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SYMPOSIUM ARTICLE

Response to My Critics

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Abstract
This article contains Daniel Sutherland's response to comments on *Kant's Mathematical World* (*KMW*) by Lanier Anderson, Jeremy Heis, and Tyke Nunez. It locates *KMW* in relation to recent debates in Kant scholarship and metaphysical versus mathematical interpretations of Kant's theoretical philosophy. It considers how best to approach a reconstruction of Kant's view that mathematics is the science of magnitudes, and it explains why and how, in Kant's view, mathematics requires intuition, providing a more fine-grained account by distinguishing the roles of intuition in mathematical concepts, judgments, and inferences. The article clarifies the relationship between empirical cognition and pure mathematical cognition, and between apprehension and construction, as well as a number of other issues.

Keywords: Kant; philosophy of mathematics; history of mathematics; magnitudes; theoretical philosophy; experience

1. Recent Debates and Metaphysical Versus Mathematical Interpretations

I begin with a few rather broad remarks about my approach to working on Kant occasioned by Anderson's and Heis' comments. My introduction to Kant's philosophy of mathematics began with the work of Michael Friedman, Charles Parsons, and Jaakko Hintikka. Parsons had explored Kant's understanding of magnitudes and Friedman had drawn a connection between Kant's treatment of magnitudes and the Euclidean tradition, but I realized that Kant's theory of magnitudes not only expressed a deeply different way of thinking about mathematics from our views today and that it had far greater implications for his philosophy of mathematics and his account of all our experience than had been appreciated. I then reconstructed Kant's views informed by the long tradition stretching back to Euclid, quite intentionally setting aside some of the primary issues in recent debates concerning Kant's philosophy of mathematics, in particular, the role of the singularity and immediacy of intuition, Kant's arguments for the syntheticity of mathematics, and the role of intuition in mathematical inferences. I let Kant's understanding of magnitudes rather than those debates guide my investigation with the aim of gaining a fresh perspective. The point was to allow Kant's theory of magnitudes to reframe Kant's account of mathematical cognition and the nature of human experience to provide a new orientation and more accurate interpretation. I believe this approach reveals a previously overlooked role for intuition in representing magnitudes and a much tighter connection between mathematics and the objects of experience than had been appreciated. It also leads to a fundamental rethinking of the singularity of intuition and lays the groundwork for an enriched understanding of the various roles of intuition in mathematics and our cognition of experience.

My original aim in writing *KMW* was to first reconstruct Kant's theory of magnitudes and to then consider Kant's accounts of geometry, arithmetic, algebra, and the foundations of

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mathematical physics, but the first task took more space than expected and I had to postpone the latter for a further book. Since a full account of the syntheticity of mathematical judgments and the role of intuition in mathematical inferences requires taking into account the differences between these four subfields of mathematics, I also postponed a systematic treatment of these issues until the second book. While *KMW* identifies multiple roles for intuition in mathematical cognition that are illuminated by his theory of magnitudes and provides material to address these issues, it does not focus on them or provide all the details. For someone well versed in recent debates, such as Heis, *KMW* might appear not only simultaneously original and refreshing but also less than fully satisfying. Heis takes me to task for not focusing on the syntheticity of mathematical judgments and the justification of mathematical inferences. In a closely related criticism, he finds my account of the role of intuition "blurry." I will attempt to provide a clearer picture in what follows without delving too deeply into details that will be addressed in the second book.

Anderson makes a claim about the broad character of my approach and asks me to comment on it. He sees a debate in the coming decades between, on the one hand, a resurgent metaphysical turn in Kant interpretation, and on the other, a mathematical strand of Kant interpretation that eschews first-order metaphysical theorizing and instead focuses on transcendental arguments concerning the conditions of the possibility of the exact sciences and cultural practices. Anderson sees my work as in the latter camp.

There is an important sense in which this is correct. My interest in Kant's philosophy has first and foremost concerned the presuppositions of mathematics and natural science and I was inspired to tackle these issues in no small part by Michael Friedman, whose work continues the mathematical strand of interpretation that began in the late-19th and early-20th centuries and to which Anderson alludes. Moreover, although I do not think, as some philosophers such as Strawson have, that one can simply jettison Kant's transcendental idealism and "transcendental psychology" in order to salvage his most valuable insights, I also think there is much of value in Kant even for those who are not committed to his transcendental idealism or transcendental psychology. It is also the case that I was not particularly drawn to resolving the thorny textual and interpretive challenges presented by his transcendental idealism. So, I attempted to focus on Kant's philosophy of mathematics and science while leaving open what the best interpretation of his transcendental idealism might be.

While these aspects of my relation to Kant's philosophy naturally align me with what Anderson calls the mathematical strand of Kant interpretation, my initial lack of engagement with Kant's transcendental idealism was not an expression of the *anti*-metaphysical reading that is central to that strand. In fact, I think the development of two-aspect metaphysical views and the revival of (broadly) phenomenalist interpretations in recent years is a valuable and important area of research into Kant's metaphysics. I see no tension taking Kant's metaphysics seriously while appreciating his work on the exact sciences and I think that they can be mutually informative.

There is another important way in which my approach differs from the mathematical strand Anderson identifies. The latter emphasizes Kant's transcendental method employed to uncover the presuppositions of mathematics, natural sciences, and cultural practices, applying that method to later developments in our understanding of those fields. It focuses on the presuppositions themselves, rather than an account of how they are grounded in our most basic cognitive capacities and how they make knowledge possible. As I have noted, I see Kant as providing a foundation of mathematical cognition that aims to do just that, which means that, despite radical differences in method and aims, his project partially overlaps with contemporary, empirically based cognitive science. I do not think that the strand of interpretation Anderson endorses has been sympathetic to this aspect of Kant's project. Nevertheless, I believe that it is in fact Kant's central project, that one must take it seriously if one wishes to understand Kant's philosophy of mathematics and its

connection to experience, and finally, that Kant's careful theorizing about mathematical cognition still has lessons for us today.

2. Heis' Master Argument

Heis takes on board the centrality of Kant's theory of magnitudes for which I have argued but then reframes its relation to Kant's philosophy of mathematics. Heis' presentation provides a different way of thinking about Kant's theory of magnitudes from my interpretation, and engaging with it is instructive, despite (and indeed because) I do not recognize my own understanding of Kant's project in it. An account of why I do not helps situate my own interpretation.

Heis identifies a "Master Argument" consisting of two halves, which establish that mathematics is the science of quantity and that the science of quantity is mathematics. Heis believes that I focus on the second half of the Master Argument, so he directs his attention there. Heis provides an eight-step reconstruction of that half of Kant's Master Argument in accord with my interpretation of Kant's theory of magnitudes, which invites challenges to both Kant and my interpretation.

Unfortunately, Heis thinks that my appeal to Kant's theory of magnitudes in support of the second half of the Master Argument is deficient in two closely related ways. He thinks it falls flat because I fail to make use of two additional features of Kant's philosophy of mathematics mentioned in the previous section: the syntheticity of mathematical judgments and the logical role of intuition in mathematical proofs. Heis finds my not focusing on these topics surprising, since they have been two major points of debate in Kant scholarship, while I think *not* focusing on them, at least not at first, has benefits. But as Heis frames the issues, he thinks that each of these features are potentially viable candidates for supplementing the otherwise deficient Master Argument. Heis also believes that my account of the role of intuition in the theory of magnitudes is "blurry." Clarifying its role would explain whether and how it can support the claims that mathematical judgments are synthetic and that mathematical inference depends on intuition. These are the two underlying criticisms of my interpretation, which I am eager to address. Before doing so, however, I would like to make several points about Heis' Master Argument, because I think his reconstruction misrepresents Kant and my interpretation in several ways and bringing them out is instructive.

