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Singular measures with absolutely continuous convolution squares

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1. *Introduction.* A famous construction of Wiener and Wintner ((13)), later refined by Salem ((11)) and extended by Schaeffer ((12)) and Ivašev-Musatov ((8)), produces a non-negative, singular, continuous measure μ on $[-\pi, \pi[$ such that

$$\int_{-\pi}^{\pi} e^{-inx}\, d\mu(x) = O(n^{-\frac{1}{2}+\epsilon}) \tag{1·1}$$

for every $\epsilon > 0$. It is plain that the convolution $\mu * \mu$ is absolutely continuous and in fact has Lebesgue–Radon–Nikodým derivative f such that $\int_{-\pi}^{\pi} |f|^p < \infty$ for all $p \geqslant 1$. For general locally compact Abelian groups, no exact analogue of (1·1) seems possible, as the character group may admit no natural order. However, it makes good sense to ask if $\mu * \mu$ is absolutely continuous and has pth power integrable derivative. We will construct continuous singular measures μ on all non-discrete locally compact Abelian groups G such that $\mu * \mu$ is absolutely continuous and for which the Lebesgue–Radon–Nikodým derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all real $p > 1$.

Our construction thus contributes to answering the vague question: 'How small can a singular Fourier–Stieltjes transform be?' In addition to the authors cited above, Men'šov ((10)) and Littlewood ((9)) have contributed to the study of this problem. There is a related question: 'How large can a singular Fourier–Stieltjes transform be?' We will make a few remarks on this question in 6·3.

In section 2 we describe our notation and state some preliminary facts. A fairly general method of constructing measures on compact Abelian groups appears in section 3. Our measures for some groups are constructed in section 4. Section 5 contains the extension to all non-discrete locally compact Abelian groups. Some conclusions and open questions are given in section 6.

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2. *Preliminaries.* 2·1. *Notation.* Throughout this paper, $G, H, \ldots$ will denote locally compact Abelian groups and $\mathsf{X}, \mathsf{Y}, \Delta, \ldots$ character groups or subsets of character groups. The word 'character' means 'continuous character'. The symbol $o(\chi)$ denotes the order of the character $\chi \in \mathsf{X}$: the least positive integer n such that $\chi^n = 1$ if such an integer exists, and ∞ otherwise. The symbol λ always means Haar measure on whatever group G is under consideration. If G is compact, we shall take $\lambda(G) = 1$. The expression 'almost everywhere' means 'λ-almost everywhere'. The symbol card (S)

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denotes the cardinal number of the set S. All notation and terminology used without explanation is as in the monograph ((5)).

2·2. *The groups R and T.* The symbol R denotes the additive group of real numbers, and T the circle group. We will take T to be the set $[-\pi, \pi[\subset R$, with the group operation addition modulo 2π. Every character of T has the form $x \to e^{inx}$ for some integer n, and so the character group of T can be identified with the additive group Z of all integers. We shall make this identification wherever it is convenient.

2·3. *The algebra* $\mathbf{M}(G)$. The symbol $\mathbf{M}(G)$ denotes the Banach algebra under convolution of all complex measures on G. Every measure $\nu \in \mathbf{M}(G)$ has a unique resolution $\nu = \nu_d + \nu_s + \nu_a$, where ν_d is a countable linear combination of point measures, ν_s is continuous and singular with respect to λ, and ν_a is absolutely continuous with respect to λ. Thus there is a function $f \in \mathfrak{L}_1(G)$ such that $d\nu_a = f d\lambda$. For brevity we will call f the *derivative of* ν_a, instead of the Lebesgue–Radon–Nikodým derivative. Thus f is the derivative of ν if ν is absolutely continuous [always with respect to λ]. The algebra $\mathbf{M}(G)$ is a direct sum

$$\mathbf{M}(G) = \mathbf{M}_d(G) \oplus \mathbf{M}_s(G) \oplus \mathbf{M}_a(G),$$

consisting of the discontinuous, singular, and absolutely continuous measures. The symbol $\mathbf{M}^+(G)$ denotes the real-valued, non-negative measures in $\mathbf{M}(G)$; similarly for $\mathbf{M}_s^+(G)$, etc. Exact definitions of the entities described here, and complete proofs of the assertions made, will be found in (5), sections 14 and 19.

We shall need the following fact.

2·4. THEOREM. *Let G be a compact Abelian group with character group X, let p be a real number such that $1 < p < 2$, and let $p' = p/(p-1)$. Let ϕ be any function in $l_p(\mathsf{X})$. Then there is a function $h \in \mathfrak{L}_{p'}(G)$ such that $\hat{h} = \phi$. Furthermore, we have*

$$\|h\|_{p'} \leqslant \left(\sum_{\chi \in \mathsf{X}} |\phi(\chi)|^p \right)^{1/p}.$$

This is an abstract form of a theorem of F. Riesz. For a proof, see, for example, (14), vol. II, p. 102. The modifications needed in the proof given by Zygmund are obvious.

3. *Construction of a class of measures.* 3·1. Let G be a compact Abelian group, finite or infinite, and let X be the (discrete) character group of G. A finite subset $\{\chi_1, \ldots, \chi_m\}$ (all χ_j's distinct) of X is said to be *dissociate* if it does not contain 1 and if the equality

$$\chi_1^{\epsilon_1} \chi_2^{\epsilon_2} \cdots \chi_m^{\epsilon_m} = 1,$$

where each $\epsilon_j \in \{-2, -1, 0, 1, 2\}$, implies the equalities

$$\chi_1^{\epsilon_1} = \chi_2^{\epsilon_2} = \cdots = \chi_m^{\epsilon_m} = 1.$$

For convenience, $\varnothing$ is also taken as dissociate. An infinite subset Δ of X is said to be *dissociate* if every finite subset of Δ is dissociate. Note that dissociation is a weakened version of independence (see, for example, (5), (A. 10)), and that dissociation can be

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defined for subsets of an arbitrary Abelian group. We need the concept here only for sets of characters.

For a complex number β and $\epsilon \in \{-1, 0, 1\}$, define $\beta^{(\epsilon)}$ as β if $\epsilon = 1$, as $\bar{\beta}$ if $\epsilon = -1$, and as 1 if $\epsilon = 0$.

3·2. THEOREM. *Let G and X be as in* (3·1) *and let Δ be a dissociate subset of X. Let β be a complex-valued function defined on Δ, the value of β at $\chi \in \Delta$ being denoted by β_χ. Suppose that $|\beta_\chi| \leqslant \frac{1}{2}$ if $o(\chi) > 2$, and that $|\beta_\chi| \leqslant 1$ and β_χ is real if $o(\chi) = 2$. Then there is a unique measure $\mu \in \mathbf{M}^+(G)$ such that:*

(i) $$\hat{\mu}(\chi_1^{\epsilon_1} \cdots \chi_n^{\epsilon_n}) = \beta_{\chi_1}^{(\epsilon_1)} \cdots \beta_{\chi_n}^{(\epsilon_n)}$$

for all subsets $\{\chi_1, \ldots, \chi_n\}$ of Δ and sequences $(\epsilon_1, \ldots, \epsilon_n)$ with values in $\{-1, 0, 1\}$;

(ii) $$\hat{\mu}(\psi) = 0$$

if $\psi \in \mathsf{X}$ and ψ is not of the form $\chi_1^{\epsilon_1} \ldots \chi_n^{\epsilon_n}$;

(iii) $$\hat{\mu}(1) = 1.$$

Proof. For each finite subset Φ of Δ, let

$$\Phi_0 = \{\chi \in \Phi : o(\chi) = 2\} \quad \text{and} \quad \Phi_1 = \Phi \cap \Phi_0'.$$

Define the function p_Φ on G by

$$p_\Phi = \prod_{\chi \in \Phi_0} (1 + \beta_\chi \chi) \prod_{\chi \in \Phi_1} (1 + \beta_\chi \chi + \bar{\beta}_\chi \chi^{-1}) \tag{3·2·1}$$

(void products are equal to 1). It is obvious that p_Φ is a real-valued, non-negative trigonometric polynomial on G. Write $\Phi_0 = \{\chi_1, \ldots, \chi_k\}$ and $\Phi_1 = \{\chi_{k+1}, \ldots, \chi_m\}$. Then (3·2·1) can be rewritten as

$$p_\Phi = \sum_{\boldsymbol{\epsilon}} \beta_1^{(\epsilon_1)} \beta_2^{(\epsilon_2)} \cdots \beta_m^{(\epsilon_m)} \chi_1^{\epsilon_1} \chi_2^{\epsilon_2} \cdots \chi_m^{\epsilon_m}, \tag{3·2·2}$$

where the sum is taken over all sequences $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$ such that $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$ and $\epsilon_{k+1}, \ldots, \epsilon_m \in \{-1, 0, 1\}$. The dissociation of Δ implies that the $2^k 3^{m-k}$ characters $\chi_1^{\epsilon_1} \cdots \chi_m^{\epsilon_m}$ are all distinct. It follows that

$$\hat{p}_\Phi(\psi) = \begin{cases} \beta_1^{(\epsilon_1)} \cdots \beta_m^{(\epsilon_m)} & \text{if} \quad \psi = \chi_1^{\epsilon_1} \cdots \chi_m^{\epsilon_m}, \\ 1 & \text{if} \quad \psi = 1, \\ 0 & \text{for all other } \psi \in \mathsf{X}. \end{cases}$$

Now consider the set of measures $p_\Phi \lambda$ in $\mathbf{M}^+(G)$, as Φ runs through all finite subsets of Δ. Plainly each of these measures has norm 1, and they admit the natural direction $\prec$ under which $p_\Phi \lambda \prec p_{\Phi^*} \lambda$ means that $\Phi \subset \Phi^*$. Since the unit sphere in $\mathbf{M}^+(G)$ is weakly compact, the net $(p_\Phi \lambda)$ admits a subnet converging in the weak topology of $\mathbf{M}(G)$ to a measure $\mu \in \mathbf{M}^+(G)$. Plainly $\mu(G) = 1$, and for every finite subset Φ of Δ and every weak neighbourhood $\mathbf{U}$ of μ, there are infinitely many $p_\Psi \lambda$ for which $\Psi \supset \Phi$ and $p_\Psi \lambda \in \mathbf{U}$.

