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Rokhlin dimension for actions of residually finite groups

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Abstract. We introduce the concept of Rokhlin dimension for actions of residually finite groups on C*-algebras, extending previous such notions for actions of finite groups and the integers by Hirshberg, Winter and the third author. We are able to extend most of their results to a much larger class of groups: those admitting box spaces of finite asymptotic dimension. This latter condition is a refinement of finite asymptotic dimension and has not previously been considered. In a detailed study we show that finitely generated, virtually nilpotent groups have box spaces with finite asymptotic dimension, providing a large class of examples. We show that actions with finite Rokhlin dimension by groups with finite-dimensional box spaces preserve the property of having finite nuclear dimension when passing to the crossed product C*-algebra. We then establish a relation between Rokhlin dimension of residually finite groups acting on compact metric spaces and amenability dimension of the action in the sense of Guentner, Willett and Yu. We show that for free actions of infinite, finitely generated, nilpotent groups on finite-dimensional spaces, both these dimensional values are finite. In particular, the associated transformation group C*-algebras have finite nuclear dimension. This extends an analogous result about $\mathbb{Z}^m$-actions by the first author to a significantly larger class of groups, showing that a large class of crossed products by actions of such groups fall under the remit of the Elliott classification programme. We also provide results concerning the genericity of finite Rokhlin dimension, and permanence properties with respect to the absorption of a strongly self-absorbing C*-algebra.

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1. *Introduction*

Since its very beginning, the theory of operator algebras has been strongly influenced and even motivated by ideas of dynamical nature. The study of C*-dynamical systems, i.e. group actions of locally compact groups on C*-algebras, is interesting in many ways. An ample reason for this is certainly the crossed product construction, which naturally associates a new C*-algebra to a C*-dynamical system, reflecting the structure of the group, the coefficient algebra, and the action. This construction has proved to be a virtually inexhaustible source of interesting examples of C*-algebras. As such, it is not surprising that the crossed product construction is harder to understand than most other standard constructions to create new C*-algebras. Moreover, crossed products are often simple C*-algebras of importance in the Elliott classification programme. In view of the astonishing progress of C*-algebra classification over the last few years [**13**, **14**, **19**, **74**], which is largely driven by the discovery of various regularity properties (see, for example, [**5**, **37**, **49**, **51**, **60**, **66**, **67**, **76**, **82**–**85**]), a natural and important question appears, which can be regarded as the main motivation of this paper.

Question. Given a C*-dynamical system $\alpha : G \curvearrowright A$, under what conditions on α does regularity pass from A to $A \rtimes_\alpha G$?

Of course, the vague term ‘regularity’ leaves some room for interpretation, but this question is interesting for all possible versions of regularity. In this paper we will mainly focus on the regularity property of having finite nuclear dimension, which is an approximation property stronger than nuclearity. We will also have a secondary focus on $\mathcal{D}$-stability in §10, for a chosen strongly self-absorbing C*-algebra $\mathcal{D}$.

General results on nuclear dimension of crossed products are hard to come by. What seems to be required is a certain regularity property of the action. As evidenced by recent developments, Rokhlin-type properties appear to be very good candidates for such a regularity property. The idea of defining a Rokhlin property stems from a basic early result in the dynamics of group actions on measure spaces, the so-called Rokhlin lemma. This result states that a measure-preserving, aperiodic integer action can be approximated by

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cyclic shifts in a suitable sense. A von Neumann theoretic translation of this approximation was used successfully by Connes as a key technique towards classifying automorphisms on the hyperfinite II_1-factor up to outer conjugacy (see [**9**, **10**]). Since then, lots of generalizations and reinterpretations have emerged within the realm of von Neumann algebras. Preliminary variants of the Rokhlin property emerged in works of Herman and Jones [**23**] and Herman and Ocneanu [**24**] for cyclic group actions on uniformly hyperfinite (UHF) algebras. Eventually, Kishimoto was successful in fleshing out the Rokhlin property for integer actions on C*-algebras (see, for instance, [**38**–**42**]). Then Izumi started his pioneering work on finite group actions with the Rokhlin property (see [**30**, **31**]). For a selection of other papers related to Rokhlin-type properties for C*-dynamical systems, see [**6**, **18**, **27**, **43**, **48**, **50**, **52**, **55**, **58**, **59**, **64**, **65**] (this list is not exhaustive).

As powerful as these earlier Rokhlin properties are, most variants share a common disadvantage. As their definition involves Rokhlin towers consisting of projections, they impose strong restrictions on the C*-algebra in question, ruling out important examples such as the Jiang–Su algebra $\mathcal{Z}$ or any other sufficiently projectionless C*-algebra. To circumvent this problem, Hirshberg, Winter and the third author introduced the concept of Rokhlin dimension for actions of finite groups and the integers in [**28**]. See also [**71**] for an extension to $\mathbb{Z}^m$-actions. The concept of Rokhlin dimension yields a generalization of many of the commonly used Rokhlin-type properties, motivated by the idea of covering dimension. Instead of requiring one (multi-)tower consisting of projections to reflect a shift-type behaviour in the given dynamical system, one allows boundedly many such towers consisting of positive elements, the bound defining the dimensional value. It turns out that this generalization provides a much more flexible concept, while still being strong enough to be compatible with the regularity property of having finite nuclear dimension. For a selection of other papers that, at least in large part, emerged out of the ideas within [**28**], see [**2**, **7**, **16**, **17**, **25**, **26**, **29**, **44**, **45**, **71**].

The main results in [**28**] are a bound on the nuclear dimension of crossed products by single automorphisms (i.e. $\mathbb{Z}$-actions) of finite Rokhlin dimension, a bound for the Rokhlin dimension for minimal actions on finite-dimensional compact metric spaces, as well as a permanence result for $\mathcal{Z}$-stability and a genericity result for finiteness of Rokhlin dimension in the set of all actions on a given algebra. The first two results were generalized by the first author to $\mathbb{Z}^m$-actions in [**71**]; he also gave a better framework for establishing a bound for the Rokhlin dimension on compact metric spaces requiring only freeness of the action. This used some crucial topological arguments inspired by [**22**, **46**], the result of which can be regarded as a topological version of the classical Rokhlin lemma. Combined with the nuclear dimension bound, it shows that C*-algebras associated to such free and minimal dynamical systems have finite nuclear dimension and thus fall within the scope of Elliott's classification programme. Such nuclear dimension results were first obtained by Toms and Winter in [**78**] by different methods.

The main purpose of this paper is to generalize the concept of Rokhlin dimension to actions of all countable, discrete and residually finite groups and to extend the results from [**28**] and [**71**] to this setting. Moreover, we enlarge the class of C*-dynamical systems under consideration to non-unital C*-algebras, and cocycle actions instead of ordinary actions.

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Definition A. Let A be a separable C*-algebra, G a countable, residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. The (full) Rokhlin dimension of α is defined as the infimum of natural numbers d such that for any finite-index subgroup H of G, there exist equivariant completely positive contractive (c.p.c.) order-zero maps

$$\varphi_l : (\mathcal{C}(G/H),\, G\text{-shift}) \longrightarrow (F_\infty(A),\, \tilde{\alpha}_\infty) \quad \text{for } l = 0, \ldots, d,$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

Here the left-hand side involves the canonical action of G on the algebra of functions over the finite quotient G/H, while on the right-hand side, $F_\infty(A)$ is Kirchberg's central sequence algebra and $\tilde{\alpha}_\infty$ is the action on $F_\infty(A)$ naturally induced from α (see Definition 5.8). The definition can be refined by considering only a sufficiently separating sequence σ of finite-index subgroups, and the separability of A is not an essential prerequisite for the definition (see Definition 5.4). In fact, the whole setting can be generalized to include also uncountable groups. We also give an equivalent characterization involving approximate towers of positive elements on the algebraic level, in the spirit of [**28**] (see Proposition 5.5).

The benefit of this generalization is threefold: it not only allows us to study a wider class of examples, but also unifies all the arguments made (so far separately) for finite groups and the integers, and provides a clearer conceptual picture for how the Rokhlin dimension relates to C*-algebraic regularity properties such as the nuclear dimension. The crucial insight is that the applications of the Rokhlin dimension require a complementary ingredient of coarse geometric nature, namely the box space $\square_\sigma G$ (in the sense of Roe; see [**61**, 11.24]) associated to a residually finite group G and some chosen sequence of finite-index subgroups $\sigma = (G_n)_n$. It turns out that a group yields an interesting Rokhlin dimension theory if it admits a box space with finite asymptotic dimension. This insight culminates in Theorem 6.2, where we establish the following upper bound for the nuclear dimension of (twisted) crossed products:

THEOREM B. *For every* C*-*algebra A and every cocycle action $(\alpha, w) : G \curvearrowright A$, we have the upper bound*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\square_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

In the above inequality, the dimensions appearing on the right-hand side are the asymptotic dimension of the box space for σ, the Rokhlin dimension of α along σ, and the nuclear dimension of the coefficient algebra A, respectively. For notational convenience, each dimension in the formula is increased by 1, as denoted by the superscripts. By introducing box spaces in this context, this bound is improved and gives a coarse geometric interpretation of the ad-hoc constants appearing in previous such estimates.

The asymptotic dimension and the box space construction have individually played significant roles in coarse geometry and have had remarkable applications to the Baum–Connes conjecture; however, to the best of the authors' knowledge, this is the first time the effects of combining these two notions have come under careful investigation. We conduct such an investigation from a general point of view in §3, even before we can generalize the Rokhlin dimension theory. It turns out that some care has to be taken in the choice of the

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finite-index subgroups featuring in the box space but that good choices always exist (see Definition 3.5 and Corollary 3.10). We also give a key characterization of the asymptotic dimension in terms of partitions of unity in Lemma 3.13 that is geared towards our main application in Theorem 6.2.

A central question in §§3 and 4 is what groups admit box spaces with finite asymptotic dimension, a condition that is vital for the applications of our Rokhlin dimension theory. This is certainly satisfied for finite groups and the integers, and was already (implicitly) crucial in the existing theory for these groups. On the other hand, it follows from classical results in coarse geometry that such a condition implies the amenability of the group. We devote the entire §4 to proving the following result (see Theorem 4.10).

THEOREM C. *Every finitely generated, virtually nilpotent group admits a box space with finite asymptotic dimension.*

This provides a reasonably large class of groups to which Theorems B and F can be applied. Our proof is done by developing a criterion for a discrete subgroup of a locally compact group to have finite-dimensional box spaces that involves the existence of a sequence of expanding automorphisms of the group. For nilpotent groups, we then apply this to known embeddings into certain Lie groups. For further examples of groups with finite-dimensional box spaces, see a preprint by Finn-Sell and the second author [**15**], which was inspired by and improves the results on box spaces in this paper. Among other things, it is shown there that for every virtually polycyclic group, there exists some box space whose asymptotic dimension is at most the Hirsch length of the group. Note that in a recent preprint [**11**] Delabie and Tointon strengthen this result for virtually nilpotent groups by showing that for such groups the asymptotic dimension is equal to the Hirsch length of the group, independently of the choice of the box space. This also strengthens our Theorem C.

There are further interesting connections of our work to ongoing developments in dimension theory for topological dynamics. Recently, Guentner, Willett and Yu have invented various stronger versions of amenability of discrete group actions on compact metric spaces (see [**20**, **21**]). In particular, they introduced for such actions a notion that may be termed amenability dimension†. This is a dynamical analogue of asymptotic dimension, the finiteness of which may be regarded as a strong form of amenability of the action. The original motivation to introduce amenability dimension was to facilitate the computation of K-theoretic invariants, and in particular to prove the Farrell–Jones conjecture or the Baum–Connes conjecture in certain cases. In fact, their work is in part inspired by the pioneering work of Bartels, Lück and Reich on the Farrell–Jones conjecture for hyperbolic groups in [**4**]. Somewhat surprisingly, there is a theorem among Guentner, Willett and Yu's results that showcases the compatibility of amenability dimension with nuclear dimension, in a similar fashion as for Rokhlin dimension. This raises the question to what extent amenability dimension and Rokhlin dimension are related concepts. After

† Our choice of this terminology 'amenability dimension' is based on an early draft of Guentner, Willett and Yu, where they used the term 'd-amenable' to denote what we call having amenability dimension at most d. Since then, they have generalized the notion to what they call 'd-BLR', which is closely related to the dynamic asymptotic dimension defined in the same paper (see [**21**, Remark 4.14]).

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examining amenability dimension in rather C*-algebraic terms, we show that for groups with finite-dimensional box spaces, the two dimensional invariants are of the same order of magnitude (see Theorem 7.2).

THEOREM D. *Let A be a C*-algebra, G a residually finite group admitting a box space with finite asymptotic dimension, and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Then the Rokhlin dimension of (α, w) is finite if and only if its amenability dimension is finite.*

As a major application of this result together with techniques developed in [**71**] and some geometric group theory, we extend the finiteness results for Rokhlin dimension to free actions of infinite, finitely generated, nilpotent groups on finite-dimensional compact metric spaces (see Corollary 8.6). When combined with the astounding recent progress in the Elliott programme by many hands [**13**, **14**, **19**, **74**], this shows that a large class of simple transformation group C*-algebras associated to actions of nilpotent groups is classifiable (see Theorem 8.9).

THEOREM E. *Let G be a finitely generated, infinite, nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free and minimal action. Then the transformation group C*-algebra $C(X) \rtimes_\alpha G$ is a simple approximately subhomogeneous algebra of topological dimension at most* 2.

This considerably extends results which previously have only been known for $\mathbb{Z}$ and $\mathbb{Z}^m$-actions. We point out that some time after the initial preprint version of this paper was published, Bartels [**3**] independently proved a similar finiteness result for more general actions of virtually nilpotent groups, by using slightly different methods. As a consequence, Theorem E can be extended from nilpotent groups to virtually nilpotent groups (cf. Theorem 9.8).

In the case of non-commutative coefficient algebras, we also explore generalizations of some other results from [**28**]. Firstly, we investigate the notion of Rokhlin dimension with commuting towers and its relation to another type of important C*-algebraic regularity property: *$\mathcal{D}$-stability* for a strongly self-absorbing C*-algebra $\mathcal{D}$. We show that for groups with finite-dimensional box spaces, finite Rokhlin dimension with commuting towers allows $\mathcal{D}$-stability to pass from the coefficient C*-algebra to the (twisted) crossed product (see Theorem 10.6).

THEOREM F. *Let A be a separable, $\mathcal{D}$-stable C*-algebra, let G be a countable, residually finite group admitting a box space with finite asymptotic dimension, and let $(\alpha, w) : G \curvearrowright A$ be a cocycle action having finite Rokhlin dimension with commuting towers. Then the twisted crossed product $A \rtimes_{\alpha,w} G$ is $\mathcal{D}$-stable.*

Secondly, we examine the genericity of cocycle actions with small Rokhlin dimension on $\mathcal{Z}$-stable or UHF-stable C*-algebras (see Theorems 11.10 and 11.14).

THEOREM G. *Let G be a discrete, finitely generated and residually finite group and let A be a separable C*-algebra. Then among all cocycle actions $(\alpha, w) : G \curvearrowright A$,*

(1) *those with Rokhlin dimension* 0 *are generic if A tensorially absorbs the universal UHF algebra;*

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(2) *those with Rokhlin dimension at most* 1 *are generic if A tensorially absorbs the Jiang–Su algebra.*

This paper is organized as follows. In §2 we fix notation and review the existing Rokhlin dimension theories for finite groups and $\mathbb{Z}^m$. Section 3 develops the theory for the asymptotic dimension of box spaces from a general point of view, while §4 focuses on its finiteness for finitely generated virtually nilpotent groups. Section 5 discusses the definitions and basic properties of the Rokhlin dimension for residually finite groups. Section 6 elaborates its application to the nuclear dimension of crossed products. Sections 7 and 9 study the amenability dimension and relate it to the Rokhlin dimension, while §8 applies our theory to free minimal actions of nilpotent groups on finite-dimensional metric spaces. Section 10 investigates Rokhlin dimension with commuting towers and its application to $\mathcal{D}$-stability. Section 11 establishes the genericity of finite Rokhlin dimension for actions on $\mathcal{D}$-stable algebras.

2. *Preliminaries*

Notation 2.1. Unless specified otherwise, we will stick to the following notation throughout the paper.

- A denotes a C*-algebra.
- α denotes a group action on a C*-algebra or a compact metric space.
- G denotes a countable, discrete group.
- If M is some set and $F \subset M$ is a finite subset, we write $F \Subset M$.
- For $\varepsilon > 0$ and a, b in some normed space, we write $a =_\varepsilon b$ as a shortcut for $\|a - b\| \leq \varepsilon$.
- Assume that 'dim' is one of the notions of dimension that appear in this paper, and X an object on which this dimension can be evaluated. Then we sometimes use the convenient notation $\dim^{+1}(X) = 1 + \dim(X)$.
- The abreviations c.p. and c.p.c. stand for completely positive and completely positive contractive, respectively.

First we recall the existing notion of Rokhlin dimension for finite group actions and $\mathbb{Z}^m$-actions on unital C*-algebras.

Definition 2.2. (Cf. [**28**, 1.1]) Let A be a unital C*-algebra, G a finite group and let $\alpha : G \curvearrowright A$ be an action. We say that α has Rokhlin dimension d, and write $\dim_{\mathrm{Rok}}(\alpha) = d$, if d is the smallest natural number such that, for all $F \Subset A$, $\varepsilon > 0$, $n \in \mathbb{N}$, there exist positive contractions $(f_g^{(l)})_{g\in G}^{l=0,\ldots,d}$ in A satisfying the following properties:

(a) $\mathbf{1}_A =_\varepsilon \sum_{l=0}^{d} \sum_{g\in G} f_g^{(l)}$;

(b) $\|f_g^{(l)} f_h^{(l)}\| \leq \varepsilon$ for all $l = 0, \ldots, d$ and $g \neq h$ in G;

(c) $\alpha_g(f_h^{(l)}) =_\varepsilon f_{gh}^{(l)}$ for all $l = 0, \ldots, d$ and $g, h \in G$;

(d) $\|[f_g^{(l)}, a]\| \leq \varepsilon$ for all $l = 0, \ldots, d$, $g \in G$ and $a \in F$.

If there is no such d, we write $\dim_{\mathrm{Rok}}(\alpha) = \infty$.

Notation. When considering actions of $\mathbb{Z}^m$ in this section, we denote

$$B_n = \{0, \ldots, n-1\}^m \subset \mathbb{Z}^m.$$

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Definition 2.3. (Cf. [**71**, 1.6]) Let A be a unital C*-algebra, and $\alpha : \mathbb{Z}^m \curvearrowright A$ an action. We say that α has Rokhlin dimension d, and write $\dim_{\mathrm{Rok}}(\alpha) = d$, if d is the smallest natural number such that, for all $F \Subset A$, $\varepsilon > 0$, $n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0,\ldots,d}$ in A satisfying the following properties:

(a) $\mathbf{1}_A \approx_\varepsilon \sum_{l=0}^{d} \sum_{v \in B_n} f_v^{(l)}$;

(b) $\| f_v^{(l)} f_{v'}^{(l)} \| \leq \varepsilon$ for all $l = 0, \ldots, d$ and $v \neq v'$ in B_n;

(c) $\alpha_v(f_w^{(l)}) \approx_\varepsilon f_{v+w}^{(l)}$ for all $l = 0, \ldots, d$ and $v, w \in B_n$ (note that $f_v^{(l)}$ denotes $f^{(l)}_{(v \mod n\mathbb{Z}^m)}$, whenever $v \notin B_n$);

(d) $\| [f_v^{(l)}, a] \| \leq \varepsilon$ for all $l = 0, \ldots, d$, $v \in B_n$ and $a \in F$.

If there is no such d, we write $\dim_{\mathrm{Rok}}(\alpha) = \infty$.

Using central sequences, there is an elegant reformulation of these definitions. We will omit the easy proofs for the sake of brevity†. Let ω be a free filter on $\mathbb{N}$. As usual, let $A_\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(\mathbb{N}, A)$ denote the ω-sequence algebra for a C*-algebra A. A itself embeds into A_ω as (representatives of) constant sequences. The relative commutant of A inside A_ω is called the central sequence algebra of A and is denoted by $A_\omega \cap A'$. If α is a group action of a discrete group on A, componentwise application of α yields well-defined actions on both A_ω and $A_\omega \cap A'$, both of which we will denote by α_ω. In what follows, we will mostly work with ω being the filter of all cofinite subsets in $\mathbb{N}$, and in this case we will insert the symbol ∞ in this notation.

LEMMA 2.4. *Let A be a separable, unital* C*-*algebra.*

(1) *Let $\alpha : G \curvearrowright A$ be an action of a finite group. Let $d \in \mathbb{N}$ be a natural number. Then* $\dim_{\mathrm{Rok}}(\alpha) \leq d$ *if and only if there are equivariant c.p.c. order-zero maps*

$$\varphi_l : (C(G), G\text{-shift}) \longrightarrow (A_\infty \cap A', \alpha_\infty) \quad (l = 0, \ldots, d)$$

such that $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

(2) *Let $\alpha : \mathbb{Z}^m \curvearrowright A$ be an action. Let $d \in \mathbb{N}$ be a natural number. Then* $\dim_{\mathrm{Rok}}(\alpha) \leq d$ *if and only if, for all $n \in \mathbb{N}$, there are equivariant c.p.c. order-zero maps*

$$\varphi_l : (C(\mathbb{Z}^m/n\mathbb{Z}^m), \mathbb{Z}^m\text{-shift}) \longrightarrow (A_\infty \cap A', \alpha_\infty) \quad (l = 0, \ldots, d)$$

such that $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

Remark 2.5. The perspective in Lemma 2.4 about the existing notions of finite Rokhlin dimension makes the similarities a lot more apparent. Regarding (2), it is not immediately clear why one would have to use the sequence of subgroups $n\mathbb{Z}^m$, instead of any other suitable separating sequence of finite-index subgroups. When we define Rokhlin dimension for residually finite groups in the upcoming sections, we will introduce a little more flexibility concerning this point.

We note that for integer actions, there were originally several versions of Rokhlin dimension (see [**28**, 2.3]). The above definition for $\mathbb{Z}^m$-actions [**71**, 1.6] extends the single tower version for $\mathbb{Z}$-actions (see [**28**, 2.3c), 2.9]).

† However, a more general statement is proved in detail in Proposition 5.5.

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As we will also treat cocycle actions in this paper, we remind the reader of some basic definitions.

Definition 2.6. (Cf. [**8**, 2.1] or [**56**, 2.1]) Let A be a C*-algebra and G a discrete group. A cocycle action $(\alpha, w) : G \curvearrowright A$ is a map $\alpha : G \to \mathrm{Aut}(A)$, $g \mapsto \alpha_g$ together with a map $w : G \times G \to \mathcal{U}(\mathcal{M}(A))$ satisfying

$$\alpha_r \circ \alpha_s = \mathrm{Ad}(w(r, s)) \circ \alpha_{rs}$$

and

$$\alpha_r(w(s, t)) \cdot w(r, st) = w(r, s) \cdot w(rs, t)$$

for all $r, s, t \in G$.

One may always assume $\alpha_1 = \mathrm{id}$ and $w(1, t) = w(t, 1) = \mathbf{1}$ for all $t \in G$. If $w = \mathbf{1}$ is trivial, then this just recovers the definition of an ordinary action $\alpha : G \curvearrowright A$.

Definition 2.7. (Cf. [**56**]) Let A be a C*-algebra and G a discrete group. Let $(\alpha, w) : G \curvearrowright A$ be a cocycle action. Then one defines the maximal twisted crossed product $A \rtimes_{\alpha,w} G$ to be the universal C*-algebra with the property that it contains a copy of A, and there is a map $G \to \mathcal{U}(\mathcal{M}(A \rtimes_{\alpha,w} G))$, $g \mapsto u_g$ satisfying

$$u_g a u_g^* = \alpha_g(a) \quad \text{and} \quad u_g u_h = w(g, h) \cdot u_{gh}$$

for all $a \in A$ and $g, h \in G$.

The reduced twisted crossed product $A \rtimes_{r,\alpha,w} G$ is defined as the C*-algebra inside $\mathcal{B}(\ell^2(G) \otimes H)$ generated by the following two representations. Let $A \hookrightarrow \mathcal{B}(H)$ be faithfully represented on a Hilbert space. Then consider

$$\iota : A \to \mathcal{B}(\ell^2(G) \otimes H) \quad \text{via } \iota(a)(\delta_t \otimes \xi) = \delta_t \otimes \alpha_{t^{-1}}(a)\xi$$

and

$$\lambda : G \to \mathcal{U}(\mathcal{B}(\ell^2(G) \otimes H)) \quad \text{given by } \lambda_s(\delta_t \otimes \xi) = \delta_{st} \otimes w(t^{-1}s^{-1}, s)\xi.$$

If G is amenable, then the maximal and reduced twisted crossed products always coincide.

