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On the Automorphism Groups of Dowling Geometries

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In 'Automorphisms of Dowling lattices and related geometries', J. Bonin constructed the automorphism group A of a Dowling lattice as the image of a certain semidirect product, $A = \theta(K \rtimes H)$. In this work we find necessary and sufficient conditions for this quotient to be the semidirect product $A = \theta(K) \rtimes \theta(H)$. In addition, we include a construction of A that lends itself to computation more readily than that found in Bonin's work.

1. Introduction

The present work grew out of a conversation between the author and Joseph E. Bonin in the early 1990s. Bonin had recently constructed the automorphism groups of Dowling geometries, and was looking for a way to simplify the (rather complicated) notation used to define the group operation. The primary question was whether or not the automorphism group $A_n(G)$ of the Dowling geometry $Q_n(G)$ could be written as a semidirect product in a particular way associated with the construction of $A_n(G)$ in [1]. In that construction, $A_n(G)$ is found to be a quotient of a semidirect product; the question is whether $A_n(G)$ itself inherits a semidirect product structure. Unfortunately, as we show in Theorem 6.1, it does not, unless the automorphism group of G is itself a particular semidirect product.

In the last section, we provide the reader with an alternative presentation $B_n(G)$ of the automorphism group of $Q_n(G)$ with a more natural group operation than that of $A_n(G)$, together with an isomorphism α of $B_n(G)$ onto $A_n(G)$.

In this paper, we work with the *jointless* Dowling geometry $Q'_n(G)$, first published in [5], rather than the original Dowling geometry $Q_n(G)$ defined in [2]. We do this for the following reasons, all of which are proved in [1].

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(1) For all integers $n \geqslant 3$ and nontrivial finite groups G, the action of the automorphism group $A_n(G)$ on $Q_n(G)$ stabilizes the subgeometry $Q'_n(G)$ of $Q_n(G)$.
(2) Except when $|G| = 1$ or when $n = 4$ and $|G| = 2$, each automorphism α' of $Q'_n(G)$ extends uniquely to an automorphism α of $Q_n(G)$.
(3) Except when $|G| = 1$ or when $n = 4$ and $|G| = 2$, every automorphism α of $Q_n(G)$ is an extension of an automorphism α' of $Q'_n(G)$.
(4) Finally, $Q'_n(G)$ has a more compact description than $Q_n(G)$, which results in a simpler description of $A_n(G)$.

For the case $n = 4$ and $|G| = 2$ the structure of the automorphism group $A_n(G)$ is described in [1], as is the automorphism group of $Q'_4(G)$.

2. Preliminaries

Groups

Unless explicitly stated otherwise, all groups are finite and nontrivial. For a group G we will use the following notation.

(1) $Z(G) =$ the centre of G.
(2) $\mathrm{Aut}(G) =$ the group of automorphisms of G.
(3) $\mathrm{Inn}(G) =$ the subgroup of $\mathrm{Aut}(G)$ consisting of the inner automorphisms of G. Note that $\mathrm{Inn}(G) \cong G/Z(G)$.
(4) $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) =$ the 'group of outer automorphisms' of G. We will write $a = a_G : \mathrm{Aut}(G) \to \mathrm{Out}(G)$ to denote the natural epimorphism of $\mathrm{Aut}(G)$ onto $\mathrm{Out}(G)$. Furthermore, for $\sigma \in \mathrm{Aut}(G)$ we will write $\overline{\sigma} = a(\sigma) = \sigma\,\mathrm{Inn}(G)$.
(5) Id_G is the identity automorphism $G \to G$. (We will only use this notation when we are consider Id_G as a morphism in the category of groups.)

When we say that $\sigma \in \mathrm{Aut}(G)$ is 'conjugation by $\gamma \in G$', we mean that $\sigma(g) = g^\gamma = \gamma^{-1}g\gamma$ for all $g \in G$. In this case we will write $\sigma = \tau_\gamma$. Thus, the map $\gamma \mapsto \tau_\gamma$ is an anti-homomorphism $G \to \mathrm{Inn}(G)$. Note that the homomorphisms $\gamma \mapsto \tau_\gamma^{-1}$ and a appear together in the exact sequence:

$$1 \to Z(G) \xrightarrow{\text{inclusion}} G \xrightarrow{\gamma \mapsto \tau_\gamma^{-1}} \mathrm{Aut}(G) \xrightarrow{a} \mathrm{Out}(G) \to 1. \tag{2.1}$$

If H is a subgroup (resp. normal subgroup) of G, then we write $H \leqslant G$ (resp. $H \trianglelefteq G$). If, in addition, $H \neq G$ then we write $H < G$ (resp. $H \triangleleft G$).

Semidirect products, split epimorphisms and complements

Throughout this work we use left functional notation. That is, if $f : X \to Y$ and $g : Y \to Z$, then we write $g \circ f$ for the composition 'f followed by g' sending $x \in X$ to $g(f(x)) \in Z$.