It is now easy to show that μ is unique and that its Fourier–Stieltjes transform is as in (i), (ii), and (iii). Let $\psi \in \mathsf{X}$ have the form $\chi_1^{\epsilon_1} \cdots \chi_m^{\epsilon_m}$ where $\Phi = \{\chi_1, \ldots, \chi_m\} \subset \Delta$. If

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Ψ is any finite subset of Δ containing Φ, then the character ψ enters in the sum (3·2·2) for p_Ψ with the coefficient $\beta_1^{(\epsilon_1)} \cdots \beta_m^{(\epsilon_m)}$. Thus we have

$$\hat{\mu}(\psi) = \lim_{\Psi} \hat{p}_\Psi(\psi) = \beta_1^{(\epsilon_1)} \cdots \beta_m^{(\epsilon_m)}. \tag{3·2·3}$$

If ψ does not have the form $\chi_1^{\epsilon_1} \cdots \chi_m^{\epsilon_m}$ for $\{\chi_1, \ldots, \chi_m\} \subset \Delta$, then $\hat{p}_\Psi(\psi) = 0$ for all finite subsets Ψ of Δ, and here we have

$$\hat{\mu}(\psi) = \lim_{\Psi} \hat{p}_\Psi(\psi) = 0. \tag{3·2·4}$$

Since μ is completely determined by $\hat{\mu}$, the proof is complete. □

For use below, we now construct some dissociate sets of characters.

3·3. *Example.* Let G and X be as in 3·1. Then every dissociate set in X is contained in a *maximal* dissociate set M. This is immediate from Zorn's lemma. For every $\chi \in \mathsf{X} \cap \mathsf{M}'$, we have

$$\chi = \chi_1^{\epsilon_1} \cdots \chi_n^{\epsilon_n} \quad \text{or} \quad \chi^2 = \chi_1^{\epsilon_1} \cdots \chi_n^{\epsilon_n}$$

for some $\{\chi_1, \ldots, \chi_n\} \subset \mathsf{M}$ and $\epsilon_j \in \{-2, -1, 0, 1, 2\}$.

3·4. *Example.* Let X be the additive group Z, so that $G = T$. Let δ be a real number greater than 2, and let $(n_k)_{k=1}^\infty$ be a sequence of positive integers such that

$$n_{k+1} \geqslant \delta(n_1 + \cdots + n_k) \quad \text{for all } k. \tag{3·4·1}$$

For example, $(4^{k-1})_{k=1}^\infty$ satisfies (3·4·1) with $\delta = 3$. An easy induction shows that $n_j \geqslant \delta(1+\delta)^{j-2}$ for $j = 2, 3, \ldots$, and so

$$n_1 + \cdots + n_l \geqslant (1+\delta)^{l-1}. \tag{3·4·2}$$

The characters $x \to \exp(in_k x)$ are dissociate, as an obvious argument shows. These are extremely special dissociate sets, but their special properties are needed for part of our later argument. The set $(3^{k-1})_{k=1}^\infty$ is a maximal dissociate set in Z, which fails to satisfy (3·4·1). Applying 3·2 to the set $(3^{k-1})_{k=1}^\infty \subset Z$, we see how to determine a measure $\mu \in \mathbf{M}^+(T)$ such that $\hat{\mu}(0) = 1$, $\hat{\mu}(3^{k-1}) = \beta_k$, and

$$\hat{\mu}(\epsilon_1 + \epsilon_2 3 + \cdots + \epsilon_s 3^{s-1}) = \beta_1^{(\epsilon_1)} \cdots \beta_s^{(\epsilon_s)}$$

for all sequences $(\epsilon_1, \ldots, \epsilon_s)$ with entries -1, 0, or 1. Thus $\hat{\mu}(3^{k-1})$ is subject only to the restriction $|\hat{\mu}(3^{k-1})| \leqslant \frac{1}{2}$, and then μ is completely determined.

3·5. *Example.* Let G be an infinite compact 0-dimensional metric Abelian group, so that X is a countably infinite torsion group. Suppose also that the set $\{\chi^2 : \chi \in \mathsf{X}\}$ is infinite. Let $(\Gamma_n)_{n=1}^\infty$ be a sequence of finite subgroups of X such that

$$\Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Gamma_n = \mathsf{X}.$$

Let χ_1 be any element of X such that $\chi_1^2 \notin \Gamma_1$. Suppose that $\chi_1, \ldots, \chi_k$ have been chosen. The subgroup H_k of X generated by $\chi_1, \ldots, \chi_k$ and Γ_k is finite, and we choose χ_{k+1} as any element of X such that $\chi_{k+1}^2 \notin \mathsf{H}_k$. Plainly $\{\chi_k\}_{k=1}^\infty$ is infinite and dissociate, and consists solely of elements of order > 2.

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3·6. *Example.* Let X be a countably infinite torsion group such that the set $\{\chi^2: \chi \in \mathsf{X}\}$ is finite. We have no hope of constructing an infinite dissociate set of elements of orders > 2 in this case, but we can find such a set all of whose elements have order 2, as the following simple argument shows. (Note that a dissociate set all of whose elements have order 2 is actually independent.) Since $\{\chi^2: \chi \in \mathsf{X}\}$ is finite, the endomorphism $\chi \to \chi^2$ maps X onto a finite subgroup of itself, and the orders of the elements of X are accordingly bounded. It is known that X is isomorphic with a weak direct product

$$Y = \overset{\infty}{\underset{k=1}{P^*}} Z(p_k^{r_k}),$$

where only finitely many distinct primes p_k and positive integers r_k appear. (See (5), (A. 25).) For $\{\chi^2: \chi \in \mathsf{X}\}$ to be finite, it is necessary and sufficient that $p_k = 2$ and $r_k = 1$ for all but a finite number of indices k. Thus there is an l such that $p_k = 2$ and $r_k = 1$ for all $k > l$. Each element of Y is a sequence $\mathbf{y} = (y_1, y_2, \ldots, y_n, \ldots)$, where $y_k \in Z(p_k^{r_k})$ for all k, $y_k = 0$ or 1 for all $k \geqslant l$, and y_k is ultimately 0. A moment's reflexion shows that Y admits a large number of infinite dissociate sets all elements of which have order 2. For use in section 4, we make the following special construction. (See 4·7 (*b*).) Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\lim_{n\to\infty} (a_n - n) = \infty.$$

(E.g. $a_n = 2n$, or $a_n = n + [\log\log n]$.) Let $\mathbf{x}^{(n)}$ be the element of Y such that

$$x^{(n)}_{l+a_{n-1}+1} = 1 \quad \text{and} \quad x^{(n)}_k = 0 \quad \text{for} \quad k \neq l + a_{n-1} + 1.$$

Let K_n be the subgroup of Y consisting of all sequences $\mathbf{y}$ such that $y_k = 0$ for $k > l + a_n$. Replacing Y by the isomorphic group X, we have the following. There is a strictly increasing sequence $(\mathsf{K}_n)_{n=1}^{\infty}$ of subgroups of X such that

$$\operatorname{card}(\mathsf{K}_n) = \prod_{k=1}^{l} p_k^{r_k} \cdot 2^{a_n} \tag{3·6·1}$$

and $\overset{\infty}{\underset{n=1}{\bigcup}} \mathsf{K}_n = \mathsf{X}$. There is also a dissociate set $(\chi_n)_{n=1}^{\infty}$ of characters in X such that

$$o(\chi_n) = 2, \quad \chi_n \in \mathsf{K}_n, \text{ and } \quad \chi_n \notin \mathsf{K}_{n-1}.$$

4. *Construction of singular measures.* In this section, we show that the measures μ constructed in section 3 are actually singular and continuous for appropriately chosen Δ'*s*, β'*s*, and G's. The following theorem is the analytic kernel of the whole paper.

4·1. Theorem. *Let G be an infinite compact Abelian group with character group X. Let Δ be a countably infinite dissociate set in X, enumerated in any order as $(\chi_k)_{k=1}^{\infty}$. Let $(\beta_k)_{k=1}^{\infty}$ be any sequence of complex numbers such that $|\beta_k| \leqslant \frac{1}{2}$ for all k and $\sum_{k=1}^{\infty} |\beta_k|^2 = \infty$. For each positive integer m, let*

(i) $$p_m(x) = \prod_{k=1}^{m} (1 + \beta_k \chi_k(x) + \bar{\beta}_k \chi_k^{-1}(x)).$$

Then we have

(ii) $$\lim_{m\to\infty} p_m(x) = 0$$

for almost all $x \in G$.

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Proof. For notational convenience, we write $\chi_{-k} = \chi_k^{-1}$ and $\beta_{-k} = \bar{\beta}_k$ for all positive integers k, and we also write $\beta_0 = 0$ and $\chi_0 = 1$. For all positive integers m, let

$$f_m = -\sum_{k=-m}^{m} \beta_k \chi_k, \quad \text{and let} \quad B_m = (|\beta_1|^2 + \cdots + |\beta_m|^2)^{\frac{1}{2}}.$$

Since $1 + t \leqslant e^t$ for all real t, we have

$$p_m(x) \leqslant e^{-f_m(x)} \tag{4·1·1}$$

for all $x \in G$. Hence (ii) will be established if we show that

$$\varliminf_{m\to\infty} (-f_m(x)) = -\infty,$$

or equivalently, that

$$\varlimsup_{m\to\infty} f_m(x) = \infty \tag{4·1·2}$$

for almost all $x \in G$.

Assume that (4·1·2) fails. The remainder of the proof consists in drawing a contradiction from this assumption. Then there exist a positive number α and a λ-measurable set $E \subset G$ such that $\lambda(E) > 0$ and $f_m(x) \leqslant \alpha$ for all $x \in E$ and all positive integers m. (This is proved by an easy argument, which we omit.) Since $\lim_{m\to\infty} B_m = \infty$, we thus have

$$\lim_{m\to\infty} (B_m^{-1} \max\{f_m(x), 0\}) = 0 \tag{4·1·3}$$

for all $x \in E$. By Egorov's theorem, we can therefore find a λ-measurable set D such that $\lambda(D) > 0$ and (4·1·3) holds *uniformly* for $x \in D$. Choose and fix such a set D.