Remark 2.8. In a more symbolic notation using A-valued infinite matrices (which we will use later) one can also write

$$au_s = \sum_{t \in G} e_{t,s^{-1}t} \otimes \alpha_{t^{-1}}(a) w(t^{-1}, s) \in A \rtimes_{r,\alpha,w} G$$

for all $a \in A$, $s \in G$.

3. *Box spaces and asymptotic dimension*

Definition 3.1. Let G be a countable, discrete group. Let $G_n \leq G$ be a decreasing sequence of subgroups with finite index, i.e. $[G : G_n] < \infty$ and $G_{n+1} \leq G_n$ for all $n \in \mathbb{N}$. We say that the sequence $(G_n)_n$ is separating if $\bigcap_{n \in \mathbb{N}} G_n = \{1_G\}$. G is (by definition) residually finite if and only if such a sequence exists.

Remark 3.2. A well-known fact in group theory is that for any finite-index subgroup $H \leq G$, there is a normal finite-index subgroup $N \leq H$ such that $[G : N] \leq [G : H]!$. It follows that a group G is residually finite if and only if it has a separating decreasing sequence of normal finite-index subgroups.

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Before we come to the next definition, we need to recall some facts. If (X, d) is a metric space and we have a discrete group G acting on X from the right, then the orbit space X/G is defined as the quotient of X by identifying any two points in the same orbit. Let $\pi : X \to X/G$ be the corresponding quotient map. One can define a quotient pseudometric by

$$\pi_*(d)(\pi(x_1), \pi(x_2)) = \inf\{d(x_1 \cdot g_1, x_2 \cdot g_2) \mid g_1, g_2 \in G\}.$$

In good cases, this will be a metric inducing the quotient topology, such as for geometric actions, i.e. isometric, properly discontinuous and cobounded actions (see [**12**, 3.20]).

Recall also the definition of a coarse disjoint union of finite metric spaces [**53**, 1.4.12]. Moreover, recall that a metric on a discrete set is called proper, if balls are finite sets.

Definition 3.3. (Following [**61**, 11.24] and [**35**]) Let G be a countable, discrete, residually finite group and let $\sigma = (G_n)_n$ be a decreasing sequence of subgroups of G. Let us equip G with a proper, right-invariant metric d. For each $n \in \mathbb{N}$, let $\pi_n : G \to G/G_n$ be the quotient map and $(\pi_n)_*(d)$ the quotient metric on G/G_n. Then the box space $\square_\sigma G$ associated to σ is defined as the coarse disjoint union of the sequence of finite metric spaces $(G/G_n, (\pi_n)_*(d))$. More precisely, it is a coarse space whose underlying set is the disjoint union $\bigsqcup_{n\in\mathbb{N}} G/G_n$, and whose coarse structure is determined by a metric d^B such that for all n, the metric d^B restricts to $(\pi_n)_*(d)$ on the subset G/G_n, and such that we have

$$\mathrm{dist}_{d^B}(G/G_n, G/G_m) \to \infty$$

as $n \neq m$ and $n + m \to \infty$.

Remark 3.4. Box spaces associated to a family of normal finite-index subgroups have seen a fair amount of study. It is known to coarse geometers that any countable, discrete group admits a proper right-invariant metric, and that the choice of the proper right-invariant metric d on G and the metric d^B does not affect the coarse equivalence class of the resulting box space. This justifies the absence of d and d^B in the notation $\square_\sigma G$. For a rigorous and comprehensive proof of this, we refer the reader to [**70**, §1.2, Proposition 1.3.3]. We remark that even though in the statement of [**70**, Proposition 1.3.3] the sequence σ is assumed to be separating and made up of normal subgroups, the actual proof does not make use of these conditions.

Although, for us, σ is not required to consist of normal subgroups, one needs a condition that rules out certain pathological examples where the coarse structure of the box space $\square_\sigma G$ behaves badly with regard to that of G itself.

Definition 3.5. (Cf. [**15**, 3.1]) A decreasing sequence σ of finite-index subgroups of G is called *semi-conjugacy-separating* if for any non-trivial element $g \in G$, there is $H \in \sigma$ such that $H \cap g^G H = \varnothing$, where g^G is the conjugacy class of g.

Notation 3.6. For the sake of brevity, we shall call a semi-conjugacy-separating decreasing sequence of finite-index subgroups of G a *regular residually finite approximation* of G or simply a *regular approximation*.

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LEMMA 3.7. *Let G be a group and σ a decreasing sequence of finite-index subgroups of G. Then the following are equivalent.*

(1) *σ is semi-conjugacy-separating.*
(2) $\bigcap_{H\in\sigma}\bigcup_{k\in G} kHk^{-1} = \{1\}$.
(3) *For any finite subset $F \subset G$, there is $H \in \sigma$ such that the quotient map $G \to G/H$ is injective on $F \cdot k$ for any $k \in G$.*

Proof. (1) $\Leftrightarrow$ (2). Condition (2) can be rewritten as follows: for any non-trivial element $g \in G$, there is $H \in \sigma$ such that for any $k \in G$, $g \notin kHk^{-1}$. But the statement at the end is equivalent to $k^{-1}gk \notin H$, which is in turn equivalent to $k^{-1}gkH \cap H = \varnothing$. Thus the entire statement is equivalent to (1).

(2) $\Rightarrow$ (3). Given any finite subset $F \subset G$ and $g_1, g_2 \in F$ with $g_1 \neq g_2$, then $g_1^{-1}g_2 \neq 1$, so by (2) we can find $H \in \sigma$ such that $g_1^{-1}g_2 \notin kHk^{-1}$ for all $k \in G$. With σ decreasing and F finite, we can find $H \in \sigma$ which works for all pairs of different elements in F simultaneously, thus verifying (3).

(3) $\Rightarrow$ (2). With F and H as in condition (3), it follows that $F^{-1}F \cap kHk^{-1} = \{1\}$ for all $k \in G$, which implies condition (2). □

Remark 3.8. Although we are only interested in decreasing sequences, Definition 3.5 and Lemma 3.7 work equally well for any collection of finite-index subgroups of G that is closed under forming intersections.

After we fix a proper, right-invariant metric on G and replace F by arbitrarily large balls around 1_G, the last condition in the above lemma says that the family of maps $\{G \to G/H\}_{H\in\sigma}$ achieves arbitrarily large injectivity radii. This suggests the importance of this condition regarding the coarse geometric information of the box space and the group. On the other hand, condition (2) directly implies the following corollary.

COROLLARY 3.9. *Every separating family of finite-index normal subgroups is semi-conjugacy-separating.*

Since a group is residually finite if and only it has a separating family of finite-index *normal* subgroups, we also have the following result.

COROLLARY 3.10. *A group is residually finite if and only if it has a semi-conjugacy-separating family of finite-index subgroups.*

The condition of having large injectivity radii is relevant to the coarse geometry of the box spaces, because here it actually implies the generally stronger condition of having large isometry radii, as shown in the next lemma.

LEMMA 3.11. *Let (X, d_X) be a metric space with an isometric right-action of a group H, let (Y, d_Y) be the quotient space by the group action with the induced pseudo-metric, and let $\pi : X \to Y$ be the corresponding projection. Let $x \in X$ and $R > 0$ be such that π is injective when restricted to $B_{3R}(x)$. Then $\pi|_{B_R(x)}$ is an isometry from $B_R(x)$ onto $B_R(\pi(x))$.*

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Proof. For any $x_1, x_2 \in B_R(x)$ and $h_1, h_2 \in H$ with $h_1 \neq h_2$, we have

$$d_X(x_1 \cdot h_1, x_2 \cdot h_1) = d_X(x_1, x_2) \leq 2R,$$

while $d_X(x_1 \cdot h_1, x_2 \cdot h_2) = d_X(x_1, x_2 \cdot h_2 h_1^{-1}) > 2R$ because $x_2 \cdot h_2 h_1^{-1}$, being different from but having the same image under π as x_2, must lie outside $B_{3R}(x)$. It follows that

$$d_Y(\pi(x_1), \pi(x_2)) = \inf_{h_1, h_2 \in H} d_X(x_1 \cdot h_1, x_2 \cdot h_2) = d_X(x_1, x_2).$$

On the other hand, for any $y \in B_R(\pi(x))$, we may pick some $x' \in \pi^{-1}(y)$. As

$$R \geq d_X(\pi(x), y) = \inf_{h, h' \in H} d_X(x \cdot h, x' \cdot h') = \inf_{h, h' \in H} d_X(x, x' \cdot h' h^{-1}),$$

we see that $x' \cdot h' h^{-1} \in \pi^{-1}(y) \cap B_R(x)$ and thus $\pi|_{B_R(x)}$ is onto $B_R(\pi(x))$. This completes the proof. □

Our main use of this lemma will be to perform lifts of uniformly bounded covers, a crucial step in the proof of the upcoming main lemma of this section.

Remark 3.12. It is well known that the box spaces of G, as coarse metric spaces, encode important properties of G. For instance, $\square_\sigma G$ has property A if and only if G is amenable (see [**61**, 11.39] and [**53**, 4.4.6])†. For us, the finiteness of asymptotic dimension is the more relevant condition. Let us briefly recall the relevant definition: the asymptotic dimension of a metric space (X, d) is defined to be the smallest non-negative integer n such that for every $R > 0$ there exists a uniformly bounded cover $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ of X with Lebesgue number R such that each $\mathcal{V}^{(l)}$ consists of pairwise disjoint sets (one may even assume R-disjointness, meaning that any two different sets in $\mathcal{V}^{(l)}$ have distance bounded below by R). Finiteness of asymptotic dimension always implies property A (see [**53**, 4.3.6]); hence in the case of box spaces, it also implies the amenability of G. When a box space has finite asymptotic dimension, one might be tempted to think that this value encodes the complexity of the group in some sense. This is demonstrated in the main technical result of this section.

LEMMA 3.13. *Let G be a residually finite group. Consider the action of G on itself by right multiplication and fix a proper, right-invariant metric. Let $\sigma = (G_n)_n$ be a decreasing sequence of finite-index subgroups of G. Then the following conditions are equivalent for all $s \in \mathbb{N}$.*

(1) *σ is semi-conjugacy-separating and the box space $\square_\sigma G$ has asymptotic dimension at most s.*
(2) *For any $R > 0$, there exist n and a uniformly bounded cover $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ of G with Lebesgue number at least R, such that each $\mathcal{U}^{(l)}$ is G_n-invariant (with regard to right multiplication) and has mutually disjoint members.*
(3) *For every $\varepsilon > 0$ and $M \Subset G$, there exist n and finitely supported functions $\mu^{(l)} : G \to [0, 1]$ for $l = 0, \ldots, s$ satisfying the following properties:*

† Although these authors only studied box spaces associated to families of *normal* subgroups, we remark that the proof of this result requires only a condition of large injectivity radii, as formulated in the previous lemmas, whence the proofs also work for semi-conjugacy-separating families.

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(a) *for every* $l = 0, \dots, s$, *one has*

$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)})h = \emptyset \quad \text{for all } h \in G_n \setminus \{1\};$$

(b) *for every* $g \in G$, *one has*

$$\sum_{l=0}^{s} \sum_{h \in G_n} \mu^{(l)}(gh) = 1;$$

(c) *for every* $l = 0, \dots, s$ *and* $g \in M$, *one has*

$$\|\mu^{(l)} - \mu^{(l)}(g \cdot _)\|_\infty \leq \varepsilon.$$

In the style of [**28**, §2], we will refer to a system $\{\mu^{(l)}\}_{l=0,\dots,s}$ of functions satisfying the properties in part (3) of Lemma 3.13 as a system of decay functions.

Remark. Compared with the definition of asymptotic dimension, condition (2) can be viewed as a periodic variant of asymptotic dimension. On the other hand, condition (3) reflects the standard fact that covers with large Lebesgue number give rise to very flat partitions of unity, but again in a periodic way.

Proof of Lemma 3.13. For the entire proof, let us fix the following notation. For each $n \in \mathbb{N}$, denote by $\pi_n : G \to G/G_n$ the quotient map. Let d be a proper, right-invariant metric on G and let d^B be an induced metric on $\square_\sigma G = \bigsqcup_{n \in \mathbb{N}} G/G_n$ as specified in Definition 3.3.

(1) $\Longrightarrow$ (2). Given $R > 0$, there is a cover $\mathcal{V} = \mathcal{V}^{(0)} \cup \dots \cup \mathcal{V}^{(s)}$ of $\square_\sigma G$ with Lebesgue number at least R, such that the diameters of members of $\mathcal{V}$ are uniformly bounded by some $R' \geq R$ and each $\mathcal{V}^{(l)}$ has mutually disjoint members. Let $\mathcal{V}' = \mathcal{V}'^{(0)} \cup \dots \cup \mathcal{V}'^{(s)}$ denote the induced finite cover on the finite subspace $G/G_n \subset \square_\sigma G$ for some n.

Applying Lemma 3.7(3), we may choose $n \in \mathbb{N}$ such that for any $g \in G$, π_n is injective when restricted to $B_{3R'}(g) = B_{3R'}(1_G)g$. It then follows from Lemma 3.11 that π_n restricts to an isometry between $B_{R'}(g)$ and $B_{R'}(\pi_n(g))$ for any $g \in G$. Applying Lemma 3.7(2), let us also assume that

$$B_{3R'}(1) \cap \bigcup_{k \in G} kG_nk^{-1} = \{1\}.$$

By uniform boundedness, each $V \in \mathcal{V}'$ is contained in $B_{R'}(\pi_n(g))$ for some $g \in G$, and thus we may find $U_V \subset B_{R'}(g)$ that is mapped isometrically onto V under π_n. Hence $\pi_n^{-1}(V)$ can be written as a disjoint union $\bigsqcup_{h \in G_n} U_V \cdot h$. Define

$$\mathcal{U}^{(l)} := \{U_V \cdot h \mid V \in \mathcal{V}'^{(l)},\ h \in G_n\}$$

for all $l = 0, \dots, s$. Then each $\mathcal{U}^{(l)}$ is G_n-invariant (with regard to multiplication from the right), uniformly bounded and has mutually disjoint members. Moreover, we claim that $\mathcal{U} = \mathcal{U}^{(0)} \cup \dots \cup \mathcal{U}^{(s)}$ covers G with Lebesgue number at least R. Indeed, Let $X \subset G$ be a subset with diameter less than R. Since the cover $\mathcal{V}'$ of G/G_n has Lebesgue number at least R and π_n is contractive, there is some $V \in \mathcal{V}'$ such that $\pi_n(X) \subset V$, whence

$$X \subset \pi_n^{-1}(\pi_n(X)) \subset \pi_n^{-1}(V) = \bigsqcup_{h \in G_n} U_V \cdot h.$$

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By construction, the set U_V has diameter at most R, and the subgroup G_n satisfies the equation

$$B_{3R'}(1) \cap \bigcup_{k \in G} kG_nk^{-1} = \{1\}.$$

Thus, for $h_1 \neq h_2$ in G_n and $g_1, g_2 \in U_V$, we have

$$d(g_1h_1, g_2h_2) = d(g_1h_1h_2^{-1}g_1^{-1}, g_2g_1^{-1}) > d(g_1h_1h_2^{-1}g_1^{-1}, 1_G) - 2R \geq R.$$

So we get $\mathrm{dist}_d(U_V \cdot h_1, U_V \cdot h_2) > R$. But this implies that X must be entirely contained in $U_V \cdot h \in \mathcal{U}$ for some $h \in G_n$, which shows the claim.

(2) $\Longrightarrow$ (3). Let $\varepsilon > 0$ and $M \Subset G$ be given. Choose $R > 2(2s+3)/\varepsilon$ big enough so that M is contained in $B_R(1_G)$. By assumption, there exist n and a uniformly bounded cover $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ of G with Lebesgue number at least R, such that each $\mathcal{U}^{(l)}$ is G_n-invariant and has mutually disjoint members. For each $l = 0, \ldots, s$, upon combining several members of $\mathcal{U}^{(l)}$ into one, we may assume that $\mathcal{U}^{(l)}$ is of the form $\{U^{(l)} \cdot h \mid h \in G_n\}$ for some (necessarily finite) set $U^{(l)} \subset G$ such that $U^{(l)} \cap (U^{(l)} \cdot h) = \emptyset$ for any $h \in G_n \setminus \{1\}$. Now define

$$\mu^{(l)} : G \to [0, 1], \quad g \mapsto \frac{\mathrm{dist}_d(g, G \setminus U^{(l)})}{\sum_{V \in \mathcal{U}} \mathrm{dist}_d(g, G \setminus V)}.$$

It is easy to see that $\mathrm{supp}(\mu^{(l)}) = U^{(l)}$ and $\{\mu^{(l)}(_ \cdot h) \mid h \in G_n,\ l = 0, \ldots, s\}$ forms a partition of unity for G. This proves properties (a) and (b). By [**53**, 4.3.5], each $\mu^{(l)}$ is Lipschitz with constant $2(2s+3)/R \leq \varepsilon$, i.e. (c) is satisfied.

(3) $\Longrightarrow$ (1). Given $R > 0$, pick $\varepsilon > 0$ with $\varepsilon < 1/(s+1)$. Choose n and finitely supported functions $\mu^{(l)} : G \to [0, 1]$ for $l = 0, \ldots, s$ satisfying the requirements of (3) with respect to the pair ε and $M = B_R(1_G)$. By choosing n large enough, we may also assume that the distance between any two of the subsets $\bigcup_{k=1}^{n-1} G/G_k$ and G/G_m for $m \geq n$ is at least R in $\square_\sigma G$. Define $U^{(l)} := \mathrm{supp}(\mu^{(l)})$. Then $U^{(l)} \cap (U^{(l)} \cdot h) = \emptyset$ for all $l = 0, \ldots, s$ and $h \in G_n \setminus \{1\}$, and $\{U^{(l)} \cdot h \mid h \in G_n,\ l = 0, \ldots, s\}$ covers G.

We claim that for any $g \in G$, there are $l \in \{0, \ldots, s\}$ and $h \in G_n$ such that $B_R(g) \subset U^{(l)}h$. Indeed, since g is in the support of at most $(s+1)$ members of the partition of unity $\{\mu^{(l)}(_ \cdot h) \mid h \in G_n,\ l = 0, \ldots, s\}$ (at most one for each l), it follows that there exist $h \in G_n$ and $l \in \{0, \ldots, s\}$ such that $\mu^{(l)}(g \cdot h^{-1}) > 1/(s+1)$. Thus for any $g' \in B_R(g)$,

$$\mu^{(l)}(g' \cdot h^{-1}) \geq \mu^{(l)}(g \cdot h^{-1}) - \varepsilon > \frac{1}{s+1} - \frac{1}{s+1} = 0.$$

This shows that $B_R(g) \subset U^{(l)} \cdot h$.

Note that, in particular, this claim implies that π_n is injective when restricted to $B_R(g)$. Since R and g are arbitrary and n only depends on R, it follows from Lemma 3.7(3) that σ is semi-conjugacy-separating.

Now for each $m \geq n$, define the collection $\mathcal{V}_m^{(l)} := \{\pi_m(U^{(l)} \cdot h) \mid h \in G_n\}$, which consists of $[G_n : G_m]$ disjoint subsets of G/G_m. Define a cover $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(s)}$ of $\square_\sigma G$ by

$$\mathcal{V}^{(0)} := \left\{\bigcup_{k=1}^{n-1} G/G_k\right\} \sqcup \bigcup_{m=n}^{\infty} \mathcal{V}_m^{(0)} \quad \text{and} \quad \mathcal{V}^{(l)} := \bigcup_{m=n}^{\infty} \mathcal{V}_m^{(l)} \quad \text{for } l = 1, \ldots, s.$$

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The diameters of its members are bounded by

$$\max\left\{\operatorname{diam}\left(\bigcup_{m=1}^{n-1} G/G_m\right), \operatorname{diam}(U^{(0)}), \ldots, \operatorname{diam}(U^{(s)})\right\}$$

and each $\mathcal{V}^{(l)}$ consists of disjoint sets. Finally, let us show that its Lebesgue number is at least R. Given any non-empty subset $X \subset \square_\sigma G$ with radius at most R, if we fix any $x \in X$, then $X \subset B_R(x)$. By our choice of n, we know that $B_R(x)$ falls entirely in one of the subsets $\bigcup_{k=1}^{n-1} G/G_k$ or G/G_m for $m \geq n$. In the first case, $B_R(x) \subset \bigcup_{k=1}^{n-1} G/G_k \in \mathcal{V}^{(0)}$. In the case where $B_R(x) \subset G/G_m$ for $m \geq n$, choose some $g \in G$ such that $\pi_m(g) = x$. By the claim we proved above, there are $l \in \{0, \ldots, s\}$ and $h \in G_n$ such that $B_R(g) \subset U^{(l)}h$, whence $X \subset B_R(x) \subset \pi_m(U^{(l)} \cdot h) \in \mathcal{V}^{(l)}$. Therefore $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(s)}$ is a uniformly bounded cover of $\square_\sigma G$ with Lebesgue number at least R and each $\mathcal{V}^{(l)}$ made up of disjoint members, which shows that $\operatorname{asdim}(\square_\sigma G) \leq s$. □

COROLLARY 3.14. *Let G be a countable, discrete, residually finite group and $\sigma = (G_n)_n$ a regular approximation. Let G be equipped with some proper, right-invariant metric, which gives it the structure of a coarse metric space. Then* $\operatorname{asdim}(G) \leq \operatorname{asdim}(\square_\sigma G)$.

Proof. This follows immediately from Lemma 3.13(2). □

COROLLARY 3.15. *Let G be a countable, discrete, residually finite group and $\sigma = (G_n)_n$ a regular approximation. Let $H \subset G$ be a subgroup. Then*
(1) *$\kappa = (H \cap G_n)_n$ defines a regular approximation of H;*
(2) *we have* $\operatorname{asdim}(\square_\kappa H) \leq \operatorname{asdim}(\square_\sigma G)$.

Proof. To prove (1), we observe that for any $n \in \mathbb{N}$, we have a natural metric embedding $HG_n/G_n \subset G/G_n$ and a bijection between $H/(H \cap G_n)$ and HG_n/G_n which is not always an isometry (we are indebted to Alain Valette for pointing this out to us). However, $H/(H \cap G_n)$ is clearly finite for all n, $\bigcap_n (H \cap G_n) = \{1_H\}$, and for any $h \in H$ we have $h^H \subset h^G$. Consequently, κ is also semi-conjugacy-separating by definition. Statement (2) follows directly from Lemma 3.13(2) by restricting the obtained cover to H. □

Definition 3.16. Let G be a countable, discrete and residually finite group. The set of all decreasing sequences of finite-index subgroups of G carries a natural preorder: we say that $(H_n)_n$ dominates $(G_n)_n$, and write $(G_n)_n \precsim (H_n)_n$, if for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $H_m \subset G_n$. We say that two sequences are order equivalent, and write $(G_n)_n \sim (H_n)_n$, if $(G_n)_n \precsim (H_n)_n$ and $(H_n)_n \precsim (G_n)_n$.

We denote by $\Lambda(G)$ the set of all regular approximations (cf. Notation 3.6) of G. It follows from Lemma 3.7(2) that $\Lambda(G)$ is upward closed: if $(G_n)_n \in \Lambda(G)$ and $(G_n)_n \precsim (H_n)_n$, then $(H_n)_n \in \Lambda(G)$.

We call a regular approximation $(G_n)_n \in \Lambda(G)$ dominating, if it is dominating with respect to the above order, i.e. $(H_n)_n \precsim (G_n)_n$ for all $(H_n)_n \in \Lambda(G)$.

LEMMA 3.17. *Let G be a countable residually finite group. A regular approximation $(G_n)_n \in \Lambda(G)$ is dominating if and only if for every subgroup $H \subset G$ of finite index, there exists n such that $G_n \subset H$.*

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Proof. It suffices to show that every subgroup $H \subset G$ of finite index is part of a regular approximation. By Corollary 3.9, this is true when H is normal. That we can drop normality follows from the well-known fact that any finite-index subgroup contains a normal finite-index subgroup. Indeed, let $H \subset G$ be a finite-index subgroup and let $F_H \subset G$ be a finite set of representatives for G/H. Then the map $\varphi : F_H \times H \to G$, given by $(x, h) \mapsto xh$, is bijective. Moreover,

$$\bigcap_{g\in G} gHg^{-1} = \bigcap_{x\in F_H, h\in H} (xh)H(xh)^{-1} = \bigcap_{x\in F_H} xHx^{-1}$$

is contained in H and has finite index in G as a finite intersection of finite-index subgroups. □

Example 3.18. Let us consider $G = \mathbb{Z}$ from the point of view of the above definition. Every element in $\Lambda(\mathbb{Z})$ has the form $(k_n\mathbb{Z})_n$ for an increasing sequence of natural numbers with $k_n | k_{n+1}$. One has $(k_n\mathbb{Z})_n \precsim (l_n\mathbb{Z})_n$ if and only if for all n, there exists m such that $k_n | l_m$.

