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Rewriting systems for the surface classification theorem

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The work reported in this paper refers to Massey's proof of the surface classification theorem based on the standard word-rewriting treatment of surfaces. We arrange this approach into a formal rewriting system $\mathscr{R}$ and provide a new version of Massey's argument. Moreover, we study the computational properties of two subsystems of $\mathscr{R}$: $\mathscr{R}_{\text{or}}$ for dealing with words denoting orientable surfaces and $\mathscr{R}_{\text{nor}}$ for dealing with words denoting non-orientable surfaces. We show how such properties induce an alternative proof for the surface classification in which the basic homeomorphism between the connected sum of three projective planes and the connected sum of a torus with a projective plane is not required.

1. Introduction

We follow the standard terminology and use the term *surfaces* to mean compact and connected 2-manifolds. The well-known triangulation theorem says that any surface can be triangulated and that, given a triangulation for a surface $\mathscr{U}$, there is an algorithm for constructing a polygon with $2n$ edges, organised by pairs of edges, such that by identifying the corresponding edges, the surface $\mathscr{U}$ is recovered (up to homeomorphisms). As a consequence, when surfaces are considered modulo homeomorphisms, their geometrical information can be encoded by a specific class of finite words. Such a word-based approach finds its most relevant mathematical application in the proof of the classification theorem, which establishes that any surface is homeomorphic to exactly one of the following surfaces: a sphere; a finite connected sum of tori; or a finite connected sum of projective planes. The sphere and the connected sum of tori are orientable, whereas the connected sum of projective planes is non-orientable (Massey 1991).

In this paper, we consider the classical demonstration of the surface classification theorem proposed by Massey (Massey 1991); although this result was known well before the publication of Massey's textbook, specialists in the field usually make reference to his standard proof.

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Roughly speaking, Massey's proof consists of:

1 providing an algorithm for rewriting each $2n$-gon into an equivalent one (that is, preserving the homeomorphism type of the represented surface) in so-called 'canonical form'[†];
2 establishing the basic homeomorphism between the connected sum of a torus with a projective plane and the connected sum of three projective planes.

As a first contribution, we reorganise the word-based approach stressed by Massey into the formal rewriting system $\mathscr{R}$. From a technical point of view, while Massey's transformations preserve the topology, $\mathscr{R}$-reductions may change the homeomorphism type: from the initial representation, the reductions cut a sphere, a torus or a projective plane until reaching a sphere, and the homeomorphism type is then read from the sequence of the rules applied. In other words, the normalisation process in $\mathscr{R}$ formally 'mimics' the process of forming, step by step, the quotient surface represented by a certain polygon through the identification of paired edges. We investigate the computational properties of $\mathscr{R}$ and show how to recover topological properties through computational ones. This leads to a more perspicuous version of Massey's proof. In particular, while formal (rewriting) and topological arguments are freely mixed in Massey (1991), here we try to separate these two aspects in the revisited proof. Moreover, we provide a very clear and concise proof for the lemma saying that the sum of three projective planes is homeomorphic to a torus plus a projective plane.

As a second contribution, we single out two subsystems of $\mathscr{R}$: $\mathscr{R}_{\mathrm{or}}$ for dealing with words denoting orientable surfaces and $\mathscr{R}_{\mathrm{nor}}$ for dealing with words denoting non-orientable surfaces. We show the computational properties of both $\mathscr{R}_{\mathrm{or}}$ and $\mathscr{R}_{\mathrm{nor}}$, and deduce from them an alternative proof for the classification in which the basic homeomorphism stressed by Massey is not required.

2. The systems $\mathscr{R}$, $\mathscr{R}_{\mathrm{or}}$ and $\mathscr{R}_{\mathrm{nor}}$

2.1. *Basic definitions*

Consider an alphabet $\mathscr{A} \cup \bar{\mathscr{A}}$, where $\mathscr{A} = \{a, b, c, \ldots\}$ and $\bar{\mathscr{A}} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$. The bar operation is an involution without fixed point: for any $x \in \mathscr{A} \cup \bar{\mathscr{A}}$, $\bar{\bar{x}} = x$ and $x \neq \bar{x}$. Finite words from $\mathscr{A} \cup \bar{\mathscr{A}}$ are denoted by small Greek letters: in particular, ϵ denotes the empty word. We use $|\alpha|$ to denote the multiset[‡] of letters occurring in α. We denote the concatenation of two words by simply juxtaposing them: $\alpha \star \beta = \alpha\beta$.

Definition 2.1 (TW, OTW and NTW). The set TW of topological words is the smallest set containing ϵ and such that if $\alpha\beta\gamma \in \mathtt{TW}$, $x \in \mathscr{A} \cup \bar{\mathscr{A}}$ and $x \notin |\alpha\beta\gamma|$, then $\alpha x \beta \bar{x} \gamma \in \mathtt{TW}$ and $\alpha x \beta x \gamma \in \mathtt{TW}$. The set OTW of orientable topological words gathers all the $\alpha \in \mathtt{TW}$ such that

[†] A polygon in canonical form represents a surface whose homeomorphism type corresponds to a sphere, a connected sum of tori or a connected sum of tori and projective planes.