Argument reconstruction is a useful tool for reconstructing Kant's views; indeed, as Chapter 3 of *KMW* illustrates, I rely on it heavily. And of course, it is useful for then evaluating those views—whether in relation to Kant's other philosophical commitments (which is my emphasis in Chapter 3 and elsewhere), or from the perspective of his rationalist predecessors and contemporaries, or from the perspective of Russell or Frege, or from our modern standpoint. It puts pressure on specific claims that are essential to Kant's account, and responding to that pressure is fruitful. But I did not recognize Kant or my interpretation in Heis' reconstruction, and I think it paints a misleading picture of both, for several reasons. First, I did not focus on what Heis calls the Master Argument because Kant does not. Of course, Heis does not claim that Kant presents his position as a Master Argument; it is a reconstruction of an argument on his behalf in accordance with my interpretation. But I do not see Kant as starting from some independent characterizations of mathematics on the one hand and a science of quantity on the other and then arguing for a biconditional connecting them. Rather, I think Kant takes as his *starting point* the claim that mathematics is the science of magnitudes and their measure and attempts to fill out a positive theory of what this amounts to in a way that both shapes and takes into consideration his account of human cognition more generally. That positive theory emerged in the critical period to include, for example, the claim that mathematics is rational cognition by means of construction. I think that in working out his account, Kant was not primarily focused on arguing against his rationalist predecessors and contemporaries or on providing an argument that they would not find, by their own lights, question-begging, even if by the time he wrote the *Critique of Pure Reason* he did mount arguments against their views. If his theory as a whole provides a more satisfactory and plausible account that would itself count in its

favor.[1] As a result, I concentrated on what I took to be an already sufficiently challenging task whose completion has until now proved elusive, which was simply to reconstruct Kant's positive views as faithfully and nonanachronistically as possible to arrive at his actual views.[2] I also think that Kant's positive views are more philosophically interesting than whether he has an argument his rationalist adversaries would accept. So, I focused on explicating Kant's theory of magnitudes and its role in mathematical and human cognition more generally, while postponing a full evaluation of his philosophy of mathematics and account of cognition until we have a clearer picture of Kant's views in their entirety. And the *starting point* of the mathematical part of his project is the claim that mathematics is the science of magnitudes and their measure. In order to understand why, it is helpful to back up to the development of Kant's critical philosophy.

As I understand it, the essays Kant wrote in the *Prize Essay* period from roughly 1762–3, that is, *The Only Possible Argument in Support of the Demonstration of the Existence of God*, *An Attempt to introduce the Concept of Negative Magnitudes in Philosophy*, and *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (a.k.a. the *Prize Essay*), left him convinced that the conceptual rationalist program of Leibniz and Wolff could not succeed. More particularly, *Negative Magnitudes* reflects his conviction that real opposition of magnitudes, such as the opposition of forces or motions, could not be accounted for by means of the logical opposition of concepts. Nearly simultaneously, writing the *Inquiry* convinced him that mathematical method differed deeply from philosophical method and that close attention to basic cases of mathematical cognition, especially geometrical cognition, reveal a dependence on the representation of concrete particulars in intuition.[3] Kant was, in this respect, sensitive to mathematical practice; he reflected carefully on what it is like to engage in quite basic mathematical acts of cognition and on the grounds of their certainty, especially in geometry. This further persuaded him that mathematics was not adequately explained by the rationalist accounts of concepts and truth. I believe Kant's focus in the *Inquiry* on the *certainty* of mathematical demonstrations suggested to him that intuition was playing a substantive role in grounding and justifying mathematical judgments, not just a heuristic role that acquaints us in a confused way with truths that reduced to relations among concepts.

Moreover, Kant would have been acquainted with Wolff's attempt to radically reform Euclidean geometry in his *Elementa Matheseos Universae* by introducing a philosophical definition of similarity and attempting to derive the truths of geometry from definitions. Wolff gives Leibniz credit for suggesting the use of this definition of similarity to overhaul Euclid. Nevertheless, a close reading of his attempt reveals a *non sequitur* in the proof of the very first proposition, revealing the inherent inadequacy of Wolff's grounding of geometry in philosophical definitions.[4] While this failure was not a decisive proof that the Leibnizian and Wolffian (and in particular the Leibnizian) approach could not be made to work, I believe it gave Kant further reason to think that rationalists could not remain consistent with their principles and explain mathematical cognition.

Kant's understanding of intuition was at this point still nascent; he had not yet taken the step of regarding space and time as ideal or intuition as *a priori*, had not yet worked out his mature formulation of the analytic/synthetic distinction, nor had he fleshed out his understanding of our cognition of magnitudes, but he had broken with Leibnizian and Wolffian rationalism and was now on the path toward his critical philosophy. I believe that Kant's reflections on magnitudes

[1] One could thus construe Kant as, at least in part, providing the grounds for an inference to the best explanation.

[2] The fact that it has eluded so many for so long seems surprising. I give good reasons it has been overlooked in *KMW* Section 1.5.

[3] A note about terminology. Kant distinguishes singular from particular functions of judgment, where the former picks out just one thing. Kant distinguishes intuitions from concepts by stating that the former are singular and the latter are general. I thus prefer to use "singular" to explicate Kant's notion of singular representation. In our contemporary philosophical discourse, particular is often used to mean just one thing, for example, a particular perception. I sometimes use "particular," and when I do, I am not using it in Kant's sense or making a distinction between singular and particular representations.

[4] See See Sutherland (2010) for a detailed discussion of Wolff's geometry.

and mathematics in the Prize Essay period deeply influenced the development of his critical philosophy and spurred him to work out a new and revolutionary theory of human cognition informed by his understanding of mathematical cognition and his conviction that it relied on intuition.[5]

Finally, in rethinking the nature of mathematical cognition, Kant begins with the view that at its foundations mathematics is a science of magnitudes and their measure because it was a commonly shared view among mathematicians. As I have argued and Heis notes, it was a familiar and uncontroversial position before 1850; this understanding of mathematics was expressed in particular by Abraham Kästner, Leonhard Euler, and by Christian Wolff. This was not surprising, since it was a venerable view with a long history tracing back to in Euclid's *Elements.* Nevertheless, it had been neither seriously examined nor adequately developed with the aim of explaining mathematical cognition, so it became part of Kant's task to take on that work. Kant was starting with a widely shared though underdeveloped understanding of mathematics with a two-millennia pedigree.

In summary, I think Kant was convinced that the rationalist program was deeply flawed, but rather than focusing on finding nonquestion-begging arguments that his rationalist predecessors and contemporaries would accept, he aimed to develop a new, positive account to replace it. An overall account of mathematical and human cognition that is more coherent and plausible than the theories the rationalists offered, that more closely reflected mathematical practice, and that was grounded in a widely held view of the relation between mathematics and magnitudes with its roots in Euclid's *Elements* all would speak in its favor. Kant builds a case with his overall account, not in linear step-by-step arguments tracing two directions of a biconditional. And in this way, I think Heis' Master Argument is misleading. By the critical period, of course, Kant also argues against his rationalist predecessors, in particular, by attempting to leverage his mature understanding of the analytic/synthetic distinction and his claim that we have synthetic *a priori* cognition. Whether his arguments against them are successful or whether they would have found them question-begging is a further matter.

A few comments on Lanier Anderson's valuable book *The Poverty of Conceptual Truth* are called for at this point, since it directly bears on whether Kant attempts to provide nonquestion-begging arguments against his rationalist predecessors and contemporaries. Anderson rightly points out the striking fact that in 1781 the Leibnizian–Wolffian philosophy was quite firmly entrenched in Germany, and yet by 1790 it had been almost entirely abandoned in favor of a Kantian framework. He believes that explaining this relatively sudden change requires a nonquestion-begging argument against the Leibnizian–Wolffian philosophy, and he structures his book about around this demand.