Recall that a supernatural number $\mathfrak{p}$ is a formal product $\mathfrak{p} = \prod p^{n_p}$, where the product runs over all primes and $n_p \in \mathbb{N} \cup \{0, \infty\}$. Another supernatural number $\mathfrak{q} = \prod p^{m_p}$ divides $\mathfrak{p}$ if $m_q \leq n_p$ for all p. We can then establish the correspondence

$$(\Lambda(G)/\sim, \precsim) \cong (\text{supernatural numbers, dividing relation})$$

by sending $(k_n\mathbb{Z})_n$ to the supernatural number $\mathfrak{p}$ defined by

$$p^m | \mathfrak{p} \iff p^m | k_n \quad \text{for some } n,$$

where p is any prime number. Hence, a sequence $(k_n\mathbb{Z})_n \in \Lambda(\mathbb{Z})$ is dominating if and only if every prime power divides some member of the sequence $(k_n)_n$. An immediate example of this is $(n! \cdot \mathbb{Z})_n$.

Example 3.19. We may generalize the last statement of the previous example to any $m \in \mathbb{N}$: the sequence $(n! \cdot \mathbb{Z}^m)_n$ is a dominating regular approximation of $\mathbb{Z}^m$.

Proof. It suffices to show that any finite-index subgroup of $\mathbb{Z}^m$ contains a subgroup of the form $n\mathbb{Z}^m$ for some $n \in \mathbb{N}$. So let $H \subset \mathbb{Z}^m$ be a subgroup with finite index. Then H is free abelian with rank m. Let $v_1, \ldots, v_m$ denote a set of generators for H. If we view H and $\mathbb{Z}^m$ as subsets of $\mathbb{Q}^m$, then we claim that the vectors $v_1, \ldots, v_m$ form a $\mathbb{Q}$-basis. Indeed, let $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$ be given with $0 = \lambda_1 v_1 + \cdots + \lambda_m v_m$. Then we can find a positive integer $k > 0$ such that $k\lambda_1, \ldots, k\lambda_m \in \mathbb{Z}$, and we have $0 = (k\lambda_1)v_1 + \cdots + (k\lambda_m)v_m$. Since H is free abelian of rank m, it follows that $k\lambda_i = 0$ for all i, which implies $\lambda_i = 0$ for all i. This shows that the vectors $v_1, \ldots, v_m \in \mathbb{Q}^m$ are linearly independent in $\mathbb{Q}^m$, and thus form a basis.

Let $e_1, \ldots, e_m$ denote the standard generators of $\mathbb{Z}^m$. We claim that for each $i = 1, \ldots, m$, there is a positive integer k_i such that $k_i e_i \in H$. Indeed, we can find some $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$ with $e_i = \lambda_1 v_1 + \cdots + \lambda_m v_m$. If we choose k_i large enough such that $k_i \lambda_j \in \mathbb{Z}$ for all $j = 1, \ldots, m$, then $k_i e_i = (k_i\lambda_1)v_1 + \cdots + (k_i\lambda_m)v_m \in H$.

After having found these numbers, let n be the least common multiple of $k_1, \ldots, k_m$. Then for all $i = 1, \ldots, m$, it follows that $n/k_i \in \mathbb{Z}$ and thus $ne_i = n/k_i \cdot k_i \cdot e_i \in H$. In particular, we have $n\mathbb{Z}^m \subset H$. □

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In applications, the existence of a dominating regular approximation can be quite useful. In the case of finitely generated groups, this turns out to be automatic.

PROPOSITION 3.20. *Let G be a finitely generated and residually finite group. Then G has a dominating regular approximation consisting of normal subgroups.*

Proof. Let $S \Subset G$ be a finite generating set. Given some finite group E, there are at most as many group homomorphisms from G to E as there are maps from S to E, of which there are $|E|^{|S|}$. Now up to isomorphism, there are only countably many finite groups. Since any normal subgroup of G with finite index arises as a kernel of a homomorphism into a finite group, this implies that G has at most countably many normal subgroups of finite index. Let $\{N_n\}_{n\in\mathbb{N}}$ be an enumeration of this set, and define $(G_n)_n \in \Lambda(G)$ recursively via $G_1 = N_1$ and $G_{n+1} = G_n \cap N_{n+1}$ for all $n \in \mathbb{N}$. By construction, any normal subgroup must contain a member of this sequence as a subgroup, so this sequence is indeed dominating. □

Remark 3.21. Let $(G_n)_n, (H_n)_n \in \Lambda(G)$ be two sequences with $(G_n)_n \precsim (H_n)_n$. Then it follows from Lemma 3.13(2) that $\operatorname{asdim}(\square_{\{H_n\}}G) \leq \operatorname{asdim}(\square_{\{G_n\}}G)$. In other words, the map

$$\Lambda(G) \longrightarrow \mathbb{N}, \quad \sigma \longmapsto \operatorname{asdim}(\square_\sigma G)$$

is order-reversing.

This implies that the values of asymptotic dimension for all box spaces associated to dominating sequences are the same, and they take the lowest value among all possible box spaces associated to decreasing, separating sequences of finite-index subgroups.

If $(H_n)_n \in \Lambda(G)$ is any dominating sequence, we will call the associated box space $\square_{\{H_n\}}G$ a *standard* box space, and will sometimes denote it by $\square_s G$ when G is finitely generated. While there might a priori be some ambiguity, we will exclusively be interested in the asymptotic dimension of these spaces, so the above argument shows that there is no ambiguity concerning the asymptotic dimension of a standard box space.

It turns out that this dimension remains the same when passing to finite-index subgroups, as the following result shows.

PROPOSITION 3.22. *Let G be a countable residually finite group, let $(G_n)_n \in \Lambda(G)$ be a dominating residually finite approximation of G and let $H \subset G$ be a subgroup of finite index in G. Then the induced residually finite approximation $\kappa = (H \cap G_n)_n$ of H is dominating for H and $\operatorname{asdim}(\square_\kappa H) = \operatorname{asdim}(\square_\sigma G)$. In particular, we have $\operatorname{asdim}(\square_s H) = \operatorname{asdim}(\square_s G)$.*

Proof. That the approximation $\kappa = (H \cap G_n)_n$ of H is dominating for H follows from Lemma 3.17. The inequality $\operatorname{asdim}(\square_\kappa H) \leq \operatorname{asdim}(\square_\sigma G)$ follows from Lemmas 3.13 and 3.17; for instance, we can use condition (2) of Lemma 3.13 and restrict the cover $\mathcal{U}$ there to H.

To establish $\operatorname{asdim}(\square_\sigma G) \leq \operatorname{asdim}(\square_\kappa H)$, denote $s = \operatorname{asdim}(\square_\kappa H)$, and let d be a proper, right-invariant metric on G. Its restriction to H obviously yields a proper, right-invariant metric on H. Let $F_H \subset G$ be a finite set of representatives for G/H, i.e. the map $F_H \times H \to G$, $(x, h) \mapsto xh$ is bijective. Let $R > 0$. Without loss of generality, assume

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that R is large enough such that $F_H \subset B_R^d(1_G)$. Use Lemma 3.13(2) in order to find n and collections $\mathcal{W}^{(0)}, \ldots, \mathcal{W}^{(s)}$ of subsets of H such that $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(s)}$ is a uniformly bounded cover of H with Lebesgue number at least $8R$, and such that for each $l = 0, \ldots, s$, the collection $\mathcal{W}^{(l)}$ consists of pairwise disjoint sets and is $H \cap G_n$-invariant with respect to multiplication from the right. Since $(G_k)_k$ is dominating, we may choose, by Lemma 3.17, n large enough such that $G_n \subset H$, which implies $G_n \cap H = G_n$. For $l = 0, \ldots, s$, define

$$\mathcal{V}^{(l)} = \{B_{-4R}(W) \mid W \in CW^{(l)}\},$$

where

$$B_{-4R}(W) = \{h \in H \mid B_{4R}^{d|_H}(h) \subset W\}.$$

Then for each $l = 0, \ldots, s$, the collection $\mathcal{V}^{(l)}$ is again G_n-invariant with respect to multiplication from the right, and the distance between any two distinct sets is greater than $4R$. Moreover, since $\mathcal{W}$ has Lebesgue number at least $8R$, any ball of radius $4R$ in H is contained in a member of $\mathcal{W}$. Therefore, it follows from the construction of $\mathcal{V}$ that it is a cover. Now for $l = 0, \ldots, s$, define

$$\mathcal{U}^{(l)} = \{B_{2R}^d(1_G)V \mid V \in \mathcal{V}^{(l)}\}.$$

Since the distance between any two distinct sets in $\mathcal{V}^{(l)}$ is greater than $4R$, it follows that any two distinct sets in $\mathcal{U}^{(l)}$ are disjoint. Moreover, it is obvious that $\mathcal{U}^{(l)}$ is G_n-invariant with respect to multiplication from the right. Next we claim that $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ is a cover with Lebesgue number at least R. Indeed, given $g \in G$, there are $x \in F_H$ and $h \in H$ with $g = xh$. Then $h \in V$ for some $V \in \mathcal{V}$. Since $x \in F_H \subset B_R^d(1_G)$, it follows that $g \in B_R^d(1_G)V$, and thus $B_R^d(g) = B_R^d(1_G)g \subset B_R^d(1_G)B_R^d(1_G)V \subset B_{2R}^d(1_G)V \in \mathcal{U}$. Since every set of diameter less than R is trivially contained in some ball of radius R, it follows that the Lebesgue number of $\mathcal{U}$ is indeed at least R. This finishes the proof. □

4. *Box spaces of nilpotent groups*

In this section we show that the standard box space of a finitely generated, virtually nilpotent groups has finite asymptotic dimension. Recall that a group G is called nilpotent if it has a central series of finite length, i.e. there is a sequence of subgroups $\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$ such that $[G, G_i] \leq G_{i-1}$ for $i = 1, \ldots, r$. Furthermore, a group is called virtually nilpotent if it contains a nilpotent subgroup of finite index.

We begin this section by recording a few technical observations.

LEMMA 4.1. *Let G be a locally compact group with a proper, right-invariant metric d. Let $H \subset G$ be a discrete, cocompact subgroup. Let G have finite covering dimension $m \in \mathbb{N}$. Then there exists a uniformly bounded, open cover $\mathcal{W}$ of G with positive Lebesgue number and a decomposition $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$, such that for each $l = 0, \ldots, m$, the collection $\mathcal{W}^{(l)}$ consists of mutually disjoint members and is H-invariant with respect to multiplication from the right.*

Proof. Consider the quotient map $\pi : G \to G/H$, which is a local homeomorphism. Then $m = \dim(G/H)$. Since $H \subset G$ is discrete, there exists $\eta > 0$ with $d(h, 1_G) \geq 3\eta$ for all $h \in H \setminus \{1_G\}$. By right-invariance of d, this implies $d(h_1, h_2) \geq 3\eta$ for all $h_1 \neq h_2$ in H.

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For every $x \in G/H$, choose $g \in G$ with $x = \pi(g)$, and then choose an open neighbourhood U_x of g with diameter at most η. Now the collection of the images of these sets $\{V_x\}_{x \in G/H}$ is an open cover of G/H. By compactness, there exists some finite subcover. Using $m = \dim(G/H)$, we can choose an open cover $\mathcal{U}$ of G/H refining $\{V_x\}_{x \in G/H}$ with the following properties.

- There is a decomposition $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(m)}$ such that for each $l = 0, \ldots, m$, one has $U \cap V = \emptyset$ for all $U \neq V$ in $\mathcal{U}^{(l)}$.
- For all $V \in \mathcal{U}$, there exists an open set $U_V \subset G$ with $\pi(U_V) = V$ and $\operatorname{diam}(U_V) \leq \eta$.

Since G/H is compact, we can apply the Lebesgue number theorem (see [**68**, 7.2.12]) to deduce that this cover has some positive Lebesgue number $\mu > 0$ with respect to the push-forward metric $\pi_*(d)$. Set $r = \min(\mu, \eta)$.

Now consider the uniformly bounded, open cover $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$ of G given by

$$\mathcal{W}^{(l)} = \{U_V \cdot h \mid V \in \mathcal{U}^{(l)}, h \in H\}$$

for all $l = 0, \ldots, m$.

Claim 1. For each $l = 0, \ldots, s$, the members in $\mathcal{W}^{(l)}$ are mutually disjoint.

Indeed, let $W_1 \neq W_2$ in $\mathcal{W}^{(l)}$. Let $V_1, V_2 \in \mathcal{U}^{(l)}$ and $h_1, h_2 \in H$ with $W_1 = U_{V_1} h_1$ and $W_2 = U_{V_2} h_2$.

Case 1: $V_1 \neq V_2$. In this case, we get $\pi(W_1) = V_1$ and $\pi(W_2) = V_2$, which are disjoint, so W_1 and W_2 have to be disjoint.

Case 2: $h_1 \neq h_2$. Then $d(h_1, h_2) \geq 3\eta$. Now W_1 is an open neighbourhood of h_1 with diameter at most η, and likewise W_2 is an open neighbourhood of h_2 with diameter at most η. It follows by the triangle inequality that W_1 and W_2 have distance at least η, so they must be disjoint.

Claim 2. The Lebesgue number of $\mathcal{W}$ is at least r.

Let $X \subset G$ be some set of diameter less than r with respect to d. Then $\pi(X)$ has diameter less than r with respect to $\pi_*(d)$. Since $r \leq \mu$, there is some $V \in \mathcal{U}$ with $\pi(X) \subset V$. But then we have

$$X \subset \pi^{-1}(\pi(X)) \subset \pi^{-1}(V) = \bigcup_{h \in H} U_V \cdot h.$$

As we observed earlier, for all $h_1 \neq h_2$, the distance between $U_V h_1$ and $U_V h_2$ is at least η with respect to d. Since $r \leq \eta$, it follows that X must be contained in exactly one set of the form $U_V h$, which is a member of $\mathcal{W}$. □

The following is the key technical lemma that will allow us to prove that finitely generated virtually, nilpotent groups have finite-dimensional box spaces.

LEMMA 4.2. *Let G be a locally compact group G with a proper, right-invariant metric d. Let $H \subset G$ be a discrete and cocompact subgroup. Suppose that there exists a sequence of continuous automorphisms $\sigma_n \in \operatorname{Aut}(G)$ satisfying:*

(a) *for all $n \in \mathbb{N}$, the map σ_n restricts to an endomorphism on H;*

(b) *for every compact set K and open neighbourhood U of 1_G, there exists $n \in \mathbb{N}$ with $K \subset (\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_0)(U)$.*

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Then H is residually finite and admits a regular approximation σ with $\operatorname{asdim}(\Box_\sigma H) \leq \dim(G)$. *In particular, if H is finitely generated, then* $\operatorname{asdim}(\Box_s H) \leq \dim(G)$.

Proof. Observe that each subgroup $H_n = (\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_0)(H) \subset H$ must have finite index in H. Indeed, since H_n is the image of H under an automorphism on G, the subgroup H_n is cocompact in G. Let $K \subset G$ be a compact subset with $G = \bigcup_{h \in H_n} K \cdot h$. Let $\pi_n : G \to G/H_n$ be the quotient map. Then the restriction of $\pi_n|_K$ is surjective. Now we may identify H/H_n as a subset of G/H_n in the obvious way, and then the restriction of π_n yields a surjective map from $K \cap H$ onto H/H_n. But since $K \cap H$ is the intersection of a compact set and a discrete set, it is finite, and thus H/H_n is finite. Since H is discrete, there exists an open neighbourhood $U \subset G$ of the unit with $U \cap H = \{1_G\}$. Then given any predescribed finite set $F \Subset H$ containing the unit, there exists n with $F \subset (\sigma_n \circ \cdots \circ \sigma_0)(U)$, from which it follows that $F \cap H_n = \{1\}$. As F is arbitrary, it follows that $\bigcap_{n \in \mathbb{N}} H_n = \{1\}$, and in particular H is residually finite.

Without loss of generality, assume that G has finite covering dimension. Set $m = \dim(G)$. Apply Lemma 4.1 to choose a uniformly bounded, open cover $\mathcal{W}$ of G with positive Lebesgue number $\delta > 0$ and a decomposition $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$, such that for each $l = 0, \ldots, s$, the collection $\mathcal{W}^{(l)}$ consists of mutually disjoint members and is H-invariant with respect to multiplication from the right.

For each n, consider the uniformly bounded cover $\mathcal{W}_n = \mathcal{W}_n^{(0)} \cup \cdots \cup \mathcal{W}_n^{(m)}$ of G given by $\mathcal{W}_n^{(l)} = (\sigma_n \circ \cdots \circ \sigma_0)(\mathcal{W}^{(l)}) = \{(\sigma_n \circ \cdots \circ \sigma_0)(U) \mid U \in \mathcal{W}^{(l)}\}$ for all $l = 0, \ldots, m$. Then any two distinct members of $\mathcal{W}_n^{(l)}$ are disjoint. The first property of $(\sigma_n)_n$ ensures that for all n and $l = 0, \ldots, m$, the collection $\mathcal{W}_n^{(l)}$ is right-invariant with respect to $H_n = (\sigma_n \circ \cdots \circ \sigma_0)(H) \subset H$. Now let $R > 0$. Since the metric d is proper, the second property of the sequence $(\sigma_n)_n$ ensures that we find some n such that

$$B_R(1_G) \subset (\sigma_n \circ \cdots \circ \sigma_0)(B_{\delta/3}(1_G)).$$

Since $\mathcal{W}$ has Lebesgue number δ, any ball of radius $\delta/3$ is contained in a member of $\mathcal{W}$. By our choice of n, any ball of radius R is therefore contained in a member of $\mathcal{W}_n$, which shows that the Lebesgue number of $\mathcal{W}_n$ is at least R.

If we now restrict the cover $\mathcal{W}_n = \mathcal{W}_n^{(0)} \cup \cdots \cup \mathcal{W}_n^{(m)}$ to a cover on H, we see that condition (2) of Lemma 3.13 is met for R. Since $R > 0$ was arbitrary, we see that $\operatorname{asdim}(\Box_{\{H_n\}} H) \leq \dim(G)$. If H is finitely generated, the last part of the claim then follows from Remark 3.21. □

COROLLARY 4.3. *For all $m \in \mathbb{N}$, we have* $\operatorname{asdim}(\Box_s \mathbb{Z}^m) = m$.

Proof. Apply Lemma 4.2 for $G = \mathbb{R}^m$, $H = \mathbb{Z}^m$ and the sequence of automorphisms given by $\sigma_n(x) = n \cdot x$. This shows $\operatorname{asdim}(\Box_s \mathbb{Z}^m) \leq m$. On the other hand, we have $\operatorname{asdim}(\Box_s \mathbb{Z}^m) \geq \operatorname{asdim}(\mathbb{Z}^m) = m$. □

Remark 4.4. Let us now list a few well-known facts about (virtually) nilpotent groups, which in particular make them interesting from the point of view of the previous section.

(1) Finitely generated, virtually nilpotent groups are residually finite (see [**80**, 2.10, 2.13]).

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(2) A subgroup or quotient of a finitely generated, (virtually) nilpotent group is finitely generated and (virtually) nilpotent (see [**63**, 5.35 and 5.36]).
(3) Any finitely generated, virtually nilpotent group contains a torsion-free nilpotent subgroup of finite index (see [**57**, 10.2.4] or [**80**, 2.6, p. 22]).
(4) The centre of an infinite, finitely generated, nilpotent group is infinite (see [**57**, 11.4.3(ii)]).

By Proposition 3.20, it follows that a finitely generated, virtually nilpotent group admits a dominating regular approximation, and thus has a standard box space.

Let us now consider a prominent class of examples for nilpotent groups.

Definition 4.5. Let R be a commutative, unital ring and $d \in \mathbb{N}$. The unitriangular matrix group of size d over R is defined by

$$U_d(R) = \{x = (x_{i,j})_{1 \le i,j \le d} \mid x_{i,j} \in R,\ x_{i,j} = 0 \text{ for } i > j \text{ and } x_{i,i} = 1\}.$$

The multiplication is given by

$$(x \cdot y)_{i,j} = \sum_{m=i}^{j} x_{i,m} y_{m,j} \quad \text{for all } 1 \le i < j \le d,$$

as the usual matrix product. In a slight abuse of notation we will write $U_d(R) = \{x = (x_{i,j})_{1 \le i < j \le d} \mid x_{i,j} \in R\}$.

These groups are known to be finitely generated and nilpotent, with nilpotency class equal to $d - 1$. Some of these groups will play an important role for the remainder of this section. This is due to the following embedding theorem.

THEOREM 4.6. (See [**33**, 5.2] and [**69**]) *Let G be a finitely generated, torsion-free nilpotent group. Then G embeds as a subgroup into $U_d(\mathbb{Z})$ for some $d \ge 2$.*

Definition 4.7. Let $r > 0$ be a real number. Then the map $\alpha_r : U_d(\mathbb{R}) \to U_d(\mathbb{R})$ defined by

$$\alpha_r(x)_{i,j} = r^{j-i} x_{i,j} \quad \text{for all } 1 \le i < j \le d$$

yields a well-defined endomorphism. Moreover, it is immediate that $\alpha_{rs} = \alpha_r \circ \alpha_s$ for all $r, s > 0$ and $\alpha_1 = \mathrm{id}$, thus

$$\alpha : \mathbb{R}^{>0} \to \mathrm{Aut}(U_d(\mathbb{R})), \quad r \mapsto \alpha_r$$

yields a group action on $U_d(\mathbb{R})$.

Remark 4.8.
(1) Observe that if $r > 0$ is an integer, then α_r restricts to an endomorphism on $U_d(\mathbb{Z})$.
(2) It is a well-known fact that $U_d(\mathbb{Z})$ is a discrete and cocompact subgroup in $U_d(\mathbb{R})$. In fact, one has $U_d(\mathbb{R}) = K \cdot U_d(\mathbb{Z})$ for the compact set

$$K = \{x = (x_{i,j})_{1 \le i < j \le d} \in U_d(\mathbb{R}) \mid |x_{i,j}| \le \tfrac{1}{2}\}$$

in $U_d(\mathbb{R})$ (see [**1**, 8.2.2]).

THEOREM 4.9. *One has* $\mathrm{asdim}(\square_s U_d(\mathbb{Z})) = d(d-1)/2$ *for all* $d \ge 2$.

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Proof. We shall apply Lemma 4.2 for $G = U_d(\mathbb{R})$, $m = d(d-1)/2$, $H = U_d(\mathbb{Z})$ and $\sigma_n = \alpha_{n+1}$. Indeed, since $U_d(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{d(d-1)/2}$ by evaluation of the matrix components, it has covering dimension $d(d-1)/2$. Moreover, $U_d(\mathbb{Z})$ is a discrete, cocompact subgroup such that each automorphism σ_n of $U_d(\mathbb{R})$ restricts to an endomorphism on $U_d(\mathbb{Z})$. Lastly, let $K \subset U_d(\mathbb{R})$ be any compact set and $U \subset U_d(\mathbb{R})$ any open neighbourhood of the unit. Then there are positive numbers $R, \delta > 0$ such that

$$K \subset \{x = (x_{i,j})_{1 \le i < j \le d} \in U_d(\mathbb{R}) \mid |x_{i,j}| \le R\}$$

and

$$\{x = (x_{i,j})_{1 \le i < j \le d} \in U_d(\mathbb{R}) \mid |x_{i,j}| \le \delta\} \subset U.$$

Then any natural number n with $(n+1)! \ge R/\delta$ clearly satisfies $K \subset (\sigma_n \circ \cdots \circ \sigma_0)(U)$. Thus, we have verified the conditions in Lemma 4.2, and thus the claim follows. □

THEOREM 4.10. *Let G be a finitely generated, virtually nilpotent group. Then* $\operatorname{asdim}(\Box_s G) < \infty$.

Proof. Let G' be a nilpotent subgroup of G with finite index. By 4.4, G' contains a torsion-free group H of finite index, which is then necessarily also finitely generated and nilpotent. By combining Theorems 4.6 and 4.9 and Proposition 3.22, we obtain $\operatorname{asdim}(\Box_s H) < \infty$. Since H also has finite index as a subgroup in G, it follows from Proposition 3.22 that $\operatorname{asdim}(\Box_s G) < \infty$. □

Remark 4.11. Using a somewhat more involved approach, one can show by induction that for any elementary amenable group G, the dimension $\operatorname{asdim}(\Box_s G)$ is bounded above by the Hirsch length of G. So, for example, it is finite for all virtually polycyclic groups, or the lamplighter group. This result is a considerable improvement of Theorem 4.10 and is shown by Finn-Sell and the second author in [**15**].

5. *Rokhlin dimension for residually finite groups*

In this section we define the Rokhlin dimension for cocycle actions of residually finite groups. First, we will recall the definition of a (corrected) relative central sequence algebra in the non-unital case, based on Kirchberg's pioneering work [**36**] on central sequences of C*-algebras.