[‡] The notion of a multiset extends the notion of a set by including classes of objects in which several copies of the same object may occur; for example, the collection $\{a, b, a, c\}$ is a multiset: it contains two different copies of a and its cardinality is 4.

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if $x \in |\alpha|$, then $\bar{x} \in |\alpha|$. The set NTW of non-orientable topological words is the complement of OTW, that is, NTW = TW\OTW.

Example 2.1. $ab\bar{a}\bar{b} \in$ OTW and $abab, abc\bar{a}\bar{b}c \in$ NTW.

Definition 2.2 (cycle). When a word is included between round brackets, it denotes the equivalence class gathering the totality of words obtained from α by circular permutation of its letters; such equivalence classes are simply called *cycles*.

Example 2.2. $abc, cab, bca \in (abc)$.

Remark 2.1. OTW and NTW are stable under the equivalence relation introduced in Definition 2.2, that is, if $\alpha, \beta \in (\gamma)$, then $\alpha \in$ OTW $\Rightarrow \beta \in$ OTW and $\alpha \in$ NTW $\Rightarrow \beta \in$ NTW.

Definition 2.3 (OTC and NTC). A cycle (α) is orientable, $(\alpha) \in$ OTC, if $\alpha \in$ OTW; it is non-orientable, $(\alpha) \in$ NTC, if $\alpha \in$ NTW.

Example 2.3. $(\epsilon), (a\bar{a}b\bar{b}) \in$ OTC and $(aab\bar{b}), (aabb) \in$ NTC.

Definition 2.4. The bar operation is extended from single letters to entire words as follows: if $\alpha = a_1 a_2 \ldots a_n$, then $\bar{\alpha} = \bar{a}_n \bar{a}_{n-1} \ldots \bar{a}_1$.

Definition 2.5 ($\mathscr{R}$, $\mathscr{R}_{\mathbf{or}}$, $\mathscr{R}_{\mathbf{nor}}$). Consider the following four transformations on topological cycles:

— *contraction*: $(\alpha x \bar{x}) \rightarrow_{\texttt{cont}} (\alpha)$
— *permutation*: $(\alpha x \beta \gamma \bar{x}) \rightarrow_{\texttt{perm}} (\alpha x \gamma \beta \bar{x})$
— *torus*: $(x\alpha y \beta \bar{x} \gamma \bar{y} \delta) \rightarrow_{\texttt{torus}} (\alpha \delta \gamma \beta)$
— *projective plane*: $(\alpha x \beta x) \rightarrow_{\texttt{pjp}} (\alpha \bar{\beta})$.

The three rewriting systems $\mathscr{R}$, $\mathscr{R}_{or}$ and $\mathscr{R}_{nor}$ are defined in the following table:

system	terms	transformations
$\mathscr{R}$	TC	cont, torus, pjp
$\mathscr{R}_{or}$	OTC	cont, torus
$\mathscr{R}_{nor}$	TC	cont, perm, pjp

Remark 2.2. For any reduction $(\alpha) \rightarrow (\beta)$, the word β is uniquely determined by α together with the applied rule, so the system $\mathscr{R}$ is well defined.

Remark 2.3. It is easy to check that $\mathscr{R}$, $\mathscr{R}_{\text{or}}$ and $\mathscr{R}_{\text{nor}}$ are indeed three rewriting systems: it is sufficient to note that the set TC is closed under the four rules listed in Definition 2.5 and that the set OTC is closed under the two $\mathscr{R}_{\text{or}}$-transformations cont and torus. On the other hand, the system $\mathscr{R}_{\text{nor}}$ cannot be limited to terms from NTC, the set NTC is in fact not stable under $\mathscr{R}_{\text{nor}}$-transformations (for instance, $(cabcba) \rightarrow_{\texttt{pjp}} (\bar{b}\bar{a}ba)$).

Definition 2.6 (segment). We say that a word β is a segment of (α), $\beta \sqsubseteq (\alpha)$, if there is a γ such that $(\alpha) = (\beta\gamma)$. We write $\beta \sqsubset (\alpha)$ to specify that $\gamma \neq \epsilon$.

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Example 2.4. $\epsilon, a, b, ab, ba, aba, bab \sqsubset (abab)$ and $abab, baba \sqsubseteq (abab)$.

Definition 2.7 (block, blocked cycle). A word $\beta \neq \epsilon$ constitutes a *block* if $\beta \in \mathtt{OTW}$ and (β) cannot be transformed into (ϵ) by using `cont` transformations exclusively. A cycle $(\alpha) \in \mathtt{TC}$ is *blocked* if there is a $\beta \sqsubseteq (\alpha)$ such that β is a block.

Example 2.5. $(ab\bar{a}\bar{b})$ and $(ab\bar{a}\bar{b}cc)$ are both blocked.