Definition 1. Let $\beta : E \to F$ be an epimorphism of groups. Any homomorphism $\pi : F \to E$ with $\beta \circ \pi = \mathrm{Id}_F$ will be called a *splitting* of β. If such a splitting exists then we will say that β *splits*. We will say that a short exact sequence

$$1 \to D \to E \xrightarrow{\beta} F \to 1$$

of groups *splits* if β splits.

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Definition 2. Let $D \trianglelefteq E$. Any subgroup $F \leqslant E$ such that $DF = E$ and $D \cap F = 1$ is called a *complement of D in E*.

Definition 3. Let D and F be groups and let F act on D by exponentiation. The *semidirect product* $D \rtimes F$ of D and F is the group consisting of ordered pairs $(d, f) \in D \times F$ together with the operation

$$(d_1, f_1)(d_2, f_2) = (d_1 d_2^{f_1^{-1}}, f_1 f_2).$$

The following result is standard and may be found in any text on group theory.

Proposition 2.1. *Let $\beta : E \to F$ be a split epimorphism of groups with kernel $D \trianglelefteq E$, and let $\pi : F \to E$ be a splitting of β. Then $\pi(F)$ is a complement of D in E, and E is isomorphic to the semidirect product $D \rtimes F$ where F acts on D by $d^f = d^{\pi(f)}$ for all $d \in D$ and $f \in F$.*

Conversely, if $E = D \rtimes F$ is a semidirect product, then the homomorphism $\beta : E \to F$ sending (d, f) to $\beta(d, f) = f$ is a split epimorphism, with splitting $\pi : F \to E$ sending $f \in F$ to $\pi(f) = (1, f) \in E$. Furthermore, the image $\pi(F)$ of F in E is a complement of $D \times 1 = \{(d, 1) : d \in D\} \leqslant E$.

Let E be a semidirect product $D \rtimes F$. As is usual with semidirect products, we identify D and F with their images $(D, 1)$ and $(1, F)$, respectively, in E. Then $E = DF = FD$ and we may use a slightly less cumbersome notation for multiplication in E. That is, writing df for $(d, f) = (d, 1)(1, f)$ in E, we have (with parentheses for clarity only)

$$(d_1 f_1)(d_2 f_2) = (d_1 d_2^{f_1^{-1}})(f_1 f_2),$$

or

$$(f_1 d_1)(f_2 d_2) = (f_1 f_2)(d_1^{f_2} d_2)$$

for all $d_1, d_2 \in D$ and $f_1, f_2 \in F$.

Let us briefly consider the epimorphism $a : \mathrm{Aut}(G) \to \mathrm{Out}(G)$. In general, $\mathrm{Out}(G)$ is *not* a group of automorphisms of G, but rather a group of equivalence classes of automorphisms modulo the normal subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G. That is, an outer automorphism is not really an automorphism at all – unless the epimorphism a splits. When the epimorphism a splits, each splitting $s : \mathrm{Out}(G) \to \mathrm{Aut}(G)$ allows us to define an action of $\mathrm{Out}(G)$ on G (although different splittings can produce vastly different actions of $\mathrm{Out}(G)$ on G). There are many groups G for which it is known that the epimorphism $a : \mathrm{Aut}(G) \to \mathrm{Out}(G)$ splits.

- Since $\mathrm{Inn}(G) \cong G/Z(G)$, if G is abelian then $\mathrm{Aut}(G) = \mathrm{Out}(G)$ and the epimorphism a is an isomorphism. Thus $s = a^{-1} : \mathrm{Out}(G) \to \mathrm{Aut}(G)$ splits a.
- If $\mathrm{Aut}(G) = \mathrm{Inn}(G)$, as is the case with $\mathfrak{S}_n$ for $n \neq 6$, then the epimorphism a is the trivial homomorphism which always splits.
- Let $G = \mathrm{GL}(n, p^d)$ be the group of $n \times n$ nonsingular matrices over the field of p^d elements, p a prime. For all but a few pairs (n, p^d), the group $\mathrm{Out}(G)$ is the direct product of a cyclic group of order 2 and a cyclic group of order d. The group of order

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2 can be generated by the map sending a matrix A to $(A^{-1})^T$, the transpose of its inverse. The cyclic group of order d can generated by the Frobenius automorphism sending a matrix (a_{ij}) to the matrix (a_{ij}^p). These automorphisms generate the image of $\mathrm{Out}(G)$ in $\mathrm{Aut}(G)$ under a suitable splitting.

In general, most simple (or 'nearly' simple) finite groups admit such a splitting. On the other hand, non-abelian p-groups (hence nilpotent groups in general) often do *not* admit a splitting s of a.

The jointless Dowling geometry $Q'_n(G)$

For the basic theory of matroids and geometries we refer the reader to [3]. If n is a positive integer, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

The Dowling geometry $Q_n(G)$ is defined in [2] for any finite group G and any positive integer n. It is line-closed; that is, every closed set in $Q_n(G)$ is the union of the lines contained in it. Thus, $Q_n(G)$ is completely determined by its points and lines.