Definition 5.1. (Cf. [**36**, 1.1]) Let A be a C*-algebra. Let $D \subset A_\infty$ be a C*-subalgebra. Consider

$$A_\infty \cap D' = \{x \in A_\infty \mid xd = dx \text{ for all } d \in D\}$$

and

$$\operatorname{Ann}(D, A_\infty) = \{x \in A_\infty \mid xd = dx = 0 \text{ for all } d \in D\}.$$

Then $\operatorname{Ann}(D, A_\infty) \subset A_\infty \cap D'$ is an ideal, and one defines

$$F(D, A_\infty) = (A_\infty \cap D')/\operatorname{Ann}(D, A_\infty),$$

the central sequence algebra of A relative to D. One writes $F_\infty(A) = F(A, A_\infty)$ for brevity.

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Remark 5.2. Let A be a C*-algebra and $D \subset A_\infty$ a separable C*-subalgebra.

(1) Using the fact that D has a countable approximate unit, one can apply a standard reindexation argument and see that there is some positive contraction $e \in A_\infty$ with $ed = d = de$ for all $d \in D$. In particular, the image of e defines a unit of $F(D, A_\infty)$, hence this is a unital C*-algebra.

(2) If $(\alpha, w) : G \curvearrowright A$ is a cocycle action and $D \subset A$ is α-invariant and closed under multiplication with the cocycles $\{w(g, h) \mid g, h \in G\}$, then componentwise application of the family of automorphisms $\{\alpha_g\}_{g\in G}$ on representing bounded sequences defines a genuine action $\tilde{\alpha}_\infty : G \curvearrowright F(D, A_\infty)$. To see this, we observe for $x \in A_\infty \cap D'$, $d \in D$ and $g, h \in G$ that

$$w(g, h)xw(g, h)^*d = w(g, h)x(w(g, h)^*d) = dx = xd.$$

In particular, $x - w(g, h)xw(g, h)^* \in \mathrm{Ann}(D, A_\infty)$, showing that the automorphism on $F(D, A_\infty)$ induced by $\mathrm{Ad}(w(g, h))$ is indeed trivial.

Remark 5.3. (Cf. [**36**, 1.1]) Let A be a C*-algebra and $D \subset A$ a separable C*-subalgebra. Then there is a canonical $*$-homomorphism

$$F(D, A_\infty) \otimes_{\max} D \to A_\infty \quad \text{via } (x + \mathrm{Ann}(D, A_\infty)) \otimes a \mapsto x \cdot a.$$

The image of $\mathbf{1} \otimes a$ under this $*$-homomorphism is a for all $a \in D$.

Now if we assume additionally that $\alpha : G \curvearrowright A$ is an action of a discrete group such that D is α-invariant, then the above map becomes an equivariant $*$-homomorphism from $(F(D, A_\infty) \otimes_{\max} D, \tilde{\alpha}_\infty \otimes \alpha|_D)$ to $(A_\infty, \alpha_\infty)$. In particular, it makes sense to multiply elements of $F(D, A_\infty)$ with elements of D to obtain an element in A_∞, and this multiplication is compatible with the naturally induced actions by α. We will implicitly make use of this observation throughout this section.

We shall now turn to the definition of Rokhlin dimension for cocycle actions of residually finite groups.

Definition 5.4. Let A be a C*-algebra, G a countable group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Let $H \subset G$ be a finite-index subgroup. Let $d \in \mathbb{N}$ be a natural number. We say that α has Rokhlin dimension at most d relative to H, and write $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$, if, for all separable, α-invariant C*-subalgebras $D \subset A$ closed under multiplication with $\{w(g, h) \mid g, h \in G\}$, there exist equivariant c.p.c. order-zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F(D, A_\infty), \tilde{\alpha}_\infty) \quad (l = 0, \ldots, d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$. The value $\dim_{\mathrm{Rok}}(\alpha, H)$ is defined to be the smallest $d \in \mathbb{N}$ such that $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$, or ∞ if no such d exists.

Let us consider a few equivalent reformulations of the above definition.

PROPOSITION 5.5. *Let A be a C*-algebra, G a countable group and $\alpha : G \curvearrowright A$ an action. Let $H \subset G$ be a subgroup with finite index and $d \in \mathbb{N}$ a natural number. Then the following statements are equivalent.*

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(1) $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$.

(2) *For every separable, α-invariant C*-subalgebra $D \subset A$, there exist positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\ldots,d}$ in $A_\infty \cap D'$ satisfying:*

(2a) $(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)}) \cdot a = a$ *for all* $a \in D$;

(2b) $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} \in \mathrm{Ann}(D, A_\infty)$ *for all* $l = 0, \ldots, d$ *and* $\bar{g} \neq \bar{h}$ *in* G/H;

(2c) $\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) - f_{\overline{gh}}^{(l)} \in \mathrm{Ann}(D, A_\infty)$ *for all* $l = 0, \ldots, d$, $\bar{h} \in G/H$ *and* $g \in G$.

(3) *For all $\varepsilon > 0$ and finite sets $M \Subset G$ and $F \Subset A$, there are positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\ldots,d}$ in A satisfying:*

(3a) $(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)}) \cdot a =_\varepsilon a$ *for all* $a \in F$;

(3b) $\|f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} a\| \leq \varepsilon$ *for all* $a \in F$, $l = 0, \ldots, d$ *and* $\bar{g} \neq \bar{h}$ *in* G/H;

(3c) $\alpha_g(f_{\bar{h}}^{(l)}) \cdot a =_\varepsilon f_{\overline{gh}}^{(l)} \cdot a$ *for all* $a \in F$, $l = 0, \ldots, d$, $\bar{h} \in G/H$ *and* $g \in M$;

(3d) $\|[f_{\bar{g}}^{(l)}, a]\| \leq \varepsilon$ *for all* $a \in F$, $l = 0, \ldots, d$ *and* $\bar{g} \in G/H$.

Moreover, it suffices to check these conditions for M being contained in a given generating set of G.

Proof. (1) $\Longrightarrow$ (2). Suppose $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$. Choose equivariant c.p.c. order-zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F(D, A_\infty), \tilde{\alpha}_\infty) \quad (l = 0, \ldots, d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$. Then, by definition, each positive contraction of the form $\varphi_l(\chi_{\{\bar{g}\}}) \in F(D, A_\infty)$ for $\bar{g} \in G/H$ has a representing element $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ with $f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty) = \varphi_l(\chi_{\{g\}})$. By functional calculus, we may assume that each $f_{\bar{g}}^{(l)}$ is a positive contraction. Then we have

$$\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty),$$

and thus the property $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$ translates to condition (2a) by Remark 5.3. For each $l = 0, \ldots, d$ and $\bar{g} \neq \bar{h}$ in G/H, the characteristic functions $\chi_{\bar{g}}, \chi_{\bar{h}} \in \mathcal{C}(G/H)$ are orthogonal. Since φ_l is an order-zero map, we thus have $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} + \mathrm{Ann}(D, A_\infty) = \varphi_l(\chi_{\bar{g}})\varphi_l(\chi_{\bar{h}}) = 0 + \mathrm{Ann}(D, A_\infty)$. This implies condition (2b). Since φ_l is equivariant with respect to the G-shift on G/H, we get for all $g \in G$, $\bar{h} \in G/H$ that

$$\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) + \mathrm{Ann}(D, A_\infty) = \tilde{\alpha}_{\infty,g}(\varphi_l(\chi_{\{\bar{h}\}})) = \varphi_l(\chi_{\{\overline{gh}\}}) = f_{\overline{gh}}^{(l)} + \mathrm{Ann}(D, A_\infty).$$

This implies condition (2c).

(2) $\Longrightarrow$ (1). Let $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\ldots,d}$ be positive contractions in $A_\infty \cap D'$ satisfying conditions (2a), (2b) and (2c). Then the same calculations as above, only read in reverse order, show that conditions (2b) and (2c) imply that for $l = 0, \ldots, d$, the linear map $\varphi_l : \mathcal{C}(G/H) \to F(D, A_\infty)$ given by $\varphi_l(\chi_{\{\bar{g}\}}) = f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty)$ is c.p.c. order zero and equivariant with respect to the G-shift and $\tilde{\alpha}_\infty$. Moreover, condition (2a) implies that $\mathbf{1} = \varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1})$ in $F(D, A_\infty)$.

(2) $\Longrightarrow$ (3). Let $\varepsilon > 0$, $M \Subset G$ and $F \Subset A$ be given. Let $D \subset A$ be a separable, α-invariant C*-subalgebra containing F. Choose positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\ldots,d}$

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in $A_\infty \cap D'$ satisfying conditions (2a), (2b) and (2c). For each $l = 0, \dots, d$ and $\bar{g} \in G/H$, the element $f_{\bar{g}}^{(l)}$ has a representing sequence $(f_{\bar{g},n}^{(l)}) \in \ell^\infty(\mathbb{N}, A)$, which we may assume by functional calculus to consist of positive contractions. Then conditions (2a), (2b) and (2c) translate to:

- $(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g},n}^{(l)}) \cdot a \overset{n\to\infty}{\longrightarrow} a$ for all $a \in D$;
- $f_{\bar{g},n}^{(l)} f_{\bar{h},n}^{(l)} a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \dots, d$ and $\bar{g} \neq \bar{h}$ in G/H;
- $(\alpha_g(f_{\bar{h},n}^{(l)}) - f_{\overline{gh},n}^{(l)}) \cdot a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \dots, d,\ \bar{h} \in G/H$ and $g \in G$.

Moreover, the fact that $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ translates to

- $[f_{\bar{g},n}^{(l)}, a] \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \dots, d$ and $\bar{g} \in G/H$.

Since F is a subset of D, and both F and M are finite, we can choose some large number n such that the elements $(f_{\bar{g},n}^{(l)})_{\bar{g}\in G/H}^{l=0,\dots,d}$ satisfy conditions (3a), (3b), (3c) and (3d).

(3) $\Longrightarrow$ (2). Let $S \subset G$ a be a generating set of G. Assume that (3) holds for all finite subsets M of S. Since S is countable, choose an increasing sequence of finite sets $M_n \Subset S$ with $S = \bigcup_{n\in\mathbb{N}} M_n$.

Let $D \subset A$ be a separable, α-invariant C*-subalgebra. Let $F_n \Subset D$ be an increasing sequence of finite subsets whose union $\bigcup_{k\in\mathbb{N}} F_n \subset D$ is dense. For every n, choose positive contractions $(f_{\bar{g},n}^{(l)})_{\bar{g}\in G/H}^{l=0,\dots,d}$ in A satisfying conditions (3a), (3b), (3c) and (3d) for the triple $(\varepsilon, M, F) = (1/n, M_n, F_n)$. Set $f_{\bar{g}}^{(l)} = [(f_{\bar{g},n}^{(l)})_n] \in A_\infty$ for each $l = 0, \dots, d$ and $\bar{g} \in G/H$. Then by choice of the sequence $(f_{\bar{g},n}^{(l)})_n$, we have:

- $(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g},n}^{(l)}) \cdot a \overset{n\to\infty}{\longrightarrow} a$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k$;
- $f_{\bar{g},n}^{(l)} f_{\bar{h},n}^{(l)} a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d$ and $\bar{g} \neq \bar{h}$ in G/H;
- $(\alpha_g(f_{\bar{h},n}^{(l)}) - f_{\overline{gh},n}^{(l)}) \cdot a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d,\ \bar{h} \in G/H$ and $g \in \bigcup_{k\in\mathbb{N}} M_k = S$;
- $[f_{\bar{g},n}^{(l)}, a] \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d$ and $\bar{g} \in G/H$.

For the corresponding elements in the sequence algebra, we thus have:

(2a′) $(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)}) \cdot a = a$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k$;

(2b′) $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} a = 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d$ and $\bar{g} \neq \bar{h}$ in G/H;

(2c′) $\alpha_g(f_{\bar{h}}^{(l)}) \cdot a = f_{\overline{gh}}^{(l)} \cdot a$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d,\ \bar{h} \in G/H$ and $g \in \bigcup_{k\in\mathbb{N}} M_k = S$;

(2d′) $[f_{\bar{g}}^{(l)}, a] = 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0, \dots, d$ and $\bar{g} \in G/H$.

Since the union $\bigcup_{k\in\mathbb{N}} F_k \subset D$ is dense, it follows by continuity of multiplication that $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ for all $l = 0, \dots, d$ and $\bar{g} \in G/H$, and that the elements $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\dots,d}$ satisfy conditions (2a), (2b) and (2c). Since condition (2c′) holds for all g in a generating set and this condition passes to products of elements, it follows that condition (2c′) even holds for all $g \in G$. □

Applying Proposition 5.5(3), we can now see that our definition of Rokhlin dimension indeed extends the original definitions (cf. Definitions 2.2 and 2.3) due to Hirshberg, Winter and the third author.

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Remark 5.6. Let A, G and $(\alpha, w) : G \curvearrowright A$ be as above. In the case where A is separable, we make two observations.

(1) The map $g \mapsto \tilde{\alpha}_{g,\infty} \in \mathrm{Aut}(F_\infty(A))$ defines a genuine action, which we denote by $\tilde{\alpha}_\infty$.

(2) If H is a finite-index subgroup in G, then $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$ if and only if there exist equivariant c.p.c. order-zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F_\infty(A), \tilde{\alpha}_\infty) \quad (l = 0, \ldots, d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

Proof of (2). In this case, A is trivially the maximal separable, α-invariant C*-subalgebra in itself. The 'only if' part is obvious. For the 'if' part, choose positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\ldots,d}$ in $A_\infty \cap A'$ satisfying conditions (2a), (2b) and (2c) from Proposition 5.5 for A in place of D. Given any other separable, α-invariant C*-subalgebra D of A, we then have $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ and conditions (2a), (2b) and (2c) hold for this subalgebra. Hence $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$. □

LEMMA 5.7. *Let A, G and $(\alpha, w) : G \curvearrowright A$ be as before. If $H_2 \subset H_1 \subset G$ are two finite-index subgroups, then* $\dim_{\mathrm{Rok}}(\alpha, H_1) \leq \dim_{\mathrm{Rok}}(\alpha, H_2)$.

Proof. Since there exists an equivariant and unital $*$-homomorphism

$$(\mathcal{C}(G/H_1), G\text{-shift}) \hookrightarrow (\mathcal{C}(G/H_2), G\text{-shift}),$$

this follows from the definition. □

Definition 5.8. Let A be a C*-algebra, G a countable, residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Let $\sigma = (G_n)_n \in \Lambda(G)$ be a regular approximation. We define the Rokhlin dimension of α along σ as

$$\dim_{\mathrm{Rok}}(\alpha, \sigma) = \sup_{n\in\mathbb{N}} \dim_{\mathrm{Rok}}(\alpha, G_n).$$

Moreover, we define the (full) Rokhlin dimension of α as

$$\dim_{\mathrm{Rok}}(\alpha) = \sup\{\dim_{\mathrm{Rok}}(\alpha, H) \mid H \leq G, [G : H] < \infty\}.$$

Remark 5.9. It follows from Lemma 5.7 that if σ is dominating in the sense of Definition 3.16, then $\dim_{\mathrm{Rok}}(\alpha, \sigma) = \dim_{\mathrm{Rok}}(\alpha)$.

The next example shows that in general, the value $\dim_{\mathrm{Rok}}(\alpha, \sigma)$ can indeed depend on σ. Later in §7, however, we will see that this dependence is rather mild for certain groups.

Example 5.10. Let $p \geq 2$ be a natural number. Consider the UHF algebra

$$M_{p^\infty} = \bigotimes_{\mathbb{N}} M_p \cong \bigotimes_{n\in\mathbb{N}} \bigotimes_{\mathbb{N}} M_{p^n}.$$

For all $n \in \mathbb{N}$, let $u_n \in M_{p^n}$ be the shifting unitary given by

$$u_n = e_{1,p^n} + \sum_{k=2}^{p^n} e_{k,k-1}.$$

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Consider

$$\alpha \in \mathrm{Aut}(A) \quad \text{via} \quad \alpha = \bigotimes_{n\in\mathbb{N}} \bigotimes_{\mathbb{N}} \mathrm{Ad}(u_n).$$

This automorphism defines an action of $G = \mathbb{Z}$. For the sequence of subgroups $G_n = p^n\mathbb{Z}$, one has $\dim_{\mathrm{Rok}}(\alpha, G_n) = 0$. One can see this by using the equivariant, unital diagonal inclusion $(\mathcal{C}(\mathbb{Z}/p^n\mathbb{Z}), G\text{-shift}) \subset (M_{p^n}, \mathrm{Ad}(u_n))$; the rest follows from the construction of the action. However, if $q \geq 2$ is a number that does not divide p, it is easy to see that $\dim_{\mathrm{Rok}}(\alpha, q\mathbb{Z}) > 0$, since otherwise the unit would be divisible by q in the K_0-group of M_{p^∞}. But this is not true. In particular, one has $\dim_{\mathrm{Rok}}(\alpha, H) > 0$ for all $H \neq p^n\mathbb{Z}$.

However, it will be a consequence of Theorem 7.2 that $\dim_{\mathrm{Rok}}(\alpha) = 1$. We remark that this fact also follows from an early unpublished result by Matui and Sato (see the introduction of [**28**]). Moreover, this would also follow from a more recent result due to Liao [**45**].

6. *Nuclear dimension of the crossed product* C*-*algebra*

In this section we show a permanence property of cocycle actions with finite Rokhlin dimension with respect to C*-algebras of finite nuclear dimension, under the assumption that the acting group has a finite-dimensional box space. This extends important results from [**28**] and [**71**]. But first, we need a slightly more flexible characterization of nuclear dimension.

PROPOSITION 6.1. (Cf. [**75**, 2.5] and [**28**, A.4]) *Let A be a* C*-*algebra and $r \in \mathbb{N}$ a natural number. Then* $\dim_{\mathrm{nuc}}(A) \leq r$, *if and only if, for all $F \Subset A$ and $\delta > 0$, there exist a finite-dimensional* C*-*algebra $\mathcal{F}$, a c.p.c. map $\psi : A \to \mathcal{F}$ and c.p.c. order-zero maps $\varphi^{(0)}, \ldots, \varphi^{(r)} : \mathcal{F} \to A_\infty$ such that*

$$a =_\delta \sum_{l=0}^{r} \varphi^{(l)} \circ \psi(a) \quad \textit{for all } a \in F.$$

Proof. Since the 'only if' part is trivial, we show the 'if' part. Let $F \Subset A$ and $\delta > 0$ be arbitrary. Find $\mathcal{F}, \psi, \varphi^{(0)}, \ldots, \varphi^{(r)}$ as in the assertion. Since the cone over $\mathcal{F}$ is projective [**47**, 10.1.11 and 10.2.1], we can find sequences of order-zero maps $\varphi_n^{(l)} : \mathcal{F} \to A$ for $l = 0, \ldots, r$ with $\varphi^{(l)}(x) = [(\varphi_n^{(l)}(x))_n]$ for all $x \in \mathcal{F}$. (This follows from the structure theorem for order-zero maps [**86**, 2.3 and 3.1] or [**81**, 1.2.4].) In particular, it follows that

$$\limsup_{n\to\infty} \left\| a - \sum_{l=0}^{r} \varphi_n^{(l)} \circ \psi(a) \right\| \leq \delta \quad \text{for all } a \in F.$$

So there exists n with

$$a =_{2\delta} \sum_{l=0}^{r} \varphi_n^{(l)} \circ \psi(a) \quad \text{for all } a \in F.$$

Since F and δ were arbitrary, this shows $\dim_{\mathrm{nuc}}(A) \leq r$. □

THEOREM 6.2. (Cf. [**28**, 4.1] and [**71**, 1.10]) *Let A be a* C*-*algebra, G a countable, residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Let $\sigma = (G_n)_n$ be a regular*

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approximation of G. Then the following estimate holds:

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\Box_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

In particular, if G is finitely generated, we get the estimate

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\Box_s G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

Proof. We may assume that $s = \mathrm{asdim}(\Box_\sigma G)$, $r = \dim_{\mathrm{nuc}}(A)$ and $d = \dim_{\mathrm{Rok}}(\alpha, \sigma)$ are all finite, or else the statement is trivial. As a first consequence of this assumption, G is seen to be amenable by Remark 3.12, and thus $A \rtimes_{\alpha,w} G$ is naturally isomorphic to $A \rtimes_{r,\alpha,w} G$.

Let $F \Subset A \rtimes_{\alpha,w} G$ and $\delta > 0$ be given. In order to show the assertion, we show that there exists a finite-dimensional C*-algebra $\mathcal{F}$ with a c.p. approximation $(\mathcal{F}, \psi, \varphi)$ for F up to δ on $A \rtimes_{\alpha,w} G$ as in Proposition 6.1.

Since the purely algebraic crossed product is dense in $A \rtimes_{\alpha,w} G$ and is linearly generated by terms of the form au_g for $a \in A$ and $g \in G$, we may assume without loss of generality that $F \subset \{au_g \mid a \in F', g \in M\}$ for two finite sets $F' \Subset A_1$ and $M \Subset G$.

The square root function $\sqrt{\cdot} : [0, 1] \to [0, 1]$ is uniformly continuous. Hence, we may choose $\varepsilon > 0$ so small that $|\sqrt{s} - \sqrt{t}| \leq \delta$, whenever $|s - t| \leq \varepsilon$. Recall that the sequence $(G_n)_n$ admits decay functions in the sense of Lemma 3.13(3). Hence there exist $n \in \mathbb{N}$ and functions $\mu^{(j)} : G \to [0, 1]$ for $j = 0, \ldots, s$ such that

$$\mathrm{supp}(\mu^{(j)}) \cap \mathrm{supp}(\mu^{(j)})h = \emptyset \quad \text{for all } j = 0, \ldots, s \text{ and } h \in G_n \setminus \{1\}, \tag{6A}$$

$$\sum_{j=0}^{s} \sum_{h \in G_n} \mu^{(j)}(gh) = 1 \quad \text{for all } g \in G, \tag{6B}$$

$$\|\mu^{(j)} - \mu^{(j)}(g^{-1} \cdot _)\|_\infty \leq \varepsilon \quad \text{for all } j = 0, \ldots, s \text{ and } g \in M. \tag{6C}$$

Consider the finite sets

$$B_n^{(j)} = \mathrm{supp}(\mu^{(j)}),\ B_n = \bigcup_{j=0}^{s} B_n^{(j)} \Subset G \tag{6D}$$

and

$$\tilde{F} = \{\alpha_{h^{-1}}(a) w(h^{-1}, g) \mid a \in F', g \in M, h \in B_n\} \Subset A.$$

Let A act faithfully on a Hilbert space H and let $A \rtimes_{\alpha,w} G \cong A \rtimes_{r,\alpha,w} G$ be embedded into $\mathcal{B}(\ell^2(G) \otimes H)$ as in Remark 2.8. Let $Q_j \in \mathcal{B}(\ell^2(G) \otimes H)$ be the projection onto the subspace $\ell^2(B_n^{(j)}) \otimes H$. Then $x \mapsto Q_j x Q_j$ defines a c.p.c. map $\Psi_j : A \rtimes_{r,\alpha,w} G \to M_{|B_n^{(j)}|}(A)$. More specifically, we have for all $a \in A$, $g \in G$ and $j = 0, \ldots, s$ that

$$\begin{aligned}
\Psi_j(au_g) &= Q_j \Bigg[\sum_{h \in G} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) w(h^{-1}, g) \Bigg] Q_j \\
&= \sum_{\substack{h \in B_n^{(j)}: \\ g^{-1}h \in B_n^{(j)}}} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) w(h^{-1}, g) \\
&= \sum_{h \in B_n^{(j)} \cap g B_n^{(j)}} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) w(h^{-1}, g).
\end{aligned} \tag{6E}$$

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For $j = 0, \dots, s$, define the diagonal matrices $D_j \in M_{|B_n|}(\mathbb{C})$ by $(D_j)_{h,h} = \mu^{(j)}(h)$. By our choice of ε and (6C), we have

$$\left\| \sqrt{\mu^{(j)}} - \sqrt{\mu^{(j)}(g^{-1} \cdot _)} \right\|_\infty \leq \delta \quad \text{for all } g \in M. \tag{6F}$$

It follows for all $g \in M$ and $a \in A$ that

$$\begin{aligned} &\|[\sqrt{D_j},\, Q_j a u_g Q_j]\| \\ &\overset{(6E)}{=} \left\| \sum_{h \in B_n \cap gB_n} \left(\sqrt{\mu^{(j)}(h)} - \sqrt{\mu^{(j)}(g^{-1}h)} \right) e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) w(h^{-1}, g) \right\| \\ &\leq \max \left\{ \left| \sqrt{\mu^{(j)}(h)} - \sqrt{\mu^{(j)}(g^{-1}h)} \right| \,\middle|\, h \in B_n \cap gB_n \right\} \cdot \|a\| \\ &\overset{(6F)}{\leq} \delta \|a\|. \end{aligned}$$

Define the c.p.c. map $\theta_j : A \rtimes_{r,\alpha,w} G \to M_{|B_n^{(j)}|}(A)$ by $\theta_j(x) = \sqrt{D_j} \Psi_j(x) \sqrt{D_j}$. By the previous calculation, we have

$$\|\theta_j(a u_g) - D_j Q_j a u_g Q_j\| \leq \delta \cdot \|a\| \quad \text{for all } g \in M \text{ and } a \in A. \tag{6G}$$

By (6E), we have for each $a u_g \in F$ that the matrix coefficients of $\theta_j(a u_g) \in M_{|B_n^{(j)}|}(A)$ are all in $\tilde{F}$. Choose an r-decomposable c.p. approximation $(\mathcal{F}, \psi, \varphi)$ for $\tilde{F}$ up to $\delta/[G : G_n]$, i.e. a finite-dimensional C*-algebra $\mathcal{F}$ and c.p. maps $A \overset{\psi}{\to} \mathcal{F} \overset{\varphi^{(i)}}{\to} A$ for $i = 0, \dots, r$ such that ψ is c.p.c., the maps $\varphi^{(i)}$ are c.p.c. order zero and

$$\|x - (\varphi \circ \psi)(x)\| \leq \frac{\delta}{[G : G_n]} \quad \text{for all } x \in \tilde{F}, \tag{6H}$$

where φ denotes the sum $\varphi = \varphi^{(0)} + \cdots + \varphi^{(r)}$.