Definition 2.8 (blocking `pjp`). A projective plane transformation $(\alpha) \rightarrow_{\mathtt{pjp}} (\alpha')$ is said to be *blocking* if (α) is non-blocked and (α') is blocked.

Example 2.6. $(abcacb) \rightarrow_{\mathtt{pjp}} (\bar{c}\bar{b}cb)$ is a blocking transformation.

Definition 2.9 (simple words). A word $\mu \in \mathtt{TW}$ is *simple* if for each $\beta \sqsubset (\mu)$, either $\beta = \epsilon$ or $\beta \notin \mathtt{TW}$.

Example 2.7. $ab\bar{a}\bar{b}$ is an orientable simple word and $ab\bar{c}acb$ is a non-orientable simple word.

2.2. *Computational properties*

Consider a generic rewriting system $\mathfrak{R}$ standing for $\mathscr{R}$, $\mathscr{R}_{\mathrm{or}}$ or $\mathscr{R}_{\mathrm{nor}}$ and a certain $\mathfrak{R}$-transformation $\mathfrak{r}$:

— $(\alpha) \rightarrow_{\mathfrak{R}} (\alpha')$ means that (α') is obtained by rewriting (α) through a single transformation in $\mathfrak{R}$.
— $(\alpha) \rightarrow^{*}_{\mathfrak{R}} (\alpha')$ means that there exists a chain (possibly empty) of $\mathfrak{R}$-transformations rewriting (α) into (α').
— $(\alpha) \rightarrow^{*}_{\mathfrak{r}} (\alpha')$ means that there exists a chain (possibly empty) made by successive repetitions of the transformation $\mathfrak{r}$ rewriting (α) into (α').

Moreover, we write $\rightarrow_{\mathtt{pjp}[x]}$ to denote the `pjp` transformation specifically involving the x-letters, for instance, $(\alpha x \beta x) \rightarrow_{\mathtt{pjp}[x]} (\alpha\bar{\beta})$.

Definition 2.10 (normal form, weak and strong normalisation). A term is said to be in *normal form* with respect to a system $\mathfrak{R}$ if it does not admit any $\mathfrak{R}$-transformation. A system $\mathfrak{R}$ enjoys the *weak normalisation* property if for every term (α), there is at least one chain of $\mathfrak{R}$-transformations from (α) to a normal form. $\mathfrak{R}$ enjoys the *strong normalisation* property if for every term (α), all chains of $\mathfrak{R}$-transformations (sufficiently prolonged) lead to a normal form (Girard 1989).

Theorem 2.1. Systems $\mathscr{R}$ and $\mathscr{R}_{or}$ enjoy strong normalisation and have a common, unique normal form (ϵ).

Proof. We show that if $\alpha \neq \epsilon$, then (α) admits at least one of the three transformations forming $\mathscr{R}$. If $(\alpha) \in \mathtt{NTC}$, by Definition 3, there is a letter x such that $\{x, x\} \subseteq |\alpha|$, so the transformation $\rightarrow_{\mathtt{pjp}[x]}$ is clearly admitted. For $(\alpha) \in \mathtt{OTC}$, we proceed by induction. If $card(|\alpha|) = 2$, then $(\alpha) = (x\bar{x})$, so we can apply a `cont` transformation. Now, if $card(|\alpha|) > 2$, we let $(\alpha) = (x\beta\bar{x}\gamma)$ and distinguish two cases: if $\beta \notin \mathtt{TW}$, then (α) admits a

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`torus` transformation; otherwise, $\beta, \gamma \in \mathtt{OTW}$, so we can call on the induction hypothesis. The conclusion then follows almost immediately from this result. □

The rest of this section is devoted to a study of the computational properties of $\mathscr{R}_{\mathrm{nor}}$.

Lemma 2.1. If $\mu, \nu \sqsubset (\alpha)$ are two simple words, then $\mu = \nu$ or $|\mu| \cap |\nu| = \varnothing$.

Proof. Suppose, in order to show a contradiction, that $\mu \neq \nu$ and $|\mu| \cap |\nu| \neq \varnothing$. Since $\mu, \nu \sqsubset (\alpha)$, they will overlap on a segment β, say $\mu = \mu'\beta$ and $\nu = \beta\nu'$ with $|\mu'| \cap |\nu'| = \varnothing$. By Definition 2.9, $\mu'\beta, \beta\nu' \in \mathtt{TW}$, so $|\beta| \cap (|\mu'| \cup |\nu'|) = \varnothing$, namely $\beta \in \mathtt{TW}$, which contradicts the assumption that both μ and ν are simple. □

Remark 2.4. Any blocked cycle also has at least a simple block, namely a block consisting of a simple word.

Lemma 2.2. For any blocked cycle $(\alpha) \in \mathtt{NTC}$ there is a permutation rule $(\alpha) \to_{\mathtt{perm}} (\alpha')$ such that the simple blocks in (α') are exactly those of (α) minus one.