Anderson deftly outlines the metaphysical program of Leibniz's and Wolff's philosophies and shows how, in reaction to Leibnizian and Wolffian rationalism, Kant's analytic/synthetic distinction developed through both methodological and epistemological versions before settling on a logical–semantic version. In particular, Anderson underscores the way in which the Leibnizian and

[5]There is quite a bit more to the story of Kant's break with the rationalist tradition, including the role of *The Only Possible Argument* and the influence of Crucius' criticisms of rationalist metaphysics. There is also Hume's wake-up call and Kant's attention to the concept and principles of causation in the precritical period. My claim about the centrality of Kant's reflections on mathematics on his path to the critical philosophy is compatible with his famous statement that remembering Hume interrupted his dogmatic slumber and gave a completely different direction to his research. There are several possibilities here. Kant may be referring to a remembrance of having read the *Enquiry*, a translation of which he might have read as early as 1755 and most likely read by the mid-1760s. He may also be referring to a remembrance of a significantly later reading of a translation of parts of the *Treatise* in the 1770s, after the *Prize Essay* period. Perhaps his work on the *Prize Essay* rekindled his memory of a reading of the *Enquiry*, which broadened his concerns about metaphysics. Perhaps the *Enquiry's* attack on the connection of cause and effect sparked a broader questioning of Leibniz' and Wolff's philosophies that in turn prompted Kant's developments in the *Prize Essay* period. Or perhaps Kant read the *Enquiry* not long after the *Prize Essay* period or read parts of the *Treatise* in the 1770s, which might have prompted him to expand his criticism of the Leibnizian and Wolffian metaphysics. Regardless of the details of the development of Kant's views, I believe his reflections on mathematical cognition had a crucial, formative influence on Kant's commitment to the role of intuition, in particular pure intuition, in all theoretical cognition.

Wolffian philosophy was thoroughly conceptualist and grounded truth in concept containment, which renders all truths analytic in Kant's mature understanding of the analytic/synthetic distinction. He also shows how Wolff aspired to arrange all concepts into a hierarchy and to assimilate all truths to containment relations between concepts in that hierarchy. Anderson describes how Kant came to appreciate the inadequacy of their approach to account for truths Kant would classify as synthetic, especially relations of equality expressed in judgments such as 7 + 5 = 12. I think Anderson's work on these themes is excellent, though I differ on one point of emphasis. One can get the impression from his work that the development of Kant's critical philosophy was first made possible by his mature formulation of the analytic/synthetic distinction, which according to Anderson emerged in about 1772, while I see that development beginning much earlier, spurred by an appreciation of the inadequacy of Wolff's reform of geometry, the essays of the Prize Essay period of 1762–3, and reflections on Hume, while his settling on the mature analytic/synthetic distinction in 1772 allowed him to express his own view more clearly and to sharply contrast it to the Leibnizian–Wolffian philosophy.

Anderson notes that to Kant's rationalist opponents, Kant's pronouncements in the *Critique of Pure Reason* that mathematical judgments are synthetic despite being *a priori* and hence necessary will sound like mere table-thumping. Moreover, Kant's negative argument faces a problem because they have a ready response to his claim that mathematical judgments are synthetic, which Anderson calls the "Recuperation Strategy": evidence that a predicate belongs with necessity to a subject in a judgment is simply dispositive evidence that the predicate is contained in the subject, so that any purportedly *a priori* synthetic proposition is analytic after all, even if we did not at first recognize it to be so. From a Kantian perspective, the Recuperation Strategy appears *ad hoc.* Nevertheless, Eberhard makes use of it in his sustained attacks on Kant's philosophy in the *Philosophisches Magazin* beginning in 1788, to which Kant attempts to respond in *On a Discovery According to which Any New Critique of Pure Reason Has Been Made Superfluous by an Earlier One*, published in 1790. Anderson does not think that the rationalist will be moved by Kant's insistence in the *Critique* and elsewhere that, for example, search as we may, the concept of 12 cannot be found in the concept of 7 + 5, so that Kant requires a further argument. Anderson finds two such arguments on Kant's behalf. One is specific to Wolff's insistence that mathematical propositions such as 7 + 5 = 12 have a place in a hierarchy of conceptual truths; as Anderson shows, it appears impossible to find a place for such a truth in the Wolffian hierarchy. The other argument is broader and deeper in targeting not just the hierarchy of conceptual truths on which Wolff insisted, but the Leibnizian account of truth in terms of concept containment. Here, Anderson draws on my argument that intuition is required for the representation of the mathematical properties of magnitudes because it depends on the representation of strictly homogeneous manifolds and their properties and relations. Anderson focuses on equality and in particular the equality expressed in equations such as 7 + 5 = 12, providing his own illuminating argument that it cannot be reduced to the relation between concepts, an argument that Anderson thinks the rationalist would have to accept.[6]

I believe that looking for arguments that various philosophers at different times would find non-question-begging is valuable insofar as it allows us to formulate challenging questions and hold Kant's feet to the fire. It is also valuable because it helps us relate Kant not just to his rationalist predecessors, but to later mathematicians and philosophers—such as Russell and Frege—who resisted Kant's claim that arithmetic requires intuition and believed that conceptual representation and logic, properly understood, were sufficient for arithmetic. These later criticisms rejected Kant's claim that mathematical judgments are synthetic in light of developments in logic and a shifting understanding of analyticity that is not confined to concept containment; it also led Russell, Hintikka, and Friedman to more closely consider the role of the singularity of intuition in

[6]Note that this second argument presupposes the role of intuition in representing a strictly homogeneous manifold, and hence it could not be used to supplement an argument for the latter. Thanks to Heis for pointing this out.

mathematical proofs. Anderson and Heis have this larger context in mind, which provides a broader motivation to search in Kant for a nonquestion-begging argument that mathematics is not merely conceptual. Nevertheless, I think insisting on a nonquestion-begging argument in reconstructing Kant's views can occlude Kant's positive views, which have been my focus.

Returning to Heis' Master Argument, I think there are a few other features that are potentially misleading. Heis refers to the science of *quantity.* The vocabulary in Kant's day was not settled; moreover, the terms "magnitude" and "quantity" were used ambiguously to refer to something that can be measured and to the result of its measurement—referring, for example, to a piece of lumber itself as a magnitude and to its length, for example, 6 feet, as its magnitude—where the former is paradigmatically some concrete particular. Kant attempted to resolve the ambiguity by distinguishing between two senses of magnitude, *quanta* and *quantitas*, where "*quanta*" refers to what can be measured and "*quantitas*" refers to its measurement. Thus, "*quanta*" gives a name to the concrete sense of magnitudes already in play. In Kant's terminology, the science of magnitudes and their meaure become the the science of *quanta* and their *quantitas* which is the starting point of his investigation. Kant then asked himself what it is about such *quanta* that allows of measurement and distinguishes them from those things that do not. As Heis makes clear in footnote 3 [**XX**], by "science of quantity," he means the science of *quantitas* alone, which he identifies with the categories of quantity. Heis drops out reference to *quanta*, though he goes on to say that an alternative that seems Kantian and consistent with *KWM* is that mathematics is the science of *quantitas* "inasmuch as it is" the quantity of *quanta.* This is almost there, but it still singles out *quantitas.* I believe Heis focuses on *quantitas* because he wants a characterization of the science of quantity that does not already implicitly invoke representations of concrete particulars; he wants Kant to argue that they are required.[7] I will address this challenge next, but it would have been closer to Kant's views to use "magnitudes" rather than quantity, and to take his starting point to be the view that mathematics is the science of *quanta* and their *quantitas.* Henceforth I will refer to the science of magnitudes and their measurement simply as the science of magnitude, and substitute "magnitude" where Heis uses "quantity."