Now consider the characteristic function ξ_D of the set D. Since $\lambda(D) \leqslant 1$, ξ_D is in $\mathfrak{L}_2(G)$, and Parseval's identity yields

$$\sum_{\chi \in \mathsf{X}} |\hat{\xi}_D(\chi)|^2 = \int_G \xi_D^2 \, d\lambda = \lambda(D) \leqslant 1.$$

For every positive number ϵ_0, there is therefore a finite subset Γ of X such that $\Gamma = \Gamma^{-1}$ and

$$\sum_{\chi \in \Gamma'} |\hat{\xi}_D(\chi)|^2 < \epsilon_0^2. \tag{4·1·4}$$

We use the set Γ to choose a positive integer m_0 with the following properties:

$$\text{if} \quad k > m_0, \quad \text{then} \quad \chi_k \notin \Gamma; \tag{4·1·5}$$

and

$$\text{if} \quad k > m_0 \quad \text{and} \quad j > m_0 \quad \text{and} \quad k \neq j, \quad \text{then} \quad \chi_k \chi_j^{-1} \notin \Gamma. \tag{4·1·6}$$

Condition (4·1·5) is easily met, since $\Gamma \cap \Delta$ is finite (possibly void). Condition (4·1·6) is likewise satisfied for suitable m_0, because of the dissociation of Δ. If (k,j) and (p,q) are pairs of positive integers such that $k \neq j$ and $p \neq q$, then $\chi_k^{-1}\chi_j \neq \chi_q^{-1}\chi_p$ except in the case $k = q$ and $j = p$ (or, if $\chi_j, \chi_k, \chi_p, \chi_q$ have order 2 also in the case $k = p$ and $j = q$). This is easily verified from the dissociation of Δ: we omit the argument. Since Γ is finite, it intersects $\{\chi_k^{-1}\chi_j : k \neq j\}$ in a finite set. The indices k and j for which $\chi_k^{-1}\chi_j$ lies in Γ must both be bounded, and this enables us to choose m_0 satisfying (4·1·6).

Consider next the functions $f_m - f_{m_0}$ $(m > m_0)$. Since

$$\max\{f_m - f_{m_0}, 0\} \leqslant \max\{f_m, 0\} + |f_{m_0}|,$$

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and since $\lim_{m\to\infty} B_m = \infty$, we infer from (4·1·3) that

$$\lim_{m\to\infty} (B_m^{-1} \max\{f_m(x) - f_{m_0}(x), 0\}) = 0$$

uniformly for all $x \in D$. That is, for every positive number ϵ_1, we can choose a positive integer m_1 such that $m_1 > m_0$ and

$$B_m^{-1} \max\{f_m(x) - f_{m_0}(x), 0\} < \epsilon_1$$

for all $x \in D$ and all $m > m_1$. We therefore have

$$\begin{aligned}\int_D |f_m - f_{m_0}| d\lambda &\leqslant \int_D |f_m - f_{m_0} - \epsilon_1 B_m| d\lambda + \epsilon_1 B_m \lambda(D) \\ &= 2\epsilon_1 B_m \lambda(D) - \int_D (f_m - f_{m_0})\, d\lambda \\ &\leqslant 2\epsilon_1 B_m \lambda(D) + \left| \int_G \xi_D (f_m - f_{m_0})\, d\lambda \right|. \end{aligned} \tag{4·1·7}$$

Let us estimate the last integral in (4·1·7):

$$\begin{aligned}\left| \int_G \xi_D (f_m - f_{m_0})\, d\lambda \right| &= \left| \left(\sum_{k=-m}^{-m_0-1} + \sum_{k=m_0+1}^{m} \right) - \beta_k \int_G \xi_D \chi_k d\lambda \right| \\ &= \left| \left(\sum_{k=-m}^{-m_0-1} + \sum_{k=m_0+1}^{m} \right) - \beta_k \hat{\xi}_D(\chi_k^{-1}) \right| \\ &\leqslant \{ \sum_{m_0 < |k| \leqslant m} |\beta_k|^2 \}^{\frac{1}{2}} \{ \sum_{m_0 < |k| \leqslant m} |\hat{\xi}_D(\chi_k)|^2 \}^{\frac{1}{2}} \\ &\leqslant \{2B_m^2 - 2B_{m_0}^2\}^{\frac{1}{2}} \epsilon_0, \end{aligned} \tag{4·1·8}$$

the last line of (4·1·8) being inferred from (4·1·4) and (4·1·5). Combine (4·1·7) and (4·1·8) and use the inequality $\lambda(D) \leqslant 1$ to write

$$\int_D |f_m - f_{m_0}| d\lambda \leqslant 2\epsilon_1 B_m + 2^{\frac{1}{2}} (B_m^2 - B_{m_0}^2)^{\frac{1}{2}} \epsilon_0 \tag{4·1·9}$$

for $m > m_1$.

We next make some estimates of $\int_D (f_m - f_{m_0})^2 d\lambda$. For notational convenience, we write Σ' to denote the sum over all pairs of integers (j, k) such that $m_0 < |j| \leqslant m$, $m_0 < |k| \leqslant m$, and $j \neq k$. Thus:

$$\int_D (f_m - f_{m_0})^2 d\lambda = \sum_{m_0 < |k| \leqslant m} |\beta_k|^2 \int_D \chi_k \chi_k^{-1} d\lambda + \Sigma' \beta_k \bar{\beta}_j \int_D \chi_k \chi_j^{-1}\, d\lambda. \tag{4·1·10}$$

Use Cauchy's inequality on the last sum in (4·1·10):

$$\left| \Sigma' \beta_k \bar{\beta}_j \int_D \chi_k \chi_j^{-1}\, d\lambda \right| \leqslant \{\Sigma' |\beta_k|^2 |\beta_j|^2\}^{\frac{1}{2}} \{\Sigma' |\hat{\xi}_D(\chi_k^{-1} \chi_j)|^2\}^{\frac{1}{2}}. \tag{4·1·11}$$

From (4·1·4) and (4·1·6), we infer that

$$\Sigma' |\hat{\xi}_D(\chi_k^{-1} \chi_j)|^2 < \epsilon_0^2, \tag{4·1·12}$$

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and we combine (4·1·10), (4·1·11) and (4·1·12) to write

$$\int_D (f_m - f_{m_0})^2 \, d\lambda \geqslant 2\lambda(D)\,(B_m^2 - B_{m_0}^2) - 2\epsilon_0(B_m^2 - B_{m_0}^2). \tag{4·1·13}$$

We also need the following estimates. We write

$$\begin{aligned}\int_D (f_m - f_{m_0})^4 \, d\lambda &\leqslant \int_G (f_m - f_{m_0})^4 \, d\lambda \\ &= \sum_p{}'' \sum_q{}'' \sum_r{}'' \sum_s{}'' \beta_p \beta_q \bar{\beta}_r \bar{\beta}_s \int_G \chi_p \chi_q \chi_r^{-1} \chi_s^{-1} \, d\lambda,\end{aligned} \tag{4·1·14}$$

where $\sum_j{}''$ denotes the sum over all integers j such that $m_0 < |j| \leqslant m$. The integral $\int_G \chi_p \chi_q \chi_r^{-1} \chi_s^{-1} \, d\lambda$ is 1 or 0 as the character $\chi_p \chi_q \chi_r^{-1} \chi_s^{-1}$ is 1 or not. Since Δ is dissociate, it is easy to check that $\chi_p \chi_q \chi_r^{-1} \chi_s^{-1}$ is 1 if $p = r$ and $q = s$; or if $p = s$ and $q = r$; (also if $p = q$ and $r = s$ in case χ_p and χ_r have order 2).

For all other values, we have $\chi_p \chi_q \chi_r^{-1} \chi_s^{-1} \neq 1$. Thus the quadruple sum in (4·1·14) does not exceed

$$3 \sum_p{}'' \sum_q{}'' |\beta_p|^2 |\beta_q|^2 = 12(B_m^2 - B_{m_0}^2)^2,$$

and so we may replace (4·1·14) by

$$\int_D (f_m - f_{m_0})^4 \, d\lambda \leqslant 12(B_m^2 - B_{m_0}^2)^2. \tag{4·1·15}$$

Finally, we use Hölder's inequality with $p = \frac{3}{2}$ and $p' = 3$ to write

$$\int_D (f_m - f_{m_0})^2 \, d\lambda \leqslant \left\{ \int_D |f_m - f_{m_0}| \, d\lambda \right\}^{\frac{2}{3}} \left\{ \int_D (f_m - f_{m_0})^4 \, d\lambda \right\}^{\frac{1}{3}}. \tag{4·1·16}$$

Use (4·1·13), (4·1·19), and (4·1·15) in (4·1·16). After an easy calculation, this yields

$$\lambda(D) \leqslant \epsilon_0 + 3^{\frac{1}{3}} \left(2^{\frac{1}{2}} \epsilon_1 \left(1 - \frac{B_{m_0}^2}{B_m^2} \right)^{-\frac{1}{2}} + \epsilon_0 \right)^{\frac{2}{3}}, \tag{4·1·17}$$

for all $m > \max\{m_0, m_1\}$. Since $\lim_{m \to \infty} B_m = \infty$, (4·1·17) shows that $\lambda(D)$ is arbitrarily small. This contradiction establishes (4·1·2). □

Theorem 4·1 has a counterpart for sequences $(\beta_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |\beta_k|^2 < \infty$, as follows.