For all $j = 0, \dots, s$ and $i = 0, \dots, r$ let

$$\psi_j = \mathrm{id}_{|B_n^{(j)}|} \otimes \psi : M_{|B_n^{(j)}|} \otimes A \to M_{|B_n^{(j)}|} \otimes \mathcal{F} \tag{6I}$$

and

$$\varphi_{j,i} = \mathrm{id}_{|B_n^{(j)}|} \otimes \varphi^{(i)} : M_{|B_n^{(j)}|} \otimes \mathcal{F} \to M_{|B_n^{(j)}|} \otimes A \tag{6J}$$

denote the amplifications of ψ and $\varphi^{(i)}$.

Let $D \subset A$ be a separable, α-invariant C*-algebra containing $\tilde{F}$, the image of $\varphi^{(i)}$ for each $i = 0, \dots, r$ and such that D is closed under multiplication with $\{w(g, h) \mid g, h \in G\}$. Since $\dim_{\mathrm{Rok}}(\alpha, G_n) \leq d$, we can use Proposition 5.5(2) and find positive contractions $(f_{\bar{h}}^{(l)})_{\bar{h} \in G/G_n}^{l=0,\dots,d}$ in $A_\infty \cap D'$ satisfying the relations

$$\left(\sum_{l=0}^{d} \sum_{\bar{h} \in G/G_n} f_{\bar{h}}^{(l)} \right) a = a \quad \text{for all } a \in D, \tag{6K}$$

$$f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} a = 0 \quad \text{for all } a \in D,\ \bar{g} \neq \bar{h} \text{ and } l = 0, \dots, d, \tag{6L}$$

$$\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) a = f_{\overline{gh}}^{(l)} a \quad \text{for all } a \in D,\ l = 0, \dots, d \text{ and } g, h \in G. \tag{6M}$$

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Now fix any $j \in \{0, \dots, s\}$ and $l \in \{0, \dots, d\}$. Define the map†

$$\sigma_{j,l} : M_{|B_n^{(j)}|}(D) \to (A \rtimes_{\alpha,w} G)_\infty \quad \text{by } \sigma_{j,l}(e_{g,h} \otimes a) = u^*_{g^{-1}} f_1^{(l)} a u_{h^{-1}}. \tag{6N}$$

Note that it is c.p. since $\sigma_{j,l}(x) = v_{j,l} x v_{j,l}^*$ for the $1 \times |B_n^{(j)}|$ matrix $v_{j,l} = (u^*_{h^{-1}} f_1^{(l)1/2})_{h \in B_n^{(j)}}$. We now show that $\sigma_{j,l}$ is order zero. We denote $f^{(l)} = \sum_{\bar{h} \in G/G_n} f_{\bar{h}}^{(l)}$, which, by virtue of (6M), is an element fixed by α_∞ upon multiplying with an element in D. Let $h_1, h_2, h_3, h_4 \in B_n^{(j)}$ and $a, b \in A$. We have

$$\begin{aligned}
&\sigma_{j,l}(e_{h_1,h_2} \otimes a)\sigma_{j,l}(e_{h_3,h_4} \otimes b) \\
&\quad = u^*_{h_1^{-1}} f_1^{(l)} a u_{h_2^{-1}} \cdot u^*_{h_3^{-1}} f_1^{(l)} b u_{h_4^{-1}} \\
&\quad = u^*_{h_1^{-1}} u_{h_2^{-1}} (\alpha_{h_2^{-1}})^{-1}_\infty (a f_1^{(l)}) \cdot (\alpha_{h_3^{-1}})^{-1}_\infty (f_1^{(l)} b) u^*_{h_3^{-1}} u_{h_4^{-1}} \\
&\quad \overset{(6M)}{=} u^*_{h_1^{-1}} u_{h_2^{-1}} (\alpha_{h_2^{-1}})^{-1}(a) f_{\bar{h}_2}^{(l)} f_{\bar{h}_3}^{(l)} (\alpha_{h_3^{-1}})^{-1}(b) u^*_{h_3^{-1}} u_{h_4^{-1}} \\
&\quad \overset{(6L)}{=} \delta_{\bar{h}_2,\bar{h}_3} \cdot u^*_{h_1^{-1}} u_{h_2^{-1}} (\alpha_{h_2^{-1}})^{-1}(a) \cdot (f_{\bar{h}_2}^{(l)})^2 \cdot (\alpha_{h_2^{-1}})^{-1}(b) u^*_{h_2^{-1}} u_{h_4^{-1}} \\
&\quad \overset{(6M)}{=} \delta_{\bar{h}_2,\bar{h}_3} \cdot u^*_{h_1^{-1}} u_{h_2^{-1}} (\alpha_{h_2^{-1}})^{-1}_\infty (a (f_1^{(l)})^2) \cdot (\alpha_{h_2^{-1}})^{-1}(b) u^*_{h_2^{-1}} u_{h_4^{-1}} \\
&\quad = \delta_{\bar{h}_2,\bar{h}_3} \cdot u^*_{h_1^{-1}} (f_1^{(l)})^2 a u_{h_2^{-1}} \cdot u^*_{h_2^{-1}} b u_{h_4^{-1}} \\
&\quad \overset{(6M)}{=} \delta_{\bar{h}_2,\bar{h}_3} \cdot (f_{\bar{h}_1}^{(l)})^2 u^*_{h_1^{-1}} a b u_{h_4^{-1}} \\
&\quad \overset{(6L)}{=} \delta_{\bar{h}_2,\bar{h}_3} \cdot f^{(l)} f_{\bar{h}_1}^{(l)} u^*_{h_1^{-1}} a b u_{h_4^{-1}} \\
&\quad \overset{(6M)}{=} \delta_{\bar{h}_2,\bar{h}_3} \cdot f^{(l)} u^*_{h_1^{-1}} f_1^{(l)} a b u_{h_4^{-1}} \\
&\quad \overset{(6M),(6A),(6D)}{=} f^{(l)} \cdot \sigma_{j,l}((e_{h_1,h_2} \otimes a)(e_{h_3,h_4} \otimes b)).
\end{aligned}$$

Note that for the last step of the above calculation, we have used (6A) in that the canonical map $G \twoheadrightarrow G/G_n$ is injective on each set $B_n^{(j)}$. Since this calculation involves linear generators of the C*-algebra $M_{|B_n^{(j)}|}(D)$, we obtain the equation

$$\sigma_{j,l}(x)\sigma_{j,l}(y) = f^{(l)} \cdot \sigma_{j,l}(xy) \quad \text{for all } x, y \in M_{|B_n^{(j)}|}(D).$$

Hence each $\sigma_{j,l}$ is order zero and so is $\sigma_{j,l} \circ \varphi_{j,i}$ for all $i = 0, \dots, r$, $j = 0, \dots, s$ and $l = 0, \dots, d$. Note that this composition makes sense because of our assumption that D contains the images of each $\varphi^{(i)}$, and so the image of each $\varphi_{j,i}$ is contained in the domain of $\sigma_{j,l}$.

As the next step, we would like to show that the maps in the diagram

$$\begin{array}{ccc}
A \rtimes_{\alpha,w} G & \longrightarrow & (A \rtimes_{\alpha,w} G)_\infty \\
& {\scriptstyle \bigoplus_j \psi_j \circ \theta_j =: \Theta} \searrow \quad \nearrow {\scriptstyle \sum_{j,l,i} \sigma_{j,l} \circ \varphi_{j,i} =: \Phi} & \\
& \bigoplus_{j=0}^{s} M_{|B_n^{(j)}|}(\mathcal{F}) &
\end{array} \tag{6O}$$

† Keep in mind that in a twisted crossed product, the unitaries u_g and $u^*_{g^{-1}}$ do not necessarily coincide.

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give rise to a good c.p. approximation of F. For this, we first calculate for every contraction $x \in D$ and every $g \in M$ that

$$\begin{aligned}
&\Big\| \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} x \Big\| \\
&\overset{(6D)}{\leq} (s+1) \cdot \max_{0 \leq j \leq s} \Big\| \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} x \Big\| \\
&\overset{(6L)}{\leq} (s+1) \cdot \max\{\mu^{(j)}(h) \mid h \in B_n^{(j)} \setminus g B_n^{(j)},\ j = 0, \ldots, s\} \\
&\overset{(6D)}{=} (s+1) \cdot \max\{\|\mu^{(j)} - \mu^{(j)}(g^{-1} \cdot _)\|_\infty \mid j = 0, \ldots, s\} \\
&\overset{(6C)}{\leq} (s+1)\varepsilon \leq (s+1)\delta. \qquad (6P)
\end{aligned}$$

Observe that by the properties of the functions $\mu^{(j)}$, we have for all l that

$$f^{(l)} = \sum_{\bar{g} \in G/G_n} f_{\bar{g}}^{(l)} \overset{(6B)}{=} \sum_{\bar{g} \in G/G_n} \sum_{j=0}^{s} \sum_{h \in \bar{g}} \mu^{(j)}(h) f_{\bar{g}}^{(l)} \overset{(6D)}{=} \sum_{j=0}^{s} \sum_{h \in B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)}. \qquad (6Q)$$

Note that the last equation follows from the fact that for each $j = 0, \ldots, s$, the set $B_n^{(j)}$ is contained in a finite set of representatives of G/G_n by (6A) and (6D). It follows for all $g \in M$ and $x \in D$ that

$$\begin{aligned}
&\Big\| x - \Big(\sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \cap g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} \Big) \cdot x \Big\| \\
&\overset{(6Q),(6K)}{=} \Big\| \Big(\sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} \Big) \cdot x \Big\| \\
&\overset{(6P)}{\leq} (s+1)(d+1)\delta. \qquad (6R)
\end{aligned}$$

Now let $au_g \in F$ and recall the definition of the maps Θ and Φ from the approximation diagram (6O). To simplify notation in the following calculation, the sums over the indices l, j, and i always stand for $\sum_{l=0}^{d}$, $\sum_{j=0}^{s}$ and $\sum_{i=0}^{r}$, while the symbol $\sum_h$ stands for $\sum_{h \in B_n^{(j)} \cap g B_n^{(j)}}$. With such conventions, we have

$$\begin{aligned}
&\Phi \circ \Theta(au_g) \\
&\overset{(6O)}{=} \sum_{l,j,i} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j \circ \theta_j)(au_g) \\
&\overset{(6G)}{=} {}_{(s+1)(d+1)(r+1)\delta} \sum_{l,j,i} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j)(D_j Q_j au_g Q_j) \\
&\overset{(6E)}{=} \sum_{l,j,i} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j) \Big(\sum_h \mu^{(j)}(h) \cdot e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) w(h^{-1}, g) \Big) \\
&\overset{(6I),(6J)}{=} \sum_{l,j,i} \sigma_{j,l} \Big(\sum_h \mu^{(j)}(h) \cdot e_{h,g^{-1}h} \otimes (\varphi^{(i)} \circ \psi)[\alpha_{h^{-1}}(a) w(h^{-1}, g)] \Big)
\end{aligned}$$

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$$\overset{(6N)}{=} \sum_{l,j,i}\sum_{h} \mu^{(j)}(h)\cdot u_{h^{-1}}^{*} f_1^{(l)}[(\varphi^{(i)}\circ\psi)(\alpha_{h^{-1}}(a)w(h^{-1},g))]u_{h^{-1}g}$$

$$\overset{(6M)}{=} \sum_{l,j}\sum_{h} \mu^{(j)}(h) f_{\bar{h}}^{(l)}\cdot u_{h^{-1}}^{*}[(\varphi\circ\psi)(\alpha_{h^{-1}}(a)w(h^{-1},g))]u_{h^{-1}g}$$

$$\overset{(6H)}{=}_{((s+1)(d+1)|B_n^{(j)}|\cdot\delta/[G:G_n])} \sum_{l,j}\sum_{h} \mu^{(j)}(h) f_{\bar{h}}^{(l)}\cdot \underbrace{u_{h^{-1}}^{*}\alpha_{h^{-1}}(a)w(h^{-1},g)u_{h^{-1}g}}_{=au_g}$$

$$= \left(\sum_{l,j}\sum_{h} \mu^{(j)}(h) f_{\bar{h}}^{(l)}\right)\cdot au_g$$

$$\overset{(6R)}{=}_{(s+1)(d+1)\delta} au_g.$$

Summing up these approximation steps and using (6A) in the form of the inequality $|B_n^{(j)}| \le [G:G_n]$, it follows for all $au_g \in F$ that

$$au_g =_{3(s+1)(d+1)(r+1)\delta} \Phi\circ\Theta(au_g).$$

Now let us summarize what we have. We have constructed a c.p. approximation

$$\left(\bigoplus_{j=0}^{s} M_{|B_n^{(j)}|}(\mathcal{F}),\ \Theta,\ \Phi\right)$$

of tolerance $3(s+1)(d+1)(r+1)\delta$ on F, where the map

$$\Phi = \sum_{i=0}^{r}\sum_{l=0}^{d}\sum_{j=0}^{s} \sigma_{j,l}\circ\varphi_{j,i} : \bigoplus_{j=0}^{s} M_{|B_n^{(j)}|}(\mathcal{F}) \to (A \rtimes_{\alpha,w} G)_\infty$$

is a sum of $(s+1)(d+1)(r+1)$ c.p.c. maps of order zero. Since d, s, r are constants and $F \Subset A \rtimes_{\alpha,w} G$ and $\delta > 0$ were arbitrary, it follows from Proposition 6.1 that

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \le (s+1)(d+1)(r+1),$$

which is what we wanted to show. □

We now present a nice test case for the above theorem.

Remark 6.3. Let G be a countably infinite, residually finite and amenable group. Let $\sigma = (G_n)_n$ be a regular approximation of G consisting of normal subgroups. Then the generalized Bunce–Deddens algebra associated to G and σ (see [**54**]) has nuclear dimension at most $\mathrm{asdim}(\square_\sigma G)$.

This is because the associated generalized Bunce–Deddens algebra arises as the crossed product of the dynamical system

$$(\mathcal{C}(X),\alpha) = \varinjlim(\mathcal{C}(G/G_n),\ G\text{-shift}),$$

where the connecting maps are induced by the natural maps $G/G_n \twoheadrightarrow G/G_{n+1}$ and X turns out to be homeomorphic to the Cantor set. From the definition, it is trivial that the

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resulting action α has Rokhlin dimension 0 along σ. Hence it follows from Theorem 6.2 that

$$\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha G) \leq \mathrm{asdim}^{+1}(\square_\sigma G) \cdot 1 \cdot 1 - 1 = \mathrm{asdim}(\square_\sigma G).$$

In particular, this is independent of the other results within [**54**]. On the other hand, it is easy to see that generalized Bunce–Deddens algebras are separable, unital, nuclear, quasidiagonal, have a unique tracial state and satisfy the UCT (Universal Coefficient Theorem). A combination of these facts with [**51**, 6.2] gives an alternative proof that Bunce–Deddens algebras, associated to regular approximations yielding a box space of finite asymptotic dimension, are classifiable.

However, we should mention that it is now known that *all* generalized Bunce–Deddens algebras are classifiable. It follows from the detailed study of the transformation groupoid, which was carried out in [**54**], that all generalized Bunce–Deddens algebras have strict comparison of positive elements. This, in turn, is sufficient to deduce from [**51**, 6.2] that they are classifiable.

7. *Dependence on the approximation sequence*

In this section we discuss the dependence of the Rokhlin dimension on the approximation sequence. In order to do so, we introduce a related invariant $\dim_{\mathrm{am}}(\alpha)$ which does not depend on any residual finite approximation. Later, in §9, we shall see that for topological actions, this invariant corresponds to the notion of amenability dimension that appeared in the work of Guentner, Willett and Yu [**21**].

Definition 7.1. Let A be a C*-algebra, G a countable discrete group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. We define $\dim_{\mathrm{am}}(\alpha)$ to be the smallest natural number d with the property that for any $\varepsilon > 0$ and any finite sets $M \Subset G$ and $F \Subset A$, there exist finitely supported maps $\mu^{(l)} : G \to A_{1,+}$, $g \mapsto \mu_g^{(l)}$ for $l = 0, \ldots, d$ satisfying:

(a) $(\sum_{l=0}^{d} \sum_{g \in G} \mu_g^{(l)}) \cdot a =_\varepsilon a$ for any $a \in F$;

(b) $\mu_g^{(l)} \mu_h^{(l)} = 0$ for all $l = 0, \ldots, d$ and $g \neq h$ in G;

(c) $\|\sum_{h \in E} (\alpha_g(\mu_h^{(l)}) - \mu_{gh}^{(l)}) \cdot a\| \leq \varepsilon$ for all $a \in F$, $l = 0, \ldots, d$, $g \in M$ and every finite subset $E \Subset G$;

(d) $\|[\sum_{g \in E} \mu_g^{(l)}, a]\| \leq \varepsilon$ for all $a \in F$, $l = 0, \ldots, d$ and every finite subset $E \Subset G$.

If no such d exists, then we set $\dim_{\mathrm{am}}(\alpha) = \infty$.

Note that the sum in the first condition is finite because each function $\mu^{(l)}$ is finitely supported. Meanwhile, the third condition expresses a kind of almost equivariance for each $\mu^{(l)}$, and in particular has the consequence that the function $\mu^{(l)}$ tapers off near the edge of its support. As in Definition 5.8, we drop the cocycle w in the notation because the definition of $\dim_{\mathrm{am}}$ only depends on α.

Note also that the definition of $\dim_{\mathrm{am}}(\alpha)$ does not require residual finiteness of the group. If the group is residually finite, this notion allows us to compare the Rokhlin dimension associated to various regular approximation sequences, via the following intertwining inequalities.

THEOREM 7.2. *Let A be a C*-algebra, G a residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Let $\sigma = (G_n)_n \in \Lambda(G)$ be a regular approximation of G. Then we have*

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the estimates

$$\dim_{\mathrm{Rok}}^{+1}(\alpha) \leq \dim_{\mathrm{am}}^{+1}(\alpha) \leq \mathrm{asdim}^{+1}(\square_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma).$$

In particular, if the right-hand side is finite for some regular approximation σ, then it follows that $\dim_{\mathrm{Rok}}(\alpha) < \infty$.

Proof. Denote $d = \dim_{\mathrm{am}}(\alpha)$, $s = \mathrm{asdim}(\square_\sigma G)$ and $r = \dim_{\mathrm{Rok}}(\alpha, \sigma)$. Let us prove the first inequality, assuming that $d < \infty$. Let $M \Subset G$, $F \Subset A$ and $\varepsilon > 0$ be given. By Definition 7.1, we can find finitely supported maps $\mu^{(l)} : G \to A_{1,+}$ for $l = 0, \ldots, d$ with the given properties depending on (M, F, ε). Let $H \subset G$ be a subgroup of finite index. For $l = 0, \ldots, d$ and $\bar{g} \in G/H$, we define $f_{\bar{g}}^{(l)} = \sum_{h \in \bar{g}} \mu_h^{(l)}$, where the sum is in fact finite. Then one can easily verify that Proposition 5.5(3) is satisfied for the parameters (M, F, ε): for any $a \in F$, we have

$$\left(\sum_{l=0}^{d} \sum_{\bar{g} \in G/H} f_{\bar{g}}^{(l)} \right) \cdot a =_\varepsilon a; \tag{7A}$$

and for all $l = 0, \ldots, d$, we have

$$f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} = 0 \quad \text{for all } \bar{g} \neq \bar{h} \text{ in } G/H, \tag{7B}$$

$$\left\| (\alpha_g(f_{\bar{h}}^{(l)}) - f_{\overline{gh}}^{(l)}) \cdot a \right\| \leq \varepsilon \quad \text{for all } \bar{h} \in G/H \text{ and } g \in M, \tag{7C}$$

$$\| [f_{\bar{g}}^{(l)}, a] \| \leq \varepsilon \quad \text{for all } \bar{g} \in G/H. \tag{7D}$$

As H, M and ε were arbitrary, we can deduce $\dim_{\mathrm{Rok}}(\alpha) \leq d$ by Proposition 5.5.

Let us prove the second inequality, assuming that $s, r < \infty$. Let $M \Subset G$ and $\varepsilon > 0$ be given. By Lemma 3.13(3), there is some n such that there exist finitely supported functions $\nu^{(j)} : G \to [0, 1]$, for $j = 0, \ldots, d$, that satisfy the three conditions in Lemma 3.13(3) in place of $\mu^{(j)}$, with regard to the pair (M, ε). Moreover, given a finite subset $F \subset A_1$ and setting $N = [G : G_n]$, we apply Proposition 5.5(3) and choose positive contractions $f_{\bar{g}}^{(l)}$ in $A_{1,+}$ for $l = 0, \ldots, r$ and $\bar{g} \in G/G_n$ such that for every $a \in F$, we have

$$\sum_{l=0}^{d} \sum_{\bar{g} \in G/G_n} f_{\bar{g}}^{(l)} \cdot a =_\varepsilon a; \tag{7E}$$

and for all $l = 0, \ldots, d$, we have

$$\| f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} \cdot a \| \leq \frac{\varepsilon}{N} \quad \text{for all } \bar{g} \neq \bar{h} \text{ in } G/G_n, \tag{7F}$$

$$\bar{\alpha}_g(f_{\bar{h}}^{(l)}) \cdot a =_{\varepsilon/N} f_{\overline{gh}}^{(l)} \cdot a \quad \text{for all } \bar{h} \in G/G_n, \tag{7G}$$

$$\| [f_{\bar{g}}^{(l)}, a] \| \leq \frac{\varepsilon}{N} \quad \text{for all } \bar{g} \in G/H. \tag{7H}$$

Now the cone over $\mathbb{C}^N$ is projective by [**47**, 10.1.7]. Since in the relations (7E)–(7H), ε may be arbitrarily small, we can use a projectivity argument and strengthen (7F) to

$$f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} = 0 \quad \text{for all } l = 0, \ldots, d \text{ and } \bar{g} \neq \bar{h} \text{ in } G/G_n.\dagger \tag{7F$'$}$$

† Fix the index l. For a shrinking sequence of ε, the elements $f_{\bar{g}}^{(l)}$ give rise to a $*$-homomorphism from $\mathcal{C}_0(0, 1]^{\oplus N}$ to the sequence algebra A_∞. Due to projectivity, this lifts to a sequence of $*$-homomorphisms to A, meaning that we may find honestly orthogonal elements close to the original ones.

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Now consider the functions $\mu^{(l,j)}$ for $l = 0, \ldots, r$ and $j = 0, \ldots, s$ such that

$$\mu_g^{(l,j)} = \nu_g^{(j)} \cdot f_{\bar{g}}^{(l)} \quad \text{for all } g \in G.$$

Notice that the support of each $\mu^{(l,j)}$ is contained in the support of $\nu^{(j)}$, which consists of no more than N elements. Thus one directly computes that the conditions of Definition 7.1 are satisfied for M, F and 3ε. For example, to check (3), we compute

$$\begin{aligned}
&\left\| \sum_{h \in E} (\alpha_g(\mu_h^{(l,j)}) - \mu_{gh}^{(l,j)}) \cdot a \right\| \\
&= \left\| \sum_{h \in E} (\alpha_g(\nu_h^{(j)} \cdot f_{\bar{h}}^{(l)}) - \alpha_g(\nu_h^{(j)}) \cdot f_{\bar{gh}}^{(l)} + \alpha_g(\nu_h^{(j)}) \cdot f_{\bar{gh}}^{(l)} - \nu_{gh}^{(j)} \cdot f_{\bar{gh}}^{(l)}) \cdot a \right\| \\
&\leq \sum_{h \in E \cap \mathrm{supp}(\nu^{(j)})} \|\alpha_g(\nu_h^{(j)})\| \|(\alpha_g(f_{\bar{h}}^{(l)}) - f_{\bar{gh}}^{(l)}) \cdot a\| \\
&\quad + \sum_{h \in E \cap (\mathrm{supp}(\nu^{(j)}) \cup g \cdot \mathrm{supp}(\nu^{(j)}))} \|\alpha_g(\nu_h^{(j)}) - \nu_{gh}^{(j)}\| \|f_{\bar{gh}}^{(l)} \cdot a\| \\
&\leq N \cdot \frac{\varepsilon}{N} + 2N \cdot \frac{\varepsilon}{N} = 3\varepsilon.
\end{aligned}$$

The other conditions are checked similarly, and in fact these conditions hold with respect to the tolerance ε instead of 3ε. Since M, F and ε were arbitrary, we get $d + 1 \leq (s + 1)(r + 1)$. □

This shows that the mere finiteness of the Rokhlin dimension is, in a sense, independent of the regular approximation. In particular, for a system (A, α, w), whenever Theorem 6.2 applies non-trivially (i.e. G admits a regular approximation σ such that the right-hand side of the first inequality, $\mathrm{asdim}^{+1}(\square_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma) \cdot \dim_{\mathrm{nuc}}^{+1}(A)$, is finite), all versions of Rokhlin dimension are automatically finite.