A second potentially misleading feature concerns the conclusion of the Master Argument and hence the scope of what Heis states is the main claim of my book. I see Kant as developing a general theory of magnitudes, which includes the synthetic *a priori* metaphysical principles that objects of experience are extensive magnitudes and that all of their real properties have an intensive magnitude. The general theory and these principles in particular support the view that the world is mathematical. In formulating the conclusion of the Master Argument as the claim that mathematics is the science of magnitude, Heis drops the second half of what he quotes as the main claim of my book, that is, that the world of experience is a world of magnitudes; shortening it to "mathematics is the science of magnitude" threatens to obscure the broader theory and its centrality to Kant's views. It all depends in how we understand "mathematics" and "science" in this claim. I would just like to dispel any mistaken impression one might have that my main claim and Heis' formulation of the conclusion of the Master Argument only concern *pure* mathematics and a *proper* science of magnitudes, where a proper science might be erroneously construed as restricted to magnitudes constructed *a priori* in pure intuition, such as a triangle in pure geometry. That is not my intent. Nor is it Heis'.[8]

[7]Heis states in footnote 3 **[XX]** that he might be mischaracterizing my view by excluding *quanta.* He defends the exclusion by pointing out that *quantitas* is directly tied to all three categories of quantity. Mathematics requires representations of *quantitas* as well as equality, but that is no reason to exclude *quanta.* Heis also states that it would be question-begging for Kant to assume that arithmetic and algebra concern *quanta.* I respond to that charge next.

[8]The first three steps of the Master Argument concern all *quanta* in general, and *pure* spatial regions only enter at step four. Heis is explicit that by spatiotemporal magnitudes he means anything that is a magnitude that occupies space and time, not just *a priori* magnitudes. And one of Heis' proposals for supplementing the Master Argument is to appeal to the notion of a proper

A final point concerns the structure of the Master Argument. Heis thinks that the second direction of the biconditional—that if a science concerns magnitude, then it is mathematics—is more difficult to argue, and that this is the direction I am following. But in light of what I said earlier, I think that Kant's unfolding of the claim that mathematics is the science of magnitudes supports a broad theory of magnitudes that, if anything, supports both directions of the biconditional, with neither direction being more difficult.[9] Moreover, I would say that my interpretation, if anything, addresses both directions. The fact that I focus on the role of intuition in making it possible to represent magnitudes may suggest that I am moving from the science of magnitudes to mathematics (as Heis states on p. 4 [**XX**]), but in focusing on representations that intuition makes possible, I am thereby in no small part articulating necessary conditions of those representations, and a chain of such necessary conditions backing up from intuition to mathematical cognition would provide an argument that mathematics is the science of magnitude—the first of Heis' biconditionals.

Despite these misgivings with Heis' Master Argument, however, his analysis leads to some deep and insightful criticisms. Broadly speaking, the first of his two main criticisms is his discomfort with the clarity of the role of intuition in mathematical cognition, including in mathematical judgments and inferences. As Heis points out, I issued promissory notes in *KMW* with the intention of cashing in those notes in a second book, because a full account needs to consider the differences between geometry, arithmetic, and algebra.[10] Sections 3–6 will provide a brief synopsis of the role of intuition in mathematical cognition in general with discussions of the particular elementary branches of mathematics. Heis' other main criticism concerns the relation between empirical cognition and pure mathematical cognition, and Nunez as well as Heis object to my account of construction. I will address these two issues in Section 7.

3. The Role of Intuition in Representing Strict Homogeneity

Heis is interested in the role of intuition in mathematical cognition quite generally, but his concern with my interpretation centers on the role of intuition that arises from Kant's theory of magnitudes. Since on my interpretation intuition is what makes it possible to represent a strictly homogeneous manifold, his discomfort is more specifically with this role. And hovering in the background are the two questions that have been the focus of recent debates: what is the role of the representation of strict homogeneity in grounding synthetic *a priori* judgments in mathematics and its role in mathematical inferences? Does the theory of magnitudes support these features of mathematical cognition, or does the theory need to appeal to them for extra support?

I will begin in this section by reprising why Kant thinks that the representation of strict homogeneity is required for mathematical cognition, but before doing so I would like to remind that though *KMW* focuses on a previously neglected role for intuition in representing a strictly homogeneous manifold, I allow that in Kant's view there are other roles for intuition in mathematical cognition as well; once Kant came to think that intuition played a central role in

science to account for apriority, while at the same time exploiting the fact that a proper science can concern empirical objects and hence magnitudes more broadly.

[9]Even by Heis' lights, I do not see why the argument for the claim that if something is mathematics, then it is a science of magnitude is the easier. In his reconstruction, it would start with the claim that mathematics is rational cognition from the construction of concepts, and since construction requires the *a priori* exhibition of an intuition corresponding to an object, it already assumes the representation of concrete individuals in intuition. But one of Heis' objections to the argument in the other direction is that we need an argument for this claim. I have other qualms about the steps in Heis' argument, but I set them aside in order to highlight what I take to be his main and most helpful criticisms.

[10]I set aside the foundations of mathematical physics, because Kant's account of the composition of motion in the Phoronomy of the *Metaphysical Foundations of Natural Science* as well as his views on infinitesimals, differentials, fluxions, and fluents build on what Kant establishes for magnitudes in general and geometry, arithmetic, and algebra.

mathematical cognition, he was open to various roles it might play both in mathematics and in human cognition more generally.

The notion of strict homogeneity derives from the Eudoxian theory of proportions found in Book V of Euclid's geometry. According to that theory, magnitudes included only the continuous spatial magnitudes: lines, areas, volumes, and angles. Lines are homogeneous with lines, areas with areas, and so on, but there is no homogeneity across these types of spatial magnitude. As I discuss in *KMW* Section 6.4, Euclid does not define magnitude, but the first five definitions of Book V of the *Elements* implicitly define homogeneous magnitudes as the sorts of things that can measure each other. I argue in Chapter 9 of *KMW* that the notion of measure is part of a general theory of pure concrete measurement that governs magnitudes. It also governs numbers conceived of as collections of units, which later in the Greek mathematical tradition are classified as discrete magnitudes. This theory of measurement presupposes that homogeneous magnitudes stand in part/whole relations and in a total ordering of comparative size relations. It also presupposes that homogenous magnitudes can be composed into larger magnitudes that are homogeneous with them. Moreover, homogeneous magnitudes can stand in ratios to one another. Finally, homogeneous magnitudes are capable of standing in the relation of equality. These presuppositions guarantee that homogeneous magnitudes are governed by the propositions concerning ratios and proportions developed in Books V and VII, which describe the mathematical relations among magnitudes. This is why it is reasonable for Kant to take as his starting point the claim that mathematics is the science of magnitude.

Since only magnitudes that are homogeneous with each other stand in these mathematical relations to each other, there is something about their homogeneity that is fundamentally tied to their mathematical character. I believe that sometime during or after the *Prize Essay* period, Kant reflected on what that something is, and first considering geometry, he concluded that underlying the homogeneity of each of the kinds of determinate space (i.e., lines, areas, etc.) is the strict homogeneity of space itself. Each of the kinds of determinate space are composed of parts that are qualitatively identical and can be composed into wholes that are qualitatively identical with those parts and hence homogeneous with them, yet at the same time are quantitatively *more* than those parts.[11] The synthesis of a homogeneous manifold results in more than what you started with but is nevertheless homogeneous with it—if you extend a straight line segment, the result is still a line; it only differs in being *more* line relative to that which you started, and this is a purely quantitative difference. The synthesis of composition is at the heart of Kant's account of determinate magnitudes, a form of synthesis unique to homogeneous magnitudes (B201n). Taking his cue from the Greek mathematical tradition, Kant extends this theory to cover discrete magnitudes as well, and hence number conceived of as collections of units, where we abstract from all qualitative features of that which we call a one. Finally, Kant holds that concepts on their own cannot represent a strictly homogeneous manifold, while intuition makes that first possible. (See *KMW* Chapter 7 for the full argument and textual support.) This is the most fundamental role for intuition in the representation of magnitudes and hence in mathematics—it makes it possible for us to represent *quanta* and their quantitative determinations, that is, their *quantitas.*

[11]There are a number of issues that require further elaboration; I mention two here. First, in Kant's view, lines, areas, and volumes are qualitatively different because of the difference in their dimensionality, so that lines, areas, and volumes are each strictly homogeneous in virtue of the strict homogeneity of space itself, while they are only homogeneous within their kinds in virtue of the difference in the dimensionality of the strictly homogeneous manifold of which they are composed. Second, one can make qualitative distinctions among different homogeneous magnitudes, for example, between straight and curved lines, but these are specifically qualities of *quanta* that depend on and are derived from our representation of *quanta.* Matters are further complicated by the fact that Kant uses "quality" in two different senses. I discuss this issue in *KMW* Chapter 8 and section 8.5 in particular, and return to it in section 8 below. Finally, angles always fit uneasily with the other spatial magnitudes; the sense in which they are additive provides a particular challenge. See *KMW* p. 285, fn. 214.