Proof. Let (α) be a blocked cycle. By the previous remark, we know there is a word $x\mu_1\bar{x}\mu_2 \sqsubset (\alpha)$ consisting of a simple block. Now put $(\alpha) = (y\alpha_1 x\mu_1\bar{x}\mu_2\alpha_2 y\alpha_3)$ and rewrite it as follows:

$$(y\alpha_1 x\mu_1\bar{x}\mu_2\alpha_2 y\alpha_3) \to_{\mathtt{perm}} (y\alpha_1\mu_2\alpha_2 y\alpha_3 x\mu_1\bar{x}).$$

The statement then follows easily by Lemma 2.1. □

Example 2.8. The following example illustrates this de-blocking procedure:

$$(xabc\bar{a}\bar{c}\bar{b}xde\bar{d}\bar{e}) \to_{\mathtt{perm}} (x\bar{c}\bar{b}xabc\bar{a}de\bar{d}\bar{e}) \to_{\mathtt{perm}} (xde\bar{d}\bar{c}\bar{b}xabc\bar{a}\bar{e}).$$

Lemma 2.3. If $(\alpha) \in \mathtt{NTC}$ is non-blocked, then (α) admits a non-blocking `pjp` transformation.

Proof. Suppose that (α) admits a blocking `pjp` transformation. By Definition 2.8, there are two words $\beta, \gamma \sqsubset (\alpha)$ such that $\bar{\beta}\gamma$ forms a block (and, clearly, $\beta, \gamma \neq \epsilon$, otherwise (α) would be blocked). The proof consists of showing that there is a letter $t \in |\beta| \cap |\gamma|$ such that $\to_{\mathtt{pjp}[t]}$ is non-blocking. We let $(\alpha) = (\beta\delta\gamma\eta)$ and proceed by induction on $card(|\bar{\beta}\gamma|)$:

— *Base case*: Since $\bar{\beta}\gamma$ forms a block, $card(|\bar{\beta}\gamma|) \geqslant 4$. Consider the following three cases:

- $\beta = tq$ and $\gamma = qt$:
  We perform the transformation
  $$(tq\delta qt\eta) \to_{\mathtt{pjp}} (t\bar{\delta}t\eta).$$
  $\bar{\delta}$ and η do not contain blocks (otherwise (α) would be blocked), so $\to_{\mathtt{pjp}[q]}$ is non-blocking.
- $\beta = t$ and $\gamma = qt\bar{q}$ (or *vice versa*):
  We perform the transformation
  $$(t\delta qt\bar{q}\eta) \to_{\mathtt{pjp}} (\bar{q}\bar{\delta}\bar{q}\eta),$$
  so $\to_{\mathtt{pjp}[t]}$ turns out to be non-blocking as well.

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– $\beta = \gamma = tq$:
This case can be rejected since the word $\bar{q}\bar{t}tq$ is not a block; for the same reason, we can reject the case in which $\beta = t$ and $\gamma = tq\bar{q}$ (or *vice versa*).

— *Inductive step*: The non-blocked *status* of (α) ensures the presence of a letter $t \in |\beta| \cap |\gamma|$. Let $(\alpha) = (\beta_1 t \beta_2 \delta \gamma_1 t \gamma_2 \eta)$, where $\beta = \beta_1 t \beta_2$ and $\gamma = \gamma_1 t \gamma_2$. We then transform (α) using

$$(\beta_1 t \beta_2 \delta \gamma_1 t \gamma_2 \eta) \rightarrow_{\mathtt{pjp}} (\beta_1 \bar{\gamma}_1 \bar{\delta} \bar{\beta}_2 \gamma_2 \eta)$$

and suppose $\rightarrow_{\mathtt{pjp}[t]}$ to be blocking. Since $\bar{\beta}_2 \bar{t} \bar{\beta}_1 \gamma_1 t \gamma_2$ forms a block, we have that:

(i) $z \in |\beta_1 \bar{\gamma}_1| \Rightarrow \bar{z} \notin |\bar{\beta}_2 \gamma_2|$ (that is, if a block occurs in $\beta_1 \bar{\gamma}_1 \bar{\beta}_2 \gamma_2$, then it occurs in $\beta_1 \bar{\gamma}_1$ or it occurs in $\bar{\beta}_2 \gamma_2$); and

(ii) $z \in |\beta_1 \bar{\gamma}_1 \bar{\beta}_2 \gamma_2| \Rightarrow \{z, \bar{z}\} \cap |\bar{\delta}\gamma| = \varnothing$.

Moreover, note that because (α) is non-blocked, the six segments $\beta_1, \bar{\gamma}_1, \bar{\delta}, \bar{\beta}_2, \gamma_2$ and η cannot contain blocks. A consequence of this is that if $\beta_1 \bar{\gamma}_1$ (or $\bar{\beta}_2 \gamma_2$) contains a block ρ, such a block will be 'shared' by β_1 and $\bar{\gamma}_1$ (or $\bar{\beta}_2$ and γ_2), that is, $|\rho| \not\subseteq |\beta_1|$ and $|\rho| \not\subseteq |\bar{\gamma}_1|$ (or $|\rho| \not\subseteq |\bar{\beta}_2|$ and $|\rho| \not\subseteq |\gamma_2|$). It is now easy to show that a block occurs in $\beta_1 \bar{\gamma}_1$ or in $\bar{\beta}_2 \gamma_2$. We consider the following four cases:

– $\boldsymbol{\beta_1, \bar{\gamma}_1, \bar{\beta}_2, \gamma_2 \neq \epsilon}$:
By the previous remarks, $\beta_1 \bar{\gamma}_1$ or $\bar{\beta}_2 \gamma_2$ contains a block.

