Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 157 SEPTEMBER 2014 PART 2

Math. Proc. Camb. Phil. Soc. (2014), **157**, 189–207 © *Cambridge Philosophical Society* 2014
doi:10.1017/S0305004114000255
First published online 2 June 2014

A decomposition for Hardy martingales II

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(*Received* 29 *September* 2013)

Abstract

We prove Davis and Garsia Inequalities for dyadic perturbations of Hardy martingales and show that those inequalities play a substantial role in the proof of Bourgain's [**1**] embedding $L^1 \hookrightarrow L^1/H^1_0$. This paper continues [**17**] on Davis and Garsia Inequalities (DGI).

1. *Introduction*

The introduction is divided into three separate parts. We first list preliminary material, standard notations and tools. Then we state our main result, the Davis and Garsia inequality (DGI) for dyadic perturbations of Hardy martingales. We single out the essential complex analytic component of our proof. Thereafter we revisit Bourgain's embedding [**1**] and give the motivation for working with dyadic perturbations of Hardy martingales.

We recall that Hardy martingales arise in complex and functional analysis. For instance, in renorming problems [**5**], isomorphic classification [**2**, **15**], factorization problems [**18**], similarity problems [**19**], boundary convergence of analytic functions [**7**, **13**, **12**], Jensen measures [**4**] and unconditional convergence [**8**, **9**].

† Supported by the Austrian Science foundation (FWF) Pr.Nr. FWFP23987000. Participant of the NSF sponsored *Workshop in Analysis and Probability* at Texas A & M University 2012 and 2013.

https://doi.org/10.1017/S0305004114000255 Published online by Cambridge University Press

1·1. *Preliminaries, notation, conventions*

Classical martingale spaces. Let $(\Omega, (\mathcal{G}_n), \mathbb{P})$ be a filtered probability space. For $F \in L^1(\Omega, \mathbb{P})$ the conditional expectation of F with respect to $\mathcal{G}_n$ we denote by $\mathbb{E}_{\mathcal{G}_n}(F)$ and (when suitable) we shorten $\mathbb{E}_{\mathcal{G}_n}(F)$ to $\mathbb{E}_n(F)$. Corresponding to the fixed filtration we define the well known spaces of martingales H^1, $\mathcal{P}$ and $\mathcal{A}$ by specifying their norms as follows. Let $G = (G_k)$ be an integrable $(\mathcal{G}_n)$ martingale. Define the previsible norm $\mathcal{P}$,

$$\|G\|_{\mathcal{P}} = \mathbb{E}\left(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2\right)^{1/2}, \tag{1·1}$$

the H^1 norm, and the absolutely summing norm $\mathcal{A}$,

$$\|G\|_{H^1} = \mathbb{E}\left(\sum_{k=1}^{n} |\Delta G_k|^2\right)^{1/2} \quad \text{and} \quad \|G\|_{\mathcal{A}} = \mathbb{E}\left(\sum_{k=1}^{n} |\Delta G_k|\right). \tag{1·2}$$

Clearly $\mathcal{A} \subseteq H^1$. By the Burkholder–Gundy inequality

$$\|G\|_{H^1} \leqslant 2\|G\|_{\mathcal{P}}, \tag{1·3}$$

we have as well $\mathcal{P} \subseteq H^1$. By contrast, the Davis and Garsia inequality for general martingales gives $H^1 \subseteq \mathcal{P} + \mathcal{A}$, that is, each $F \in H^1$ can be written as $F = G + B$ so that

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leqslant C\|F\|_{H^1}. \tag{1·4}$$

and

$$|\Delta G_k| \leqslant C \max_{m \leqslant k-1} |F_m|. \tag{1·5}$$

The B. Davis inequality asserts that $C^{-1}\|G\|_{H^1} \leqslant \mathbb{E}\sup_{k\leqslant n}|G_k| \leqslant C\|G\|_{H^1}$, hence $H^1 \subseteq L^1$.

We use [**11**] as our reference to martingale inequalities.

Martingales on $\mathbb{T}^{\mathbb{N}}$. Let $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$ be the torus equipped with the normalized angular measure. Let $\mathbb{T}^{\mathbb{N}}$ be its countable product equipped with its product Haar measure $\mathbb{P}$. We let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

Denote by $\mathcal{F}_n$ the sigma-algebra on $\mathbb{T}^{\mathbb{N}}$ generated by the cylinder sets $\{(A_1, \ldots, A_n, \mathbb{T}^{\mathbb{N}})\}$, where A_i, $i \leqslant n$ are measurable subsets of $\mathbb{T}$. Thus $(\mathbb{T}^{\mathbb{N}}, (\mathcal{F}_k), \mathbb{P})$ becomes a filtered probability space. We let $\mathbb{E}_n$ denote the conditional expectation with respect to the $\sigma-$algebra $\mathcal{F}_n$. Let $F = (F_k)$ be an $L^1(\mathbb{T}^{\mathbb{N}})-$bounded martingale. Conditioned on $\mathcal{F}_{k-1}$ the martingale difference $\Delta F_k = F_k - F_{k-1}$ defines an element in $L^1_0(\mathbb{T})$, the Lebesgue space of integrable, functions with vanishing mean.

Hardy martingales. Let $H^1_0(\mathbb{T}) \subset L^1_0(\mathbb{T})$ consist of those integrable functions for which the harmonic extension to the unit disk is analytic. And put $H^\infty_0(\mathbb{T}) = H^1_0(\mathbb{T}) \bigcap L^\infty(\mathbb{T})$. See [**10**].

An $L^1(\mathbb{T}^{\mathbb{N}})$ bounded $(\mathcal{F}_n)$ martingale $F = (F_k)$ is called a Hardy martingale if conditioned on $\mathcal{F}_{n-1}$ the martingale difference

$$\Delta F_n = F_n - F_{n-1}$$

defines an element in $H^1_0(\mathbb{T})$. See [**8**, **9**].

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Davis and Garsia inequalities for Hardy martingales. We review here results from [**1**, **3**, **17**]. The L^1 norm of a Hardy martingale $F = (F_k)_{k=1}^n$ is equivalent to the L^1 norm of its square function. This result of J. Bourgain [**1**, **3**] and a strengthened version thereof [**17**] are the starting point for this work. We have

$$\|F\|_{H^1} \leqslant C\|F\|\|_{L^1}, \tag{1·6}$$

and there exist Hardy martingales $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ so that

$$F = G + B, \tag{1·7}$$

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leqslant C\|F\|_{H^1}, \tag{1·8}$$

and

$$|\Delta G_k| \leqslant C|F_{k-1}|. \tag{1·9}$$

We refer to (1·6) as the square function inequality and to (1·7) — (1·9) as the Davis and Garsia inequalities for Hardy martingales.

