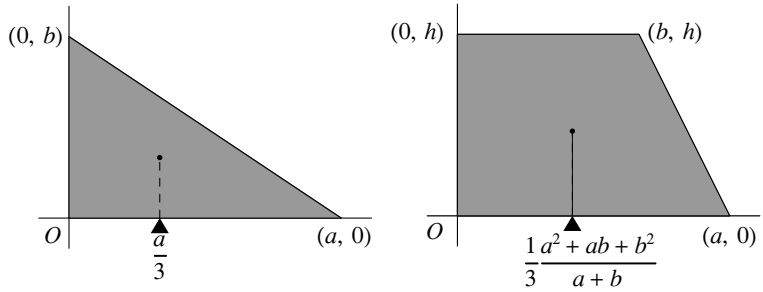


FIGURE 5

Proposition 1 readily yields a proof of Proposition 2. We leave the details to the reader.

Proposition 2: For a right-angled trapezium whose vertices are (0, 0), $(a, 0)$, $(0, h)$ and (b, h), the x-mean is $\bar{x} = \frac{1}{3}\frac{a^2 + ab + b^2}{a + b}$. This is shown in Figure 5(b).

Now consider any finite sequence of positive numbers $(c_j)_{j=1}^n$ whose partial sums $(S_j)_{j=0}^n$ are given by $S_0 = 0$ and $S_j = \sum_{k=1}^j c_k$. We now construct a right-angled triangle of base 1 and altitude S_n as shown in Figure 6 below. We then partition the altitude into segments whose lengths are given by the sequence $(c_j)_{j=1}^n$.

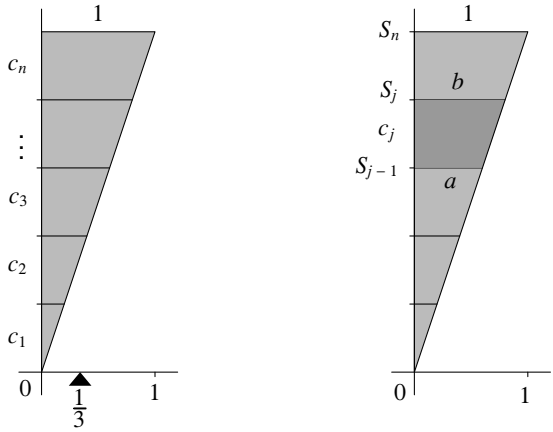


FIGURE 6

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Consider trapezium j indicated in Figure 6(b). Its parallel sides have lengths a and b that can be determined using similar triangles:

$$\frac{a}{S_{j-1}} = \frac{1}{S_n} \Rightarrow a = \frac{S_{j-1}}{S_n},$$
$$\frac{b}{S_j} = \frac{1}{S_n} \Rightarrow b = \frac{S_j}{S_n}.$$

Therefore the area of trapezium j is

$$A_j = \frac{c_j}{2}(a + b) = \frac{c_j}{2S_n}\left(S_{j-1} + S_j\right).$$

Then, as the total area of the triangle is equal to the sum of its trapezoidal parts, we obtain

$$\sum_{j=1}^{n} A_j = \frac{1}{2}S_n$$
$$\Rightarrow \sum_{j=1}^{n} \frac{c_j}{2S_n}\left(S_{j-1} + S_j\right) = \frac{1}{2}S_n.$$

From this we can deduce Proposition 3.

Proposition 3: For any sequence of positive real numbers $(c_j)_{j=1}^n$ whose partial sums are $(S_j)_{j=0}^n$ we have

$$\sum_{j=1}^{n} c_j\left(S_{j-1} + S_j\right) = S_n^2. \tag{12}$$

We have given Proposition 3 as it yields a second simple derivation of the formula for $S_3(n)$.

Example 6.1: If we let $c_j = j$ then $S_j = S_1(j) = \frac{1}{2}j(j+1)$ and

$$c_j\left(S_{j-1} + S_j\right) = j\left(\frac{j(j-1)}{2} + \frac{j(j+1)}{2}\right) = j^3.$$

Therefore (12) yields

$$S_3(n) = S_1^2(n).$$

We can deduce a counterpart to (12) by considering the x-mean of the same configuration in Figure 6. This will ultimately enable us to find the formula for $S_5(n)$. In Figure 6(a) we see that the x-mean of the triangle is located at

$$\bar{x} = \frac{1}{3}. \tag{13}$$

On the other hand, the jth trapezium in Figure 6(b) has area $A_j = \frac{c_j}{2S_n}\left(S_{j-1} + S_j\right)$. Using Proposition 2 its balancing point is given by