– $\boldsymbol{\beta_1 \bar{\gamma}_1 = \epsilon}$:
As we have already seen, $\bar{\beta}_2 \bar{t} \bar{\beta}_1 \gamma_1 t \gamma_2$ forms a block, namely $\bar{\beta}_2 \bar{t} t \gamma_2$ is a block and so is $\bar{\beta}_2 \gamma_2$.

– $\boldsymbol{\bar{\beta}_2 \gamma_2 = \epsilon}$:
This is the same as for the previous point since $\bar{\beta}_2 \bar{t} \bar{\beta}_1 \gamma_1 t \gamma_2$ is a block, $\bar{t} \bar{\beta}_1 \gamma_1 t$ is a block and so are $\bar{\beta}_1 \gamma_1$ and $\beta_1 \bar{\gamma}_1$.

– $\boldsymbol{\beta_1 = \epsilon}$ and $\boldsymbol{\gamma_1 \neq \epsilon}$ (or *vice versa*):
$\bar{\beta}_2 \gamma_2$ contains a block (recall that β_1 and γ_1 cannot contain blocks); we argument in the same way for $\beta_2 = \epsilon$ and $\gamma_2 \neq \epsilon$ (or *vice versa*).

In summary, for each of these situations, there are two segments $\beta_i', \gamma_i' \sqsubset (\alpha)$, with $i \in \{1, 2\}$, such that $\bar{\beta}'_i \gamma'_i$ forms a block and, since $card(|\bar{\beta}_i \gamma_i|) < card(|\bar{\beta}\gamma|)$, $|\beta_i'| \subseteq |\beta_i|$ and $|\gamma_i'| \subseteq |\gamma_i|$, we have $card(|\bar{\beta}'_i \gamma'_i|) < card(|\bar{\beta}\gamma|)$. □

Theorem 2.2. If $(\alpha) \in \mathtt{NTC}$, then $(\alpha) \rightarrow^*_{\mathscr{R}_{nor}} (\epsilon)$.

Proof. First note that by Definition 2.7, if $(\alpha) \in \mathtt{OTC}$ does not form a block, then $(\alpha) \rightarrow^*_{\mathtt{cont}} (\epsilon)$. For any $(\alpha) \in \mathtt{NTC}$, we can produce a chain $(\alpha) \rightarrow^*_{\mathscr{R}_{\mathrm{nor}}} (\epsilon)$ by combining Lemmas 2.2 and 2.3, as shown in Figure 1. □

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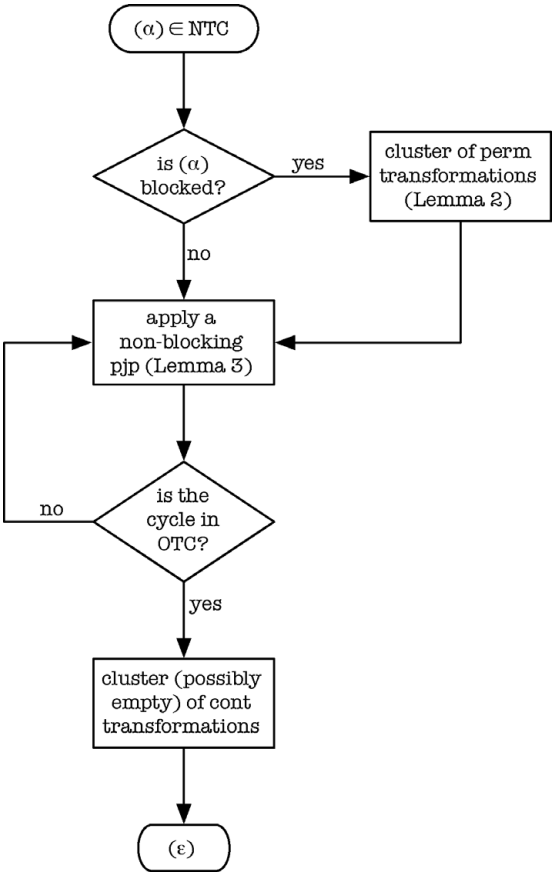


Fig. 1. The normalisation algorithm for non-orientable cycles.