The Hilbert transform. The Hilbert transform on $L^2(\mathbb{T})$ is defined as Fourier multiplier by

$$H(e^{in\theta}) = -i\operatorname{sign}(n)e^{in\theta}.$$

Let $h \in H_0^2(\mathbb{T})$ and let $y = \Im h$. The Hilbert transform recovers h from its imaginary part y, we have $h = -Hy + iy$. and $\|h\|_2 = \sqrt{2}\|y\|_2$. For $w \in \mathbb{C}$, $|w| = 1$ we have therefore

$$\|h\|_2 = \sqrt{2}\|\Im(w \cdot h)\|_2.$$

Dyadic martingales. The dyadic sigma-algebra on $\mathbb{T}^{\mathbb{N}}$ is defined with Rademacher functions. For $x = (x_k) \in \mathbb{T}^{\mathbb{N}}$ define $\cos_k(x) = \Re x_k$ and

$$\sigma_k(x) = \operatorname{sign}(\cos_k(x)).$$

We let $\mathcal{D}$ be the sigma-algebra generated by $\{\sigma_k, k \in \mathbb{N}\}$ and call it the dyadic sigma-algebra on $\mathbb{T}^{\mathbb{N}}$. The dyadic filtration on $\mathbb{T}^{\mathbb{N}}$ is given by $\mathcal{D}_k = \mathcal{F}_k \bigcap \mathcal{D}, k \in \mathbb{N}$. Let $D = (D_k)$ be an integrable $(\mathcal{F}_k)$ martingale in $\mathbb{T}^{\mathbb{N}}$. If each D_k is measurable with respect to $\mathcal{D}$, then we say that $D = (D_k)$ is a dyadic martingale on $\mathbb{T}^{\mathbb{N}}$. For $F \in L^1(\mathbb{T}^{\mathbb{N}})$ we denote the conditional expectation of F with respect to the dyadic σ-algebra by $\mathbb{E}_{\mathcal{D}}(F)$.

1·2. *The main result*

In the preliminary section we formulated in (1·4) - (1·5) the Davis and Garsia inequalities for a general martingale, and recalled that a Hardy martingale decomposes into the sum of two Hardy martingales for which the estimates (1·7) - (1·9) hold true. By inspection (1·7) - (1·9) are stronger than (1·4) - (1·5).

Now, suppose we perturb a Hardy martingale by a dyadic one, then the resulting sum is no longer a Hardy martingale, hence all we could say is that the perturbation satisfies the weaker estimates (1·4) - (1·5). While the class of Hardy martingales is not stable under dyadic perturbations we prove in this paper that for DGI (1·7) - (1·9) a certain degree of stability holds true. Our main result asserts that for dyadic perturbations of Hardy martingales there hold DGI that are the *geometric mean* of (1·4) - (1·5) and (1·7) - (1·9). (The precise statement is below.) We refer to Section 1·3 for a through discussion motivating our close analysis of dyadic perturbations of Hardy martingales.

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Let $F = (F_k)_{k=1}^n$ be a Hardy martingale, let $D = (D_k)_{k=1}^n$ be a dyadic martingale on $\mathbb{T}^{\mathbb{N}}$, and define the associated martingale transform as

$$T(H) = \mathfrak{I}\Big[\sum w_{k-1}\Delta H_k\Big], \qquad w_k = (\overline{F_k - D_k})/|F_k - D_k|. \tag{1·10}$$

The central result this paper are the Davis and Garsia inequalities for *dyadic perturbations* of Hardy martingales:

THEOREM. *There exist Hardy martingales G and B so that* $F = G + B$,

$$\|T(G-D)\|_{\mathcal{P}} \leqslant C\|F-D\|_{L^1}^{1/2}\|F-D\|_{H^1}^{1/2}, \tag{1·11}$$

$$\|B\|_{\mathcal{A}} \leqslant C\|F-D\|_{L^1} \tag{1·12}$$

and

$$\|G\|_{\mathcal{P}} \leqslant C\|F\|_{L^1} + C\|D\|_{H^1}. \tag{1·13}$$

See Theorem 4·4.

Martingale decomposition. The proof of (1·11) and (1·12) uses the following martingale decomposition obtained in Theorem 4·3: for each Hardy martingale $F = (F_k)_{k=1}^n$ and each dyadic martingale $D = (D_k)_{k=1}^n$ there exists a Hardy martingale $G = (G_k)_{k=1}^n$ so that

$$|\Delta G_k| \leqslant C_0|F_{k-1} - D_{k-1}| \tag{1·14}$$

and

$$|F_{k-1} - D_{k-1}| + \frac{1}{8}\mathbb{E}_{k-1}|\Delta B_k| \leqslant \mathbb{E}_{k-1}|F_k - D_k|, \tag{1·15}$$

where

$$B = F - G. \tag{1·16}$$

Specialization to $D = \mathbb{E}_{\mathcal{D}}F$

The main application of DGI for dyadic perturbations of Hardy martingales arises when $D = \mathbb{E}_{\mathcal{D}}(F)$. For ease of reference we now restate our main result for this case. We specialize the decomposition (1·14)– (1·16) and the estimates (1·11), (1·12), (1·13) to the case when $D = \mathbb{E}_{\mathcal{D}}(F)$:

For each Hardy martingale $F = (F_k)_{k=1}^n$ there exists a Hardy martingale $G = (G_k)_{k=1}^n$ so that

$$\|T(G - \mathbb{E}_{\mathcal{D}}G)\|_{\mathcal{P}} \leqslant C\|F - \mathbb{E}_{\mathcal{D}}F\|_{L^1}^{1/2}\|F\|_{L^1}^{1/2}, \tag{1·17}$$

where T is the martingale transform given by (1·10), and

$$\|B\|_{\mathcal{A}} \leqslant C\|F - \mathbb{E}_{\mathcal{D}}F\|_{L^1}, \quad \text{where} \quad B = F - G. \tag{1·18}$$

Moreover (1·17) gives

$$\|G\|_{\mathcal{P}} \leqslant C\|F\|_{L^1}. \tag{1·19}$$

1·3. *Motivation: the embedding* $L^1 \hookrightarrow L^1/H_0^1$

In this section we revisit Bourgain's embedding $L^1 \hookrightarrow L^1/H_0^1$. We show that DGI for dyadic perturbations of Hardy martingales play a substantial part in the proof of the embedding theorem.

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The embedding theorem of J. Bourgain [**1**] states that $L^1(\mathbb{T})$ is isomorphic to a subspace of $L^1(\mathbb{T})/H_0^1(\mathbb{T})$. The following L^1 estimate, separating a dyadic martingale from the space of Hardy martingales, is *the key* to Bourgain 's embedding [**1**]:

The separation estimate. There exists $A > 0$ such that for each dyadic martingale D,

$$\|D\|_{L^1(\mathbb{T}^{\mathbb{N}})} \leqslant A \inf \|D - F\|_{L^1(\mathbb{T}^{\mathbb{N}})}, \tag{1·20}$$

where the infimum is taken over all integrable Hardy martingales F. The proof of (1·20) consists of two structurally independent parts described next.