4·2. THEOREM. *Let G be a compact Abelian group and $(\chi_k)_{k=1}^{\infty}$ a sequence of characters of G. Let $(\beta_k)_{k=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{k=1}^{\infty} |\beta_k|^2 < \infty$. Define p_m as in 4·1 (i). Suppose that $\sum_{k=1}^{\infty} \beta_k \chi_k + \bar{\beta}_k \chi_k^{-1}$ converges at a point $x \in G$. Then $\lim_{m \to \infty} p_m(x)$ exists and is finite at x, and is zero if and only if some factor $1 + \beta_k \chi_k(x) + \bar{\beta}_k \chi_k^{-1}(x)$ is zero.*

Proof. Let $a_k = \beta_k \chi_k(x) + \bar{\beta}_k \chi_k^{-1}(x)$. There is a positive integer m_0 such that

$$|a_k| < \tfrac{1}{2} \quad \text{for} \quad k \geqslant m_0.$$

For these k's, we have $\log(1 + a_k) = a_k + a_k^2 \theta_k$,

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where $|\theta_k| < 2$, as is easily seen from the power series

$$\log(1+t) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{t^l}{l}.$$

Then
$$\prod_{k=m_0}^{n} (1+a_k) = \exp\left(\sum_{k=m_0}^{n} a_k\right) \exp\left(\sum_{k=m_0}^{\infty} a_k^2 \theta_k\right).$$

The series $\sum_{k=1}^{\infty} a_k^2$ evidently converges, and so $\sum_{k=m_0}^{\infty} a_k^2 \theta_k$ exists. Consequently

$$\lim_{n\to\infty} \prod_{k=m_0}^{n} (1+a_k) = \exp\left(\sum_{k=m_0}^{\infty} a_k\right) \exp\left(\sum_{k=m_0}^{\infty} a_k^2 \theta_k\right) > 0. \quad \square$$

4·3. *Note.* We will see in 4·6 that $\sum_{k=1}^{\infty} \beta_k \chi_k + \bar{\beta}_k \chi_k^{-1}$ does in fact converge almost everywhere on G for certain G's and dissociate χ_k's .

We now specialize our groups G and dissociate characters in order to be able to apply some known summability theorems to the polynomials p_m.

4·4. THEOREM. *Let G be either T or a 0-dimensional infinite compact metric Abelian group. Let Δ be as in 3·4, 3·5, or 3·6. Write $\Delta = (\chi_k)_{k=1}^{\infty}$, the enumeration being as in 3·4, 3·5, or 3·6. Let β be a function on Δ as in 3·2; write β_k for β_{χ_k}. Let μ be the measure in $\mathbf{M}(G)$ constructed as in 3·2.*

(i) *The measure μ is continuous.*

(ii) *If $\sum_{k=1}^{\infty} |\beta_k|^2 = \infty$, then μ is purely singular.*

(iii) *Let r be a real number such that $1 < r \leqslant 2$, and let $r' = r/(r-1)$. If $\sum_{k=1}^{\infty} |\beta_k|^r < \infty$, then μ is absolutely continuous, and its derivative is in $\mathfrak{L}_{r'}(G)$.*

Proof. In order to discuss together the groups listed in 3·4, 3·5, and 3·6, we define, for each positive integer q, the set A_q as: $\{-q, -q+1, \ldots, q-1, q\}$ if $\mathsf{X} = Z$; the subgroup H_q if X is as in 3·5; K_q if X is as in 3·6. It is known ((6)) that

$$\lim_{q\to\infty} \frac{1}{\operatorname{card}(\mathsf{A}_q)} \sum_{\chi \in \mathsf{A}_q} |\hat{\nu}(\chi)|^2 = \sum_{x \in G} |\nu(\{x\})|^2 \qquad (4\cdot4\cdot1)$$

for all $\nu \in \mathbf{M}(G)$.

To prove (i), consider first the case $\mathsf{X} = Z$, and take any sequence $(n_k)_{k=1}^{\infty}$ as in 3·4. If $\epsilon_1 n_1 + \cdots + \epsilon_s n_s$, where each $\epsilon_j \in \{-1, 0, 1\}$ and $\epsilon_s \neq 0$, has the property that

$$-(n_1 + \cdots + n_t) \leqslant \epsilon_1 n_1 + \cdots + \epsilon_s n_s \leqslant n_1 + \cdots + n_t, \quad \text{then} \quad s \leqslant t.$$

Write $q_t = n_1 + \cdots + n_t$. Setting $\nu = \mu$ and $q = q_t$ in (4·4·1) and using 3·2(i) and 3·2(ii), we obtain

$$\sum_{\chi \in \mathsf{A}_{q_t}} |\hat{\mu}(\chi)|^2 = \Sigma\, |\beta_1^{(\epsilon_1)}|^2 \cdots |\beta_t^{(\epsilon_t)}|^2, \qquad (4\cdot4\cdot2)$$

the sum on the right in (4·4·2) being taken over all sequences $(\epsilon_1, \ldots, \epsilon_t)$ such that each $\epsilon_j \in \{-1, 0, 1\}$. There are $2t$ of these sequences in which exactly one ϵ_j has absolute value 1. Since $|\beta_j| \leqslant \frac{1}{2}$, these sequences contribute at most $2t(\frac{1}{2})^2$ to the right side of (4·4·2). Similarly, there are $2^j\binom{t}{j}$ ϵ-sequences having exactly j entries of absolute value 1.

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These contribute at most $2^j\binom{t}{j}2^{-2j}$ to the right side of (4·4·2). Adding for $j = 0, \ldots, t$, we find that the right side of (4·4·2) does not exceed $(\frac{3}{2})^t$. Using this, (4·4·2), and (3·4·2), we find that

$$\frac{1}{\operatorname{card}(\mathsf{A}_{q_t})} \sum_{\chi \in \mathsf{A}_{q_t}} |\hat{\mu}(\chi)|^2 \leqslant \frac{3}{2^{t+1}}, \tag{4·4·3}$$

from which it is clear that the left side of (4·4·3) has limit 0 as $t \to \infty$. Since the limit in (4·4·1) does exist, it follows that (4·4·1) is 0. That is, μ is a continuous measure.

We next prove (i) for the groups described in 3·5. Again we will use (4·4·1). It is easy to see from 3·5 that

$$\operatorname{card}(\mathsf{A}_q) \geqslant 3^q. \tag{4·4·4}$$

Substitute μ for ν in (4·4·1), use 3·2(i) and 3·2(ii), and compute as follows:

$$\begin{aligned} \sum_{\chi \in \mathsf{A}_q} |\hat{\mu}(\chi)|^2 &= \sum_{\epsilon_1=-1}^{1} \cdots \sum_{\epsilon_q=-1}^{1} |\beta_1^{(\epsilon_1)}|^2 \cdots |\beta_q^{(\epsilon_q)}|^2 \\ &= \prod_{j=1}^{q} (|\beta_j^{(-1)}|^2 + |\beta_j^{(0)}|^2 + |\beta_j^{(1)}|^2) \\ &\leqslant (\tfrac{3}{2})^q. \end{aligned} \tag{4·4·5}$$

From (4·4·4) and (4·4·5), we see that (4·4·1) is 0.

The group examined in 3·6 is a little different, since we use only characters of order 2. It is clear from (3·6·1) that

$$\operatorname{card}(\mathsf{A}_q) \geqslant 2^{a_q}. \tag{4·4·6}$$

Since all β_k are real (note 3·2), we apply 3·2(i), 3·2(ii), and the last statement of 3·6 to write:

$$\sum_{\chi \in \mathsf{A}_q} |\hat{\mu}(\chi)|^2 = \sum_{\epsilon_1=0}^{1} \cdots \sum_{\epsilon_q=0}^{1} \beta_1^{2\epsilon_1} \cdots \beta_q^{2\epsilon_q} \leqslant 2^q. \tag{4·4·7}$$

From (4·4·6), (4·4·7), and the condition $a_q - q \to \infty$ stated in 3·6, we see that (4·4·1) is again 0. This completes the proof of (i).

To prove (ii), consider first the case $\mathsf{X} = Z$ and $G = T$. Let ν be any measure in $\mathbf{M}(T)$ and let f_ν be the Lebesgue–Radon–Nikodým derivative of the absolutely continuous part of ν. It is known that

$$\lim_{q \to \infty} \sum_{k=-q}^{q} \left(1 - \frac{|k|}{q+1}\right) \exp(ikx)\, \hat{\nu}(k) = f_\nu(x) \tag{4·4·8}$$

for almost all $x \in T$ (see (14), vol. I, ch. III, Theorem (8·1), p. 105, or (1), Corollary (3·6), p. 200). Furthermore, the condition (3·4·1) implies that

$$p_t(x) = \sum_{k=-q_t}^{q_t} \hat{\mu}(k)\, e^{ikx}$$

for all t. The condition (3·4·1) also implies that the series

$$\sum_{k=-\infty}^{\infty} \hat{\mu}(k)\, e^{ikx} \tag{4·4·9}$$

has infinitely many gaps in the sense of (14), vol. I, p. 79. Since (4·4·8) holds, it follows that (4·4·9) converges almost everywhere, to the function $f_\mu(x)$. By (4·1), we have

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$\lim_{t\to\infty} p_t(x) = 0$ a.e.; consequently the series (4·4·9) converges to 0 a.e.; and so $f_\mu = 0$. That is, μ is a singular measure.

The 0-dimensional cases 3·5 and 3·6 are even simpler. For any 0-dimensional compact metric group G and any increasing sequence $(\mathsf{B}_m)_{m=1}^{\infty}$ of finite subgroups of X such that $\bigcup_{m=1}^{\infty} \mathsf{B}_m = \mathsf{X}$, and $\nu \in \mathbf{M}(G)$, the derivative f_ν of the absolutely continuous part of ν is given by

$$\lim_{m\to\infty} \sum_{\chi \in \mathsf{B}_m} \hat{\nu}(\chi)\,\chi(x) = f_\nu(x) \tag{4·4·10}$$

a.e. on G. This is immediate from (1), Corollary (4·5), p. 206. Again $p_m(x)$ is equal to $\sum_{\chi \in \mathsf{A}_m} \hat{\mu}(\chi)\,\chi(x)$, as 3·5 and 3·6 imply. Also the sets A_m satisfy the above conditions on B_m. Again by 4·1 we see that $f_\mu(x) = 0$ a.e. on G. This completes the proof of (ii).