Remark 2.5. Unlike $\mathscr{R}$ and $\mathscr{R}_{\mathrm{or}}$, the system $\mathscr{R}_{\mathrm{nor}}$ does not strongly normalise. This is because $\mathscr{R}_{\mathrm{nor}}$ includes the `perm` transformation whose effect simply consists in permuting some letters inside a cycle without decreasing the cardinality. Moreover, in $\mathscr{R}_{\mathrm{nor}}$ we also lose the uniqueness of the normal form: for example, consider

$(aabc\bar{b}\bar{c}) \to_{\mathtt{pjp}} (bc\bar{b}\bar{c})$,
$(aabc\bar{b}\bar{c}) \to_{\mathtt{perm}} (abc\bar{b}a\bar{c}) \to_{\mathtt{pjp}} (b\bar{c}\bar{b}\bar{c}) \to_{\mathtt{pjp}} (bb) \to_{\mathtt{pjp}} (\epsilon)$.

The following table summarises the results obtained in this section:

system	normalisation	normal form	reduction chain
$\mathscr{R}$	strong	unique	$\to^*_{\mathtt{pjp}} \to^*_{\mathtt{torus}} \to^*_{\mathtt{cont}}$
$\mathscr{R}_{\mathrm{or}}$	strong	unique	$\to^*_{\mathtt{torus}} \to^*_{\mathtt{cont}}$
$\mathscr{R}_{\mathrm{nor}}$	weak	not unique	$\to^*_{\mathtt{perm}} \to^*_{\mathtt{pjp}} \to^*_{\mathtt{cont}}$

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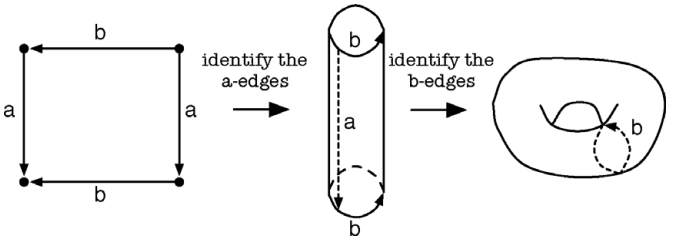
3. Geometrical interpretation

The following instructions show how we can associate to any $(\alpha) \in \mathtt{TC}$ a polygon having the edges labelled and oriented:

1 Consider a polygon having $card(|\alpha|)$ edges.
2 Let $\alpha^\star$ be the sequence obtained from α by ignoring the bar everywhere. Start from an arbitrary edge and label all the edges in clockwise order: if $\alpha^\star = a_1 a_2 \dots a_n$, label the first edge a_1, the second a_2, and so on.
3 For all $1 \leqslant i \leqslant n$, if a_i occurs in α without the bar, orient the a_i-edge according to the clockwise direction; otherwise, follow the anticlockwise direction.

Using standard terminology, we will simply refer to compact and connected 2-manifolds as *surfaces*; in the rest of this paper, surfaces will always be considered modulo homeomorphisms. We write $\mathscr{U}_{(\alpha)}$ to denote the *quotient surface* associated with the cycle/polygon (α); this is the surface obtained, as usual, through identification of paired edges according to their orientation (Massey 1991).

Example 3.1. We associate with the topological cycle $(ab\bar{a}\bar{b})$ the square shown below, which shows that the resulting quotient surface $\mathscr{U}_{(ab\bar{a}\bar{b})}$ is a torus.



We now recall two basic results from algebraic topology (Massey 1991):

— For any $(\alpha) \in \mathtt{TC}$, different gluing strategies yield the same surface $\mathscr{U}_{(\alpha)}$ (up to homoeomorphism).
— The set $\{\mathscr{U}_{(\alpha)} : (\alpha) \in \mathtt{TC}\}$ ranges over the whole class of surfaces.

We use $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{P}$ to denote the sphere, the torus and the projective plane, respectively. We use $\mathscr{U}\#\mathscr{V}$ to denote the connected sum of the two surfaces $\mathscr{U}$ and $\mathscr{V}$, and we use $\mathscr{U}^n$, with $n \in \mathbb{N}^+$, to denote the connected sum of n copies of $\mathscr{U}$. The set of topological surfaces together with the connected sum operation form a monoid in which $\mathscr{S}$ plays the role of the neutral element: for any $\mathscr{U}$, we have $\mathscr{U}\#\mathscr{S} = \mathscr{U}$ (Massey 1991).

The next theorem provides a geometrical interpretation of the rules listed in Definition 2.5: the basic idea is that the process of normalisation (in $\mathscr{R}$, $\mathscr{R}_{\mathrm{or}}$ and $\mathscr{R}_{\mathrm{nor}}$) represents, step by step, the process of forming the quotient surface associated with a certain cycle/polygon through the identification of paired edges.

Theorem 3.1.

— If $(\alpha) \rightarrow_{\mathtt{cont}} (\alpha')$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha')}$.
— If $(\alpha) \rightarrow_{\mathtt{perm}} (\alpha')$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha')}$.

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— If $(\alpha) \to_{\texttt{torus}} (\alpha')$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha')} \# \mathscr{T}$.
— If $(\alpha) \to_{\texttt{pjp}} (\alpha')$, then $\mathscr{U}_{(\alpha)} = \mathscr{U}_{(\alpha')} \# \mathscr{P}$.