Part A. The *first* part of the proof utilizes general martingale inequalities (Lepingle's inequality and the Garnett–Jones theorem on the closure of L^∞ in BMO). It may be summarized in the following way, (see [**1**]):

For each Hardy martingale F and *each* adapted sequence $W = (w_k)$ satisfying $|w_k| = 1$,

$$\|\mathbb{E}_{\mathcal{D}} F\|_{L^1(\mathbb{T}^{\mathbb{N}})} \leqslant C \left(\|F - \mathbb{E}_{\mathcal{D}} F\|_{L^1(\mathbb{T}^{\mathbb{N}})}^{\beta} + \|T_W(F - \mathbb{E}_{\mathcal{D}} F)\|_{\mathcal{P}(\mathbb{T}^{\mathbb{N}})}^{\beta} \right) \|F\|_{\mathcal{P}(\mathbb{T}^{\mathbb{N}})}^{1-\beta}, \qquad \beta = 1/4, \tag{1·21}$$

where T_W is defined by

$$T_W(H) = \mathfrak{I}\left[\sum w_{k-1} \Delta H_k\right].$$

[**16**, appendix I] contains the detailed proof of (1·21). Note that (1·21) yields separation in $\mathcal{P}$ as follows

$$\|D\|_{L^1(\mathbb{T}^{\mathbb{N}})} \leqslant A \inf_W \inf_F \|T_W(D - F)\|_{\mathcal{P}(\mathbb{T}^{\mathbb{N}})}$$

where the infimum is extended over all Hardy martingales F and all adapted sequences $W = (w_k)$ with $|w_k| = 1$.

Part B. The *second* part exploits complex analytic tools. With the DGI as in (1·18) and (1·17) we are able to strengthen (1·21) as follows: For any Hardy martingale F,

$$\|\mathbb{E}_{\mathcal{D}} F\|_{L^1} \leqslant C \|F - \mathbb{E}_{\mathcal{D}}(F)\|_{L^1}^{\alpha} \|F\|_{L^1}^{1-\alpha}, \qquad \alpha = 1/8. \tag{1·22}$$

To emphasize, with DGI the term $\|T(F - \mathbb{E}_{\mathcal{D}} F)\|_{\mathcal{P}}$ will be removed from the right-hand side of (1·21) and $\|F\|_{\mathcal{P}}$ will be replaced by $\|F\|_{L^1}$.

By routine and elementary arguments, the L^1 estimate (1·20), separating dyadic martingales from Hardy martingales, is a consequence of (1·22).

Proving (1·22). We use the Davis and Garsia inequalities (1·18), (1·17) to pass from (1·21) to (1·22). The martingale transform (1·10) is the crucial link.

(1) Let F be a Hardy martingale. By homogenity, assume that $\|F\|_{L^1} = 1$, and

$$\|F - \mathbb{E}_{\mathcal{D}}(F)\|_{L^1} = \epsilon, \tag{1·23}$$

with $\epsilon \ll 1$, since otherwise there is nothing to prove.

(2) Determine Hardy martingales G, B, satisfying (1·18) and (1·17) and so that

$$F = G + B. \tag{1·24}$$

(3) By (1·23), (1·24) and (1·17)– (1·19) we have,

$$\|G\|_{\mathcal{P}} \leqslant C, \quad \|G - \mathbb{E}_{\mathcal{D}} G\|_{L^1} \leqslant C\epsilon, \quad \|\mathbb{E}_{\mathcal{D}} B\|_{L^1} \leqslant C\epsilon. \tag{1·25}$$

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(4) Apply the martingale inequality (1·21) to the Hardy martingle G and use (1·25) to get

$$\|\mathbb{E}_{\mathcal{D}}F\|_{L^1} \leqslant \|\mathbb{E}_{\mathcal{D}}G\|_{L^1} + \|\mathbb{E}_{\mathcal{D}}B\|_{L^1} \leqslant C\|T(G - \mathbb{E}_{\mathcal{D}}G)\|_{\mathcal{P}}^{1/4} + C\epsilon^{1/4}, \tag{1·26}$$

where T is the martingale transform operator arising in (1·17).

(5) By (1·17),

$$\|T(G - \mathbb{E}_{\mathcal{D}}G)\|_{\mathcal{P}} \leqslant C\epsilon^{1/2}. \tag{1·27}$$

(6) Combining (1·27) and (1·26) we get

$$\|\mathbb{E}_{\mathcal{D}}F\|_{L^1} \leqslant C\epsilon^{1/8}.$$

We thus derived the interpolatory estimate (1·22), by combining (1·21) with the inequalites (1·17)–(1·19). Being able to do this was our motivation for proving Davis and Garsia inequalities for dyadic perturbations of Hardy martingales.

Organization. In Section 2 we prove martingale transform estimates. In Section 3 we prepare our complex analytic tools. In Section 4 we put together the results of Section 2 and Section 3. We prove there the Davis and Garsia inequalities for dyadic perturbations of Hardy martingales.

2. *Martingale transforms*

Let $(\Omega, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space. Consider the martingale transforms

$$T(G) = \mathfrak{I}\left[\sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k\right], \tag{2·1}$$

where w_k is complex valued, adapted, and $|w_k| \leqslant 1$. Clearly T is a contraction on H^1 as well as on $\mathcal{P}$. The purpose of this section is to prove point-wise L^1 estimates for T as needed in Theorem 4·3 and Theorem 4·4.

2·1. *The transform estimate*

Fix the filtered probability space $(\Omega, (\mathcal{F}_n), \mathbb{P})$. The corresponding spaces L^1, H^1, $\mathcal{P}$ and $\mathcal{A}$ are defined in (1·1) and (1·2).

THEOREM 2·1. *Let $F = (F_k)$ be a martingale. Let $A > 0$. If the martingale $G = (G_k)$ satisfies*

$$|\Delta G_k| \leqslant A|F_{k-1}|, \tag{2·2}$$

then the transform T defined as

$$T(G) = \mathfrak{I}\left[\sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k\right] \qquad \textit{where} \quad w_{k-1} = \frac{\overline{F_{k-1}}}{|F_{k-1}|}, \qquad k \leqslant n \tag{2·3}$$

satisfies the point-wise estimate

$$\|T(G)\|_{\mathcal{P}} \leqslant C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}}, \tag{2·4}$$

where $C = C(A)$, and $B = F - G$.

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Comments

(1) The appearance of the $\|F\|_{L^1}^{1/2}$ on the right-hand side of (2·4) makes Theorem 2·1 our basic tool.
(2) We will apply Theorem 2·1 only to those decompositions $F = G + B$ for which $\|B\|_{\mathcal{A}}$ is properly under control.
(3) The sequence (w_k) defining T in (2·3) depends on the martingale F appearing on the right-hand side of (2·4). Hence we speak of pointwise estimates for the operator T.
(4) The unusual imaginary part in the definition of T is the price to pay for having the L^1- factor in (2·4).

Iteration

The following iteration method [**1**] is the framework for proving square function estimates as in Theorem 2·1.

THEOREM 2·2. *Let $n \in \mathbb{N}$. Given non-negative and integrable $M_1, \dots, M_n$, $V_1, \dots, V_n$, and integrable $w_1, \dots, w_n$ so that the following estimates hold*:

$$\mathbb{E}\left(M_{k-1}^2 + V_k^2\right)^{1/2} + \mathbb{E}w_k \leqslant \mathbb{E}M_k, \quad \text{for } 1 \leqslant k \leqslant n. \tag{2·5}$$

Then

$$\mathbb{E}\left(\sum_{k=1}^n V_k^2\right)^{1/2} + \mathbb{E}\sum_{k=1}^n w_k \leqslant 2(\mathbb{E}M_n)^{1/2}\left(\mathbb{E}\max_{k\leqslant n} M_k\right)^{1/2} \tag{2·6}$$

See [**17**] for a proof of Theorem 2·2. We use it here to get Theorem 2·1.