8. *Free topological actions of nilpotent groups*

In this section we study free actions of finitely generated, nilpotent groups on finite-dimensional spaces, and show that their Rokhlin dimension is finite. First, we have to establish an important technical property of nilpotent groups.

LEMMA 8.1. *Let G be a finitely generated, nilpotent group. Let ℓ be the Hirsch length of G and set $m = 3^\ell$. Then for any finite subset $M \Subset G$, there is a finite subset $F \Subset G$ containing the unit and $g_1, \ldots, g_m \in G$, such that for any $g \in F^{-1}F$, we have $Mg \subset g_j F$ for some $j \in \{1, \ldots, m\}$.*

Proof. Before we delve into the somewhat technical proof, let us summarize the approach. It is easy to see that this is true for $G = \mathbb{Z}$. If we pretend for a moment that G is torsion-free and nilpotent, then the idea boils down to representing G as an iterated central extension by copies of $\mathbb{Z}$, and inductively defining F in such a way that, in the direction of a newly added generator, each *new* F is long enough that commutators of the old generators have relatively minute impact. This process describes an induction argument by the Hirsch length of G.

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So let us apply induction by ℓ. If $\ell = 0$, then G is finite, and we may simply take $F = G$ and $m = 1$.

Now suppose G has Hirsch length $\ell + 1$ for some $\ell \geq 0$, and the statement has been proved for all nilpotent groups with Hirsch length at most ℓ. By Remark 4.4, G has a central element $t \in G$ of infinite order. Set $H = G/\langle t \rangle$. This yields a finitely generated, nilpotent group of Hirsch length ℓ. Denote by $\pi : G \to H$ the quotient map and set $m = 3^{\ell}$.

Now let $M \Subset G$ be a finite subset. We apply our induction hypothesis on H to get a finite set $F_0 \Subset H$ containing the identity and $h_1, \ldots, h_m \in H$ such that for each $h \in F_0^{-1} F_0$, we have $\pi(M)h \subset h_j F_0$ for some $j \in \{1, \ldots, m\}$.

Pick some cross section $\sigma : H \to G$ such that $\sigma(1_H) = 1_G$. Decompose M into

$$M = \bigsqcup_{k \in \pi(M)} M_k \cdot \sigma(k) \tag{8A}$$

for some finite subsets $M_k \Subset \langle t \rangle$. Define the finite sets

$$T = \bigcup_{k \in \pi(M)} \bigcup_{k_1, k_2 \in F_0} \sigma(k k_1^{-1} k_2)^{-1} \sigma(k) \sigma(k_1)^{-1} \sigma(k_2) M_k \tag{8B}$$

and

$$S = \bigcup_{j=1}^{m} \bigcup_{k \in \pi(M)} \bigcup_{k_1, k_2 \in F_0} \sigma(h_j^{-1} k k_1^{-1} k_2)^{-1} \sigma(h_j)^{-1} \sigma(k k_1^{-1} k_2). \tag{8C}$$

Since σ is a cross section, we have $S, T \subset \langle t \rangle$. Moreover, ST is finite, so there exists $n \in \mathbb{N}$ such that

$$ST \subset \{t^i \mid -n \leq i \leq n\}. \tag{8D}$$

We set

$$F_1 = \{t^i \mid -3n \leq i \leq 3n\}, \quad F = \sigma(F_0) \cdot F_1 \tag{8E}$$

and

$$g_{i,j} = t^{4ni} \sigma(h_j) \tag{8F}$$

for $i = -1, 0, 1$ and $j = 1, \ldots, m$. We will show that the set F and the elements $g_{i,j}$ satisfy the desired property of the assertion.

Let us choose an element $x \in F^{-1} F$. By (8E), this element has the form

$$x = t^{l_1 - l_2} \sigma(k_1)^{-1} \sigma(k_2) \tag{8G}$$

for certain $k_1, k_2 \in F_0$ and $-3n \leq l_1, l_2 \leq 3n$. By assumption, we have

$$\pi(M) k_1^{-1} k_2 \subset h_{j_0} F_0 \tag{8H}$$

for some $j_0 \in \{1, \ldots, m\}$. For the number

$$i_0 = \begin{cases} -1 & \text{if } -6n \leq l_2 - l_1 < -2n, \\ 0 & \text{if } -2n \leq l_2 - l_1 \leq 2n, \\ 1 & \text{if } 2n < l_2 - l_1 \leq 6n, \end{cases}$$

we have

$$l_2 - l_1 + \{-n, \ldots, n\} \subset 4ni_0 + \{-3n, \ldots, 3n\}. \tag{8I}$$

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Combining all this, we observe that

$$\begin{aligned}
Mx &\overset{(8A),(8G)}{=} \bigsqcup_{k\in\pi(M)} M_k\cdot\sigma(k)t^{l_1-l_2}\sigma(k_1)^{-1}\sigma(k_2)\\
&= \bigsqcup_{k\in\pi(M)} \sigma(k)\sigma(k_1)^{-1}\sigma(k_2)M_k t^{l_1-l_2}\\
&= \bigsqcup_{k\in\pi(M)} \sigma(kk_1^{-1}k_2)\sigma(kk_1^{-1}k_2)^{-1}\sigma(k)\sigma(k_1)^{-1}\sigma(k_2)M_k t^{l_1-l_2}\\
&\overset{(8B)}{\subset} \bigsqcup_{k\in\pi(M)} \sigma(kk_1^{-1}k_2)Tt^{l_1-l_2}\\
&= \bigsqcup_{k\in\pi(M)} \sigma(h_{j_0})\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)^{-1}\sigma(h_{j_0})^{-1}\sigma(kk_1^{-1}k_2)Tt^{l_1-l_2}\\
&\overset{(8C)}{\subset} \bigsqcup_{k\in\pi(M)} \sigma(h_{j_0})\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)STt^{l_1-l_2}\\
&\overset{(8H)}{\subset} \sigma(h_{j_0})\sigma(F_0)STt^{l_1-l_2}\\
&\overset{(8D),\,(8I),\,(8E)}{\subset} \sigma(h_{j_0})\sigma(F_0)F_1t^{4ni_0}\\
&\overset{(8E),\,(8F)}{=} g_{i_0,j_0}\cdot F.
\end{aligned}$$

By the definition of the elements $g_{i,j}$ in (8F), we see that there are $3m = 3^{\ell+1}$ lower indices. This finishes the induction step and the proof. □

Recall the following marker property lemma.

LEMMA 8.2. (See [**71**, 3.8, 4.4]) *Let G be a countable, discrete group and X a compact metric space of finite covering dimension $d\in\mathbb{N}$. Let $\alpha: G\curvearrowright X$ be a free action. Let $F\Subset G$ be a finite set, and let $g_1,\dots,g_d\in G$ be elements such that the sets*

$$F^{-1}F,\ g_1F^{-1}F,\ \dots,\ g_dF^{-1}F\Subset G$$

are pairwise disjoint. Set $g_0=1_G$. Then there exists an open set $Z\subset X$ such that

$$X=\bigcup_{l=0}^{d}\bigcup_{g\in F^{-1}F}\alpha_{gl g}(Z)$$

and

$$\alpha_g(\overline{Z})\cap\alpha_h(\overline{Z})=\emptyset\quad \textit{for all } g\neq h \textit{ in } F.$$ □

We need one more lemma characterizing the value $\dim_{\mathrm{am}}(-)$ in the case of a topological action.

LEMMA 8.3. *Let G be a countable, discrete group, X a locally compact Hausdorff space, and $d\in\mathbb{N}$ a natural number. Let $\alpha: G\curvearrowright X$ be a free action and $\bar{\alpha}: G\curvearrowright C_0(X)$ the induced action on the C^*-algebra. Then* $\dim_{\mathrm{am}}(\bar{\alpha})\le d$ *if and only if for every $\varepsilon>0$, every finite subset $M\Subset G$ and every compact subset $K\subset X$, there exist finitely supported maps $\mu^{(l)}: G\to C_c(X)_{1,+}$ for $l=0,\dots,d$ satisfying:*

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(a) $\sum_{l=0}^{d}\sum_{g\in G}\mu_g^{(l)} \le \mathbf{1}$ *and* $\sum_{l=0}^{d}\sum_{g\in G}\mu_g^{(l)}|_K = 1$*;*
(b) $\mu_g^{(l)}\mu_h^{(l)} = 0$ *for all* $l = 0, \dots, d$ *and* $g \neq h$ *in* G*;*
(c) $\mu_h^{(l)} \circ \alpha_{g^{-1}} =_\varepsilon \mu_{gh}^{(l)}$ *for all* $l = 0, \dots, d$, $g \in M$ *and* $h \in G$.
When X *is compact, it suffices to check the conditions for* $K = X$.

Proof. Suppose $\dim_{\mathrm{am}}(\bar{\alpha}) \le d$. Then given any $\varepsilon > 0$, any finite subset $M \Subset G$ and any compact subset $K \subset X$, there exists a quasicentral approximate unit for $\mathcal{C}_0(X) \rtimes G$ inside $\mathcal{C}_c(X)$, so there is $a \in \mathcal{C}_c(X)_{1,+}$ such that $a|_K = 1$ and $a \circ \alpha_{g^{-1}} =_\varepsilon a$ for all $g \in M$. Let us assume without loss of generality that $\varepsilon \le 1/4$. Setting $F = \{a\}$, we obtain finitely supported maps $\nu^{(l)} : G \to \mathcal{C}_c(X)_{1,+},\ g \mapsto \nu_g^{(l)}$ for $l = 0, \dots, d$ satisfying the conditions in Definition 7.1 for M, ε and F. In particular, we have $\sum_{l=0}^{d}\sum_{g\in G}\mu_g^{(l,j)}|_K =_\varepsilon 1$. Now we define, for $l = 0, \dots, d$, the functions

$$\mu^{(l)} = \frac{\nu^{(l)} \cdot a}{\max\{1/2, \sum_{j=0}^{d} \nu^{(j)} \cdot a\}}.$$

A direct computation shows that the maps $\mu^{(l)}$ satisfy (a) and (b) in the statement of the lemma. As for (c), we see that

$$(\nu^{(l)} \cdot a) \circ \alpha_{g^{-1}} =_\varepsilon \nu^{(l)} \circ \alpha_{g^{-1}} \cdot a =_\varepsilon \nu^{(l)} \cdot a$$

and similarly

$$\max\left\{\frac{1}{2}, \sum_{j=0}^{d} \nu^{(j)} \cdot a\right\} \circ \alpha_{g^{-1}} =_{(d+2)\varepsilon} \max\left\{\frac{1}{2}, \sum_{j=0}^{d} \nu^{(j)} \cdot a\right\},$$

which implies that

$$\begin{aligned}
\mu_h^{(l,j)} \circ \alpha_{g^{-1}} &= \frac{(\nu^{(l)} \cdot a) \circ \alpha_{g^{-1}}}{\max\{1/2, \sum_{j=0}^{d} \nu^{(j)} \cdot a\} \circ \alpha_{g^{-1}}} \\
&=_{4\varepsilon} \frac{\nu^{(l)} \cdot a}{\max\{1/2, \sum_{j=0}^{d} \nu^{(j)} \cdot a\} \circ \alpha_{g^{-1}}} \\
&=_{4(d+2)\varepsilon} \frac{\nu^{(l)} \cdot a}{\max\{1/2, \sum_{j=0}^{d} \nu^{(j)} \cdot a\}} \\
&= \mu_h^{(l,j)}.
\end{aligned}$$

Note that in the second estimate, we used the fact that the denominator is at least $\frac{1}{2}$. Since ε was arbitrary, we see that condition (c) is also satisfied.

To prove the converse, again let $\varepsilon > 0$ and finite subsets $M \Subset G$ and $F \Subset \mathcal{C}_0(X)$ be given. Without loss of generality, we may assume $F \subset \mathcal{C}_0(X)_1$. Choose a compact set $K \subset X$ such that for any $a \in F$ and $x \in X \setminus K$, we have $|a(x)| < \varepsilon$. By our assumption, we may find finitely supported maps $\mu^{(l)} : G \to \mathcal{C}_c(X)_{1,+}$ for $l = 0, \dots, d$ satisfying conditions (a), (b), and (c) in the statement of the lemma. It is clear that the functions $\mu^{(l)}$, for $l = 0, \dots, d$, satisfy conditions (a), (b) and (d) in Definition 7.1 (the last condition is automatic). For condition (c), we see that for any $l \in \{0, \dots, d\}$, any finite subset $E \Subset G$ and any $x \in X$, the set

$$\{h \in E \mid \mu_{gh}^{(l)}(x) \neq 0\}$$

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consists of at most one element because of orthogonality. Thus

$$\left|\sum_{h\in E}(\bar{\alpha}_g(\mu_h^{(l)})-\mu_{gh}^{(l)})(x)\right| \le \sum_{h\in E_x\cup E_{\alpha_{g^{-1}}(x)}} |(\bar{\alpha}_g(\mu_h^{(l)})-\mu_{gh}^{(l)})(x)| \le 2\varepsilon.$$

Since ε was arbitrary, we conclude that $\dim_{\mathrm{am}}(\bar{\alpha}) \le d$. □

Combining these lemmas, we can obtain the following result.

THEOREM 8.4. *Let G be a finitely generated, infinite, nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free action. Then the induced* C*-*algebraic action* $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$ *satisfies*

$$\dim_{\mathrm{am}}^{+1}(\bar{\alpha}) \le 3^{\ell_{\mathrm{Hir}}(G)} \cdot \dim^{+1}(X).$$

Proof. Let d be the covering dimension of X and ℓ the Hirsch length of G, and denote $m = 3^\ell$. Suppose we are given $\varepsilon > 0$ and finite sets $M \Subset G$ and $F \Subset \mathcal{C}(X)$. Without loss of generality, we may assume that F consists of contractions. We will show that there exist finitely supported maps $\mu^{(l,j)} : G \to \mathcal{C}(X)_{1,+}$ for $l = 0, \ldots, d$ and $j = 1, \ldots, m$ satisfying:

(a) $\sum_{l=0}^{d}\sum_{j=1}^{m}\sum_{g\in G}\mu_g^{(l,j)} = \mathbf{1}$;

(b) $\mu_g^{(l,j)}\mu_h^{(l,j)} = 0$ for all $l = 0, \ldots, d$, $j = 1, \ldots, m$ and $g \ne h$ in G;

(c) $\mu_h^{(l,j)} \circ \alpha_{g^{-1}} =_\varepsilon \mu_{gh}^{(l,j)}$ for all $l = 0, \ldots, d$, $j = 1, \ldots, m$, $g \in M$ and $h \in G$.

By Lemma 8.3, these conditions are sufficient to imply $\dim_{\mathrm{am}}^{+1}(\bar{\alpha}) \le m(d+1)$. Since G is amenable, we can choose a Følner set

$$J \Subset G \text{ with } |J\Delta gJ| \le \varepsilon|J| \quad \text{for all } g \in M. \tag{8J}$$

By Lemma 8.1, we can find a finite subset $F \Subset G$ containing the unit and $h_1, \ldots, h_m \in G$ such that for every $x \in F^{-1}F$, we have $Jx \subset h_jF$ for some $j \in \{1, \ldots, m\}$. Since G is infinite and nilpotent, so is its center by Remark 4.4. Thus we can find $g_1, \ldots, g_d$ in the centre of G such that the sets

$$F^{-1}F,\ g_1F^{-1}F,\ \ldots,\ g_dF^{-1}F \Subset G$$

are pairwise disjoint. By Lemma 8.2, there exists an open set $Z \subset X$ with

$$\alpha_g(\overline{Z}) \cap \alpha_h(\overline{Z}) = \emptyset \quad \text{for all } g \ne h \text{ in } F \tag{8K}$$

and

$$X = \bigcup_{l=0}^{d}\ \bigcup_{g\in F^{-1}F} \alpha_{g_lg}(Z). \tag{8L}$$

For $l = 0, \ldots, d$ and $j = 1, \ldots, m$, define the finite sets

$$N^{(l,j)} = \{g \in G \mid Jg \subset g_lh_jF\} \Subset G.$$

For each $g \in N^{(l,j)}$, define the open set

$$Z^{(l,j,g)} = \alpha_g(Z) \subset X.$$

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We now claim that

$$X = \bigcup_{l=0}^{d} \bigcup_{j=1}^{m} \bigcup_{g \in N^{(l,j)}} Z^{(l,j,g)}.$$

For any $x \in X$, we can apply (8L) and find some $l \in \{0, \dots, d\}$ and $g \in F^{-1}F$ with $x \in \alpha_{g_l g}(Z)$. By our choice of F and the elements $h_1, \dots, h_m$, we can find $j \in \{1, \dots, m\}$ with $Jg \subset h_j F$. But then $Jg_l g \subset g_l h_j F$, so we see that $g_l g \in N^{(l,j)}$ and $x \in Z^{(l,j,g_l g)}$.

We may find a partition of unity

$$\{\nu^{(l,j,g)} \mid g \in N^{(l,j)}, l = 0, \dots, d, j = 1, 2, \dots, m\}$$

of X subordinate to the open cover†

$$\{Z^{(l,j,g)} \mid g \in N^{(l,j)}, l = 0, \dots, d, j = 1, 2, \dots, m\}.$$

For the sake of convenience, let us set $\nu^{(l,j,g)} = 0$ for all $l = 0, \dots, d$ and $j = 1, \dots, m$, whenever $g \notin N^{(l,j)}$.

For every $l = 0, \dots, d$ and $j = 1, \dots, m$, we now define the finitely supported maps $\mu^{(l,j)} : G \to \mathcal{C}(X)_{1,+}$ via

$$\mu_g^{(l,j)} = \frac{1}{|J|} \sum_{h \in J} \nu^{(l,j,h^{-1}g)} \circ \alpha_{h^{-1}}.$$

We claim that these satisfy the desired properties. Firstly, the support of each map $\mu^{(l,j)}$ is contained in $J \cdot N^{(l,j)}$, and is hence finite. Secondly, since each map $\nu^{(l,j,g)}$ has values in $[0, 1]$, so does each $\mu_g^{(l,j)}$ by the triangle inequality. We have

$$\begin{aligned}
\sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \mu_g^{(l,j)} &= \sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \frac{1}{|J|} \sum_{h \in J} \nu^{(l,j,h^{-1}g)} \circ \alpha_{h^{-1}} \\
&= \frac{1}{|J|} \sum_{h \in J} \Bigg(\underbrace{\sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \nu^{(l,j,h^{-1}g)}}_{=\mathbf{1}} \Bigg) \circ \alpha_{h^{-1}} \\
&= \mathbf{1}.
\end{aligned}$$

It follows that condition (a) holds.

Next we observe that for all $l = 0, \dots, d$, $j = 1, \dots, m$ and $g \in G$, the open support of $\mu_g^{(l,j)}$ satisfies

$$\begin{aligned}
\operatorname{supp}(\mu_g^{(l,j)}) &\subset \bigcup_{h \in J} \alpha_h(\operatorname{supp}(\nu^{(l,j,h^{-1}g)})) \\
&\subset \begin{cases} \displaystyle\bigcup_{h \in J} \alpha_h(\alpha_{h^{-1}g}(Z)) = \alpha_g(Z) & \text{if } g \in J \cdot N^{(l,j)}, \\ \emptyset & \text{if } g \notin J \cdot N^{(l,j)}. \end{cases}
\end{aligned}$$

By definition of the set $N^{(l,j)}$, we have $J \cdot N^{(l,j)} \subset g_l h_j F$. So let $g' \in G$ be an element different from g. If $g \notin J \cdot N^{(l,j)}$ or $g' \notin J \cdot N^{(l,j)}$, then we trivially have $\mu_g^{(l,j)} \mu_{g'}^{(l,j)} = 0$

† Note that we allow repetitions with this notation.

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because one of the functions is zero already. If $g, g' \in J \cdot N^{(l,j)} \subset g_l h_j F$, then we can write $g = g_l h_j f_1$ and $g' = g_l h_j f_2$ for some $f_1 \neq f_2$ in F. By (8K), it follows that $\alpha_{f_1}(Z) \cap \alpha_{f_2}(Z) = \emptyset$, and thus $\alpha_g(Z) \cap \alpha_{g'}(Z) = \emptyset$. In particular, the two functions $\mu_g^{(l,j)}$ and $\mu_{g'}^{(l,j)}$ have disjoint open supports and are therefore orthogonal. This verifies condition (b).

Lastly, for all $l = 0, \ldots, d$, $j = 1, 2, \ldots, m$, $g \in M$ and $g' \in G$, we calculate

$$\begin{aligned}
&\|\mu_{g'}^{(l,j)} \circ \alpha_{g^{-1}} - \mu_{gg'}^{(l,j)}\| \\
&= \frac{1}{|J|}\left\| \sum_{h\in J} \nu^{(l,j,h^{-1}g')} \circ \alpha_{h^{-1}} \circ \alpha_{g^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} \right\| \\
&= \frac{1}{|J|}\left\| \sum_{h\in J} \nu^{(l,j,(gh)^{-1}gg')} \circ \alpha_{(gh)^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} \right\| \\
&= \frac{1}{|J|}\left\| \sum_{h\in gJ} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} \right\| \\
&= \frac{1}{|J|}\left\| \sum_{h\in gJ\setminus J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} - \sum_{h\in J\setminus gJ} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} \right\| \\
&\leq \frac{1}{|J|}\left(\sum_{h\in gJ\setminus J} \|\nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\| + \sum_{h\in J\setminus gJ} \|\nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\| \right) \\
&\leq \frac{|J \Delta gJ|}{|J|} \overset{(8J)}{\leq} \varepsilon.
\end{aligned}$$

This verifies condition (c) and finishes the proof. □

COROLLARY 8.5. *Let G be a finitely generated, infinite, nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free action. Then the* C*-*algebraic action* $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$ *has finite Rokhlin dimension, and in fact*

$$\dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}) \leq 3^{\ell_{\mathrm{Hir}}(G)} \cdot \dim^{+1}(X).$$

Proof. This follows directly from Lemma 8.4 and Theorem 7.2. □

COROLLARY 8.6. *Let G be a finitely generated, infinite, nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free action. Then the transformation group* C*-algebra $\mathcal{C}(X) \rtimes_\alpha G$ has finite nuclear dimension, and in fact*

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}(X) \rtimes_\alpha G) \leq 3^{\ell_{\mathrm{Hir}}(G)} \cdot \mathrm{asdim}^{+1}(\square_s G) \cdot \dim^{+1}(X)^2.$$

Proof. This follows directly from Corollary 8.5 and Theorems 4.10 and 6.2. □

Remark 8.7. We remark that one can generalize Corollaries 8.5 and 8.6 to actions on locally compact spaces. A rigorous treatment of this can be found in the first author's dissertation [**70**], which follows a very similar approach, using a generalized version of Lemma 8.2. (Note that a variant of this can be found in the appendix of [**29**].)

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Remark 8.8. Through recent groundbreaking progress in the Elliott programme by many hands, the combined main results of [**13**, **14**, **19**, **74**] have completed the classification of separable, unital, simple C*-algebras that satisfy the UCT and have finite nuclear dimension. Now transformation group C*-algebras of amenable groups are well known to satisfy the UCT due to [**79**], and they are further known to be simple if the action is free and minimal [**34**]. Therefore, the main result of this section implies the classifiability of a large class of simple transformation group C*-algebras.

THEOREM 8.9. *Let G be a finitely generated, infinite, nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free and minimal action. Then the transformation group* C*-*algebra* $C(X) \rtimes_\alpha G$ *is a simple approximately subhomogeneous algebra of topological dimension at most 2.*

9. *Amenability dimension for topological actions*

In this section we show that for topological actions, the invariant $\dim_{\mathrm{am}}(\bar{\alpha})$ from Definition 7.1 coincides with a dimension concept studied by Guentner, Willett and Yu in [**21**]†. We remark that before this work, a similar concept had been defined, at least implicitly, in the pioneering work [**4**] of Bartels, Lück and Reich on the Farrell–Jones conjecture for hyperbolic groups.