By [1, Theorem 3.1] the automorphism group of $Q_n(G)$ is isomorphic to the automorphism group of the jointless Dowling geometry $Q'_n(G)$, unless either $|G| = 1$ or $(|G|, n) = (2, 4)$. We will not consider either of these cases in the current work.

In general, $Q'_n(G)$ is *not* line-closed. However, in light of [1, Theorem 3.1] it suffices to consider the action of the automorphism group on the points and lines of $Q'_n(G)$. Therefore, we will only concern ourselves with the points and lines of $Q'_n(G)$, which we define below for the convenience of the reader.

The points

Let M be a free left $\mathbb{Z}G$-module of rank n with basis $\{e_i : i \in [n]\}$. Let X be the set of elements of M of the form $ae_i + be_j$ for $a, b \in G$ and $i, j \in [n]$ with $i \neq j$. Note that X is closed under the action of $G \subset \mathbb{Z}G$. We define an equivalence relation $\sim$ on X by $(ae_i + be_j) \sim (ce_k + de_l)$ if and only if there exists $g \in G$ such that $(ae_i + be_j) = (gce_k + gde_l)$. That is, the $\sim$-equivalence classes in X are the G-orbits under the above action. We will denote the set of $\sim$-equivalence classes by $\mathbb{P}X$. If $x \in X$ we will write $[x] \in \mathbb{P}X$ for the equivalence class containing x. The set of points of $Q'_n(G)$ is the set $\mathbb{P}X$.

The lines

Let $x = [e_i + ae_j]$ and $y = [e_k + be_l]$ be distinct elements of $\mathbb{P}X$. We will let $x \vee y$ denote the (unique) line containing x and y. Clearly, $\delta = |\{i, j\} \cap \{k, l\}|$ can take only the values 0, 1 or 2. These three cases distinguish between the three types of lines in $Q'_n(G)$.

(1) If $\delta = 0$ then $x \vee y = \{x, y\}$.

(2) If $\delta = 1$ we can, without loss of generality, assume that $i = k$. Then $x \vee y = \{x, y, [ae_j + be_l]\}$.

(3) If $\delta = 2$ then $x \vee y = \{[e_i + ge_j] : g \in G\}$.

Evidently, every pair of points lies on a line of one of the three above types.

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3. The group $\mathcal{A}_n(G)$

Fix a group G and a positive integer n. Let $G^{[n]}$ denote the group of all functions $f : [n] \to G$ with multiplication given by $(fg)(m) = f(m)g(m)$. That is, $G^{[n]}$ is isomorphic to the direct product of n copies of G. For $f \in G^{[n]}$ we will write $f_m = f(m)$. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$. We view $\mathfrak{S}_n$ and $\mathrm{Aut}(G)$ as sets of functions. Then, for any $\psi \in \mathfrak{S}_n$, any $\sigma \in \mathrm{Aut}(G)$, and any $f \in G^{[n]}$, we may form the composition of functions $\sigma \circ f \circ \psi \in G^{[n]}$.

Let W be the wreath product $G \wr \mathfrak{S}_n$, which we view as the semidirect product $G^{[n]} \rtimes \mathfrak{S}_n$. That is, each element of W may be written uniquely as ψf where $\psi \in \mathfrak{S}_n$ and $f \in G^{[n]}$. The action of $\mathfrak{S}_n$ on $G^{[n]}$ is given by $f^{\phi} = f \circ \phi$ for all $f \in G^{[n]}$ and $\phi \in \mathfrak{S}_n$. Let $\mathrm{Aut}(G)$ act on W as follows: for $\sigma \in \mathrm{Aut}(G)$ we define $f^{\sigma} = \sigma^{-1} \circ f$ and $\psi^{\sigma} = \psi$ for all $f \in G^{[n]}$ and $\psi \in \mathfrak{S}_n$. Then we may form the semidirect product $\mathcal{A}_n(G) = W \rtimes \mathrm{Aut}(G)$ with elements written uniquely as $\psi f \sigma$ for $\psi \in \mathfrak{S}_n$, $f \in G^{[n]}$ and $\sigma \in \mathrm{Aut}(G)$. Multiplication in $\mathcal{A}_n(G)$ is given by

$$(\psi f \sigma)(\phi g \tau) = (\psi\phi)(f^{\phi} g^{\sigma^{-1}})(\sigma\tau) \tag{3.1}$$

for all $f, g \in G^{[n]}$, all $\psi, \phi \in \mathfrak{S}_n$, and all $\sigma, \tau \in \mathrm{Aut}(G)$.