Proposition 2·3 is needed to establish the hypothesis (2·5) of Theorem 2·2.

PROPOSITION 2·3. *Fix a probability space $(\Omega, \mathbb{P})$. Let $A > 0$, $z \in \mathbb{C}$, $w = \overline{z}/|z|$. Assume that*

$$|g| \leqslant A|z| \quad \text{and} \quad \mathbb{E}g = 0.$$

Then

$$y = \mathfrak{I}(g \cdot w)$$

satisfies

$$(|z|^2 + \alpha^2\mathbb{E}(y^2))^{1/2} \leqslant \mathbb{E}|z + g|, \tag{2·7}$$

where $\alpha = \alpha(A)$. Consequently, for any integrable f

$$\left(|z|^2 + \alpha^2\mathbb{E}(y^2)\right)^{1/2} \leqslant \mathbb{E}|z + f| + \mathbb{E}|f - g|. \tag{2·8}$$

We first treat the case when $z = 1$ in a separate Lemma. Thereafter we return to (2·7) and (2·8).

LEMMA 2·4. *Let $A > 0$. Assume that*

$$|g| \leqslant A \quad \text{and} \quad \mathbb{E}g = 0.$$

Then $u = \mathfrak{I}(g)$ satisfies

$$(1 + \alpha^2\mathbb{E}(u^2))^{1/2} \leqslant \mathbb{E}|1 + g|, \tag{2·9}$$

where $\alpha = \alpha(A)$.

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Proof. The idea used below is taken from [**1**, p. 695]. Since $\mathbb{E}g = 0$,

$$\mathbb{E}\left|1 + \frac{g}{M}\right| \leqslant \mathbb{E}|1 + g|. \tag{2·10}$$

for any $M \geqslant 1$. Now write $g = u + iv$ where u, v are bounded real valued. Choose $M = M(A) > 2A$. Then rewrite

$$\left|1 + \frac{g}{M}\right| = \left(\left(1 + \frac{u}{M}\right)^2 + \frac{v^2}{M^2}\right)^{1/2}. \tag{2·11}$$

Pull out the factor $(1 + u/M)$ from the term on the right-hand side of (2·11). By arithmetic the right-hand side of (2·11) becomes

$$\left(1 + \frac{u}{M}\right)\left(1 + \frac{v^2}{(M+u)^2}\right)^{1/2}. \tag{2·12}$$

Since $1 + u/M > 0$ and

$$\left(1 + \frac{v^2}{(M+u)^2}\right)^{1/2} \geqslant 1 + \frac{v^2}{3(M+A)^2}, \tag{2·13}$$

we multiply $1 + u/M$ and the right-hand side of (2·13) to obtain

$$\left|1 + \frac{g}{M}\right| \geqslant 1 + \frac{u}{M} + \frac{v^2}{6(M+A)^2}. \tag{2·14}$$

Since g has vanishing mean, we have also $\mathbb{E}u = 0$, hence taking the expectation in (2·14) gives

$$\mathbb{E}\left|1 + \frac{g}{M}\right| \geqslant 1 + \frac{\mathbb{E}(v^2)}{6(M+A)^2}. \tag{2·15}$$

Finally, since the right-hand side of (2·15) is $\geqslant 1$ we may replace it by its square-root and arrive at (2·9).

Proof of Proposition 2·3. Scaling and rotation reduces matters to the special case of Lemma 2·4. Write $1/z = (1/|z|)(w)$ where $w = \overline{z}/|z|$. Then

$$\mathbb{E}|z + g| = |z|\mathbb{E}\left|1 + \frac{1}{|z|}(g \cdot w)\right|.$$

Next put $y = \Im(g \cdot w)$. Since $|g|/|z| \leqslant A$, Lemma 2·4 implies

$$|z|\left(1 + \frac{\alpha^2}{|z|^2}\mathbb{E}(y^2)\right)^{1/2} \leqslant \mathbb{E}|z + g|, \tag{2·16}$$

where $\alpha = \alpha(A)$. Note that (2·16) is just (2·7). The triangle inequality gives (2·8).

Proof of Theorem 2·1. Fix $k \leqslant n$. Condition on $\mathcal{F}_{k-1}$ and put

$$z = F_{k-1}, \quad w = \overline{F_{k-1}}/|F_{k-1}|, \quad f = \Delta F_k \quad \text{and} \quad g = \Delta G_k.$$

Apply Proposition 2·3 with the above specification. Then (2·8) implies that

$$Y_k = \Im(w_{k-1} \cdot \Delta G_k),$$

satisfies the following estimate

$$\left(|F_{k-1}|^2 + \alpha^2 \mathbb{E}_{k-1} Y_k^2\right)^{1/2} \leqslant \mathbb{E}_{k-1}|F_k| + \mathbb{E}_{k-1}|\Delta B_k|,$$

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where $\Delta B_k = \Delta F_k - \Delta G_k$. Taking expectation yields

$$\mathbb{E}\left(|F_{k-1}|^2 + \alpha^2 \mathbb{E}_{k-1} Y_k^2\right)^{1/2} \leqslant \mathbb{E}|F_k| + \mathbb{E}|\Delta B_k|$$

hence Theorem 2·2 gives

$$\mathbb{E}\left(\sum_{k=1}^{n} \mathbb{E}_{k-1} Y_k^2\right)^{1/2} \leqslant C(\mathbb{E}|F_n|)^{1/2} \cdot (\mathbb{E}\max_{k\leqslant n}|F_k|)^{1/2} + C\mathbb{E}\sum_{k=1}^{n}|\Delta B_k|.$$

Since $T(G) = \sum Y_k$ we get

$$\|T(G)\|_{\mathcal{P}} = \mathbb{E}(\sum \mathbb{E}_{k-1}|Y_k|^2)^{1/2}.$$

The theorem of B. Davis [**11**] asserts that $\mathbb{E}\max_{k\leqslant n}|F_k| \leqslant C\|F\|_{H^1}$. This completes the proof.

2·2. *Perturbation and stability*

We next perturb the martingales in Theorem 2·1 by regular martingales and obtain estimates for the resulting martingale transforms. The perturbations we consider here are not *small* in any sense, but rather structurally simple.

Regular martingales. Fix a filtered probability space $(\Omega, (\mathcal{F}_n), \mathbb{P})$. We say that $D = (D_k)$ is an $\alpha-$regular martingale if there exist $(\mathcal{F}_k)$ adapted sequences (σ_k) and (d_k) so that

$$\mathbb{E}_{k-1}(\sigma_k) = 0, \quad |\sigma_k| \leqslant 1, \quad \mathbb{E}_{k-1}(|\sigma_k|^2) > \alpha \tag{2·17}$$

and

$$\Delta D_k = d_{k-1}\sigma_k, \quad k \leqslant n. \tag{2·18}$$

Dyadic martingales are regular, and $\|D\|_{\mathcal{P}} \leqslant \|D\|_{H^1}$.