Definition 9.1. Let G be a countable, discrete group and let X be a compact metric space. Let $\alpha : G \curvearrowright X$ be an action. Then we denote by $\Delta : G \curvearrowright G \times X$ the diagonal action defined by $\Delta_g(h, x) = (hg^{-1}, \alpha_g(x))$.

Definition 9.2. (See [**21**, 4.10, 4.14]) Let G be a countable, discrete group and let X be a compact metric space. Let $\alpha : G \curvearrowright X$ be a free action. The *amenability dimension* of α, denoted by $\dim_{\mathrm{am}}(\alpha)$, is defined to be the smallest natural number d with the property that for any finite subset $M \Subset G$, there exists an open cover $\mathcal{U}$ of $G \times X$ satisfying:

(a) $\mathcal{U}$ is a so-called *free G-cover* with regard to the diagonal action Δ, that is, for any $U \in \mathcal{U}$ and $1 \neq g \in G$, we have $\Delta_g(U) \in \mathcal{U}$ and $U \cap \Delta_g(U) = \emptyset$;

(b) the multiplicity of $\mathcal{U}$ is at most $d + 1$;

(c) for any $x \in X$, the set $M \times \{x\}$ is contained in a member of $\mathcal{U}$.

If no such d exists, we write $\dim_{\mathrm{am}}(\alpha) = \infty$.

Next we shall see that this definition is a special case of Definition 7.1, thus reconciling the potential conflict of notation. To this end, let us fix some terminology.

Definition 9.3. For us, an *abstract G-simplicial complex* Z consists of:

- a set Z_0, called the set of *vertices*, which comes equipped with a G-action; and
- a G-invariant collection of its finite subsets closed under taking subsets, called the collection of *simplices*.

We often write $\sigma \in Z$ to denote that σ is a simplex of Z. The dimension of a simplex is the cardinality of the corresponding finite subset minus 1, and the dimension of the abstract G-simplicial complex is the supremum of the dimensions of its simplices. The *geometric*

† Note that instead of writing '$\dim_{\mathrm{am}}(\bar{\alpha}) \leq d$' they write '$\alpha$ is d-BLR'. Our terminology is inspired by an earlier draft of their paper. Another closely related notion they introduced in the same paper is termed 'dynamic asymptotic dimension'.

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realization of an abstract G-simplicial complex Z, denoted by $|Z|$, is the set of formal sums

$$|Z| = \left\{ \sum_{v \in \sigma} \lambda_v v \;\middle|\; \sigma \in Z \text{ and } \lambda_v \geq 0 \text{ with } \sum_{v \in \sigma} \lambda_v = 1 \right\}.$$

By convention, λ_v is defined to be zero for $v \in Z_0 \setminus \sigma$. Similarly, for a simplex σ of Z, we define its geometric realization $|\sigma|$ to be

$$\left\{ \sum_{v \in \sigma} \lambda_v v \;\middle|\; \lambda_v \geq 0 \text{ with } \sum_{v \in \sigma} \lambda_v = 1 \right\} \subset |Z|.$$

The space $|Z|$ carries a natural G-action, where

$$g \cdot \sum_{v \in \sigma} \lambda_v v = \sum_{v \in \sigma} \lambda_v (g \cdot v) \in |g \cdot \sigma|$$

for any $g \in G$, $\sigma \in Z$ and $\sum_{v \in \sigma} \lambda_v v \in |\sigma|$. If this action is free, then we call Z an abstract *free* G-simplicial complex. Usually Z is equipped with the weak topology, but for our purposes we consider the ℓ^1-topology, induced by the ℓ^1-metric $d^1 : Z \times Z \to [0, 2]$ given by

$$d^1\left(\sum_{v \in \sigma} \lambda_v v, \sum_{v \in \sigma'} \mu_v v \right) = \sum_{v \in \sigma \cup \sigma'} |\mu_v - \lambda_v|.$$

This metric is obviously G-invariant.

LEMMA 9.4. *Let G be a countable, discrete group and let X be a compact Hausdorff space. Let $\alpha : G \curvearrowright X$ be a free action and $d \in \mathbb{N}$ a natural number. Then the following statements are equivalent.*

(1) *The amenability dimension of α is at most d.*
(2) *For every $M \Subset G$ and $\varepsilon > 0$, there exists a free G-Simplicial complex Z of dimension at most d together with a continuous map $\varphi : X \to |Z|$ that is (M, ε)-approximately equivariant†, i.e. $d^1(\varphi(\alpha_g(x)), g \cdot \varphi(x)) \leq \varepsilon$ for all $x \in X$ and $g \in M$.*
(3) *For the induced action $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$, we have $\dim_{\mathrm{am}}(\bar{\alpha}) \leq d$.*

Proof. (1) $\implies$ (2). We follow the proof of [**21**, 4.6, (ii) $\implies$ (i)], though some extra care is necessary as the authors of that paper did not discuss free simplicial complexes, but rather complexes whose *vertex stabilizers* are from a fixed collection $\mathcal{F}$ of subgroups of G (e.g. the collection of all finite subgroups; see [**21**, 4.3]).

Let $M \Subset G$ and $\varepsilon > 0$ be given. Replacing M by $M \cup M^{-1} \cup \{1\}$, we may assume that M is symmetric and contains the identity element. Pick n with

$$\frac{(2d+2)(4d+6)}{n} < \varepsilon.$$

Since the amenability dimension of α is at most d, we can find a cover $\mathcal{U}$ of $G \times X$ as in Definition 9.2 with M replaced by M^n. Let Z be the nerve of this cover, i.e. $Z_0 = \mathcal{U}$ and a finite subset $\{U_1, U_2, \ldots, U_k\}$ of $\mathcal{U}$ is in Z if and only if $\bigcap_{i=1}^k U_i \neq \varnothing$. Since $\mathcal{U}$ is G-invariant, we see that Z becomes a G-simplicial complex under the G-action on the vertex set $\mathcal{U}$. The dimension of Z is simply the multiplicity of $\mathcal{U}$ minus 1 and thus is at most d. We also observe that the action of G on $|Z|$ is free. Indeed, if this were not the case, we

† In [**21**], the terminology '(M, ε)-equivariant' is used instead.

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would be able to find a non-trivial element $g \in G$ and a point z such that $g \cdot z = z$. Write $z = \sum_{v \in \sigma} \lambda_v v$ with $\lambda_v > 0$ for each $v \in \sigma$. Then the simplex σ is fixed by g, too. Thus picking an element U in σ, viewed as an element of $\mathcal{U}$, we would have $g \cdot U \cap U \neq \varnothing$, which contradicts the fact that $\mathcal{U}$ is a free G-cover. In summary, Z is a free G-simplicial complex of dimension at most d.

The construction of the continuous map $\varphi\colon X \to |Z|$ is identical to that in the proof of [**21**, 4.6, (ii) $\Longrightarrow$ (i)] and is thus omitted.

(2) $\Longrightarrow$ (3). Given any free G-simplicial complex Z of dimension no more than d, there is a canonical G-invariant cover $\mathcal{U}$ constructed as follows. For all $l = 0, \ldots, d$ and every l-dimensional simplex σ of Z, we define an open subset

$$U_\sigma = \left\{ \sum_{v \in Z_0} \lambda_v v \;\middle|\; \lambda_v > \lambda_{v'} \text{ for all } v \in \sigma \text{ and } v' \in Z_0 \setminus \sigma \right\}.$$

Then $U_{g\cdot\sigma} = g \cdot U_\sigma$ and $U_\sigma \cap U_{\sigma'} = \emptyset$ for $\sigma \neq \sigma'$ of the same dimension. Now $\mathcal{U}$ is defined to be the collection $\mathcal{U} = \{U_\sigma \mid \sigma \in Z\}$, which gives us an open cover of $|Z|$ of multiplicity at most $d + 1$.

This allows us to define a partition of unity $\{\nu_\sigma\}_{\sigma \in Z}$ subordinate to $\mathcal{U}$ by setting

$$\nu_\sigma : |Z| \to [0, 1], \quad z \mapsto \frac{d^1(z, |Z| \setminus U_\sigma)}{\sum_{\sigma' \in Z} d^1(z, |Z| \setminus U_{\sigma'})}.$$

Observe that $\nu_{g\cdot\sigma} = g \cdot \nu_\sigma$ for all $\sigma \in Z$ and $\nu_\sigma \cdot \nu_{\sigma'} = 0$, whenever $\sigma \neq \sigma'$ are two simplices of the same dimension.

Moreover, we claim that each ν_σ is Lipschitz with respect to the constant $(d + 1)(d + 2)$. Since $|Z|$ is a geodesic space, with geodesics being piecewise linear paths in $|Z|$, it suffices to show that all the directional derivatives of ν_σ along linear paths are bounded by this constant. Observe that for any $z = \sum_{v \in Z_0} \lambda_v v \in U_\sigma$, the point in $|Z| \setminus U_\sigma$ that is closest to z differs from z only by replacing one of the $(\dim(\sigma) + 1)$th greatest coordinates of z and one of the $(\dim(\sigma) + 2)$th greatest coordinates of z by the average of the two values, and thus we have

$$d^1(z, |Z| \setminus U_\sigma) = \lambda^{(\dim(\sigma)+1)} - \lambda^{(\dim(\sigma)+2)},$$

where $\lambda^{(l)}$ is the value of the lth greatest coordinate of z for all $l \in \mathbb{N}$. Hence for any $z = \sum_{v \in Z_0} \lambda_v v \in |Z|$, we have

$$\nu_\sigma(z) = \chi_{U_\sigma}(z) \cdot \frac{\lambda^{(\dim(\sigma)+1)} - \lambda^{(\dim(\sigma)+2)}}{\sum_{l=2}^{\mathrm{card}(Z_0)} (\lambda^{(l-1)} - \lambda^{(l)})} = \chi_{U_\sigma}(z) \cdot \frac{\lambda^{(\dim(\sigma)+1)} - \lambda^{(\dim(\sigma)+2)}}{\lambda^{(1)}},$$

where χ_{U_σ} is the characteristic function for U_σ and $\mathrm{card}(Z_0)$ is the cardinality of Z_0 (in fact, we are only dealing with a finite sum). Since z has no more than $d + 1$ positive terms and they sum to 1, it follows that $\lambda^{(1)} \geq 1/(d + 1)$. Thus for any finite sum $w = \sum_{v \in Z_0} \eta_v v$ with $\sum_{v \in Z_0} |\eta_v| = 1$ and $z + tw \in |Z|$ for any t in a small neighborhood of 0 in $\mathbb{R}^{\geq 0}$, we have

$$\left| \frac{d\nu_\sigma(z + tw)}{dt} \bigg|_{t=0^+} \right| \leq \frac{1}{\lambda^{(1)}} + \frac{1}{(\lambda^{(1)})^2} \leq (d + 1)(d + 2),$$

which proves what we claimed†.

† Another way to give an estimate of the Lipschitz constant is by showing that the cover $\mathcal{U}$ has Lebesgue number at least $2/(d + 1)(d + 2)$ and use [**53**, 4.3.5].

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Now given any finite subset $M \Subset G$ and $\varepsilon > 0$, we apply the assumption to get a free G-simplicial complex Z of dimension no more than d and a $(M^{-1}, \varepsilon/2(d+1)(d+2))$-approximately equivariant continuous map $\varphi : X \to |Z|$. Since X is compact, by [**21**, 4.5], we may assume without loss of generality that the image of φ intersects finitely many simplices in Z. Let $\{\nu_\sigma\}$ be the partition constructed above. For every $l = 0, \ldots, d$, pick a set $\Sigma^{(l)}$ of representatives for the G-action on the set of l-dimensional simplices in Z, and define $\nu^{(l)} = \sum_{\sigma \in \Sigma^{(l)}} \nu_\sigma$. Since there is only one non-zero summand at every point $z \in |Z|$, the sum is well defined and we see that each $\nu^{(l)}$ is also Lipschitz with respect to the constant $(d+1)(d+2)$. Additionally, we have $\nu^{(l)} \cdot (g \cdot \nu^{(l)}) = 0$ for all $g \in G \setminus \{1\}$ and

$$\sum_{l=0}^{d} \sum_{g \in G} (g \cdot \nu^{(l)})(z) = 1 \quad \text{for all } z \in |Z|.$$

We define the maps $\mu^{(l)} : G \to \mathcal{C}(X)_+$ as the pullbacks $\mu_g^{(l)} = (g \cdot \nu^{(l)}) \circ \varphi$ for all $l = 0, \ldots, d$ and $g \in G$. We claim that these maps satisfy the conditions in Lemma 8.3. As assumed above, the image of φ intersects only finitely many simplices in Z. Since, moreover, each simplex intersects only finitely many U_σs, it follows that the image of φ intersects only finitely many U_σs and thus there are only finitely many $g \in G$ such that $\mu_g^{(l)} \neq 0$. Conditions (a) and (b) follow directly from the above properties of the collection $\{\nu^{(l)}\}_{l=0,\ldots,d}$. Finally, we check that for all $l = 0, \ldots, d$, $g \in M$, $h \in G$ and $x \in X$, we have

$$\begin{aligned} \bar{\alpha}_g(\mu_h^{(l)})(x) &= \mu_h^{(l)}(\alpha_{g^{-1}}(x)) \\ &= (h \cdot \nu^{(l)}) \circ \varphi \circ \alpha_{g^{-1}}(x) \\ &=_{\varepsilon/2} (h \cdot \nu^{(l)})(g^{-1} \cdot \varphi(x)) \\ &= (gh \cdot \nu^{(l)}) \circ \varphi(x) = \mu_{gh}^{(l)}(x). \end{aligned}$$

Note that for the intermediate approximation step, we have used the fact that the function $h \cdot \nu^{(l)}$ is Lipschitz with respect to the constant $(d+1)(d+2)$, combined with $d^1(g^{-1} \cdot \varphi(x), \varphi \circ \alpha_{g^{-1}}(x)) \leq \varepsilon/2(d+1)(d+2)$. Hence condition (c) follows.

(3) $\Longrightarrow$ (1). Given any finite subset $M \Subset G$, we use the assumption to get finitely supported maps $\mu^{(l)} : G \to \mathcal{C}(X)_+$ satisfying the conditions in Lemma 8.3 for $(M, 1/(d+1))$. For $l = 0, \ldots, d$, define an open set

$$U^{(l)} = \{(g, x) \in G \times X \mid \mu_g^{(l)}(\alpha_g(x)) > 0\}$$

and put $\mathcal{U}^{(l)} := \{\Delta_g(U^{(l)}) \mid g \in G\}$. We claim that $\mathcal{U} := \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(d)}$ is an open cover satisfying the conditions in Definition 9.2.

First, we observe that the G-invariance of $\mathcal{U}$ is built into the definition. Now it holds for any $(h, x) \in G \times X$, $l = 0, \ldots, d$ and $g \in G$ that $(h, x) \in \Delta_g(U^{(l)})$ if and only if $\mu_{hg}^{(l)}(\alpha_h(x)) > 0$. Since for two distinct $g, g' \in G$, one has $\mu_{hg}^{(l)} \mu_{hg'}^{(l)} = 0$, it follows that (h, x) cannot be in both $\Delta_g(U^{(l)})$ and $\Delta_{g'}(U^{(l)})$. This proves that for each $l = 0, \ldots, d$, the collection $\mathcal{U}^{(l)}$ is G-invariant and consists of pairwise disjoint sets.

Finally, we observe that for any $(h, x) \in G \times X$, there exist $g_0 \in G$ and $l_0 \in \{0, \ldots, d\}$ such that $\mu_{hg_0}^{(l)}(\alpha_h(x)) \geq 1/(d+1)$. This is because we have $1 = \sum_{l=0}^{d} \sum_{g \in G} \mu_g^{(l)}(\alpha_h(x))$

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and the collections of functions $\{\mu_g^{(l)} \mid g \in G\}$ consist of pairwise orthogonal functions for all $l = 0, \ldots, d$. It follows from condition (c) that for any $b \in M$,

$$\mu_{bhg_0}^{(l)}(\alpha_{bh}(x)) > \mu_{hg_0}^{(l)}(\alpha_h(x)) - \frac{1}{d+1} \geq 0.$$

But this is equivalent to $(bh, x) \in \Delta_{g_0}(U^{(l)})$. Hence it follows that $Mh \times \{x\}$ is contained in a member of $\mathcal{U}$. In particular, $\mathcal{U}$ indeed covers $G \times X$. This yields all the conditions in order to deduce $\dim_{\mathrm{am}}(\alpha) \leq d$. □

It turns out that amenability dimension for free actions is a notion that generally behaves well with respect to the nuclear dimension of the transformation group C*-algebra. The following result is due to Guentner, Willett and Yu.

THEOREM 9.5. (See [**21**, 8.6, 4.11]) *Let G be a countable, discrete group and X a compact metric space. Let $\alpha : G \curvearrowright X$ be a free action. Then*

$$\dim_{\mathrm{nuc}}^{+1}(C(X) \rtimes_\alpha G) \leq \dim_{\mathrm{am}}^{+1}(\alpha) \cdot \dim^{+1}(X).$$

This theorem allows us to provide a better estimate for Corollary 8.6 when X is compact. In view of Definition 7.1 and Lemma 9.4, it is interesting to ask in what generality this formula holds beyond the commutative case, with $\dim_{\mathrm{am}}(\alpha)$ replaced by $\dim_{\mathrm{am}}(\bar{\alpha})$.

COROLLARY 9.6. *Let G be a finitely generated, infinite nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free action. Then the transformation group C*-algebra $C(X) \rtimes_\alpha G$ has finite nuclear dimension, and in fact*

$$\dim_{\mathrm{nuc}}^{+1}(C(X) \rtimes_\alpha G) \leq 3^{\ell_{\mathrm{Hirsch}}(G)} \cdot \dim^{+1}(X)^2.$$

Proof. This follows directly from Theorem 8.4, Lemma 9.4 and Theorem 9.5. □

Remark 9.7. Some time after the initial preprint version of this paper was published, Bartels [**3**] independently proved a more general version of 8.4 with slightly different methods. In particular, his result [**3**, 1.10] in conjunction with Theorem 9.5 implies that the finiteness part of Theorem 8.4 holds for virtually nilpotent groups, although no specific estimate is obtained this way. Using his results instead of ours, we thus obtain the following improvement of Theorem 8.9.

THEOREM 9.8. *Let G be a finitely generated, infinite, virtually nilpotent group and X a compact metric space of finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free and minimal action. Then the C*-algebra $C(X) \rtimes_\alpha G$ is a simple approximately subhomogeneous algebra of topological dimension at most 2.*

10. *Rokhlin dimension with commuting towers*

In this section we consider the notion of Rokhlin dimension with commuting towers, in analogy to what has been done in [**28**, §5]. Similarly to [**28**, §5], it turns out that cocycle actions with finite Rokhlin dimension with commuting towers preserve $\mathcal{Z}$-stability. Moreover, borrowing a technique from [**26**], this turns out to be true even for $\mathcal{D}$-stability, for any strongly self-absorbing C*-algebra $\mathcal{D}$. Let us first recall some relevant notions.

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Definition 10.1. (See [**77**, 1.3]) A separable, unital C*-algebra $\mathcal{D}$ is called strongly self-absorbing if there is an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, i.e. there is a sequence of unitaries $u_n \in \mathcal{D} \otimes \mathcal{D}$ such that

$$\varphi(x) = \lim_{n\to\infty} u_n(x \otimes \mathbf{1}_{\mathcal{D}})u_n^*$$

for all $x \in \mathcal{D}$.

A C*-algebra A is called $\mathcal{D}$-stable if $A \cong A \otimes \mathcal{D}$.

Strongly self-absorbing C*-algebras are always simple and nuclear. Known examples are the Jiang–Su algebra $\mathcal{Z}$, any UHF algebra of infinite type, the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$, as well as countable tensor products of these. They play an important role in the Elliott classification programme in that $\mathcal{D}$-stability can be considered as an important regularity property. Most notably, $\mathcal{Z}$-stability appears as a prerequisite for a C*-algebra to be inside a classifiable class. We are thus led to the problem of determining cases in which $\mathcal{D}$-stability is preserved under forming crossed products.

Next we introduce the main definition of the section, which is a strengthening of Remark 5.6 and Definition 5.8.

Definition 10.2. Let A be a separable C*-algebra, G a residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Let $H \subset G$ be a subgroup of finite index. We say that α has Rokhlin dimension d with commuting towers and relative to H, and write $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, H) = d$, if d is the smallest number such that there exist equivariant order-zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \to (F_\infty(A), \tilde{\alpha}_\infty) \quad (l = 0, \ldots, d)$$

with pairwise commuting ranges and $\mathbf{1} = \varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1})$.

Let $\sigma = (G_n)_n \in \Lambda(G)$ be a regular approximation of G. We define

$$\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \sigma) = \sup_{n\in\mathbb{N}} \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, G_n)$$

and

$$\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = \sup\{\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, H) \mid H \subset G, [G : H] < \infty\}.$$

Before we can prove the main theorem of the section, we need two results from the literature. The first may be called a folklore result. Note that its unital case was already implicitly at the core of the results of [**32**, 4.6], [**48**, 4.8] and [**50**, 4.9]. The proof essentially goes by a similar method to those in [**77**, 2.3], [**36**, 4.11] or [**62**, 2.3.5, 7.2.1, 7.2.2]. A detailed treatment has been given by the first author in [**73**] for more general (cocycle) actions.

THEOREM 10.3. (See [**73**, 3.8]) *Let G be a countable, discrete group, A a separable C*-algebra and let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. Let $(\alpha, w) : G \curvearrowright A$ be a cocycle action. Then (α, w) is cocycle conjugate to $(\alpha \otimes \mathrm{id}_{\mathcal{D}}, w \otimes \mathbf{1}_{\mathcal{D}})$ if and only if there exists a unital $*$-homomorphism from $\mathcal{D}$ to the fixed point algebra $F_\infty(A)^{\tilde{\alpha}_\infty}$.*

The second is an important technical lemma for proving the existence of $*$-homomorphisms from strongly self-absorbing C*-algebras.

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LEMMA 10.4. (See [**26**, 5.8]) *Let $\mathcal{D}$ be a strongly self-absorbing* C*-*algebra. Let B be a unital* C*-*algebra. Suppose that $\psi_1, \ldots, \psi_n : \mathcal{D} \to B$ are c.p.c. order-zero maps with pairwise commuting ranges such that $\psi_1(\mathbf{1}_{\mathcal{D}}) + \cdots + \psi_n(\mathbf{1}_{\mathcal{D}}) = \mathbf{1}_B$. Then there exists a unital $*$-homomorphism from $\mathcal{D}$ to B.*

Remark 10.5. We note that one can use [**37**, 7.6] to see that Lemma 10.4 holds, if one replaces $\mathcal{D}$ either by a matrix algebra or a dimension drop algebra. This would be enough to prove Theorem 10.6 below for $\mathcal{D}$ being either $\mathcal{Z}$ or a UHF algebra of infinite type†.

The main theorem of the section is a generalization of [**28**, 5.8, 5.9].

THEOREM 10.6. *Let $\mathcal{D}$ be a strongly self-absorbing* C*-*algebra and A a separable, $\mathcal{D}$-stable* C*-*algebra. Let G be a countable, residually finite group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Assume that $\sigma \in \Lambda(G)$ is a regular approximation with* $\operatorname{asdim}(\square_\sigma G) < \infty$ *and* $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \sigma) < \infty$. *Then (α, w) is cocycle conjugate to $(\alpha \otimes \mathrm{id}_{\mathcal{D}}, w \otimes \mathbf{1}_{\mathcal{D}})$, and in particular the twisted crossed product $A \rtimes_{\alpha,w} G$ is $\mathcal{D}$-stable.*

Proof. Denote $s = \operatorname{asdim}(\square_\sigma G)$ and $d = \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \sigma)$. Denote $\sigma = (G_n)_n$. As A is $\mathcal{D}$-stable, there exists a unital $*$-homomorphism from $\mathcal{D}$ to $F_\infty(A)$. By a standard reindexation trick (such as [**36**, 1.13]), we may find unital embeddings $\iota_{l,j} : \mathcal{D} \to F_\infty(A)$ for $l = 0, \ldots, d$ and $j = 0, \ldots, s$ such that

$$[\iota_{l_1,j_1}(x), \tilde{\alpha}_{\infty,g}(\iota_{l_2,j_2}(y))] = 0 \quad \text{for all } x, y \in \mathcal{D},\ g \in G \text{ and } (l_1, j_1) \neq (l_2, j_2).$$

Let $M \Subset G$ and $\varepsilon > 0$ be arbitrary. By Lemma 3.13(3), we can find n and finitely supported functions $\nu^{(j)} : G \to [0, 1]$ for $j = 0, \ldots, s$ satisfying:

(a) for every $j = 0, \ldots, s$, one has

$$\operatorname{supp}(\nu^{(j)}) \cap \operatorname{supp}(\nu^{(j)})h = \emptyset \quad \text{for all } h \in G_n \setminus \{1\};$$

(b) for every $g \in G$, one has

$$\sum_{j=0}^{s} \sum_{h \in G_n} \nu^{(j)}(gh) = 1;$$

(c) for every $j = 0, \ldots, s$ and $g \in M$, one has

$$\|\nu^{(j)} - \nu^{(j)}(g \cdot _)\|_\infty \leq \varepsilon.$$

For ease of notation for the rest of the proof, we will write $\nu^{(j)}_h = \nu^{(j)}(h)$ for all $j = 0, \ldots, s$ and $h \in G$.