We may also view $\mathcal{A}_n(G)$ as a semidirect product $G^{[n]} \rtimes (\mathfrak{S}_n \times \mathrm{Aut}(G))$, where $f^{\psi \times \sigma} = \sigma^{-1} \circ f \circ \psi$. We see this by noting that the map $(\psi, \sigma) \mapsto \psi\sigma = \psi 1 \sigma$ splits the short exact sequence of groups

$$1 \to G^{[n]} \xrightarrow{f \mapsto f} \mathcal{A}_n(G) \xrightarrow{\psi f \sigma \mapsto (\psi, \sigma)} \mathfrak{S}_n \times \mathrm{Aut}(G) \to 1. \tag{3.2}$$

We will let $p : \mathcal{A}_n(G) \twoheadrightarrow \mathfrak{S}_n \times \mathrm{Aut}(G)$ denote the nontrivial epimorphism in the above sequence.

4. The action of $\mathcal{A}_n(G)$ on $Q'_n(G)$

For the remainder of this note, we assume that $|G| > 1$ and that $n \neq 4$ if $|G| = 2$. We also wish to inform the reader that many of our choices of notation, particularly the way $\mathcal{A}_n(G)$ acts on $Q'_n(G)$, were dictated by the notation in [1]. (From a group-theoretic point of view, this notation is somewhat unnatural, so we have included a section at the end of this note in which we give a more natural action of $\mathcal{A}_n(G)$ on $Q'_n(G)$.)

Given $\psi \in \mathfrak{S}_n$, $f \in G^{[n]}$ and $\sigma \in \mathrm{Aut}(G)$, we define $\theta(\psi f \sigma) : \mathbb{P}X \to \mathbb{P}X$ by

$$\theta(\psi f \sigma)[ae_i + be_j] = [\sigma(a) f_i^{-1} e_{\psi(i)} + \sigma(b) f_j^{-1} e_{\psi(j)}]$$

for all $i \neq j \in [n]$ and $a, b \in G$. To see that $\theta(\psi f \sigma)$ is a well-defined function $\mathbb{P}X \to \mathbb{P}X$ notice that

$$\begin{aligned}
\theta(\psi f \sigma)[ae_i + be_j] &= [\sigma(a) f_i^{-1} e_{\psi(i)} + \sigma(b) f_j^{-1} e_{\psi(j)}] \\
&= [\sigma(g)\sigma(a) f_i^{-1} e_{\psi(i)} + \sigma(g)\sigma(b) f_j^{-1} e_{\psi(j)}] \\
&= [\sigma(ga) f_i^{-1} e_{\psi(i)} + \sigma(gb) f_j^{-1} e_{\psi(j)}] \\
&= \theta(\psi f \sigma)[gae_i + gbe_j]
\end{aligned}$$

for all $g \in G$ and all $[ae_i + be_j] \in \mathbb{P}X$. By [1, Theorem 3.1], the map $\theta(\psi f \sigma)$ is an

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automorphism of $Q'_n(G)$, and every automorphism of $Q'_n(G)$ is of the form $\theta(\psi f\sigma)$ for some $\psi f\sigma \in \mathcal{A}_n(G)$. By the remarks following [1, Theorem 2.1] we see that θ is indeed an epimorphism from $\mathcal{A}_n(G)$ onto the group $A_n(G)$ of all automorphisms of $Q'_n(G)$.

Let $\Delta : G \to G^{[n]}$ be the diagonal embedding of G in $G^{[n]}$. That is, for $g \in G$ let $\Delta(g)_i = g$ for all $i \in [n]$. Now, for $g \in G$ let $\eta_g = \Delta(g^{-1})\tau_g^{-1} \in \mathcal{A}_n(G)$. (Recall that τ_g is conjugation by g.)

Proposition 4.1. *If $n \geqslant 3$, then the kernel N of $\theta : \mathcal{A}_n(G) \to A_n(G)$ is given by*

$$N = \{\eta_g : g \in G\}.$$

Proof. First we show that $N \leqslant \ker\theta$. If $g \in G$, then for all $[ae_i + be_j] \in \mathbb{P}X$ we have

$$\begin{aligned}\theta(\eta_g)[ae_i + be_j] &= \theta(\Delta(g^{-1})\tau_g^{-1})[ae_i + be_j]\\ &= [\tau_g^{-1}(a)\Delta(g^{-1})_i^{-1}e_i + \tau_g^{-1}(b)\Delta(g^{-1})_j^{-1}e_j]\\ &= [(gag^{-1})ge_i + (gbg^{-1})ge_j]\\ &= [gae_i + gbe_j]\\ &= [ae_i + be_j].\end{aligned}$$

Thus η_g acts trivially on $\mathbb{P}X$ for all $g \in G$.