THEOREM 2·5. *Let $F = (F_k)_{k=1}^N$ and $D = (D_k)$ be martingales, so that $D = (D_k)$ is $\alpha-$regular. Let $G = (G_k)_{k=1}^N$ be a martingale such that*

$$|\Delta G_k| \leqslant A|F_{k-1} - D_{k-1}|, \qquad k \leqslant N. \tag{2·19}$$

Then the transform T given by

$$T(H) = \mathfrak{I}\left[\sum w_{k-1}\Delta H_k\right], \qquad w_k = (\overline{F_k - D_k})/|F_k - D_k|,$$

satisfies

$$\|T(G-D)\|_{\mathcal{P}} \leqslant C\|F-D\|_{L^1}^{1/2}\|F-D\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}},$$

where $C = C(A, \alpha)$ and

$$B = F - G.$$

Note that the hypothesis (2·19) in Theorem 2·5 is matched by the decompositions for Hardy martingales (1·14).

The proof of Theorem 2·5 is based on the iteration principle Theorem 2·2. We use Proposition 2·6 below to verify the assumptions (2·5). Following is an extension of Proposition 2·3.

PROPOSITION 2·6. *Fix $0 < \alpha \leqslant 1$, $C \geqslant 1$ and a probability space $(\Omega, \mathbb{P})$. Let $\sigma : \Omega \to \mathbb{C}$ satisfy*

$$\mathbb{E}\sigma = 0, \quad |\sigma| \leqslant 1, \quad \mathbb{E}|\sigma|^2 > \alpha.$$

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Let $z \in \mathbb{C}$, $w = \overline{z}/|z|$, and assume $g : \Omega \to \mathbb{C}$ satisfy

$$\mathbb{E}g = 0, \quad |g| \leqslant C|z|.$$

Then, for any $b \in \mathbb{C}$,

$$y = \Im((g - b\sigma) \cdot w)$$

satisfies

$$\left(|z|^2 + \delta^2 \mathbb{E}(y^2)\right)^{1/2} \leqslant \mathbb{E}|z + g - b\sigma|,$$

where $\delta = \delta(\alpha, C)$. Consequently, for any integrable f

$$\left(|z|^2 + \delta^2 \mathbb{E}(y^2)\right)^{1/2} \leqslant \mathbb{E}|z + f - b\sigma| + \mathbb{E}|f - g|. \tag{2·20}$$

Proof. Put $A = 4C/\alpha$. Choose $b \in \mathbb{C}$. Then we distinguish between two cases.

Case 1. Let $|b| \leqslant A|z|$. In that case we have

$$|g - b\sigma| \leqslant (C + A)|z|.$$

Proposition 2·3 implies that $y = \Im((g - b\sigma) \cdot w)$ satisfies

$$\left(|z|^2 + \delta^2 \mathbb{E}(y^2)\right)^{1/2} \leqslant \mathbb{E}|z + g - b\sigma|,$$

with $\delta = \delta(\alpha, C)$

Case 2. Let $|b| \geqslant A|z|$. This case is straightforward since b dominates everything else. By rotation we assume that $w = 1$. Define the testing function $m = -\sigma b/|b|$. Note that

$$\mathbb{E}m = 0, \quad |m| \leqslant 1, \quad \mathbb{E}(b\sigma\overline{m}) = -|b|\mathbb{E}|\sigma|^2.$$

This gives

$$\mathbb{E}|z + g - b\sigma| \geqslant \mathbb{E}((z + g - b\sigma)\overline{m}) \geqslant |b|\alpha - \mathbb{E}|g|.$$

Since $|g| \leqslant C|z|$ and $|b| \geqslant A|z|$

$$\mathbb{E}|z + g - b\sigma| \geqslant |z| + |b|A^{-1}\left(A\alpha - C - 1\right).$$

Recall that we set $A = 4C/\alpha$, and $C > 1$. Hence $A\alpha - C - 1 \geqslant 1$. Using again $|g| \leqslant C|z|$ and $|b| \geqslant A|z|$ we have, that

$$|b| \geqslant \delta_0 \left(\mathbb{E}|g - b\sigma|^2\right)^{1/2},$$

where $\delta_0 = (1 + C/A)^{-1}$. Inserting gives

$$\mathbb{E}|z + g - b\sigma| \geqslant |z| + \delta \left(\mathbb{E}|g - b\sigma|^2\right)^{1/2},$$

where $\delta = \delta(\alpha, C)$.

Proof of Theorem 2·5.

Fix $k \leqslant N$. Condition on $\mathcal{F}_{k-1}$ and put

$$z = F_{k-1} - D_{k-1}, \quad w = \overline{z}/|z| \quad g = \Delta G_k \quad \text{and} \quad b\sigma = \Delta D_k.$$

Apply Proposition 2·6 with these parameters. By (2·20)

$$Y_k = \Im(w_{k-1}(\Delta G_k - d_{k-1}\sigma_k)) \quad \text{where} \quad w_{k-1} = \overline{F_{k-1} - D_{k-1}}/|F_{k-1} - D_{k-1}|$$

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satisfies

$$(|F_{k-1} - D_{k-1}|^2 + A_0^{-2}\mathbb{E}_{k-1}|Y_k|^2)^{1/2} \leqslant \mathbb{E}_{k-1}|F_k - D_k| + \mathbb{E}_{k-1}|\Delta B_k|,$$

where $\Delta B_k = \Delta F_k - \Delta G_k$. Taking expectations gives

$$\mathbb{E}(|F_{k-1} - D_{k-1}|^2 + A_0^{-2}\mathbb{E}_{k-1}|Y_k|^2)^{1/2} \leqslant \mathbb{E}|F_k - D_k| + \mathbb{E}|\Delta B_k|.$$

By Theorem 2·2, and the theorem of Davis [**11**]

$$\mathbb{E}\left(\sum \mathbb{E}_{k-1}|Y_k|^2\right)^{1/2} \leqslant C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}},$$

where $C = C(A, \alpha)$. Since $T(G - D) = \sum Y_k$ we have

$$\|T(G - D)\|_{\mathcal{P}} = \mathbb{E}\left(\sum \mathbb{E}_{k-1}|Y_k|^2\right)^{1/2}.$$

This completes the proof.

3. *Brownian motion and truncation*

This section contains the complex analytic ingredients. Our aim is Theorem 3·2. The proofs rely on estimates for outer functions and stopping time arguments based on complex Brownian motion.

3·1. *Outer functions*

Let H denote the Hilbert transform on $L^2(\mathbb{T})$. Let $p \in L^\infty(\mathbb{T})$ be real valued. Assume that $\log(1 - p)$ is well defined and bounded. Then

$$q = \exp[\log(1 - p) + iH\log(1 - p)]. \tag{3·1}$$

defines an element in $H^\infty(\mathbb{T})$. See Garnett [**10**] for background and Bourgain [**1**] for the following properties of the outer function q.