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$$\bar{x}_j = \frac{1}{3}\frac{a^2 + ab + b^2}{a + b}$$
$$= \frac{1}{3S_n}\frac{S_{j-1}^2 + S_{j-1}S_j + S_j^2}{S_{j-1} + S_j}.$$

Therefore the x-mean of the triangle can also be found as

$$\bar{x} = \frac{1}{A}\sum_{j=1}^{n} A_j \bar{x}_j$$
$$= \frac{1}{\frac{1}{2}S_n}\sum_{j=1}^{n}\left(\frac{c_j(S_{j-1} + S_j)}{2S_n}\frac{1}{3S_n}\frac{S_{j-1}^2 + S_{j-1}S_j + S_j^2}{S_{j-1} + S_j}\right)$$
$$= \frac{1}{3S_n^3}\sum_{j=1}^{n} c_j\left(S_{j-1}^2 + S_{j-1}S_j + S_j^2\right). \tag{14}$$

By comparing (12) and (13) we obtain Proposition 4.

Proposition 4: For any sequence of positive real numbers $(c_j)_{j=1}^n$ whose partial sums are $(S_j)_{j=0}^n$ we have

$$\sum_{j=1}^{n} c_j\left(S_{j-1}^2 + S_{j-1}S_j + S_j^2\right) = S_n^3. \tag{15}$$

We are now in a position to establish the formula for $S_5(n)$.

Example 6.2: If we let $c_j = j$ then $S_j = S_1(j) = \dfrac{j(j+1)}{2}$ and

$$c_j\left(S_{j-1}^2 + S_{j-1}S_j + S_j^2\right)$$
$$= j\left(\tfrac{1}{4}j^2(j-1)^2 + \tfrac{1}{4}j^2(j-1)(j+1) + \tfrac{1}{4}j^2(j+1)^2\right)$$
$$= \tfrac{3}{4}j^5 + \tfrac{1}{4}j^3.$$

Therefore (11) gives

$$\sum_{j=1}^{n}\left(\frac{3}{4}j^5 + \frac{1}{4}j^3\right) = S_1^3(n)$$
$$\Rightarrow \frac{3}{4}S_5(n) + \frac{1}{4}S_3(n) = S_1^3(n).$$

It follows that

$$S_5(n) = \frac{4}{3}S_1^3(n) - \frac{1}{3}S_3(n) = \frac{1}{12}n^2(1+n)^2\left(2n^2 + 2n - 1\right).$$

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7. *The balancing point of a right-angled triangle*

We now complete our proof of Proposition 1. This proof has been written intentionally to emulate the method fully expounded by Archimedes in [3]. The proof can be made more rigorous, but this comes at the expense of the crucial idea.

Proof: Take right-angled ΔABC in Figure 7 and suppose that $CA = a$. Let M be the midpoint of BC. Now consider $\Delta AB'C'$ such that M is also the midpoint of $C'B'$ and $CBB'X'$ is a rectangle. Suppose $C'C = x$ is chosen so that $\Delta A'B'C$ is isosceles. This occurs if, and only if,

$$2x = a - x \Leftrightarrow x = \frac{a}{3}.$$

Since $\Delta AB'C'$ is isosceles, its centroid is located along the line $X'B'$ where $CX' = \frac{a}{3}$. We now simply have to show that the centroid of ΔABC coincides with that of $\Delta A'B'C$. To prove this, take any segment GH in ΔABC parallel to BC and a corresponding segment EF in $\Delta AB'C'$ parallel to $B'C'$ and possessing the same midpoint, as shown in Figure 7. Since segments GH and EF intersect at their midpoints, their balancing points coincide. Then as ΔABC and $\Delta A'B'C$ are, respectively, the sum of all such segments, they also have the same balancing point, and this is located at the desired point.

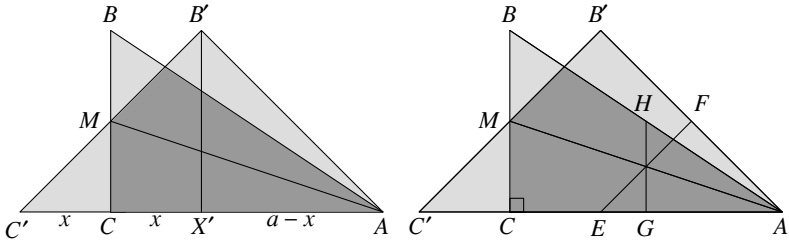


FIGURE 7: ΔABC and $\Delta AB'C'$ have the same centroid

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DAVID TREEBY
School of Mathematical Sciences, Monash University, Clayton, Australia 3800
e-mail: *david.treeby@monash.edu*

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