To prove (iii), it suffices to show that

$$\sum_{\chi \in \mathsf{X}} |\hat{\mu}(\chi)|^r < \infty, \tag{4·4·11}$$

since this inequality shows that $\hat{\mu}$ agrees with the Fourier transform $\hat{g}$ of a function g in $\mathfrak{L}_r(G)$ [see Theorem 2·4]. Since μ is determined by $\hat{\mu}$, μ is then absolutely continuous, and its derivative is g. In all three cases, it is easy to see from 3·2 (i) and 3·2 (ii) that

$$\sum_{\chi \in \mathsf{X}} |\hat{\mu}(\chi)|^r = \lim_{q\to\infty} \sum_{\chi \in \mathsf{A}_q} |\hat{\mu}(\chi)|^r. \tag{4·4·12}$$

For $G = T$, and $q = q_t$ as in (4·4·2), we have

$$\sum_{\chi \in \mathsf{A}_{q_t}} |\hat{\mu}(\chi)|^r = \prod_{j=1}^{t} \sum_{\epsilon_j=-1}^{1} |\beta_j^{(\epsilon_j)}|^r = \prod_{j=1}^{t} (1 + 2|\beta_j|^r). \tag{4·4·13}$$

Similarly, for the groups 3·5, we compute as in (4·4·5) to write

$$\sum_{\chi \in \mathsf{A}_q} |\hat{\mu}(\chi)|^r = \prod_{j=1}^{q} (1 + 2|\beta_j|^r). \tag{4·4·14}$$

For the groups 3·6, we compute as in (4·4·7) to write

$$\sum_{\chi \in \mathsf{A}_q} |\hat{\mu}(\chi)|^r = \prod_{j=1}^{q} (1 + |\beta_j|^r). \tag{4·4·15}$$

It is now clear from (4·4·12) that

$$\sum_{\chi \in \mathsf{X}} |\hat{\mu}(\chi)|^r = \begin{cases} \prod_{j=1}^{\infty} (1 + 2|\beta_j|^r) < \infty & \text{in cases 3·4 and 3·5,} \\ \prod_{j=1}^{\infty} (1 + |\beta_j|^r) < \infty & \text{in case 3·6.} \end{cases} \tag{4·4·16}$$

This completes the proof of (iii). □

4·5. Corollary. *Let G be either T or a 0-dimensional compact metric Abelian group. Then there is a measure μ in $\mathbf{M}_s^+(G)$ such that $\mu(G) = 1$, and $\mu * \mu$ is absolutely continuous, with derivative in $\mathfrak{L}_p^+(G)$ for all $p \in]1, \infty[$. We also have $\sum_{\chi \in \mathsf{X}} |\hat{\mu}(\chi)|^{2r} < \infty$ for all $r > 1$.*

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Proof. In the construction of 4·4, let $(\beta_k)_{k=1}^{\infty}$ be such that

$$\sum_{k=1}^{\infty} |\beta_k|^2 = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\beta_k|^{2+\epsilon} < \infty$$

for all $\epsilon > 0$. For example, we may take $|\beta_k| = k^{-\frac{1}{2}}$. Then the measure μ is continuous and singular (4·4(i) and 4·4(ii)). For the measure $\mu * \mu$, we have

$$\widehat{\mu * \mu}(\chi) = \hat{\mu}(\chi)^2$$

for all $\chi \in X$. Thus $\mu * \mu$ is the measure constructed as in 3·2 with β_k^2 instead of β_k. Let p be a number such that $2 \leqslant p < \infty$, and write s for $p/(p-1)$. We have

$$\sum_{k=1}^{\infty} |\beta_k^2|^s = \sum_{k=1}^{\infty} |\beta_k|^{2s} < \infty, \tag{4·5·1}$$

since $2s > 2$, and so 4·4(iii) shows that $\mu * \mu$ is absolutely continuous and that the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$. To complete the argument, note that

$$\mathfrak{L}_p(G) \subset \mathfrak{L}_s(G) \subset \mathfrak{L}_1(G) \quad \text{if} \quad p \geqslant s \geqslant 1,$$

since $\lambda(G) = 1$. The inequality $\sum_{\chi \in X} |\hat{\mu}(\chi)|^{2r} < \infty$ is checked from (4·5·1) and (4·4·16). □

4·6. *Remarks.* We can find the support of the measures $\mu * \mu$ for μ as in 4·5. Consider first the case $G = T$, and a dissociate set $(n_k)_{k=1}^{\infty}$ of integers (i.e., characters of T) as in 3·4. If

$$|\beta_k| \leqslant \tfrac{1}{2}, \quad \sum_{k=1}^{\infty} |\beta_k|^2 = \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} |\beta_k|^4 < \infty,$$

then we know that the corresponding μ is singular and continuous and that $\mu * \mu$ is absolutely continuous and has derivative g in $\mathfrak{L}_2^+(T)$. Now consider the Fourier series

$$\sum_{k=1}^{\infty} \beta_k^2 \exp(in_k x) + \bar{\beta}_k^2 \exp(-in_k x). \tag{4·6·1}$$

Since $n_{k+1}/n_k > 2$, a classical theorem asserts that (4·6·1) converges a.e. to a finite limit. (See, for example, (14), vol. I, p. 203, (6·3).) Hence we can apply 4·2 with β_k replaced by β_k^2 to assert that

$$\prod_{k=1}^{\infty} (1 + \beta_k^2 \exp(in_k x) + \bar{\beta}_k^2 \exp(-in_k x)) \tag{4·6·2}$$

converges a.e. to a finite non-negative function, which is zero if and only if

$$1 + \beta_k^2 \exp(in_k x) + \bar{\beta}_k^2 \exp(-in_k x)$$

is zero for some k. The mth partial product of (4·6·2) is the mth partial sum of the Fourier series for the function g, because of the condition $n_{k+1} > \delta(n_1 + \cdots + n_k)$. Since the Fourier series of g is $(C, 1)$ summable to g a.e., it follows that the product (4.6.1) is $g(x)$ a.e. Since $|\beta_k| \leqslant \frac{1}{2}$, no factor in (4·6·2) can vanish. Therefore g is positive a.e., and the support of $\mu * \mu$ is the whole group T.

For the groups described in 3·5, we argue as in the preceding paragraph, but using (1), Corollary (4·5), p. 206, as in the proof of 4·4(ii). This shows that the support of $\mu * \mu$ is all of G if $|\beta_k| \leqslant \frac{1}{2}$ and $\sum_{k=1}^{\infty} |\beta_k|^4 < \infty$.

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For the groups described in 3·6, the measure of the support of $\mu * \mu$ can be made to be any of the numbers 1, 2^{-1}, 2^{-2}, ..., by appropriate choice of (β_k). In fact, let all β_k be such that $-1 \leqslant \beta_k \leqslant 1$, let exactly s of the β_k be of absolute value 1, and let $\sum_{k=1}^{\infty} \beta_k^4$ be finite. Then $\sum_{k=1}^{\infty} \beta_k^2 \chi_k$ converges to a finite limit a.e., as (1), Corollary (4·5), p. 206 shows. Theorem 4·4 implies that $\prod_{k=1}^{\infty} (1+\beta_k^2 \chi_k)$ converges a.e. to a finite non-negative limit, which is zero if and only if some factor $1+\beta_k^2 \chi_k$ is zero. It is well known that G is topologically isomorphic with the complete direct product $\overset{\infty}{P}_{k=1} Z(p_k^{r_k})$. A simple computation then shows that the set

$$A = \left\{ \mathbf{x} \in \overset{\infty}{P}_{k=1} Z(p_k^{r_k}) : \prod_{k=1}^{\infty} (1+\beta_k^2 \chi_k(\mathbf{x})) = 0 \right\}$$

has measure $1 - 2^{-s}$. It follows that the support of $\mu * \mu$, which is the complement of A in G, has measure 2^{-s}.

We do not know how to find the support of μ for μ as in 4·5. Since $\mu * \mu$ can have support equal to G, the support S of μ must have the property that $S \cdot S = G$, so that S is in a sense large. For the case $G = T$, Salem ((11)) constructs a measure μ such that $\hat{\mu}(n) = O(n^{-\frac{1}{2}+\epsilon})$ and for which the support of μ is a certain perfect nowhere dense set. He says nothing about the support of $\mu * \mu$.

4·7. *Notes.* (*a*) Let G be any compact Abelian group and ν a measure in $\mathbf{M}^+(G)$ with zero absolutely continuous part: $\nu = \nu_d + \nu_s$. If $\nu * \nu$ is continuous, it is easy to see that $\nu_d = 0$. In particular, if $\nu * \nu$ is absolutely continuous, then ν itself is continuous. This is of no help in our construction of continuous singular measures μ such that $\mu * \mu$ is absolutely continuous, since the proof of 4·4(ii) apparently requires such special dissociate sets that 4·4(i) can also be proved for them.

(*b*) The dissociate sets of characters constructed in 3·4, 3·5, and 3·6 and used in 4·4, are rather special. The sequences (β_k) used in 4·4, however, are quite general. Plainly one can construct measures μ satisfying the conclusions of 4·5 with more general dissociate sets of characters, if the sequences (β_k) are restricted. For example, in 3·6, let (a_n) be any strictly increasing sequence of positive integers, and take a sequence (β_k) of real numbers for which $|\beta_k| \leqslant t < 1$, for some fixed t. Then the measure μ constructed as in 3·2 is continuous: the proof of 4·4(i) can be repeated with obvious changes. The measure μ is still singular if $\sum_{k=1}^{\infty} \beta_k^2 = \infty$: the proof of 4·4(ii) can be repeated. How much more can be done for the case $G = T$, by way of relaxing the condition $n_{k+1} > \delta(n_1 + \cdots + n_k)$, is an open question.

5. *Extension to non-discrete locally compact Abelian groups.* We first construct a class of measures on the line.