Proof. Figure 2 provides a geometrical description of the four $\mathscr{R}$-transformations – see Pulcini (2009) for the full technical details. □

Remark 3.1. Theorem 3.1 induces a very easy algorithm for computing the quotient surface $\mathscr{U}_{(\alpha)}$ represented by any cycle/polygon $(\alpha) \in \texttt{TC}$. First note that $\mathscr{U}_{(\epsilon)} = \mathscr{U}_{(x,\bar{x})} = \mathscr{S}$. Then normalise (α) through a chain $(\alpha) \to^*_{\mathscr{R}} (\epsilon)$ in which p `torus` applications and q `pjp` applications occur. By Theorem 3.1, we have $\mathscr{U}_{(\alpha)} = \mathscr{S} \# \mathscr{T}^p \# \mathscr{P}^q = \mathscr{T}^p \# \mathscr{P}^q$.

3.1. *The classification theorem*

In this section we propose a new version of Massey's proof in which the geometrical properties established by the classification are recovered as consequences of the computational properties of $\mathscr{R}$.

Lemma 3.1. $\mathscr{T} \# \mathscr{P} = \mathscr{P}^3$.

Proof. We normalise the cycle $(abcacb)$ using the following two chains:

— $(abcacb) \to_{\texttt{pjp}} (\bar{c}\bar{b}cb) \to_{\texttt{torus}} (\epsilon)$
— $(abcacb) \to_{\texttt{pjp}} (ab\bar{a}b) \to_{\texttt{pjp}} (aa) \to_{\texttt{pjp}} (\epsilon)$.

By Remark 3.1, we have $\mathscr{U}_{(abcacb)} = \mathscr{T} \# \mathscr{P}$ and $\mathscr{U}_{(abcacb)} = \mathscr{P}^3$. Then, by the uniqueness of the surface represented, $\mathscr{T} \# \mathscr{P} = \mathscr{P}^3$. □

Remark 3.2. The proof of this lemma is a delicate point in Massey's argument. In fact, he uses a sort of 'visual argument' in which he shows how to cut a surface representing the connected sum of a torus with a projective plane and a surface representing the connected sum of a projective plane with a Klein bottle (the Klein bottle is homeomorphic to the connected sum of 2 projective planes (Massey 1991)) in order to produce the same geometrical object (such a strategy is shown in Figure 3 where the cut at issue is the one performed along a). In this way, the two initial surfaces turn out to be geometrically undifferentiable.

Theorem 3.2 (classification). For any surface $\mathscr{U}$:

— $\mathscr{U} = \mathscr{S}$; or
— $\mathscr{U} = \mathscr{T}^p$, with $p \in \mathbb{N}^+$; or
— $\mathscr{U} = \mathscr{P}^q$, with $q \in \mathbb{N}^+$.

Proof. Let $p, q \in \mathbb{N}^+$. The proof follows easily by combining the following three propositions:

1 If $(\alpha) \in \texttt{OTC}$, then $\mathscr{U} = \mathscr{S}$ or $\mathscr{U} = \mathscr{T}^p$ (Remark 3.1 and Theorem 2.1).
2 if $(\alpha) \in \texttt{NTC}$, then $\mathscr{U} = \mathscr{P}^q$ or $\mathscr{U} = \mathscr{T}^p \# \mathscr{P}^q$ (Remark 3.1 and Theorem 2.1).
3 $\mathscr{T}^p \# \mathscr{P}^q = \mathscr{P}^{2p+q}$ (Lemma 3.1). □