4. Why Do Arithmetic and Algebra Require the Representation of *Quanta*?

As I noted earlier, Heis would like Kant to argue that the representation of *quanta* is required for mathematical cognition and not get it for free. Heis raises a two-level challenge against the claim that *quantitas* requires the representation of *quanta*. He first allows that geometry concerns *quanta*, such as lines, circles, areas, etc. That seems like a reasonable concession, at least for the sake of argument: synthetic geometry proceeds through construction of figures in space, which are *quanta*, and yields propositions about those *quanta*. But Heis thinks that Kant is question-begging in assuming that arithmetic and algebra do. I stated earlier that Kant is sensitive to mathematical practice, especially to geometrical practice, but it might seem that he is overgeneralizing from geometry to all of mathematics and is insensitive to arithmetical and algebraic practice. Although I had postponed a full treatment until the second book, I will begin to address these issues here, starting with arithmetic.

4.a Arithmetic and the representation of *quanta*

The Greek mathematical tradition gave Kant good reason for thinking that our representation of numbers is grounded in the representation of *quanta*. Kant did not think of the natural numbers, such as 7, in the abstract way we do today. As I discuss in *KMW* Sections 6.6–6.8, Books V and VII of Euclid's *Elements* provides a separate treatment of magnitudes and number, respectively, where number was defined as a multitude composed of units and a unit is that with respect to which each existing thing is called one. This is a conception of number as a collection of particulars.[12] The Greek mathematical tradition later assimilated number into a more general theory of magnitudes and their proportions, regarding such collections as discrete magnitudes, and numbers were so described by Kant and his contemporaries. As I discuss in *KMW* Chapter 6, the relation between number and magnitudes was a matter of controversy in the long history of mathematics and matters were further complicated as the notion of number was expanded to include irrationals. But natural numbers remained foundational for Kant and his contemporaries and were closely tied to collections of discrete magnitudes.[13] So Kant had good reason to think that arithmetic had at its foundation the representation of particular discrete magnitudes, and hence *quanta*.

Nevertheless, the relationship of number and arithmetic to *quanta* is more complicated than the relationship of geometry to *quanta*. First, the ancient Greek notion of number is of collections of units and hence corresponds to cardinal number, but Kant also thought that the notion of number included the ordering of discrete *quanta* corresponding to ordinal number, and this aspect of number depends on the intuition of time in the successive synthesis of one unit to another.[14] Second, number implicity makes reference to what counts as a unit, which can be defined in different ways, e.g., to cite a common empirical example, the units counted might be shoes or pairs of shoes. Third, Euclidean synthetic geometry is quite obviously tied to the representation of spatial *quanta* and only yields knowledge of spatial *quanta*, while number and arithmetic have a much broader application—a point Frege drives home in the *Foundations of Arithmetic*. Moreover, the number 5 itself as an object of cognition is at a further remove from the canonical spatial *quanta* envisioned in the representation of collections of units than geometry is from its *quanta*.[15] Both the

[12] In addition to the influence of Euclid's Book VII, there was a distinct arithmetical tradition represented by Archimedes, Nichomachus, Diophantus, and Psellus, as I discuss in Sutherland (2006) and will revisit in the next book. The view of number as a collection of units was common to both.

[13] Among Kant's contemporaries, Wolff, Kästner, and Euler, for example, explained mathematics as a science of magnitude by appealing to discrete magnitudes, that is, concrete particular collections such as coins. In a potentially misleading passage in the Anticipations of Perception, Kant appears to reject the notion of discrete magnitudes. But elsewhere in his writings, including the *Critique of Pure Reason*, he accepts them. See *KMW* Sections 3.5, 4.6.1, and 6.8.

[14] I discuss the cardinal and ordinal features of Kant's understanding of number in some depth in Sutherland (2014).

[15] As Parsons repeatedly emphasizes. See Parsons (1983b), p. 128, for example.

ordinal features of number and their standing at a remove from collections of units are reflected in Kant's definition of number as the schema of *quantitas* rather than simply as a collection of units; the schema is a representation of a procedure for summarizing the successive addition of one homogeneous unit to another. Nevertheless, number is a procedure for generating the representation of collections of discrete *quanta* under the categories of unity, plurality, and totality, so that the content and objective reality of number concepts is still bound up with and ultimately depends on our ability to represent *quanta.*[16]

A further serious complication arises from Kant's distinction in the Discipline of Pure Reason between ostensive and symbolic construction. Kant states that geometry rests on ostensive construction, while algebra employs symbolic construction, where the former constructs *quanta* and the latter mere *quantitas.* I will discuss the contrast between ostensive and symbolic construction shortly. What is important at present is that Kant does not clarify whether arithmetic and number rest on ostensive or symbolic construction, leaving it a matter of interpretation. In many respects, arithmetic bears a closer relation to algebra than to geometry, so would seem to rest on symbolic construction. Moreover, number and measurement concern how big something is, and hence magnitude in the sense of *quantitas* (A163-4/B204-5). On the other hand, Kant's repeated reference to the representation of units in intuition and hence *quanta* strongly suggests a reliance on the representation of discrete *quanta* as units.

[17] I will argue in my book that arithmetic employs symbolic construction to represent the *quantitas* of *quanta*; numerals such as 5 and 7 are symbols, as are the symbols for arithmetical operations, and in Kant's view, intuition plays an important role in the representation of symbols in a way that makes it possible to effectively manipulate them in arithmetic. Nevertheless, the content and objective reality of these symbolic constructions and the judgments that employ them ultimately depend on the ostensive construction of discrete *quanta*, such as dots on a page or Hilbert's stroke arithmetic. How exactly to understand what "7" and "5" refer to is of course an issue that has challenged philosophers and mathematicians since at least Plato. Addressing Kant's views on this issue requires an interpretation of the Schematism and the relationship between pure sensible concepts, schemata, and images, which I will also address in my book.

4.b Algebra and the representation of *quanta*

Algebra is yet more challenging than arithmetic. Algebra was widely thought to be a kind of general arithmetic corresponding to the Euclidean theory of proportions of both continuous and discrete magnitudes; at the same time, algebra developed within its own arithmetical tradition to which Archimedes contributed, which only reinforces the close connection between numerical and general arithmetic.[18] Kant himself repeatedly refers to algebra as general arithmetic,[19] which supports the view that algebra requires the representation of *quanta* because and insofar as arithmetic does.

Algebra, however, also differs from numerical arithmetic in a way that matters for the representation of *quanta.* Algebra shares the same signs or symbols as numerical arithmetic for arithmetical operations, and numerals are also used in algebra; the difference between algebra and arithmetic is in the signs or symbols used for variables and constants, such as the variable x or

[16]Kant's doctrine of the schematism is a vexed topic in its own right, and its role in mathematical cognition raises significant complications that I cannot address here; I will address them in my second book.

[17]I consider the relationship between arithmetic and algebra in more detail Sutherland (2006), where I argue against the view that arithmetic ultimately rests on the ostensive construction of geometrical lines, and for the view that it ultimately rests on the construction of discrete *quanta.* I did not there adequately address the role of signs or symbols, and hence the role of symbolic construction.