By definition, there are equivariant c.p.c. order-zero maps

$$\varphi_l : (\mathcal{C}(G/G_n), G\text{-shift}) \to (F_\infty(A), \tilde{\alpha}_\infty) \quad (l = 0, \ldots, d)$$

with pairwise commuting ranges and $\mathbf{1} = \varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1})$. Without loss of generality, we may assume that the image of each map φ_l commutes with the image of each map of the form $\tilde{\alpha}_{\infty,g} \circ \iota_{l',j}$ for $l, l' = 0, \ldots, d$, $j = 0, \ldots, s$ and $g \in G$.

† In fact, this was done in a previous preprint version of this paper.

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For each $\bar{g} \in G/G_n$, let $e_{\bar{g}} \in C(G/G_n)$ denote the characteristic function of $\{\bar{g}\}$. For each $j = 0, \ldots, s$ and $l = 0, \ldots, d$, define the maps $\psi_{l,j} : \mathcal{D} \to F_\infty(A)$ via

$$\psi_{l,j} = \sum_{g \in G} \nu_g^{(j)} \cdot \varphi_l(e_{\bar{g}}) \cdot \tilde{\alpha}_{\infty,g} \circ \iota_{l,j}.$$

Then each of the maps $\psi_{l,j}$ is c.p.c. order zero. They also have pairwise commuting ranges because we have

$$\|[\psi_{l_1,j_1}(x_1), \psi_{l_2,j_2}(x_2)]\| \leq \sup_{g_1,g_2 \in G} \|[\tilde{\alpha}_{\infty,g_1}(\iota_{l_1,j_1}(x_1)), \tilde{\alpha}_{\infty,g_2}(\iota_{l_2,j_2}(x_2))]\| = 0$$

for all $(l_1, j_1) \neq (l_2, j_2)$ and all $x_1, x_2 \in \mathcal{D}$. Moreover, we have

$$\begin{aligned}
\sum_{l=0}^{d} \sum_{j=0}^{s} \psi_{l,j}(\mathbf{1}) &= \sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{g \in G} \nu_g^{(j)} \cdot \varphi_l(e_{\bar{g}}) \cdot \tilde{\alpha}_{\infty,g} \circ \iota_{l,j}(\mathbf{1}) \\
&= \sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{g \in G} \nu_g^{(j)} \cdot \varphi_l(e_{\bar{g}}) \\
&= \sum_{l=0}^{d} \sum_{\bar{g} \in G/G_n} \varphi_l(e_{\bar{g}}) \cdot \sum_{j=0}^{s} \sum_{h \in \bar{g}} \nu_h^{(j)} \\
&= \sum_{l=0}^{d} \varphi_l(\mathbf{1}) = \mathbf{1}.
\end{aligned}$$

Condition (c) of the functions $\nu^{(j)}$ ensures that for all $l = 0, \ldots, d$, $j = 0, \ldots, s$ and $g \in M$, we have $\|\psi_{l,j} - \tilde{\alpha}_{\infty,g} \circ \psi_{l,j}\| \leq \varepsilon$.

Since $M \Subset G$ and $\varepsilon > 0$ were arbitrary, we can apply a standard diagonal sequence argument to construct new c.p.c. order-zero maps $\psi_{l,j} : \mathcal{D} \to F(A)^{\tilde{\alpha}_\infty}$ for $l = 0, \ldots, d$ and $j = 0, \ldots, s$ with pairwise commuting ranges and

$$\sum_{l=0}^{d} \sum_{j=0}^{s} \psi_{l,j}(\mathbf{1}) = \mathbf{1}.$$

We can now apply Lemma 10.4 to obtain the existence of some unital $*$-homomorphism from $\mathcal{D}$ to $F_\infty(A)^{\tilde{\alpha}_\infty}$. The claim now follows from Theorem 10.3. □

Remark 10.7. The usefulness of a result like Theorem 10.6 is showcased in the following two non-trivial applications.

(1) Let G be a finite group. If $\alpha : G \curvearrowright \mathcal{O}_2$ is an arbitrary action with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, then it follows from Theorem 10.6 that α absorbs the trivial G-action on $\mathcal{O}_2$. Thus by [**30**, 4.2], α is uniquely determined up to conjugacy. In fact, the same argument works for residually finite groups with $\mathrm{asdim}(\square_s G) < \infty$ due to [**72**, 5.5].

(2) In contrast to the above, nice torsion-free groups such as $G = \mathbb{Z}^n$ exhibit a different behaviour. There are results such as [**44**, **45**] showing that strongly outer actions of G on nice, classifiable finite C*-algebras have finite Rokhlin dimension; although commuting towers are not arranged in the cited articles, this will be done in

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the first author's forthcoming work. It then follows from Theorem 10.6 that equivariant $\mathcal{D}$-absorption is an automatic phenomenon for nice, torsion-free group actions on classifiable $\mathcal{D}$-stable C*-algebras, as one would expect from K-theoretic considerations; cf. [**72**, 4.12 and 4.16].

In light of the fact that all outer G-actions on a Kirchberg algebra have Rokhlin dimension at most one (see [**72**, 6.3]), the above discussion supports the idea that, as compared to the condition of finite Rokhlin dimension, finite Rokhlin dimension with commuting towers is a more rigid notion when the acting group has torsion. See [**25**], where such a phenomenon was discovered much earlier by K-theoretic methods.

11. *Genericity of finite Rokhlin dimension on $\mathcal{Z}$-stable* C*-*algebras*

In this section we investigate the genericity of cocycle actions with finite Rokhlin dimension on C*-algebras absorbing either $\mathcal{Z}$ or a certain UHF algebra of infinite type, in analogy to what has been done in [**28**, §3]. As we will see, the main argument in the proof of genericity relies on a more general principle that has nothing to do with Rokhlin dimension in particular. For instance, we also prove that for every strongly self-absorbing C*-algebra $\mathcal{D}$ and every separable, $\mathcal{D}$-stable C*-algebra A, a cocycle action on A will generically absorb the trivial action on $\mathcal{D}$ tensorially.

Let us first establish some notation.

Definition 11.1. Let G be a countable, discrete group and A a C*-algebra. Consider the set $\mathrm{CoAct}(G, A)$ of all cocycle actions $(\alpha, w) : G \curvearrowright A$. For a pair of $F \Subset A$ and $M \Subset G$, the map

$$d_{F,M} : \mathrm{CoAct}(G, A) \times \mathrm{CoAct}(G, A) \to \mathbb{R}^{\geq 0}$$

given by

$$d_{F,M}((\alpha^1, w^1), (\alpha^2, w^2)) = \max_{a \in F, g \in M} \|\alpha_g^1(a) - \alpha_g^2(a)\| + \max_{a \in F, g \in M} (\|(w_g^1 - w_g^2)a\| + \|a(w_g^1 - w_g^2)\|)$$

defines a pseudometric on $\mathrm{CoAct}(G, A)$. We equip $\mathrm{CoAct}(G, A)$ with the topology induced by all these pseudometrics. In other words, a net $(\alpha^\lambda, w^\lambda)$ of cocycle G-actions on a C*-algebra A converges to a cocycle action (α, w), if $\alpha_g^\lambda \to \alpha_g$ in point-norm and $w_g^\lambda \to w_g$ strictly, for all $g \in G$.

Remark 11.2. By choosing a countable, dense set, one can see that when A is separable, this topology is metrizable, and in fact $\mathrm{CoAct}(G, A)$ becomes a Polish space with any such metric. A subset of cocycle actions in $\mathrm{CoAct}(G, A)$ is called generic, if it contains a countable intersection of dense open subsets.

PROPOSITION 11.3. *Let G be a countable, discrete group and A a separable* C*-*algebra. Let $\mathcal{D}$ be a strongly self-absorbing* C*-*algebra. Assume that $A \cong A \otimes \mathcal{D}$. Let $d \in \mathbb{N}$ be a natural number. Then:*

(1) *The set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ that are cocycle conjugate to $(\alpha \otimes \mathrm{id}_{\mathcal{D}}, w \otimes \mathbf{1}_{\mathcal{D}})$ forms a G_δ-subset in* $\mathrm{CoAct}(G, A)$.

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(2) *For every subgroup $H \subset G$ of finite index, the set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$ forms a G_δ-subset in* $\mathrm{CoAct}(G, A)$.

(3) *Assume that G is residually finite. Given a regular approximation σ of G, the set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha, \sigma) \leq d$ forms a G_δ-subset in* $\mathrm{CoAct}(G, A)$.

(4) *Assume that G is residually finite and finitely generated. The set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha) \leq d$ forms a G_δ-subset in* $\mathrm{CoAct}(G, A)$.

Proof. (1) We make use of Theorem 10.3. Let $S_n \Subset \mathcal{D}$ be an increasing sequence of finite subsets whose union is dense in $\mathcal{D}$, and let $F_n \Subset A_1$ be an increasing sequence of finite subsets whose union is dense in the unit ball of A. Let $M_n \Subset G$ be an increasing sequence of finite subsets whose union is G. For each $n \in \mathbb{N}$, define the set $V_n \subset \mathrm{CoAct}(G, A)$ by saying that $(\alpha, w) \in V_n$, if there exists a $*$-linear map $\varphi : \mathcal{D} \to A$ satisfying:

(a) $\|[a, \varphi(d)]\| < 1/n$ for all $a \in F_n$ and $d \in S_n$;

(b) $\|(\varphi(d_1 d_2) - \varphi(d_1)\varphi(d_2))a\| < 1/n$ for all $a \in F_n$ and $d_1, d_2 \in S_n$;

(c) $\|\varphi(d)a\| < ((n+1)/n)\|d\|$ for all $a \in F_n$ and $d \in S_n$;

(d) $\|\varphi(\mathbf{1}_{\mathcal{D}})a - a\| < 1/n$ for all $a \in F_n$;

(e) $\|((\alpha_g \circ \varphi)(d) - \varphi(d))a\| < 1/n$ for all $a \in F_n$, $d \in S_n$ and $g \in M_n$.

Looking at the above conditions (in fact only the last one), it is clear that each V_n defines an open subset of $\mathrm{CoAct}(G, A)$ with respect to the topology from Definition 11.1. A cocycle action $(\alpha, w) \in \mathrm{CoAct}(G, A)$ admits a unital $*$-homomorphism $\mathcal{D} \to F_\infty(A)^{\tilde{\alpha}_\infty}$ if and only if $(\alpha, w) \in \bigcap_{n \in \mathbb{N}} V_n$.

(2) In a similar fashion to (1), this follows directly from the equivalent reformulation of $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$ in Proposition 5.5.

(3) Let $\sigma = (G_n)_n$. By definition, we have that

$$\{(\alpha, w) \in \mathrm{CoAct}(G, A) \mid \dim_{\mathrm{Rok}}(\alpha, \sigma) \leq d\}$$

coincides with

$$\bigcap_{n \in \mathbb{N}} \{(\alpha, w) \in \mathrm{CoAct}(G, A) \mid \dim_{\mathrm{Rok}}(\alpha, G_n) \leq d\}.$$

So it follows from (2) that one obtains a G_δ-set.

(4) follows from (3) by applying Proposition 3.20 and Remark 5.9 and inserting a dominating regular approximation. □

Notation 11.4. Let G be a countable, discrete group. Consider the category $\mathcal{C}_G$ of all (twisted) C*-dynamical systems (A, α, w) for separable C*-algebras A and cocycle actions $(\alpha, w) : G \curvearrowright A$, where the morphisms are equivariant, non-degenerate $*$-homomorphisms respecting the cocycles. We have a natural (maximal) tensor product between (twisted) C*-dynamical systems given by

$$(A, \alpha, w) \otimes_{\max} (B, \beta, v) = (A \otimes_{\max} B, \alpha \otimes_{\max} \beta, w \otimes v).$$

Now let $\mathcal{S}$ be a strictly full† subcategory of $\mathcal{C}_G$. We call $\mathcal{S}$ tensorially absorbing if whenever $(\alpha, w) : G \curvearrowright A$ and $(\beta, v) : G \curvearrowright B$ are cocycle actions with (B, β, v) in $\mathcal{S}$,

† This means that $\mathcal{S}$ retains all morphisms from $\mathcal{C}_G$ between its objects and that $\mathcal{S}$ is closed under isomorphism in $\mathcal{C}_G$. In this case, two systems are isomorphic if they are conjugate.

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then $(A, \alpha, w) \otimes_{\max} (B, \beta, v)$ is also in $\mathcal{S}$. Since we will only apply this in the case where B is nuclear, the choice of maximal tensor product is rather arbitrary.

Let us record a general observation that will lead to the genericity of various properties of cocycle actions.

THEOREM 11.5. *Let G be a countable, discrete group. Let $\mathcal{S}$ be a strictly full subcategory of $\mathcal{C}_G$ that is tensorially absorbing in the above sense. Let A and $\mathcal{D}$ be two separable* C**-algebras with $\mathcal{D}$ being unital and strongly self-absorbing. Assume that $A \cong A \otimes \mathcal{D}$. If there is some action $\gamma : G \curvearrowright \mathcal{D}$ such that $(\mathcal{D}, \gamma)$ is in $\mathcal{S}$, then the set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ yielding an object in $\mathcal{S}$ is dense in* $\mathrm{CoAct}(G, A)$.

Proof. Let $(\alpha, w) : G \curvearrowright A$ be a cocycle action. Let $\varphi : A \to A \otimes \mathcal{D}$ be an isomorphism that is approximately unitarily equivalent to $\mathrm{id}_A \otimes \mathbf{1}$ (see [**77**, 2.3]). Let u_n be a sequence of unitaries in the unitalization $(A \otimes \mathcal{D})^{\sim}$ with $u_n \varphi(a) u_n^* \overset{n\to\infty}{\longrightarrow} a \otimes \mathbf{1}$ for all $a \in A$. Put $\varphi_n = \mathrm{Ad}(u_n) \circ \varphi$. Then for every pair of non-degenerate $*$-endomorphisms $\psi : A \to A$ and $\theta : \mathcal{D} \to \mathcal{D}$, we have for all $x \in A$ that

$$\begin{aligned}
&\|\varphi_n^{-1} \circ (\psi \otimes \theta) \circ \varphi_n(x) - \psi(x)\| \\
&\quad \leq \|\varphi_n(x) - x \otimes \mathbf{1}\| + \|\varphi_n^{-1}(\psi(x) \otimes \mathbf{1}) - \psi(x)\| \\
&\quad = \|\varphi_n(x) - x \otimes \mathbf{1}\| + \|u_n^*(\psi(x) \otimes \mathbf{1}) u_n - \varphi(\psi(x))\| \\
&\quad \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

An analogous calculation shows that $\varphi_n^{-1}(m \otimes \mathbf{1})$ converges to m strictly for every $m \in \mathcal{M}(A)$. In particular, the sequence of cocycle actions

$$(\alpha^{(n)}, w^{(n)}) = (\varphi_n^{-1} \circ (\alpha \otimes \gamma) \circ \varphi_n, \varphi_n^{-1}(w \otimes \mathbf{1}))$$

converges to (α, w) in the topology from Definition 11.1. By the assumptions on $\mathcal{S}$, each cocycle action $(\alpha^{(n)}, w^{(n)})$ yields an object in $\mathcal{S}$. This finishes the proof. □

Remark. By the same proof as above, an analogous variant of Theorem 11.5 is true even if G is assumed to be a σ-compact, locally compact group. One merely has to pay attention to the topology on the set of all point-norm continuous cocycle G-actions on a C*-algebra, since it is defined in a slightly different way than in the discrete case (Definition 11.1).

COROLLARY 11.6. *Let G be a countable, discrete group and A a separable* C**-algebra. Let $\mathcal{D}$ be a strongly self-absorbing* C**-algebra. Assume that $A \cong A \otimes \mathcal{D}$. Then the set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ that are cocycle conjugate to $(\alpha \otimes \mathrm{id}_{\mathcal{D}}, w \otimes \mathbf{1}_{\mathcal{D}})$ is generic in* $\mathrm{CoAct}(G, A)$. *Hence so is the set of all cocycle actions $(\alpha, w) : G \curvearrowright A$ the crossed products $A \rtimes_{\alpha,w} G$ of which are $\mathcal{D}$-stable.*

Proof. The only thing left to show is that such cocycle actions are dense in $\mathrm{CoAct}(G, A)$. But this follows from Theorem 11.5. Firstly, the given property of cocycle actions is trivially invariant under conjugation, and in fact defines a strictly full and tensorially absorbing subcategory of $\mathcal{C}_G$. Secondly, the trivial action of G on $\mathcal{D}$ is clearly conjugate to its tensorial product with itself. □

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PROPOSITION 11.7. *Let G be a countable, discrete and residually finite group with a regular approximation σ. Let $d \in \mathbb{N}$ be a natural number. Then the full subcategory defined by all (twisted) C^*-dynamical systems (A, α, w) with $\dim_{\mathrm{Rok}}(\alpha, \sigma) \leq d$ is strictly full and tensorially absorbing.*

Proof. This follows directly from Remark 5.6 and Definition 5.8. □

Remark 11.8. Let G be a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a regular approximation consisting of normal subgroups. For each $n \in \mathbb{N}$, denote by $\lambda_n : \Gamma_n \to \mathcal{U}(M_{|\Gamma_n|})$ the left-regular representation of $\Gamma_n = G/G_n$. For each $n \in \mathbb{N}$, let $\pi_n : G \twoheadrightarrow \Gamma_n$ be the canonical surjection. Let $M_\sigma = \bigotimes_{n=1}^{\infty} M_{|\Gamma_n|^\infty}$ and define $\beta^\sigma : G \curvearrowright M_\sigma$ via $\beta^\sigma_g = \bigotimes_{n=1}^{\infty}(\bigotimes_{\mathbb{N}} \mathrm{Ad}((\lambda_n \circ \pi_n)(g)))$. By design, this yields an action of Rokhlin dimension zero along σ.

THEOREM 11.9. *Let G be a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a regular approximation consisting of normal subgroups. Assume that A is a separable C^*-algebra with $A \cong A \otimes M_\sigma$. Then the cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha, \sigma) = 0$ are generic in $\mathrm{CoAct}(G, A)$.*

Proof. By Proposition 11.3, we only have to show that said actions are dense. But this follows directly from Theorem 11.5, Proposition 11.7 and Remark 11.8. □

THEOREM 11.10. *Let G be a discrete, finitely generated and residually finite group. Assume that A is a separable C^*-algebra that tensorially absorbs the universal UHF algebra. Then the cocycle actions $(\alpha, w) : G \curvearrowright A$ with Rokhlin dimension zero are generic in $\mathrm{CoAct}(G, A)$.*

Definition 11.11. Let Γ be a finite group. Identifying $\mathcal{Z} \cong \bigotimes_\Gamma \mathcal{Z}$, we define the noncommutative Bernoulli shift $\gamma^\Gamma : \Gamma \curvearrowright \mathcal{Z}$ via

$$\gamma^\Gamma_g\left(\bigotimes_{h\in\Gamma} a_h\right) = \bigotimes_{h\in\Gamma} a_{gh}.$$

PROPOSITION 11.12. *For all finite groups Γ, the action γ^Γ has Rokhlin dimension 1.*

Proof. First, let us observe

$$\bigotimes_{g\in\Gamma} \mathcal{Z} \cong \bigotimes_{g\in\Gamma}\left(\bigotimes_{\mathbb{N}} \mathcal{Z}\right) \cong \bigotimes_{\mathbb{N}}\left(\bigotimes_{g\in\Gamma} \mathcal{Z}\right).$$

With this rearrangement of tensor factors, we get that γ^Γ is conjugate to $\bigotimes_{\mathbb{N}} \gamma^\Gamma$. Since γ^Γ is a faithful action, its infinite tensor power is pointwise strongly outer. Now it follows from [**48**, 3.4] that γ^Γ has the weak Rokhlin property in the sense of [**48**, **50**]. Let $\sigma : \Gamma \curvearrowright \mathcal{C}(\Gamma)$ denote the canonical Γ-shift action. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. In this particular instance, the weak Rokhlin property means that there is an equivariant c.p.c. order-zero map

$$\psi : (\mathcal{C}(\Gamma), \sigma) \to (\mathcal{Z}_\omega, \gamma^\Gamma_\omega)$$

with $\tau_{\mathcal{Z},\omega}(\psi(\mathbf{1})) = 1$. (Recall the definition of $\mathcal{Z}_\omega$, γ^Γ_ω from the first section.)

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Now notice that the fixed point algbera $\mathcal{Z}^{\gamma^\Gamma}$ is separable, unital, nuclear, simple and has a unique tracial state. Moreover, the element $\psi(\mathbf{1})$ is an element of $(\mathcal{Z}^{\gamma^\Gamma})_\omega$ and has full trace. Let $h_0 \in \mathcal{Z}$ be a positive contraction with full spectrum $[0, 1]$, and define $h_1 = \mathbf{1}_\mathcal{Z} - h_0$. It follows from [**67**, 4.3] that there are unitaries u_0 and u_1 in $(\mathcal{Z}^{\gamma^\Gamma} \otimes \mathcal{Z})_\omega$ with

$$u_i(\psi(\mathbf{1}) \otimes h_i)u_i^* = \mathbf{1} \otimes h_i \quad \text{for } i = 0, 1.$$

Viewing these as unitaries in $(\mathcal{Z} \otimes \mathcal{Z})_\omega$, we define the two c.p.c. order-zero maps

$$\psi_0, \psi_1 : \mathcal{C}(\Gamma) \to (\mathcal{Z} \otimes \mathcal{Z})_\omega \quad \text{via } \psi_i(x) = u_i(\psi(x) \otimes h_i)u_i^* \text{ for } i = 0, 1.$$

As ψ was equivariant and the unitaries u_0, u_1 are both fixed by $(\gamma^\Gamma \otimes \mathrm{id}_\mathcal{Z})_\omega$, this actually yields two equivariant c.p.c. order-zero maps

$$\psi_0, \psi_1 : (\mathcal{C}(\Gamma), \sigma) \to ((\mathcal{Z} \otimes \mathcal{Z})_\omega, (\gamma^\Gamma \otimes \mathrm{id})_\omega) \quad \text{with } \psi_0(\mathbf{1}) + \psi_1(\mathbf{1}) = \mathbf{1}.$$

Now by [**48**, 4.7, 4.8], γ^Γ is conjugate to $\gamma^\Gamma \otimes \mathrm{id}_\mathcal{Z}$. Since γ^Γ is also conjugate to its own infinite tensor power, this verifies $\dim_{\mathrm{Rok}}(\gamma^\Gamma) \leq 1$. Note that $\dim_{\mathrm{Rok}}(\gamma^\Gamma) = 0$ is impossible, since $\mathcal{Z}$ contains no non-trivial projections. □

Remark 11.13. Let G be a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a regular approximation consisting of normal subgroups. For each $n \in \mathbb{N}$, let $\Gamma_n = G/G_n$ and $\pi_n : G \twoheadrightarrow \Gamma_n$ the canonical surjection. Identifying $\mathcal{Z} \cong \bigotimes_\mathbb{N} \mathcal{Z}$, we define

$$\gamma^\sigma : G \curvearrowright \mathcal{Z} \quad \text{via } \gamma_g^\sigma = \bigotimes_{n \in \mathbb{N}} \gamma^{\Gamma_n}_{\pi_n(g)}.$$

Notice that by design (i.e. by Proposition 11.12), this action has Rokhlin dimension 1.

THEOREM 11.14. *Let G be a countable, discrete and residually finite group with a chosen regular approximation σ consisting of normal subgroups. Let A be a separable* C**-algebra with $A \cong A \otimes \mathcal{Z}$. Then the cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha, \sigma) \leq 1$ are generic in* $\mathrm{CoAct}(G, A)$. *If G is finitely generated, then the cocycle actions $(\alpha, w) : G \curvearrowright A$ with $\dim_{\mathrm{Rok}}(\alpha) \leq 1$ are generic in* $\mathrm{CoAct}(G, A)$.

Proof. By Proposition 11.3, we only have to show that said actions form a dense subset. But this follows directly from Theorem 11.5, Proposition 11.7 and Remark 11.13. □

Remark 11.15. It is very straightforward to see that if one restricts one's attention to genuine actions, then the genericity results of this section (in particular, Corollary 11.6, Proposition 11.10, Theorem 11.14 and the general principle, Theorem 11.5) remain true within the Polish space of all genuine G-actions on a separable C*-algebra A. In order to modify the proofs, one merely has to omit the cocycles.

Remark 11.16. We would like to emphasize that unlike most of the previous results concerning genericity (such as in [**28**] or [**59**]), the results of this section were proved rather directly. In particular, we did not have to appeal to a Baire category argument in order to show denseness of any subsets in $\mathrm{CoAct}(G, A)$.

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