LEMMA 3·1. *Let $p \in L^\infty(\mathbb{T})$ with $0 \leqslant p \leqslant 1/2$. Then the outer function* (3·1) *satisfies the following properties*:

(1) *Then $p + |q| = 1$.*

(2) *Let q_1 be the real part of q and q_2 its imaginary part so that $q = q_1 + iq_2$. Then*

$$\int_{\mathbb{T}} q_2 dm = 0 \tag{3·2}$$

and

$$\int_{\mathbb{T}} |1 - q_1| dm \leqslant C_1 \int_{\mathbb{T}} p dm, \tag{3·3}$$

where $C_1 = 7/8$.

(3) *If p is even, $p(e^{i\theta}) = p(e^{-i\theta})$, then q_2 is odd,*

$$q_2(e^{i\theta}) = -q_2(e^{-i\theta}). \tag{3·4}$$

Proof. Since

$$|q| = \exp[\log(1 - p)] = 1 - p,$$

we have $p + |q| = 1$. Note $\int H\log(1 - p) = 0$, gives (3·2).

We now turn to (3·3). Note that

$$q_1 = (1 - p)\cos H(\log(1 - p))$$

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and

$$1 - q_1 = (1 - \cos H(\log(1-p)) + p \cos H(\log(1-p)).$$

Since $1 - \cos(x) \leqslant x^2/2$ we obtain the point-wise estimate

$$|1 - q_1| \leqslant \frac{1}{2}|H(\log(1-p))|^2 + p.$$

By the L^2 estimates for the Hilbert transform,

$$\int_{\mathbb{T}} |H(\log(1-p))|^2 dm \leqslant 2 \int_{\mathbb{T}} |\log(1-p))|^2 dm.$$

Invoking that $0 \leqslant p \leqslant 1/2$ gives (3·3) as follows

$$\int_{\mathbb{T}} |\log(1-p))|^2 dm \leqslant C_1 \int_{\mathbb{T}} p^2 dm \leqslant C_1 \int_{\mathbb{T}} p dm.$$

If moreover p is even, then by inspection

$$q_2 = (1-p) \sin H \log(1-p)$$

is odd, hence (3·4) holds.

3·2. *Brownian motion*

On Wiener space we let (B_t) denote complex 2D-Brownian motion started at 0, and we let $((\mathcal{F}_t), \mathbb{P})$, be the associated filtered probability space. Put

$$\tau = \inf\{t > 0 : |B_t| > 1\}.$$

Given an integrable random variable Y on Wiener space we denote by $\mathbb{E}(Y|B_\tau = e^{i\theta})$ the expectation of Y with respect to Brownian motion started at 0 and conditioned to exit the unit disk at $e^{i\theta}$). See Durrett [**6**].

The following theorem is the main complex analytic tool exploited in this paper.

THEOREM 3·2. *There exists $C_0 \geqslant 1$ so that the following holds. For $h \in H_0^1(\mathbb{T})$ and $z \in \mathbb{C}$, let*

$$\rho = \inf\{t < \tau : |h(B_t)| > C_0|z|\}, \quad \text{and} \quad g(e^{i\theta}) = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}). \tag{3·5}$$

Then $g \in H_0^\infty(\mathbb{T})$,

$$|g| \leqslant C_0|z| \tag{3·6}$$

and for any $b \in \mathbb{C}$

$$|z| + \frac{1}{8}\int_{\mathbb{T}} |h - g| dm \leqslant \int_{\mathbb{T}} |z + h - b\sigma| dm, \tag{3·7}$$

where

$$\sigma(e^{i\theta}) = \text{sign}(\cos(\theta)).$$

Proof. By a result of N. Varopoulos [**20**], g defined by (3·5) is bounded, analytic with vanishing mean, hence in $H_0^\infty(\mathbb{T})$. See also [**14**]. The upper bound (3·6) results from (3·5). We get lower bounds for $\int_{\mathbb{T}} |z + h - b\sigma| dm$ by integrating against testing functions. In the case $|b| \leqslant 8|z|$ we take the outer functions of Lemma 3·1. The case $|b| \geqslant 8|z|$ is straight forward and uses simple exponentials as testing functions.

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Case 1. Assume $|b| \leqslant 8|z|$. Define

$$p(e^{i\theta}) = \frac{1}{4}\left[\mathbb{E}(1_A|B_\tau = e^{i\theta}) + \mathbb{E}(1_A|B_\tau = e^{-i\theta})\right], \quad \text{where} \quad A = \{\rho < \infty\}.$$

Clearly p is even, $0 \leqslant p \leqslant 1/2$, and

$$\int_{\mathbb{T}} p dm = \frac{1}{2}\mathbb{P}(A). \tag{3·8}$$

By definition of $A = \{\rho < \infty\}$,

$$\begin{aligned} 2\int_{\mathbb{T}} |h| p dm &= \mathbb{E}|h(B_\tau)1_A| \\ &\geqslant C_0|z|\mathbb{P}(A). \end{aligned} \tag{3·9}$$

Using that $|b| \leqslant 8|z|$, in the present case, together with (3·8) and (3·9) we get by the triangle inequality

$$\begin{aligned} \int_{\mathbb{T}} |z + h - b\sigma| p dm &\geqslant \frac{1}{2}\mathbb{E}|h(B_\tau)1_A| - (|z| + |b|)\frac{1}{2}\mathbb{P}(A) \\ &\geqslant (\frac{1}{2} - \frac{9}{2C_0})\mathbb{E}|h(B_\tau)1_A|. \end{aligned} \tag{3·10}$$

Let $q \in H^\infty(\mathbb{T})$ be the outer function defined by (3·1). Note that $q \in H^\infty(\mathbb{T})$ is orthogonal to h. By (3·4) $q_2 = \Im q$ is an odd function, hence orthogonal to σ, and to constants. This gives the identities below:

$$\begin{aligned} \int_{\mathbb{T}} |z + h - b\sigma| \cdot |q| dm &\geqslant \left|\int_{\mathbb{T}} (z + h - b\sigma) q dm\right| \\ &= \left|\int_{\mathbb{T}} (z + b\sigma) q dm\right| \\ &= \left|\int_{\mathbb{T}} (z + b\sigma) q_1 dm\right|. \end{aligned} \tag{3·11}$$

Recall (3·3), that

$$\int_{\mathbb{T}} |1 - q_1| dm \leqslant C_1\mathbb{P}(A)/2.$$

Hence writing $q_1 = 1 + (q_1 - 1)$, using (3·3) and $|b| \leqslant 8|z|$, gives

$$\begin{aligned} \left|\int_{\mathbb{T}} (z + b\sigma) q_1 dm\right| &\geqslant |z| - \int_{\mathbb{T}} |(z + b\sigma)(1 - q_1)| dm \\ &\geqslant |z| - \frac{9C_1}{2}|z|\mathbb{P}(A). \end{aligned} \tag{3·12}$$

Hence combining (3·9) with (3·12) and (3·11) gives

$$\int_{\mathbb{T}} |z + h - b\sigma| \cdot |q| dm \geqslant |z| - \frac{9C_1}{2C_0}\mathbb{E}|h(B_\tau)1_A|. \tag{3·13}$$

Take the sum of (3·10) and (3·13). Since $p + |q| = 1$ we obtain with $\alpha_0 = (1/2 - 9/2C_0 - 9C_1/2C_0)$ that

$$\int_{\mathbb{T}} |z + h - b\sigma| dm \geqslant |z| + \alpha_0\mathbb{E}|h(B_\tau)1_A|.$$

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If $C_0 > 0$ is large enough, $\alpha_0 > 1/4$. Since

$$\int_{\mathbb{T}} |h - g| dm \leqslant \mathbb{E}|h(B_\tau)1_A|,$$

this gives (3·7) in the case $|b| \leqslant 8|z|$.