Now we show that $\ker\theta \leqslant N$. Let $\psi f\sigma \in \ker\theta$. First, note that every ψ must fix every pair (i, j), $i, j \in [n]$ with $i \neq j$. Because $n \geqslant 3$, it follows that $\psi = 1_{\mathfrak{S}_n}$. Therefore $\psi f\sigma = f\sigma$ for some $f \in G^{[n]}$ and $\sigma \in \mathrm{Aut}(G)$. Now $f\sigma$ must fix every element of $\mathbb{P}X$. In particular, $f\sigma$ must fix $[e_i + e_j]$ for all $i, j \in [n]$, $i \neq j$. Since $\sigma(1_G) = 1_G$ for all $\sigma \in \mathrm{Aut}(G)$, we have

$$\begin{aligned}[e_i + e_j] &= \theta(f\sigma)[e_i + e_j]\\ &= [f_i^{-1}e_i + f_j^{-1}e_j].\end{aligned}$$

It follows immediately that $f_i = f_j$ for all $i, j \in [n]$. That is, $f = \Delta(g^{-1})$ for some $g \in G$, and $f\sigma = \Delta(g^{-1})\sigma$. Finally, for all $a \in G$ and $i, j \in [n]$ with $i \neq j$, we have

$$\begin{aligned}[e_i + ae_j] &= \theta(\Delta(g^{-1})\sigma)[e_i + ae_j]\\ &= [ge_i + \sigma(a)ge_j]\\ &= [e_i + \tau_g(\sigma(a))e_j].\end{aligned}$$

Again, it follows immediately that $\sigma = \tau_g^{-1}$. Therefore $\ker\theta \leqslant N$, completing the proof. □

The reader may easily verify that η is an isomorphism of G onto N.

5. Some structure for $A_n(G)$

For the remainder of this note, all diagrams will have exact rows and columns. In addition, we will assume that $n \geqslant 3$ so that we may use Proposition 4.1.

Since we know that $\mathcal{A}_n(G)$ is the semidirect product $G \rtimes (\mathfrak{S}_n \times \mathrm{Aut}(G))$, it seems natural to ask whether the epimorphism θ of $\mathcal{A}_n(G)$ onto $A_n(G)$ makes $A_n(G)$ a semidirect product

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of $\theta(G^{[n]})$ and $\theta(\mathfrak{S}_n \times \mathrm{Aut}(G))$. We begin by considering the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & G & & & & \\
 & & & & {\scriptstyle g\mapsto\eta_g}\downarrow & & & & \\
1 & \longrightarrow & G^{[n]} & \xrightarrow{\text{inclusion}} & \mathcal{A}_n(G) & \xrightarrow{p} & \mathfrak{S}_n \times \mathrm{Aut}(G) & \longrightarrow & 1 \\
 & & & & {\scriptstyle\theta}\downarrow & & & & \\
 & & & & A_n(G) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

Recall that $\eta_g = \Delta(g^{-1})\tau_g^{-1}$. Because $\Delta(g)$ and τ_g commute in $\mathcal{A}_n(G)$, we have $\eta_g = (\Delta(g)\tau_g)^{-1}$ and the map $g \mapsto \eta_g$ is a homomorphism $G \to \mathcal{A}_n(G)$.

First we fill in the top row. If g is an element of G such that $\eta_g \in G^{[n]}$, then $\tau_g = 1$. Thus $g \in Z(G)$, from which it follows that the restriction to $Z(G)$ of the map $g \mapsto \eta_g$ sends $z \in Z(G)$ to $\Delta(z)^{-1}$. Combining this with (2.1) we have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G) & \xrightarrow{\text{inclusion}} & G & \xrightarrow{g\mapsto\tau_g^{-1}} & \mathrm{Inn}(G) & \longrightarrow & 1 \\
 & & {\scriptstyle z\mapsto\Delta(z)^{-1}}\downarrow & & {\scriptstyle g\mapsto\eta_g}\downarrow & & \downarrow{\scriptstyle \sigma\mapsto 1\times\sigma} & & \\
1 & \longrightarrow & G^{[n]} & \xrightarrow{\text{inclusion}} & \mathcal{A}_n(G) & \xrightarrow{p} & \mathfrak{S}_n \times \mathrm{Aut}(G) & \longrightarrow & 1 \\
 & & & & {\scriptstyle\theta}\downarrow & & & & \\
 & & & & A_n(G) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

(Recall that the map $G \to \mathrm{Inn}(G)$ sending g to τ_g^{-1} is a homomorphism.) The arrow in the third column is merely the composition of inclusions $\mathrm{Inn}(G) \hookrightarrow \mathrm{Aut}(G) \hookrightarrow \mathfrak{S}_n \times \mathrm{Aut}(G)$.

Now we fill in the third row. Since $G^{[n]}$ is normal in $\mathcal{A}_n(G)$, we know that the image $\theta(G^{[n]}) \cong G^{[n]}/\Delta(Z(G))$ is normal in $\theta(\mathcal{A}_n(G)) = A_n(G)$. Now, if $\bar{p}$ is the epimorphism of $A_n(G)$ induced by p with kernel $\theta(G^{[n]})$, then the map $\overline{\theta} = \mathrm{Id}_{\mathfrak{S}_n} \times a \colon \mathfrak{S}_n \times \mathrm{Aut}(G) \to \mathfrak{S}_n \times \mathrm{Out}(G)$ sending $(\psi, \sigma) \in \mathfrak{S}_n \times \mathrm{Aut}(G)$ to $(\psi, \overline{\sigma}) \in \mathfrak{S}_n \times \mathrm{Out}(G)$ makes the following diagram commute (recall that a is the natural epimorphism $a \colon \mathrm{Aut}(G) \to \mathrm{Out}(G)$ sending