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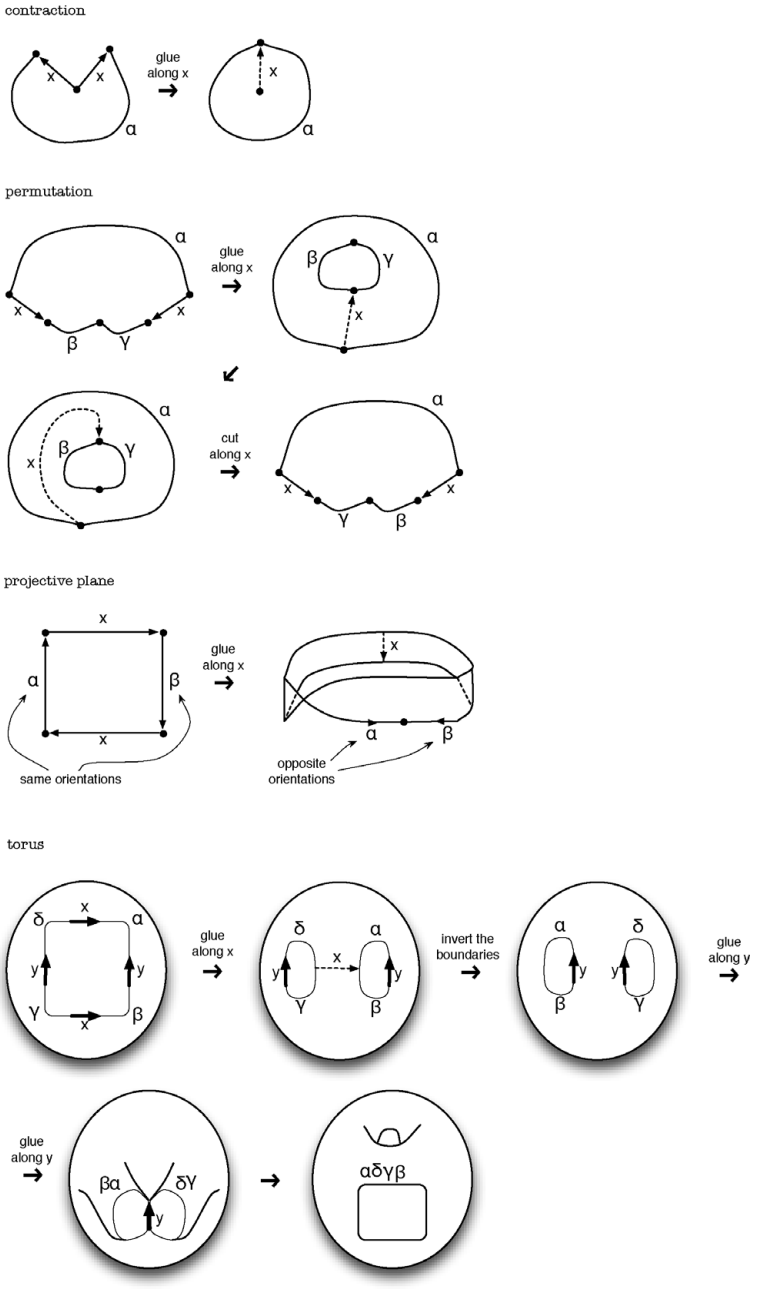


Fig. 2. A geometrical description of the $\mathscr{R}$-transformations.

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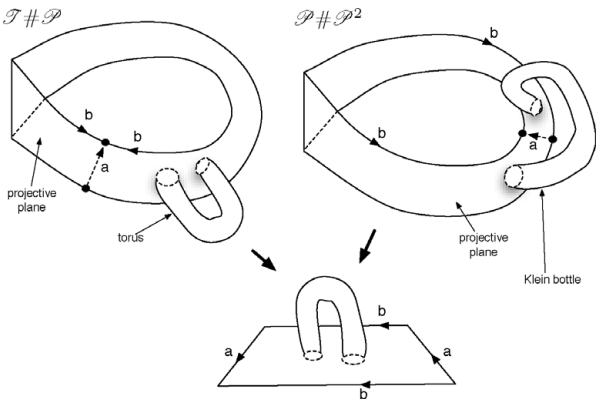


Fig. 3. The 'visual argument' proving the basic homeomorphism.

The following alternative proof of the classification theorem does not require Lemma 3.1 but uses the specific computational properties of $\mathscr{R}_{\mathrm{or}}$ and $\mathscr{R}_{\mathrm{nor}}$. Note that in this way the topological aspects present in the proof are further reduced in favour of the formal ones.

Proof. The proof follows straightforwardly from the following two propositions:

1 If $(\alpha) \in \mathtt{OTC}$, then $\mathscr{U} = \mathscr{S}$ or $\mathscr{U} = \mathscr{T}^p$ (Remark 3.1 and Theorem 2.1).
2 If $(\alpha) \in \mathtt{NTC}$, then $\mathscr{U} = \mathscr{P}^q$ (Remark 3.1 and Theorem 2.2). □

4. Conclusion

As already noted, the great mathematical relevance of the triangulation theorem arises from the fact that it allows a discretisation of the fundamental geometrical information expressed by topological surfaces, which represents a passage from continuous geometrical objects to discrete combinatorial ones.

From a conceptual point of view, pure combinatorics represents a less structured level in which this kind of information can be treated. A higher level is provided by a formal rewriting systems in which the different combinatorial behaviours are recovered and catalogued in terms of the computational properties of a suitable formal system. Both this paper and our previous one (Pulcini 2009) should be seen as an attempt to carry the proof of the classification from the combinatorial level to the more structured one of formal rewriting systems.

The level of formal rewriting systems finds its full conceptual realisation when the system at issue can be arranged into a well-behaved logical calculus, namely a calculus able to eliminate cuts and in which the rules are clearly partitioned into *negative* (reversible) and *positive* (non-reversible)[†]. Such a 'topological logic' could allow us to interpret the process

[†] Among the several advantages of dealing with a polarised calculus is that we immediately have at our disposal a denotational semantics in ludics style (Girard 2001).

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of forming the quotient surface from a given triangulation as a real logical derivation in which triangles play the role of axioms and the conclusion represents the resulting surface. Viewed in this way, triangulation may be seen in terms of proof-search, while the study of cut-elimination could help us provide the computational content of such a branch of algebraic topology (if any). The problem of defining a logic that can act in this way as a real interface between the geometry of topological surfaces and their discrete treatment is still open.

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