[18]I discuss the historical background in more detail in Sutherland (2006).

[19]For Kant's descriptions of algebra as general arithmetic, see *Inquiry* (2:278, 2:282) and his letter to Schultz of November 25, 1788 (10:555).

the constant *a*.[20] These symbols in some manner either indeterminately represent *quantitas* or represent an indeterminate *quantitas*, and in any case do not represent a *determinate quantitas* of a *particular quantum*. Nevertheless, if the content of concepts of *quantitas* depend on *quanta* in numerical arithmetic, then Kant may well hold, and I think he does hold, that the content and objective reality of the indeterminate representation of *quantitas* in algebra ultimately depends on the representation of *quanta*, despite the indeterminacy of representation.

One might reasonably remain worried about this claim. What Kant says about algebra in the Discipline of Pure Reason seems to support the view that the representation of particular *quanta* is not required. Kant ties algebra quite tightly to the theory of magnitudes, referring to magnitude six times in his one-paragraph explication, but he says that algebra constructs mere [*bloß*] magnitude (*quantitatem*), contrasting it with the construction of magnitudes (*quanta*) in geometry, and says that algebra "entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude" (A717/B745). I argued in *KMW* Section 3.5 that Kant cannot mean that we abstract *entirely* from the constitution of the object that is thought in accordance with a concept of *quantitas*, which is the quantitative determination of a *quantum*. Kant also states that:

> No one can define the concept of magnitude in general except by something like this: That it is the determination of a thing through which it can be thought how many units are posited in it (A242/B300).

His reference to the determination of a thing indicates that by the concept of magnitude in general he means here the concept of *quantitas* as the determination of a thing; thus, the concept of *quantitas* is inseparable from its being a determination of something. Since *quantitas* is the quantitative determination of a homogeneous manifold, it is part of the concept of *quantitas* that it is the quantitative determination of a *quantum*. So, it seems likely that Kant means that we entirely abstract from the constitution of the object except for the fact that *quantitas* is the determination of it insofar as it is a *quantum*.

But, as I mentioned earlier, Kant's appeal to symbolic construction further complicates matters. He states that

> … algebra achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves)… (A717/B745).

The contrast to ostensive or geometrical construction and their association with construction of "the objects themselves" strongly suggests that symbolic construction in algebra in no way depends on the ostensive construction of *quanta*.

While I cannot provide the full interpretation and argument here, I believe that Kant holds that both arithmetic and algebra employ symbolic constructions, and that intuition plays an essential role not only in representing the symbols themselves, canonically on a page and hence in space, but crucially on the rules governing the spatial relations of the symbols and the rule-governed spatial rearrangement of those symbols. The intuition of space thus plays an essential part of what makes symbol manipulation effective. At the same time, I believe that in Kant's view, the content and objective reality of the concepts of *quantitas*for which the symbols are surrogates and the judgments involving them ultimately depend on ostensive constructions, that is, the representation of discrete *quanta*.

Finally, I think Kant's claim that the symbolic construction essential to algebra constructs mere [*bloß*] magnitude (*quantitatem*) reveals that in his view, we do not treat the symbols as *mere* signs,

[20] I say "signs or symbols" in order to leave open Kant's understanding of the distinction between the two, from the 1763 Prize Essay to the 1790 *Critique of the Power of Judgment*; clarifying that distinction is required for a full understanding of symbolic construction. For the remainder, I will simply use the term "symbol."

that is, either token marks of a particular shape on a page or their types. Rather, we exploit the advantages of symbolic representation and manipulation while also keeping in mind that the symbols stand for *quantitas*. While we may slip back and forth from treating symbols as *mere* marks on a page and thinking of what they stand for, we often take into account what they stand for. And, as I argued earlier, when Kant contrasts symbolic construction of *quantitas* in algebra with the construction of *quanta* in geometry, and says that algebra "entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude" (A717/B745), Kant does not mean we abstract from the fact that the object thought in accordance with the concept of *quantitas* is a *quantum*. As a result, a full account of the content and objective reality of the concepts of *quantitas* depends on the ostensive construction of *quanta*.

5. Why Does *Any* Mathematical Cognition Require Intuition?

5.a Bringing the role of intuition into sharper focus

Having at first conceded that geometry concerns *quanta* such as lines and circles and pressing only on arithmetic and algebra, Heis takes his challenge to a deeper level. He asks why even geometry requires the representation of particular *quanta*. It might be conceded, for the sake of argument, that mathematical cognition requires the concept of *quanta*, but why does it require the representation of particular *quanta* in intuition? Why can we not make do with the concepts of *quantitas* and *quanta* or indeed with just concepts of *quantitas*? As I noted in Section 2 earlier, Heis identifies the science of quantity with *quantitas*, and then *quantitas* with the categories of quantity (see Heis, fn. 6 [**XX**]), and the thrust of his question is why mathematical cognition cannot simply employ abstract concepts without the representation of particular concrete intuitions.

A fast reply on Kant's behalf is that because *quantitas* is the determination of *quanta*, you cannot represent *quantitas* without representing a *quantum*. As I noted in Section 4.2 earlier, Kant holds that the only way to define *quantitas* is as a determination of a thing, that is, as a determination of a *quantum*. But Heis does not take this to be a sufficient answer:

> … why should we assume that even geometry should require the representation of *concrete, singular quanta* (as opposed to the concept <*quantum*> or concept of species of *quanta*, such as the concepts <line> or <circle>? Granted, of course, that every *quantitas* is the quantity of some *quantum*. But why would it follow that a science of *quantitas* must represent particular, concrete *quanta*?(Heis, pp. 7–8 [**XX**]).

These questions are of central significance in the history of the foundations of mathematics and bear comment before continuing. Many view one of the great advances of mathematics in the 19th century as the rejection of any role for intuition as mathematics was arithmetized, logicized, and axiomatized. Kant wears the black hat in this understanding of the history of mathematics, but there have been persistent holdouts that look to Kant for inspiration.[21] Whether Kant's answer is satisfying in light of developments in logic and our contemporary understanding of the nature of conceptual representation is not a challenge that I will address here. My aim is to explicate as accurately as possible what Kant's views are. What follows may seem like simply a more elaborate, Kantian way of insisting that *quantitas* requires the representation of *quanta* in intuition. Nevertheless, evaluating Kant's views requires a proper understanding of them, and even if we ultimately reject Kant's views, a correct understanding informs the rest of his account of human cognition, sets the record straight in the history of the foundations of mathematics, and I believe potentially still offer insights into the nature of mathematical cognition today.

[21]For a recent important example, see Parsons (2008).

Heis' challenge requires a more detailed account of the roles of intuition in mathematical cognition. For even if we agree that Kant's understanding of the theory of magnitudes entails that all mathematical cognition ultimately in some way *concerns quanta*, that does not pin down exactly what the necessary roles are that intuition plays, and it is here that Heis finds my interpretation "blurry."

There are three ways to sharpen the answer to Heis' question. First, intuition plays various roles in mathematical cognition, but since I have argued for the fundamental and neglected role of intuition in representing a strictly homogeneous manifold, I will focus primarily on how this role in particular is necessary for mathematical cognition. Second, I did not clearly distinguish between three ways in which intuition can contribute to mathematical cognition: supplying the content for and securing the objective reality of mathematical concepts, grounding synthetic *a priori* mathematical judgments, and justifying mathematical inferences. Third, these three different possible roles of intuition need to be addressed both for *quanta* and *quantitas* in general and also individually for geometry, arithmetic, and algebra.