5·1. Theorem. *Let μ be a measure in $\mathbf{M}^+(T)$. Define a measure $\mu_1 \in \mathbf{M}^+(R)$ by the rule $\mu_1(E) = \mu(E \cap [-\pi, \pi[)$ for all Borel sets $E \subset R$. Define $\mu_2 \in \mathbf{M}^+(R)$ as $\mu_1 * \left(\sum_{s=-\infty}^{\infty} b_s \varepsilon_{2\pi s} \right)$, the*

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convolution being taken in $\mathbf{M}(R)$ *and* $(b_s)_{s=-\infty}^{\infty}$ *being a sequence of positive real numbers such that* $\sum_{s=-\infty}^{\infty} b_s < \infty$. *Suppose that* μ *is continuous and singular and that the measure* $\mu * \mu \in \mathbf{M}^+(T)$ *is absolutely continuous and has its derivative in* $\mathfrak{L}_p(T)$, *for some* $p > 1$. *Then* μ_2 *is continuous and singular and* $\mu_2 * \mu_2 \in \mathbf{M}^+(R)$ *is absolutely continuous and has its derivative in* $\mathfrak{L}_1^+(R) \cap \mathfrak{L}_p^+(R)$. *If the support of* $\mu * \mu$ *is* T, *then the support of* $\mu_2 * \mu_2$ *is* R.

Proof. Note that in the theorem and in its proof, the convolution $\mu * \mu$ is taken in the group T; all others are in the group R. Let C be a Borel subset of $[-\pi, \pi[$ such that $\mu(C) = 0$. Then we have

$$\mu_2\left(\bigcup_{t=-\infty}^{\infty} (C + 2\pi t)\right) = 0, \tag{5·1·1}$$

since

$$\begin{aligned}
\mu_2\left(\bigcup_{t=-\infty}^{\infty} (C + 2\pi t)\right) &= \sum_{s=-\infty}^{\infty} b_s \mu_1\left(\bigcup_{t=-\infty}^{\infty} (C + 2\pi t) - 2\pi s\right) \\
&= \sum_{s=-\infty}^{\infty} b_s \mu_1\left(\bigcup_{t=-\infty}^{\infty} (C + 2\pi t)\right) \\
&= \sum_{s=-\infty}^{\infty} b_s \mu\left(\left(\bigcup_{t=-\infty}^{\infty} (C + 2\pi t)\right) \cap [-\pi, \pi[\right) \\
&= \sum_{s=-\infty}^{\infty} b_s \mu(C) \\
&= 0.
\end{aligned}$$

From (5·1·1) it is easy to see that μ_2 is continuous, since μ is continuous, and singular, since μ is singular.

Let $c_l = \sum_{k=-\infty}^{\infty} b_{l-k} b_k$. Then we have $\sum_{l=-\infty}^{\infty} c_l < \infty$ and

$$\mu_2 * \mu_2 = \mu_1 * \mu_1 * \left(\sum_{l=-\infty}^{\infty} c_l \epsilon_{2\pi l}\right). \tag{5·1·2}$$

We first show that $\mu_1 * \mu_1$ is absolutely continuous. It is clear that

$$\mu_1 * \mu_1(]-\infty, -2\pi[\cup [2\pi, \infty[) = 0.$$

Thus to show that $\mu_1 * \mu_1$ is absolutely continuous, it suffices to prove that

$$\mu_1 * \mu_1(B) = 0 \quad \text{if} \quad \lambda(B) = 0 \quad \text{and} \quad B \subset [-\pi, \pi[$$

or $B \subset [\pi, 2\pi[$ or $B \subset [-2\pi, -\pi[$. For $B \subset [-\pi, \pi[$, we have

$$\mu_1 * \mu_1(B) = \int_{[-\pi, \pi[} \mu((B - x) \cap [-\pi, \pi[)\, d\mu(x) \tag{5·1·3}$$

and

$$\mu * \mu(B) = \int_{[-\pi, \pi[} \mu((B - x \,(\mathrm{mod}\, 2\pi)) \cap [-\pi, \pi[)\, d\mu(x), \tag{5·1·4}$$

where the interpretation of the symbol $B - x \,(\mathrm{mod}\, 2\pi)$ is obvious. Since

$$(B - x) \cap [-\pi, \pi[\subset (B - x(\mathrm{mod}\, 2\pi)) \cap [-\pi, \pi[$$

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and since $\mu * \mu(B) = 0$, it follows that $\mu_1 * \mu_1(B) = 0$. In a somewhat similar way, one shows that if $B \subset [\pi, 2\pi[$, then

$$\mu_1 * \mu_1(B) \leqslant \mu * \mu(B - \pi) = 0 \tag{5·1·5}$$

and, if $B \subset [-2\pi, -\pi[$, then

$$\mu_1 * \mu_1(B) \leqslant \mu * \mu(B + \pi) = 0. \tag{5·1·6}$$

Thus $\mu_1 * \mu_1$ is absolutely continuous.

Let g_1 be the derivative of $\mu_1 * \mu_1$ on R, and g the derivative of $\mu * \mu$ on T. We will show that $g_1 \in \mathfrak{L}_p(R)$. The equality

$$g_1(x) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu_1 * \mu_1([x - \delta, x + \delta]) \tag{5·1·7}$$

holds for almost all $x \in R$, with a similar formula for $g(x)$ for almost all $x \in T$. For $x \in]-\pi, \pi[$, and δ sufficiently small, formulas (5·1·3) and (5·1·4) show that

$$\mu_1 * \mu_1([x - \delta, x + \delta]) \leqslant \mu * \mu([x - \delta, x + \delta]).$$

For $x \in]\pi, 2\pi[$, we have

$$\mu_1 * \mu_1([x - \delta, x + \delta]) \leqslant \mu * \mu([x - \pi - \delta, x - \pi + \delta]), \tag{5·1·8}$$

and for $x \in]-2\pi, -\pi[$,

$$\mu_1 * \mu_1([x - \delta, x + \delta]) \leqslant \mu * \mu([x + \pi - \delta, x + \pi + \delta]). \tag{5·1·9}$$

Combining (5·1·7), (5·1·8) and (5·1·9), we find that

$$g_1(x) \leqslant \begin{cases} g(x + \pi) & \text{if} \quad -2\pi < x < -\pi, \\ g(x) & \text{if} \quad -\pi < x < \pi, \\ g(x - \pi) & \text{if} \quad \pi < x < 2\pi, \end{cases}$$

almost everywhere, of course, and $g_1(x) = 0$ for $|x| \geqslant 2\pi$. Since $g \in \mathfrak{L}_p(T)$, it follows that $g_1 \in \mathfrak{L}_p(R)$.

We now take up the measure $\mu_2 * \mu_2$. Since $\mu_1 * \mu_1$ is absolutely continuous, (5·1·2) shows that $\mu_2 * \mu_2$ is also. Let g_2 be the derivative of $\mu_2 * \mu_2$: we claim that

$$g_2(x) = \sum_{t=-\infty}^{\infty} c_t g_1(x - 2\pi t). \tag{5·1·10}$$

For every Borel set E, (5·1·2) shows that

$$\begin{aligned} \mu_2 * \mu_2(E) &= \sum_{t=-\infty}^{\infty} c_t \mu_1 * \mu_1(E - 2\pi t) \\ &= \sum_{t=-\infty}^{\infty} c_t \int_{E - 2\pi t} g_1(x)\, dx \\ &= \sum_{t=-\infty}^{\infty} c_t \int_{E} g_1(x - 2\pi t)\, dx \\ &= \int_E \left(\sum_{t=-\infty}^{\infty} c_t g_1(x - 2\pi t) \right) dx, \end{aligned} \tag{5·1·11}$$

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the last equality following from the monotone convergence theorem. The function $x \to \sum_{t=-\infty}^{\infty} c_t g_1(x-2\pi t)$ plainly being in $\mathfrak{L}_1(R)$, and the derivative being unique, (5·1·11) proves (5·1·10). Next let us prove that $g_2 \in \mathfrak{L}_p(R)$. We have

$$\begin{aligned}\int_R g_2^p \, d\lambda &= \int_R \left(\sum_{t=-\infty}^{\infty} c_t g_1(x-2\pi t) \right)^p dx \\ &= \sum_{m=-\infty}^{\infty} \int_{2\pi m}^{2\pi(m+1)} \left(\sum_{t=-\infty}^{\infty} c_t g_1(x-2\pi t) \right)^p dx \\ &= \sum_{m=-\infty}^{\infty} \int_0^{2\pi} \left(\sum_{t=-\infty}^{\infty} c_t g_1(y-2\pi t+2\pi m) \right)^p dy. \end{aligned} \tag{5·1·12}$$

Since g_1 vanishes outside of $]-2\pi, 2\pi[$, the function $g_1(y-2\pi t+2\pi m)$ can be different from 0 for $y \in [0, 2\pi]$ only for $t = m$ and $t = m+1$. The last line of (5·1·12) is therefore equal to

$$\sum_{m=-\infty}^{\infty} \int_0^{2\pi} (c_m g_1(y) + c_{m+1} g_1(y-2\pi))^p \, dy. \tag{5·1·13}$$

Hölder's inequality and a short computation show that (5·1·13) does not exceed

$$\sum_{m=-\infty}^{\infty} (c_m^{p'} + c_{m+1}^{p'})^{p-1} \int_{-2\pi}^{2\pi} g_1(y)^p \, dy,$$

where $p' = p/(p-1)$. It is easy to see that

$$\sum_{m=-\infty}^{\infty} (c_m^{p'} + c_{m+1}^{p'})^{p-1} \leqslant 2^p \sum_{m=-\infty}^{\infty} c_m^p < \infty,$$

and so g_2 is in $\mathfrak{L}_p^+(R)$.