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$\sigma \in \mathrm{Aut}(G)$ to $a(\sigma) = \overline{\sigma} \in \mathrm{Out}(G)$):

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G) & \xrightarrow{\text{inclusion}} & G & \xrightarrow{g \mapsto \tau_g^{-1}} & \mathrm{Inn}(G) & \longrightarrow & 1 \\
 & & {\scriptstyle z \mapsto \Delta(z)^{-1}}\downarrow & & {\scriptstyle g \mapsto \eta_g}\downarrow & & \downarrow{\scriptstyle \sigma \mapsto 1 \times \sigma} & & \\
1 & \longrightarrow & G^{[n]} & \xrightarrow{\text{inclusion}} & \mathcal{A}_n(G) & \xrightarrow{p} & \mathfrak{S}_n \times \mathrm{Aut}(G) & \longrightarrow & 1 \\
 & & {\scriptstyle \theta}\downarrow & & {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \overline{\theta} = \mathrm{Id}_{\mathfrak{S}_n} \times a} & & \\
1 & \longrightarrow & \theta(G^{[n]}) & \xrightarrow{\text{inclusion}} & A_n(G) & \xrightarrow{\overline{p}} & \mathfrak{S}_n \times \mathrm{Out}(G) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{5.1}$$

Let $t : \mathfrak{S}_n \times \mathrm{Aut}(G) \to \mathcal{A}_n(G)$ be the natural splitting of p given by $t(\psi, \sigma) = \psi\sigma$ for all $(\psi, \sigma) \in \mathfrak{S}_n \times \mathrm{Aut}(G)$. Recall that $N = \ker \theta$ from Proposition 4.1.

6. Splitting $A_n(G)$

If the bottom row of (5.1) splits, there is a subgroup $H \leqslant \mathcal{A}_n(G)$ such that $G^{[n]}N \cap HN = N$ and $G^{[n]}HN = \mathcal{A}_n(G)$. A natural choice of H would be $\mathrm{im}\, t$. Unfortunately, this choice does not generally induce a splitting for the third row. To see this, let $H = \mathrm{im}\, t$. Then HN consists of all elements of $\mathcal{A}_n(G)$ of the form $\phi\Delta(\gamma)\sigma$ where ϕ and σ range over all elements of $\mathfrak{S}_n$ and $\mathrm{Aut}(G)$, respectively, and γ is any element of G. The subgroup $G^{[n]}N$ consists of all elements of the form $f\tau_g$ where f is any element of $G^{[n]}$ and g is any element of G. Although every element of $G^{[n]}N \cap HN$ has the form $\Delta(\gamma)\tau_g$, we cannot say that $\gamma = g$ and so we cannot say that $G^{[n]}N \cap HN \leqslant N$. In fact, if G has a nontrivial inner automorphism τ, then $\Delta(1)\tau$ will be an element of $G^{[n]}N \cap HN$ which is not in N. (Of course, G has a nontrivial inner automorphism if and only if G is not abelian.)

The actual conditions under which the bottom row of (5.1) splits are given by the following theorem.

Theorem 6.1. *If $n \geqslant 3$, $|G| > 1$ and $(|G|, n) \neq (2, 4)$, then the natural epimorphism $\overline{p} : A_n(G) \twoheadrightarrow \mathfrak{S}_n \times \mathrm{Out}(G)$ splits if and only if the natural epimorphism $a : \mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G)$ splits.*

Note. The reason that this fails when $n = 4$ and $|G| = 2$ is that the automorphism group of $Q_4(G)$ with $|G| = 2$ is *not* given by the construction in Section 4 (see [1]).

Proof. First we assume that $s : \mathrm{Out}(G) \to \mathrm{Aut}(G)$ is a splitting of a. We define a splitting $\overline{s}$ of $\overline{p}$ by

$$\overline{s}(\psi, \overline{\sigma}) = (\theta \circ t)(\psi, s(\overline{\sigma}))$$

for all $(\psi, \overline{\sigma}) \in \mathfrak{S}_n \times \mathrm{Out}(G)$. That is, $\overline{s}$ is the homomorphism $\theta \circ t \circ (\mathrm{Id}_{\mathfrak{S}_n} \times s)$. Since

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$\overline{p} \circ \theta = \overline{\theta} \circ p$ we have

$$\begin{aligned}(\overline{p} \circ \overline{s})(\psi, \overline{\sigma}) &= (\overline{p} \circ \theta \circ t)(\psi, s(\overline{\sigma})) \\ &= (\overline{\theta} \circ p \circ t)(\psi, s(\overline{\sigma})) \\ &= \overline{\theta}(\psi, s(\overline{\sigma})) \\ &= (\psi, \overline{\sigma})\end{aligned}$$

for all $(\psi, \overline{\sigma}) \in \mathfrak{S}_n \times \mathrm{Out}(G)$. Therefore $\overline{s}$ splits $\overline{p}$.