Case 2. Next we turn to the case when $|b| > 8|z|$. This case is straight forward. The testing functions involved are the simple exponentials. Note first that

$$\int_{\mathbb{T}} |z + h - b\sigma| dm \geqslant \left| \int_{\mathbb{T}} (z + h - b\sigma) e^{i\theta} dm \right| = \frac{2|b|}{\pi}. \tag{3·14}$$

Next by triangle inequality

$$\int_{\mathbb{T}} |z + h - b\sigma| dm \geqslant \int_{\mathbb{T}} |h| dm - (|z| + |b|) \geqslant \int_{\mathbb{T}} |h| dm - \frac{9|b|}{8}. \tag{3·15}$$

Take a weighted average of the equation (3·15) and (3·14), to get

$$\int_{\mathbb{T}} |z + h - b\sigma| dm \geqslant \frac{|b|}{8} + \frac{1}{4} \int_{\mathbb{T}} |h| dm.$$

Finally since $|z| \leqslant |b|/8$, and $\int_{\mathbb{T}} |h - g| dm \leqslant \int_{\mathbb{T}} |h| dm$, we get

$$\int_{\mathbb{T}} |z + h - b\sigma| dm \geqslant |z| + \frac{1}{8} \int_{\mathbb{T}} |h - g| dm.$$

4. *Davis and Garsia inequalities*

This Section contains our main result, namely the DGI for dyadic perturbations of Hardy martingales, see Subsection 4·2. The proof builds apon the results in Section 2 and Section 3. To illustrate the method of proof, in Subsection 4·1 we specialize our argument to the unperturbed case, and thereby obtain a direct and simultaneous proof of the square function estimates [**1**] as well as DGI [**17**] for Hardy martingales.

4·1. *Davis and Garsia inequalities for Hardy martingales*

We begin, by explaining the simultaneous proof of the square function estimate [**1**] and the DGI [**17**] for Hardy martingales. We use the complex analytic truncation Theorem 3·2. and the transform estimates in Theorem 2·1.

THEOREM 4·1. *For each Hardy martingale* $F = (F_k)_{k=1}^n$

$$\|F\|_{H^1} \leqslant C\|F\|_{L^1}, \tag{4·1}$$

and there exists a Hardy Martingale $G = (G_k)_{k=1}^n$

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leqslant C\|F\|_{L^1}, \tag{4·2}$$

where

$$B = F - G.$$

The Hardy martingale $G = (G_k)_{k=1}^n$ with the properties stated in Theorem 4·1 is obtained in the course of proving the following decomposition. This construction carries the complex analytic content of the Davis and Garsia inequalities.

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THEOREM 4·2. *There exists $C_0 > 0$ so that: For any Hardy martingale $F = (F_k)_{k=1}^n$ there is a Hardy martingale $G = (G_k)_{k=1}^n$ so that*

$$|\Delta G_k| \leqslant C_0 |F_{k-1}| \tag{4·3}$$

and

$$|F_{k-1}| + \frac{1}{8}\mathbb{E}_{k-1}|\Delta B_k| \leqslant \mathbb{E}_{k-1}|F_k|, \tag{4·4}$$

where

$$B = F - G.$$

Proof. We first define the Hardy martingale G. Fix $k \leqslant n$. Condition to $\mathcal{F}_{k-1}$. Fix $x = (x_1, \ldots, x_{k-1}) \in \mathbb{T}^{k-1}$ and $y \in \mathbb{T}$. Put

$$h(y) = \Delta F_k(x, y) \quad \text{and} \quad z = F_{k-1}(x).$$

Let

$$\rho = \inf\{t < \tau : |h(B_t)| > C_0|z|\}, \quad g = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}).$$

Then by [**20**] $g \in H_0^\infty(\mathbb{T})$ and clearly

$$|g| \leqslant C_0|z|. \tag{4·5}$$

We apply Theorem 3·2 with $b = 0$ and get

$$|z| + \frac{1}{8}\int_{\mathbb{T}} |h - g|dm \leqslant \int_{\mathbb{T}} |z + h|dm. \tag{4·6}$$

Put

$$\Delta G_k(x, y) = g(y), \qquad \text{and} \quad \Delta B_k(x, y) = h(y) - g(y).$$

so that

$$\Delta F_k = \Delta G_k + \Delta B_k.$$

Hence (4·5) gives

$$|\Delta G_k| \leqslant C_0 |F_{k-1}|$$

and by (4·6)

$$|F_{k-1}| + \frac{1}{8}\mathbb{E}_{k-1}|\Delta B_k| \leqslant \mathbb{E}_{k-1}|F_k|.$$

Proof of Theorem 4·1. Apply Theorem 4·2 to F. Let $F = G + B$ be the resulting decomposition into Hardy martingales satisfying (4·3) and (4·4). Define the rotation

$$w_{k-1} = \overline{F_{k-1}}/|F_{k-1}|$$

and the transform

$$T(G) = \mathfrak{I}\left[\sum w_{k-1} \cdot \Delta G_k\right].$$

Since $|\Delta G_k| \leqslant C_0|F_{k-1}|$, Theorem 2·1 implies that

$$\|T(G)\|_{\mathcal{P}} \leqslant C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}}, \tag{4·7}$$

where $C = C(C_0)$. Integrating (4·4) gives

$$\mathbb{E}|F_{k-1}| + \frac{1}{8}\mathbb{E}|\Delta B_k| \leqslant \mathbb{E}|F_k|,$$

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and by summing the telescoping estimates, we obtain

$$\|B\|_{\mathcal{A}} \leqslant C\|F\|_{L^1}. \tag{4·8}$$

Since G is a Hardy martingale and $|w_{k-1}| = 1$ we have

$$\mathbb{E}_{k-1}|\Delta G_k|^2 = 2\mathbb{E}_{k-1}|\Im\,[w_{k-1}\cdot \Delta G_k]\,|^2.$$

Hence

$$\|G\|_{\mathcal{P}} = \sqrt{2}\|T(G)\|_{\mathcal{P}}. \tag{4·9}$$

Inserting the estimates (4·8) and (4·9) into equation (4·7) gives

$$\|G\|_{\mathcal{P}} \leqslant C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}. \tag{4·10}$$

Next use the Burkholder Gundy inequality in combination with (4·10).

$$\|F\|_{H^1} \leqslant 2\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leqslant C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}.$$

Cancelling $\|F\|_{H^1}^{1/2}$ gives the square function estimate,

$$\|F\|_{H^1} \leqslant C\|F\|_{L^1}$$

and simultaneously the Davis and Garsia inequality

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leqslant C\|F\|_{L^1}.$$

4·2. *Dyadic perturbation and stability of DGI*

The main results of this paper are Theorem 4·3 and Theorem 4·4. These theorems determine the extent to which Theorem 4·1 persists under dyadic perturbation. The application of Theorem 4·4 to the embedding theorem [**1**] were the impetus for considering the class of martingales that consists of dyadic perturbations of Hardy martingales.