It remains to show that the support of $\mu_2 * \mu_2$ is R if the support of $\mu * \mu$ is T. Clearly it suffices to prove that $\mu_2 * \mu_2(U) > 0$ if U is an open set such that

$$\varnothing \neq U \subset](2j-1)\pi, (2j+1)\pi[$$

for some integer j. Let $\beta = \min\{b_{-1}, b_0, b_1, b_{j-1}, b_j, b_{j+1}\}$. Since the numbers b_s are positive and the measure μ_1 is non-negative, we have

$$\begin{aligned}\mu_2 * \mu_2(U) &\geqslant \{\mu_1 * (b_{-1}\varepsilon_{-2\pi} + b_0\varepsilon_0 + b_1\varepsilon_{2\pi})\} * \{\mu_1 * (b_{j-1}\varepsilon_{2\pi(j-1)} \\ &\quad + b_j\varepsilon_{2\pi j} + b_{j+1}\varepsilon_{2\pi(j+1)})\}(U) \\ &\geqslant \beta^2\{\mu_1 * (\varepsilon_{-2\pi} + \varepsilon_0 + \varepsilon_{2\pi})\}^2 * \varepsilon_{2\pi j}(U) \\ &= \beta^2\{\mu_1 * (\varepsilon_{-2\pi} + \varepsilon_0 + \varepsilon_{2\pi})\}^2 (U - 2\pi_j). \end{aligned} \tag{5·1·14}$$

The set $U - 2\pi j$ is contained in $]-\pi, \pi[$. For a Borel subset A of $]-\pi, \pi[$, it is easy to see that

$$\{\mu_1 * (\varepsilon_{-2\pi} + \varepsilon_0 + \varepsilon_{2\pi})\}^2 (A) = \mu * \mu(A)$$

(recall that $\mu * \mu$ is a convolution in T). Since $\mu * \mu(U - 2\pi j) > 0$, it follows that $\mu_2 * \mu_2(U) > 0$. □

5·2. COROLLARY. *There is a continuous singular measure μ in $\mathbf{M}^+(R)$ such that $\mu(R) = 1$, the support of $\mu * \mu$ is R, and $\mu * \mu$ is absolutely continuous and has derivative in $\mathfrak{L}_p^+(R)$ for all $p > 1$.*

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Proof. Combine 4·5, 4·6, and 5·1. □

We next construct our measures on arbitrary compact Abelian groups.

5·3. THEOREM. *Let G be an infinite compact Abelian group. There exists a measure μ in $\mathbf{M}_s^+(G)$ such that $\mu(G) = 1$, $\mu * \mu$ is absolutely continuous, the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all $p \in]1, \infty[$, and the support of $\mu * \mu$ is the entire group G.*

Proof. Consider the character group X of G. Either X contains a countably infinite torsion subgroup or X contains an infinite cyclic subgroup. Let Y be a subgroup of X of one of these two types. Let H be the annihilator of Y in G, let G_1 denote the group G/H, and let τ be the natural mapping $x \to xH$ of G onto G_1. It is a classical fact that Y is isomorphic with the character group of G_1, the characters of G_1 all having the form

$$xH \to \chi(x) \tag{5·3·1}$$

for some $\chi \in \mathrm{Y}$. It follows that G_1 is either a 0-dimensional infinite compact metric Abelian group or is topologically isomorphic with T. In both cases we apply 4·5 to find a measure $\mu_1 \in \mathbf{M}_s^+(G_1)$ such that $\mu_1(G_1) = 1$ and

$$\sum_{\chi \in \mathrm{Y}} |\hat{\mu}_1(\chi)|^{2r} < \infty$$

for all $r > 1$. Thus $\mu_1 * \mu_1$ is in $\mathbf{M}_a(G_1)$ and the derivative of $\mu_1 * \mu_1$ is in $\mathfrak{L}_p(G_1)$ for all $p > 1$. Furthermore, the support of $\mu_1 * \mu_1$ can be made to be the entire group G_1, as 4·6 shows.

Now consider the function h on all of X such that

$$h(\chi) = \begin{cases} \hat{\mu}_1(\chi) & \text{for} \quad \chi \in \mathrm{Y}, \\ 0 & \text{for} \quad \chi \notin \mathrm{Y}. \end{cases}$$

Since $\hat{\mu}_1$ is positive-definite on Y and since Y is a subgroup of X, it is easy to check that h is positive-definite on X. By the Herglotz–Bochner theorem, there is a measure $\mu \in \mathbf{M}^+(G)$ such that $\mu(G) = h(1) = 1$ and $\hat{\mu}(\chi) = h(\chi)$ for all $\chi \in \mathrm{X}$. Since

$$\sum_{\chi \in \mathrm{X}} |\hat{\mu}(\chi)|^{2r} = \sum_{\chi \in \mathrm{Y}} |\hat{\mu}_1(\chi)|^{2r} < \infty$$

for all $r > 1$, it follows that $\mu * \mu$ is absolutely continuous and has derivative in $\mathfrak{L}_p(G)$ for all $p > 1$.

We wish to show that μ is continuous and singular. To do this, let χ_1 be any character of G_1 and let χ be the character of G corresponding as in (5·3·1) to χ_1. It is evident that

$$\int_{G_1} \chi_1 d\mu_1 = \hat{\mu}_1(\chi_1^{-1}) = \hat{\mu}(\chi^{-1}) = \int_G \chi d\mu = \int_G (\chi_1 \circ \tau)\, d\mu. \tag{5·3·2}$$

Since linear combinations of characters are uniformly dense in $\mathfrak{C}(G_1)$, the identities (5·3·2) and a simple calculation show that

$$\int_{G_1} f_1 d\mu_1 = \int_G (f_1 \circ \tau)\, d\mu \tag{5·3·3}$$

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for all $f_1 \in \mathfrak{C}(G_1)$. That is, μ_1 is the continuous image of μ under the mapping τ. It is known (see, for example, (7), p. 180, Theorem (12·46)) that for every μ_1-measurable subset A_1 of G_1, the equality

$$\mu_1(A_1) = \mu(\tau^{-1}(A_1)) \tag{5·3·4}$$

obtains. Let λ and λ_1 be Haar measure on G and G_1, respectively. It is obvious that λ_1 is the image of λ under the mapping τ. The measure μ_1 is singular in $\mathbf{M}(G_1)$. Thus there is a Borel set $E_1 \subset G_1$ such that $\lambda_1(E_1) = 0$ and $\mu_1(E_1') = 0$. Applying 5·3·4, we have

$$0 = \mu(\tau^{-1}(E_1')) = \mu((\tau^{-1}(E_1))') \tag{5·3·5}$$

and

$$0 = \lambda_1(E_1) = \lambda(\tau^{-1}(E_1)). \tag{5·3·6}$$

The equalities 5·3·5 and 5·3·6 show that μ is singular. A like argument proves that μ is continuous.

Observe next that $\mu_1 * \mu_1$ (convolution in $\mathbf{M}(G_1)$) is the image under τ of $\mu * \mu$ (convolution in $\mathbf{M}(G)$). This is so because $(\mu_1 * \mu_1)^\wedge$ and $(\mu * \mu)^\wedge$ agree on Υ; and so the argument used above with μ and μ_1 can be repeated. We choose μ_1 so that G_1 is the support of $\mu_1 * \mu_1$. Let U be a non-void open subset of G. Then $\tau(U)$ is open in G_1, and so we have

$$0 < \mu_1 * \mu_1(\tau(U)) = \mu * \mu(\tau^{-1}(\tau(U))).$$

The set $\tau^{-1}(\tau(U))$ is plainly the set $H \cdot U$, so that

$$\mu * \mu(H \cdot U) > 0. \tag{5·3·7}$$

Let σ be normalized Haar measure on H, regarded as a measure in $\mathbf{M}(G)$. Since $\hat{\sigma} = \xi_\Upsilon$, we have

$$(\mu * \mu)^\wedge \hat{\sigma} = (\mu * \mu)^\wedge,$$

and so

$$\mu * \mu * \sigma = \mu * \mu.$$

Now for each $x \in H$, we have

$$\mu * \mu(xU) = \mu * \mu * \sigma(xU) = \int_H \mu * \mu(y^{-1}xU)\, d\sigma(y); \tag{5·3·8}$$

since σ is invariant on H, the integral in (5·3·8) is equal to

$$\int_H \mu * \mu(y^{-1}U)\, d\sigma(y) = \mu * \mu * \sigma(U) = \mu * \mu(U). \tag{5·3·9}$$

Now assume that $\mu * \mu(U) = 0$. Then we apply (5·3·8), (5·3·9), and a known fact ((5), p. 122, (11·15)) to write

$$\mu * \mu(HU) = \mu * \mu\left(\bigcup_{x \in H} xU\right) \leqslant \int_G \left(\sum_{x \in H} \xi_{xU}\right) d\mu * \mu$$

$$= \sum_{x \in H} \mu * \mu(xU) = 0.$$

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This contradicts (5·3·7), and so we infer that $\mu * \mu(U) > 0$. Since U is an arbitrary non-void open set in G, the support of $\mu * \mu$ is the entire group G. □

We now make another extension.

5·4. THEOREM. *Let G be a non-compact, locally compact, Abelian group containing an infinite, compact, open subgroup J. Let S be a σ-compact subset of G. There exists a measure $\mu \in \mathbf{M}_s^+(G)$ such that $\mu(G) = 1$, $\mu * \mu$ is absolutely continuous, the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all $p \in]1, \infty[$, and the support of $\mu * \mu$ contains S.*

Proof. Since S is σ-compact, a countable number of cosets of J cover S. Write A_1 for the set J, and let $(A_1, A_2, A_3, \ldots)$ be a countably infinite family of distinct cosets of J such that $S \subset \bigcup_{n=1}^{\infty} A_n$. Let $a_1 = e$ and let a_n be an arbitrary element of A_n for $n = 2, 3, \ldots$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n = 1$.

Let μ_0 be a measure in $\mathbf{M}_s^+(J)$ satisfying all of the conditions laid down in 5·3. Let μ_1 be the measure in $\mathbf{M}(G)$ such that

$$\int_G f d\mu_1 = \int_J f d\mu_0$$

for all $f \in \mathfrak{C}_0(G)$. Since J is a compact open subgroup of G, it is easy to see that $\mu_1 \in \mathbf{M}_s^+(G)$, that $\mu_1(G) = 1$, that $\mu_1 * \mu_1 \in \mathbf{M}_a(G)$, that the derivative of $\mu_1 * \mu_1$ is in $\mathfrak{L}_p(G)$ for all $p > 1$, and that the support of $\mu_1 * \mu_1$ is J. We omit the argument.