Now assume that $\overline{s}$ is a splitting of $\overline{p}$. Let

$$H = G^{[n]} \mathrm{Aut}(G) = G^{[n]} \rtimes \mathrm{Aut}(G) \trianglelefteq \mathcal{A}_n(G).$$

Write $G^{[n-1]} = (G^{[n-1]} \times 1)$ for the subgroup of all $f \in G^{[n]} \leqslant \mathcal{A}_n(G)$ with $f_n = 1$. Then $G^{[n-1]}N \leqslant G^{[n]} \mathrm{Inn}(G) \leqslant H$.

Notice that this $G^{[n-1]}$ is normalized by $\mathrm{Aut}(G)$. So $K = G^{[n-1]} \mathrm{Aut}(G)$ is a subgroup of H. We easily compute that $K \cap N = 1$ and $KN = H$. So $H = N \rtimes K$. Now let $\overline{H} = \theta(H) = H/N$. Then $K \cong \overline{H} = G^{[n]} \mathrm{Aut}(G)/N$. Therefore there exists a unique isomorphism $\theta^* : G^{[n]} \mathrm{Aut}(G)/N \to K = G^{[n-1]} \mathrm{Aut}(G)$ such that $\theta \circ \theta^* = \mathrm{Id}_{G^{[n]} \mathrm{Aut}(G)/N}$.

Since $\overline{p} \circ \overline{s} = \mathrm{Id}_{\mathfrak{S}_n \times \mathrm{Out}(G)}$ we know that $\overline{s}(\mathrm{Out}(G)) \leqslant \overline{p}^{-1}(\mathrm{Out}(G))$. But

$$\begin{aligned}\overline{p}^{-1}(\mathrm{Out}(G)) &\leqslant \theta\left((\overline{p} \circ \theta)^{-1}(\mathrm{Out}(G))\right) \\ &= \theta\left((\overline{\theta} \circ p)^{-1}(\mathrm{Out}(G))\right) \\ &= \theta(G^{[n]} \mathrm{Aut}(G)) \\ &= \theta(H) \\ &= \overline{H}.\end{aligned}$$

Therefore θ^* is defined on $\overline{s}(\mathrm{Out}(G)) \leqslant \overline{H}$. So we may define $\overline{\theta}^* : \mathfrak{S}_n \times \mathrm{Out}(G) \to \mathfrak{S}_n \times \mathrm{Aut}(G)$ as follows:

$$\overline{\theta}^*(\psi, \overline{\sigma}) = p\left(\psi \cdot \left((\theta^* \circ \overline{s})(\overline{\sigma})\right)\right)$$

for all $\psi \in \mathfrak{S}_n$ and $\sigma \in \mathrm{Aut}(G)$. (We use the 'dot' notation for multiplication to avoid confusion as we do not intend to mean that we are applying ψ to the expression which follows it.) Then

$$\begin{aligned}(\overline{\theta} \circ \overline{\theta}^*)(\psi, \overline{\sigma}) &= (\overline{\theta} \circ p)\left(\psi \cdot \left((\theta^* \circ \overline{s})(\overline{\sigma})\right)\right) \\ &= (\overline{p} \circ \theta)\left(\psi \cdot \left((\theta^* \circ \overline{s})(\overline{\sigma})\right)\right) \\ &= (\psi, (\overline{p} \circ \theta \circ \theta^* \circ \overline{s})(\overline{\sigma})) \\ &= (\psi, (\overline{p} \circ \overline{s})(\overline{\sigma})) \\ &= (\psi, \overline{\sigma}).\end{aligned}$$

So $\overline{\theta}^*$ is a splitting of $\overline{\theta}$. Clearly $\overline{\theta}^*(\mathrm{Out}(G)) \leqslant \mathrm{Aut}(G) \leqslant \mathfrak{S}_n \times \mathrm{Aut}(G)$. Thus $\overline{\theta}^*$ induces a unique splitting $s : \mathrm{Out}(G) \to \mathrm{Aut}(G)$ of a. □

In the proof we provided a 'partial' splitting $\theta^* : H/N \to \mathcal{A}_n(G)$ of the restriction of θ to the subgroup H of $\mathcal{A}_n(G)$. One should not therefore assume that θ^* can be extended to a splitting of θ. In particular, the image of $G^{[n]}/N$ under θ^* is $G^{[n-1]}$ which is *not*

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normalized by $\mathfrak{S}_n$ in $\mathcal{A}_n(G)$. If anything, the existence of θ^* merely points out the fact that the action of $\mathfrak{S}_n$ is not relevant to our result.

