ON $A^4+B^4+C^4+D^4=E^4$

By JOHN LEECH

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Euler (see (1), p. 648) conjectured that the sum of three fourth powers cannot be a fourth power, at least four being necessary, although no one had been able to exhibit four such. It was not until 1911 that a solution was exhibited by Norrie (2) who gave the first of the solutions listed below. In order to find if this was the smallest solution and to try to find others, I recently carried out a search for solutions, which was carried to $E = 4303$ and which exhibited the eight solutions given below, showing Norrie's to be the smallest.

A	B	C	D	E
30	120	272	315	353
240	340	430	599	651
435	710	1384	2420	2487
1130	1190	1432	2365	2501
850	1010	1546	2745	2829
2270	2345	2460	3152	3723
350	1652	3230	3395	3973
205	1060	2650	4094	4267

The search is remarkable for the extent to which the range of search can be reduced by taking advantage of certain congruence properties of fourth powers. Any fourth power is congruent to 0 or 1 modulo 16; hence (rejecting imprimitive solutions with a common factor) three of A, B, C, D are even and we need only consider odd values of E. Further, if D is the odd number on the left-hand side, $(E^4-D^4)/16$ must itself be representable as the sum of three fourth powers, namely of $\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$, and we can reject values of D, E such that $(E^4-D^4)/16 \not\equiv 0, 1, 2, 3 \pmod{16}$. The modulus 5 is even more advantageous. Any fourth power is congruent to 0 or 1 modulo 5, whence three of A, B, C, D are multiples of 5 and the other is not. If D is the non-multiple (not necessarily the same as the D selected above as being odd), we must have E^4-D^4 divisible by 625, that is $D \equiv \pm E$ or $\pm 182E \pmod{625}$ since $182^2 \equiv -1 \pmod{625}$. This effects an immense reduction in the number of values of D that need be considered for each value of E. When both these moduli are taken into account, the range of search is reduced by a factor exceeding 10^6, and, despite the search being four-dimensional, it was feasible to search to the limit $E = 4303$. This limit was imposed by the method of search, which consisted of examining, for each value of E prime to 10, the integer values of $(E^4-D^4)/625$ for representability as the sum of three fourth powers, these latter being held in a table for the range $N^4 < 2^{39}$. The search procedure was slightly different according as these three fourth powers were all even or two even and one odd.

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The search was carried out on the electronic computer EDSAC 2 at the University Mathematical Laboratory, Cambridge. I am greatly indebted to Mr M. F. Mitchell who carried out the coding of the problem.

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EMBEDDINGS OF REAL PROJECTIVE SPACES

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Let P^m denote real projective m-space, where $m \geqslant 1$. Following Hopf (2) we denote by $D(m)$ the least number q such that P^m can be mapped homeomorphically into a q-element. In other words, $D(m)$ is the least dimension in which a Euclidean model of P^m can be constructed. We have at once that

$$D(m) \leqslant D(m+1), \tag{1}$$

because P^m can be embedded as a subspace of P^{m+1}. Also $D(1) = 2$, since P^1 is a 1-sphere. If $m \geqslant 2$ then $D(m) = r$, where r is the least number such that P^m can be mapped homeomorphically into an r-sphere. Let $m \geqslant 2$. In (2) Hopf proves by explicit construction that

$$\left.\begin{aligned} D(m) &< 2m \quad (m \text{ odd}), \\ D(m) &\leqslant 2m \quad (m \text{ even}). \end{aligned}\right\} \tag{2}$$

Hopf also proves that

$$D(m) \geqslant m+2, \tag{3}$$

by using Alexander duality. Hence $D(2) = 4$ and $D(3) = 5$. More recently, the methods of cohomology theory have established that

$$D(m) = 2m \quad \text{if} \quad m \text{ is a power of 2.} \tag{4}$$

This relation can be obtained from Theorem III.25 of (3), for example, by a straightforward computation. In some cases (3) can be improved by combining (1) and (4). Apart from this the above summary includes all the results I have been able to find in the literature related to the problem of determining $D(m)$.

Our present purpose is to point out the connexion between this problem and the one which Hopf investigates in (1). The latter problem gives rise to a numerical

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