Finally we define $\mu \in \mathbf{M}(G)$ as

$$\mu = \mu_1 * \left(\sum_{n=1}^{\infty} \alpha_n \varepsilon_{a_n} \right). \tag{5·4·1}$$

It is evident that $\mu(G) = 1$, that μ is in $\mathbf{M}^+(G)$, and that μ is continuous and singular. Since

$$\mu * \mu = \mu_1 * \mu_1 * \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_m \alpha_n \varepsilon_{a_m a_n} \right),$$

it follows that $\mu * \mu$ is absolutely continuous. From (5), p. 292, Theorem (20·12), we infer that the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all $p \in]1, \infty[$. It remains only to show that the support of μ contains S. Let U be a non-void open subset of $a_k J$ for some $k = 1, 2, 3, \ldots$. It suffices to prove that $\mu * \mu(U) > 0$. Compute as follows:

$$\begin{aligned} \mu * \mu(U) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_m \alpha_n \mu_1 * \mu_1(U a_n^{-1} a_m^{-1}) \\ &\geqslant \alpha_k \alpha_1 \mu_1 * \mu_1(U a_k^{-1}). \end{aligned}$$

Since $U a_k^{-1}$ is a non-void open subset of J, $\mu_1 * \mu_1(U a_k^{-1})$ is positive. □

For our final theorem, a lemma is needed.

5·5. LEMMA. *Let G_1 and G_2 be arbitrary locally compact groups, and let ρ_j and σ_j be measures in $\mathbf{M}(G_j)$ $(j = 1, 2)$. Then*

$$\text{(i)} \qquad (\rho_1 \times \rho_2) * (\sigma_1 \times \sigma_2) = (\rho_1 * \sigma_1) \times (\rho_2 * \sigma_2),$$

the convolutions being in $\mathbf{M}(G_1 \times G_2)$, $\mathbf{M}(G_1)$ and $\mathbf{M}(G_2)$, respectively.

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Proof. Let $f_j \in \mathfrak{C}^r_{00}(G_j)$ $(j = 1, 2)$ and let ϕ be the function $(x_1, x_2) \to f_1(x_1)f_2(x_2)$ on $G_1 \times G_2$. The following equalities follow from the definitions of product measure and convolution and from a simple case of Fubini's theorem:

$$\begin{aligned}
&\int_{G_1 \times G_2} \phi d(\rho_1 \times \rho_2) * (\sigma_1 \times \sigma_2) \\
&\quad = \int_{G_1 \times G_2} \int_{G_1 \times G_2} f_1(x_1 y_1) f_2(x_2 y_2)\, d\rho_1 \times \rho_2(x_1, x_2)\, d\sigma_1 \times \sigma_2(y_1, y_2) \\
&\quad = \int_{G_2} \int_{G_1} \int_{G_2} \int_{G_1} f_1(x_1 y_1) f_2(x_2 y_2)\, d\rho_1(x_1)\, d\rho_2(x_2)\, d\sigma_1(y_1)\, d\sigma_2(y_2) \\
&\quad = \int_{G_1} \int_{G_1} f_1(x_1, y_1)\, d\rho_1(x_1)\, d\sigma_1(y_1) \cdot \int_{G_2} \int_{G_2} f_2(x_2 y_2)\, d\rho_2(x_2)\, d\sigma_2(y_2) \\
&\quad = \int_{G_1} f_1 d\rho_1 * \sigma_1 \cdot \int_{G_2} f_2 d\rho_2 * \sigma_2 \\
&\quad = \int_{G_1 \times G_2} \phi d((\rho_1 * \sigma_1) \times (\rho_2 * \sigma_2)). \qquad (5{\cdot}5{\cdot}1)
\end{aligned}$$

That is, the two measures in (i) give the same integral for functions ϕ. Linear combinations of these functions are uniformly dense in $\mathfrak{C}_0(G)$, and so (5·5·1) holds if ϕ is replaced by any function in $\mathfrak{C}_0(G)$. This implies (i), since measures may be regarded as bounded linear functionals on $\mathfrak{C}_0(G)$. □

5·6. THEOREM. *Let G be a non-discrete locally compact Abelian group, and let S be a σ-compact subset of G. There exists a measure $\mu \in \mathbf{M}^+_s(G)$ such that $\mu(G) = 1$, $\mu * \mu$ is absolutely continuous, the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all $p \in]1, \infty[$, and the support of $\mu * \mu$ contains S.*

Proof. A classical structure theorem asserts that G is topologically isomorphic with $R^a \times G_0$, where a is a non-negative integer and G_0 is a locally compact Abelian group containing a compact open subgroup J_0. Thus we can write elements of G as

$$(\mathbf{x}, y) = (x_1, \ldots, x_a, y),$$

where each $x_j \in R$ and $y \in G_0$. It is plain that $S \subset R^a \times S_0$, where S_0 is a σ-compact subset of G_0. Since G is non-discrete, we have either $a > 0$ or J_0 is infinite.

If $a > 0$, use 5·2 to construct a measure μ_1 on R with the properties ascribed to μ in 5·2. For $j = 2, 3, \ldots, a$, let μ_j be $(2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x_j^2)$ times Lebesgue measure on R. Choose the Haar measure λ_0 on G_0 such that $\lambda_0(J_0) = 1$. Let $(a_n J_0)$ be a pairwise disjoint countable family of cosets of J_0 such that $\cup a_n J_0 \supset S_0$. Let (α_n) be a sequence of positive numbers such that $\Sigma\alpha_n = 1$. Let f_0 be the function $\Sigma\alpha_n \xi_{a_n J_0}$ on G_0, and let μ_0 be f_0 times λ_0. Then define μ by

$$\mu = \mu_1 \times \cdots \times \mu_a \times \mu_0.$$

From (7), p. 349, Theorem (21·29), we infer that μ is singular in $\mathbf{M}(G)$; plainly μ is non-negative and $\mu(G) = 1$. By 5·5, we have

$$\mu * \mu = (\mu_1 * \mu_1) \times \cdots \times (\mu_a * \mu_a) \times (\mu_0 * \mu_0);$$

and (7), p. 394, Theorem (21·29), shows that $\mu * \mu \in \mathbf{M}_a(G)$ and that the derivative of $\mu * \mu$ is the product of the derivatives of $\mu_1 * \mu_1, \ldots, \mu_a * \mu_a, \mu_0 * \mu_0$. As each of these

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derivatives is in $\mathfrak{L}_p$ for all $p \in]1, \infty[$, we infer that the derivative of $\mu * \mu$ is in $\mathfrak{L}_p(G)$ for all $p \in]1, \infty[$.

Finally, if $a = 0$, we are in the case treated in 5·4. □

6. *Remarks.* 6·1. In (2) Glicksberg gives a condition on μ which ensures that $\mu * \mu$ is absolutely continuous. Theorem 5·6 shows that for no non-discrete G can one infer from Glicksberg's theorem that μ itself is absolutely continuous.

6·2. Readers familiar with Wiener and Wintner's theorem ((13)) will see that we could have used their measure in Theorem 4·4 for the case $G = T$, instead of the measure that we construct. We would not have known the support of $\mu * \mu$, however, and in any case we think it worth while to give a complete construction in the present paper. A similar comment applies to the possible use of Salem's or Ivašev-Musatov's construction.

6·3. The measures constructed in 3·2 evidently can be made to yield 'large' Fourier–Stieltjes transforms. For example, for the groups considered in 3·4 and 3·5, we can take $|\beta_k| = \frac{1}{2}$ for all k, and for the groups in 3·6 we can take $\beta_k = \pm 1$. The subject of large Fourier–Stieltjes transforms of singular continuous measures has a long history. It is touched upon by Salem ((11)), and is treated in detail by Hewitt and Kakutani ((3), (4)).

6·4. Theorem 5·6 shows that the Banach algebra $\mathbf{A} = \mathbf{M}(G)/\mathbf{M}_a(G)$ has a fairly large radical if G is a non-discrete locally compact Abelian group. We have shown that there are always non-zero elements in $\mathbf{A}$ whose square is 0, and these of course are in the radical. An easy modification of our construction yields the following. For each integer $k > 1$, there is a non-zero measure μ in $\mathbf{M}_s^+(G)$ such that

$$\mu^2, \mu^3, \ldots, \mu^{k-1} \in \mathbf{M}_s(G) \quad \text{and} \quad \mu^k \in \mathbf{M}_a(G).$$

For such a μ, $\mu + \mathbf{M}_a(G)$ is also plainly in the radical of $\mathbf{A}$. A glance at 3·2 and 4·4 shows that there may be very large sets of pairwise orthogonal elements in $\mathbf{A}$ each with square 0. Let G be a compact Abelian group with character group X. For $\Gamma \subset \mathsf{X}$, let $h(\Gamma)$ be the set of all characters $\chi_1^{\epsilon_1} \chi_2^{\epsilon_2} \cdots \chi_m^{\epsilon_m}$, where $\chi_1, \ldots, \chi_m$ are distinct elements of Γ and $\epsilon_j \in \{-1, 0, 1\}$. To construct orthogonal elements in $\mathbf{A}$ with square 0, one needs only a family $\{\Delta_\iota\}_{\iota \in I}$ of infinite subsets of X such that each Δ_ι is dissociate and

$$h(\Delta_\iota) \bigcap h(\Delta_\kappa)$$

is finite if $\Delta_\iota \neq \Delta_\kappa$. Then $\mathbf{A}$ contains a subalgebra having card (I) independent elements x_ι such that $x_\iota^2 = x_\iota x_\kappa = 0$. Such a family $\{\Delta_\iota\}_{\iota \in I}$ with cardinal number $2^{\aleph_0}$ is easily found in an infinite weak product of finite Abelian groups. It seems likely that Z also admits such a family with cardinal number $2^{\aleph_0}$. We hope to return to this topic in another communication.

6·5. Plainly Theorem 5·3 can be stated for a compact non-Abelian group G. We can prove this theorem for some such groups, in particular for the 3-dimensional rotation group. Our argument is very like that used in the Abelian case, and depends on obtaining a set of irreducible continuous unitary representations of G that behave under conjugation and decomposition of tensor products like dissociate sets of characters. We hope to take up this matter as well in another communication.

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