7. An alternative presentation of $A_n(G)$

As we noted above, we have followed [1] in defining the action of $\mathcal{A}_n(G)$ on $Q'_n(G)$. The reader may have noticed that this action of $G^{[n]}$ on $\mathbb{P}X$ is unnatural in the sense that one must invert and then multiply. This results in a cumbersome multiplication in $\mathcal{A}_n(G)$ and hence $A_n(G)$. We will now construct a group $\mathcal{B}_n(G) \cong \mathcal{A}_n(G)$ which acts on $\mathbb{P}X$ on the right.

We begin by changing the action of $\mathrm{Aut}(G)$ on G from functional notation to exponential. Now, if τ_g denotes conjugation by g, then $x^{\tau_g} = x^g$ for all $x, g \in G$. Therefore the map $g \mapsto \tau_g$ is now a homomorphism, rather than an anti-homomorphism. Also, we will use the right action of $\mathfrak{S}_n$ on $[n]$ so that i^ψ will denote the image of $i \in [n]$ under the action of $\psi \in \mathfrak{S}_n$. For $\mathrm{Aut}(G)$ and $\mathfrak{S}_n$, passing from the left action to the right action is merely a matter of inverting. That is, for $\sigma \in \mathrm{Aut}(G)$, what we now write as g^σ is the element which we previously denoted by $\sigma^{-1}(g)$ and similarly for $\mathfrak{S}_n$.

Now we define actions of $G^{[n]}$, $\mathfrak{S}_n$, and $\mathrm{Aut}(G)$ on $\mathbb{P}X$. Let $[ae_j + be_j]$ be any element of $\mathbb{P}X$. For $f \in G^{[n]}$ we define

$$[ae_i + be_j]^f = [af_i e_i + bf_j e_j].$$

For $\sigma \in \mathrm{Aut}(G)$ we define

$$[ae_i + be_j]^\sigma = [a^\sigma e_i + b^\sigma e_j].$$

Finally, for $\psi \in \mathfrak{S}_n$ we define

$$[ae_i + be_j]^\psi = [ae_{i^\psi} + be_{j^\psi}].$$

Clearly the above three actions are automorphic on $Q'_n(G)$; in particular, we see that

$$\begin{aligned}[ae_i + be_j]^{(\psi\sigma f)} &= [a^\sigma f_{i^\psi} e_{i^\psi} + b^\sigma f_{j^\psi} e_{j^\psi}] \\ &= \theta(\psi^{-1}(f^{-1})^{\psi^{-1}}\sigma^{-1})[ae_i + be_j]\end{aligned}$$

in our old notation, for any $\psi \in \mathfrak{S}_n$, $\sigma \in \mathrm{Aut}(G)$, and $f \in G^{[n]}$. Now, define $\mathcal{B}_n(G)$ to be the group of elements of the form $\psi\sigma f$ with $\psi \in \mathfrak{S}_n$, $\sigma \in \mathrm{Aut}(G)$ and $f \in G^{[n]}$. Multiplication in $\mathcal{B}_n(G)$ is tentatively given by

$$(\psi\sigma f)(\phi\tau g) = (\psi\phi)(\sigma\tau)(f^{\phi\tau} g)$$

for $\psi, \phi \in \mathfrak{S}_n$, $\sigma, \tau \in \mathrm{Aut}(G)$ and $f, g \in G^{[n]}$. We say 'tentatively' because we must define f^ϕ and f^τ for $f \in G^{[n]}$, $\phi \in \mathfrak{S}_n$, and $\tau \in \mathrm{Aut}(G)$. In order to make $\psi\sigma g$ have the same effect on $Q'_n(G)$ as the automorphism denoted previously by $\theta(\psi^{-1}(f^{-1})^{\psi^{-1}}\sigma^{-1})$, we define $(f^\phi)_i = f_{i^\phi}$ and $(f^\tau)_i = (f_i)^\tau$ for $i \in [n]$. The reader may easily verify that this produces the desired result. Indeed, the map $\alpha : \mathcal{B}_n(G) \to \mathcal{A}_n(G)$ given by $\alpha(\psi\sigma f) = \psi^{-1}(f^{-1})^{\psi^{-1}}\sigma^{-1}$ is an isomorphism. In addition, we see that $N_B = \{\tau_g \Delta(g^{-1}) : g \in G\}$ is the kernel of the action of $\mathcal{B}_n(G)$ on $\mathbb{P}X$.

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Clearly, the quotient $B_n(G) = \mathcal{B}_n(G)/N_B$ is isomorphic to the group $A_n(G)$ of automorphisms of $Q'_n(G)$. Furthermore, the isomorphism α provides an induced action of $B_n(G)$ on $Q'_n(G)$. Therefore, we may identify the elements of $B_n(G)$ with the classes of elements $\psi\sigma f$ of $\mathcal{B}_n(G)$ modulo the relations

$$\psi\sigma\tau_g f = \psi\sigma\Delta(g)f$$

for all $\psi \in \mathfrak{S}_n$, all $\sigma \in \mathrm{Aut}(G)$, all $f \in G^{[n]}$ and all $g \in G$.

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