There are several reasons I was not clearer that should be kept in mind in what follows. As I have mentioned, the aim of *KMW* was to reconstruct Kant's theory of magnitudes as a framework for understanding his account of mathematical cognition and human cognition generally, and I had to set aside the various roles for intuition in mathematical cognition that inevitably draw in the differences between the subfields of mathematics I will discuss in the next book. A second reason is that mathematical cognition is defined by its form and method, which is construction, that is, the *a priori* exhibition of an object or figure corresponding to a concept. Construction supplies content to and establishes the objective reality of concepts of magnitude; I focused on the role of intuition in supplying the content of concepts of magnitude, that is, in representing a homogeneous manifold, because no mathematical cognition at all is possible without that content, and that is already sufficient to establish that intuition plays a necessary, fundamental, and heretofore unrecognized role in mathematical cognition. Finally, Kant himself emphasizes the role of intuition in the construction of mathematical concepts, and as I will argue, he does not clearly distinguish the role of intuition in supplying content to and establishing the objective reality of mathematical concepts from its role in grounding synthetic *a priori* mathematical judgments or its role in justifying mathematical inferences.

5.b The content and objective reality of the concepts of *quanta* and *quantitas*

I begin by considering why the representation of a strictly homogeneous manifold in intuition is required for the content and objective reality of the concepts of *quantitas* and *quanta* in general. Kant's account of the dependence of the representation of *quantitas* on the representation of *quanta* emerges from consideration of what it would be to represent *quantitas* without representing particular, concrete *quanta*. As noted, Heis identifies quantity with *quantitas*, and then *quantitas* with the categories of quantity. The categories of quantity are unity, plurality, and totality; they are at the conceptual foundation of the concept of *quantitas* as a totality of a plurality of units. But at the level of the categories, we have at best a representation of a mereology of unities, pluralities, and totalities with a partial ordering of part within parts, not mathematics; no amount of conceptual reasoning from these mereological concepts will ground even basic mathematical claims. We need first to schematize these concepts to provide them sensible content by means of the schema of *quantitas*, that is number, which is the representation of a successive synthesis of "one to one (homogeneous [i.e., *quanta*])," and is the "unity of the synthesis of the manifold of homogeneous *intuition* in general" (A142-3/B182, my italics). This gives sensible content to the concept of *quantitas* by bringing it into relation to homogeneous manifolds, that is, *quanta*; it thereby establishes the possibility and objective reality of the concepts of both *quanta* and *quantitas*.

While schematizing the concept of *quantitas* secures its content and objective reality, this on its own is not sufficient for mathematics. The theory of magnitudes also crucially depends on the concept of equality, that is the identity of *quantitas* between distinct discrete homogeneous *quanta*; that is, we require a just-as-many relation between collections of units.[22] And, since *quantitas* is the quantitative determination of continuous as well as discrete *quanta*, we need an equality relation among distinct continuous homogeneous *quanta* as well.[23] Thus, intuition is also required to give sensible content to and secure the objective reality of the concept of equality.

I emphasized the role of intuition in supplying content and objective reality to the sensible concept of *quanta* in *KMW* Section 7.4. I pointed out that one could define the concept of a homogeneous manifold using the concepts of quality, identity, and distinctness, but whatever content these concepts might individually have, the concept of a qualitatively identical yet numerically diverse manifold would have no content or objective reality without intuition; we could not have cognition of *quanta*, and hence mathematical cognition.

There is, however, a further important point. While intuition is required for the content of the representation of any *quanta* as such, there is an additional requirement for the representation of *determinate quanta*: the synthesis of composition of the homogeneous manifold. Kant elaborates the role of intuition in mathematical cognition in the *Discipline of Pure Reason*, where he states that mathematics is distinguished by its form and method, which is construction, and to construct a concept is to exhibit *a priori* the object corresponding to it (B714/B742). So far, this leaves it open what role intuition might play, and it has served as the starting point for many investigations into the role of intuition in Kant's philosophy of mathematics, particularly for the role of the singularity and immediacy of intuition, which Kant highlights in the famous *Stufenleiter* passage (A319-20/B376-7). But immediately after characterizing mathematics by its method of construction, Kant states that.

> The form of mathematical cognition is the cause of its pertaining solely to *quanta.* For only the concept of magnitudes can be constructed, i.e., exhibited *a priori* in intuition, while qualities cannot be exhibited in anything but empirical intuition (B714/B742).[24]

Thus, it is not merely that pure intuition is required in some unspecified way; the conditions for exhibiting an *a priori* intuition for a concept entail that only the representation of a strictly homogeneous manifold can be constructed. And construction, in turn, requires the synthesis of composition of a homogeneous manifold (see B201n, B202). The content of any concept of a determinate *quantum* is of a homogeneous manifold synthesized by means of the synthesis of composition, and its objective reality is established by synthesizing such a manifold. In *KMW* I described various more particular roles for intuition, but I emphasized this role because it is the *conditio sine qua non* of any mathematical cognition whatsoever.

Unfortunately, Heis is not impressed by the role of intuition in supplying content, and hence objective reality, to concepts such as *quantitas*, *quanta*, and related mathematical concepts, such as equality, line, circle, and so on; he calls it "small potatoes" (Heis, p.17).[25] He also states that:

[22]I discuss the just-as-many relation and its role in the cardinal features of Kant's concept of number in Sutherland (2017).

[23]I discuss the role of the relation of equality in some detail in *KMW* Chapter 9.

[24]The first sentence of this passage in German is "Die Form der mathematischen Erkenntnis ist die Ursache, daß diese lediglich auf Quanta gehen kann." Note Kant's insistence that mathematics can only "pertain to" *quanta*; he does not allow that it can pertain to *quantitas* on its own, even though he states two paragraphs later that mathematics can construct mere magnitude (*quantitatem*), as in algebra (A717/B745). Even though algebraic magnitude (*quantitas*) can be constructed, it still ultimately pertains to *quanta.*

[25]Heis singles out objective reality rather than content, but as I understand Kant, providing sensible content for these concepts is what establishes their objective reality.

> … one very minimal role for intuition would be to demonstrate the possibility of *quanta*, and thus the objective reality of the concept <magnitude>. This role, though important, is still minimal, since on Kant's view, intuition is required to prove the objective reality of any concept whatsoever, and thus mathematics would be no more an intuitive science than any other (Heis, p. 11).

While it is correct that all sensible concepts ultimately depend on deliverances of sensibility for their content and objective reality—the content of the concept <red> depends on having had the sensation of red, for example—it matters a great deal *which* content is supplied. The content of the concept *quanta* requires the representation of a manifold that is qualitatively identical while numerically diverse, a manifold on which the synthesis of composition can act in order to represent a determinate *quantum* and the size of which can be determined through the schematized concept of *quantitas.* Thus, the role of intuition in supplying this specific content is a necessary condition of any mathematical cognition.

Even if, however, one were to admit that supplying the content to and establishing the objective reality of the concepts of *quantitas*, *quanta*, and equality is big potatoes, and I think it is, I think one ought to continue to press for more clarity concerning the role of intuition by distinguishing between its role in supplying content to and establishing the objective reality of mathematical concepts, its role in grounding synthetic *a priori* mathematical judgments, and its role in justifying mathematical inferences. Although, as I will argue next, Kant does not himself clearly distinguish these roles, a proper understanding and evaluation of Kant's views demands that we do so. For, to begin with the relation between the content of concepts and the grounding of judgments, suppose we have provided the concept of *quantitas* its sensible content through the representation of *quanta*; why does the justification of the *judgments* that constitute the science of *quantitas* require the representation of particular *quanta*? Why cannot we make judgments based on the pure sensible concepts of *quanta*, *quantitas*, and equality without a representation of particular *quanta*? These questions concerning *quanta* and *quantitas* concern all mathematical cognition, but one can, and should, raise these same questions about the more specific account Kant has of geometry, arithmetic, and algebra. I address these questions in what follows.