Dyadic martingales. We recall the definition of the dyadic σ algebra on $\mathbb{T}^{\mathbb{N}}$. It is defined by means of the independent Rademacher functions

$$\sigma_k(x) = \operatorname{sign}(\cos_k(x)), \qquad x = (x_k) \in \mathbb{T}^{\mathbb{N}},$$

where $\cos_k(x) = \Re x_k$. Let $\mathcal{D}$ be the $\sigma-$algebra on $\mathbb{T}^{\mathbb{N}}$ generated by $\sigma_1, \ldots, \sigma_k, \ldots$.

Stopping time decomposition. We next fix two martingales, $F = (F_k)_{k=1}^n$ is Hardy and $D = (D_k)_{k=1}^n$ is dyadic. Fix $k \leqslant n$. Condition to $\mathcal{F}_{k-1}$. That is, fix $(x_1, \ldots, x_{k-1}) \in \mathbb{T}^{k-1}$. Put

$$h(y) = \Delta F_k(x_1, \ldots, x_{k-1}, y) \quad \text{and} \quad z = F_{k-1}(x_1, \ldots, x_{k-1}) - D_{k-1}(x_1, \ldots, x_{k-1}).$$

Let

$$\rho = \inf\{t < \tau : |h(B_t)| > C_0|z|\}, \quad g = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}) \tag{4·11}$$

and put

$$\Delta G_k(x_1, \ldots, x_{k-1}, y) = g(y) \qquad \text{and} \quad \Delta B_k(x_1, \ldots, x_{k-1}, y) = h(y) - g(y), \tag{4·12}$$

such that

$$\Delta F_k = \Delta G_k + \Delta B_k. \tag{4·13}$$

We turn now to proving the key properties of the Hardy martingale $G = (G_k)_{k=1}^n$ defined by the stopping times (4·12).

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THEOREM 4·3. *For every Hardy martingale* $F = (F_k)_{k=1}^n$ *and dyadic martingale* $D = (D_k)$ *the Hardy martingale* $G = (G_k)_{k=1}^n$ *defined by* (4·12) *satisfies the following estimates*

$$|\Delta G_k| \leqslant C_0 |F_{k-1} - D_{k-1}| \tag{4·14}$$

and

$$|F_{k-1} - D_{k-1}| + \frac{1}{8}\mathbb{E}_{k-1}|\Delta B_k| \leqslant \mathbb{E}_{k-1}|F_k - D_k|, \tag{4·15}$$

where

$$B = F - G.$$

Proof. Fix $k \leqslant n$. Condition to $\mathcal{F}_{k-1}$ and put

$$z = F_{k-1} - D_{k-1}, \quad h = \Delta F_k, \quad g = \Delta G_k \quad b\sigma = \Delta D_k.$$

By (4·11) $|g| \leqslant C_0|z|$, hence (4·14) holds. Theorem 3·2 gives

$$|z| + \frac{1}{8}\int_{\mathbb{T}} |h - g| dm \leqslant \int_{\mathbb{T}} |z + h - b\sigma| dm. \tag{4·16}$$

Since

$$z + h - b\sigma = F_k - D_k \quad \text{and} \quad h - g = \Delta B_k,$$

we translate back the estimate (4·16) and get

$$|F_{k-1} - D_{k-1}| + \frac{1}{8}\mathbb{E}_{k-1}|\Delta B_k| \leqslant \mathbb{E}_{k-1}|F_k - D_k|.$$

Following are the consequences of Theorem 4·3. For a given Hardy martingale $F = (F_k)_{k=1}^n$ and dyadic martingale $D = (D_k)$ let T be defined as

$$T(H) = \mathfrak{I}\Big[\sum w_{k-1}\Delta H_k\Big], \qquad w_k = (\overline{F_k - D_k})/|F_k - D_k|. \tag{4·17}$$

The next theorem states Davis and Garsia inequalities for a perturbed Hardy martingale. In its proof we exploit Theorem 4·3 and Theorem 2·5.

THEOREM 4·4. *For every Hardy martingale* $F = (F_k)_{k=1}^n$ *and dyadic martingale* $D = (D_k)$ *the Hardy martingale* $G = (G_k)_{k=1}^n$ *defined by* (4·12) *and* T *defined by* (4·17) *satisfy*

$$\|T(G - D)\|_{\mathcal{P}} \leqslant C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2} \tag{4·18}$$

and

$$\|B\|_{\mathcal{A}} \leqslant C\|F - D\|_{L^1}, \qquad \textit{where} \qquad B = F - G. \tag{4·19}$$

Moreover,

$$\|G\|_{\mathcal{P}} \leqslant C\|F\|_{L^1} + C\|D\|_{H^1}. \tag{4·20}$$

Proof. Invoke the estimates of Theorem 4·3. Taking expectations in (4·15). gives

$$\mathbb{E}|F_{k-1} - D_{k-1}| + \frac{1}{8}\mathbb{E}|\Delta B_k| \leqslant \mathbb{E}|F_k - D_k|.$$

Summing the telescoping series gives

$$\mathbb{E}\sum_{k=1}^{n} |\Delta B_k| \leqslant 4\mathbb{E}|F_n - D_n|, \tag{4·21}$$

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or (4·19). Next use (4·14) and apply Theorem 2·5 to T (defined in (4·17)). This gives

$$\|T(G-D)\|_{\mathcal{P}} \leqslant C\|F-D\|_{L^1}^{1/2}\|F-D\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}},$$

Invoking (4·21) we get $\|B\|_{\mathcal{A}} \leqslant C\|F-D\|_{L^1}$, hence

$$\|T(G-D)\|_{\mathcal{P}} \leqslant C\|F-D\|_{L^1}^{1/2}\|F-D\|_{H^1}^{1/2},$$

as claimed. The remaining estimate (4·20) is a simple consequence of the above. We get first,

$$\|T(G)\|_{\mathcal{P}} \leqslant C\|F\|_{L^1} + C\|D\|_{H^1} + C\|T(D)\|_{\mathcal{P}}. \tag{4·22}$$

Since G is a Hardy martingale we have

$$\mathbb{E}_{k-1}|\Delta G_k|^2 = 2\mathbb{E}_{k-1}|\Im(w_{k-1}\cdot\Delta G_k)|^2, \tag{4·23}$$

whenever w_{k-1} is $\mathcal{F}_{k-1}$ measurable and $|w_{k-1}| = 1$. Hence $\|G\|_{\mathcal{P}} = \sqrt{2}\|T(G)\|_{\mathcal{P}}$. Note also that for regular martingales

$$\|T(D)\|_{\mathcal{P}} \leqslant \|D\|_{\mathcal{P}} \leqslant \|D\|_{H^1}.$$

This gives (4·20) as claimed.

Acknowlegement. It is my pleasure to thank M. Schmuckenschläger and P. Wojtaszczyk for many helpful discussions concerning the topics of this paper.

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