6. The Role of Intuition: Concepts, Judgments, and Inferences

In the previous section, I defended the claim that the representation of *quanta* in intuition is required to supply sensible content to the concepts of *quantitas* and *quanta* in general. In Section 6.a I briefly consider the same role for the representation of *quanta* in the subfields of geometry, arithmetic, and algebra. The discussion is brief because the content and objective validity of geometrical concepts constitute Kant's paradigm case for the need for the representation of *quanta* in intuition and is the most plausible, and I already discussed the role of intuition in providing sensible content and objective validity to the concepts of arithmetic and algebra in Section 4. In Section 6.2, I turn to the role of intuition in grounding synthetic mathematical judgments, first for *quanta* and *quantitas* in general and then for geometry, arithmetic, and algebra. I then point out a few important additional roles for intuition in mathematical cognition in Section 6.3, before considering some difficulties in distinguishing between the role of intuition in providing sensible content to concepts and grounding synthetic judgments in Section 6.4. Section 6.5 addresses the role of intuition in justifying nonanalytic inferences in mathematics in general and then for geometry, arithmetic, and algebra. We will see that determining Kant's views and making the case for a role for intuition become increasingly difficult; Section 6.6 argues that Kant's views are opaque because he continually returns to the notion of the construction of a concept whenever he argues for the role of intuition in mathematics.

6.a Mathematical concepts

What I have said in Section 5.2 about the role of intuition in providing content to and establishing the objective reality of the concepts of *quanta* and *quantitas* in general carries over fairly directly to Euclid's geometry. It is difficult to imagine how Euclidean proofs could be carried out without employing particular spatial representations of lines, circles, triangles, etc., that is, without representing *quanta.* Moreover, the role of the synthesis of composition of *quanta* is ubiquitous in geometry. Space is an indeterminate *quantum* and determinate spatial *quanta* represented in it are determined through the synthesis of composition of the homogeneous manifold that space provides. The construction of lines and circles at the basis of Euclidean geometry are instances of the synthesis of composition, and without the synthesis of composition of spatial *quanta*, there would be no way to construct the figures required to do geometry. As I mentioned in the Introduction, this is Kant's strongest case for the need for the representation of *quanta* for providing content to and establishing the objective reality of mathematical concepts.

What I said about arithmetic and algebra in Section 4 concerned the content and objective reality of their concepts; as indicated they are more complicated and challenging for Kant than geometry, because they are at a remove from objects in a way that geometry is not and because of symbolic construction. As I noted, the indirectness of the dependence on the ostensive construction of *quanta* in intuition invites challenges to Kant's account, and more work is required to fill it out, but I believe it is Kant's view that they do.

6.b Synthetic mathematical judgments

We can now turn to the following challenge: even if we grant that intuition is required to provide content to and establish the objective reality of the pure sensible concepts of *quanta*, *quantitas*, and equality without a representation of particular *quanta*, why cannot we make judgments involving them without the representation of particular *quanta* in intuition? Kant's answer is that what you can extract from the concepts of *quantitas*, *quanta*, and equality, even with their sensible content, is nothing but propositions grounded in the relations among the concepts contained in those concepts, which will not get you far. I can ground, for example, the *a priori* analytic judgments that any *quantitas* is a totality of a plurality of units, and that equals added to or subtracted from equals give an equal, or that if two magnitudes are equal, their magnitude production [*Größenerzeugung*] is equal (See A164/B204-5). But we will not get much further and will definitely not get to the science of mathematics. That is, even with their sensible content, the judgments one can justify from the concepts of *quantitas* and equality on their own are mere analytic truths; the fact that these truths have sensible content is not sufficient to make them synthetic.[26] And although analytic judgments do play a role in mathematics, one cannot ground synthetic *a priori* judgments in mathematics on a consideration of the relations of the concepts and their contents on their own. In the Introduction of the *Critique of Pure Reason*, Kant states that synthetic judgments must go beyond what is contained in the concepts for their grounding (A6-10/B10-14), and in the case of the synthetic *a priori* judgments of mathematics, that which provides the grounding must be pure intuition. As was the case for Kant's opening description of construction in the Discipline of Pure Reason, this initial framing leaves it open what role pure intuition might play, and here too the framing invites consideration of the roles of the singularity and immediacy of intuition described in the *Stufenleiter* passage. Nevertheless, just as the synthesis of composition of a homogeneous manifold is required to supply the sensible content and establish the objective reality of concepts of magnitudes, it is also required to ground many if not most of the synthetic *a priori* judgments concerning magnitudes.

[26]For example, the propositions that the whole is greater than its part and t that equals added to or subtracted from equals are equal are analytic. I discuss their analyticity in Sutherland (2005).

As I noted, the role of the synthesis of composition of *quanta* is ubiquitous in geometry, which includes an essential role in grounding synthetic judgments in geometry. Kant explicitly states that synthetic *a priori* axioms such as "Between two points only one straight line is possible" and "Two lines do not enclose a space" are grounded in the successive synthesis of productive imagination (A163/B204), which is the synthesis of composition of a homogeneous manifold. The same synthesis of composition of a homogeneous manifold also grounds synthetic *a priori* judgments other than axioms. For example, in Kant's favorite Euclidean proof, Book I Proposition 32, there are several steps in which two angles are represented as together composing a larger angle, and the synthetic *a priori* judgment of this part-whole relation depends on the synthesis of composition of the two angles.

Kant's emphasis on the role of the synthesis of composition in grounding mathematical judgments stands out particularly clearly in arithmetic. In his B15–16 argument in the *Critique of Pure Reason* that 7 + 5 = 12 is synthetic, Kant does not just say that we cannot get the concept of 12 out of an analysis of or from applying the principle of contradiction to the concepts of 7and 5. He states that we cannot get it from analysis of or from applying the principle of contradiction to the concept of the *sum* of 7 + 5. This concept contains "nothing more than the unification [*Vereinigung*] of both numbers in a single one," and the concept of twelve is

> … by no means already thought by my merely thinking of the unification [*Vereinigung*] of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it (B15).

What is required is to seek assistance in the intuition corresponding to the fingers and to add [*hinzutut*] discrete *quanta* in intuition, adding [*antun*] one unit after another through the synthesis of composition. This is what allows us to see what the single number is which "comprehends [*zusammenfaßt*]" 7 and 5.

Kant is arguing that there is a special sort of synthesis even the sensible concept of which is insufficient on its own to allow us to ground the judgment. Uncovering the justification of the judgment requires carrying out the synthesis of composition of discrete homogeneous *quanta.* [27] Kant's argument depends on his assertion that the judgments that one can justify by means of concepts on their own, even concepts with sensible content, is limited. No mere synthesis of concepts together, no mere unification, can represent the composition of a homogeneous manifold, nor can the mere synthesis of concepts in a judgment ground that judgment without appeal to the synthesis of composition of a homogeneous manifold. Kant's argument relies on his claims about the limits of conceptual representation, which is based on the best account of such representation available at the time, but the argument includes a positive account, an alternative explanation rooted in the venerable and at least tacitly accepted framework of the theory of magnitudes that comports with mathematical practice and which thereby supports his claim that 7 + 5 = 12 is a synthetic *a priori* proposition.[28]

I noted in Section 4 that the case for the role of the representation of *quanta* in supplying the content and establishing the objective reality of the content of algebraic concepts is more difficult to see than its role in geometric and arithmetic concepts. Not surprisingly, the role of intuition is also

[27]Kant of course does not mean that we have to carry out the synthesis of composition of unit to unit for every arithmetical judgment token or type. Kant is giving an account of what, ultimately, justifies such a claim. Kant's discussion of 7 + 5 = 12 at B15–16 is a very special case of inviting us to reflect on the justification of a basic arithmetical proposition, not a run-of-the-mill instance of arithmetic.

[28]While addition is the most basic operation on *quanta*, it is important to keep in mind that Kant would think of multiplication as iterated addition and that he has a broad understanding of synthesis that includes decomposition and hence subtraction. See Kant's letter to Schultz of November 25, 1788. See also Marschall (2019) and Rosefeldt (2022) for further discussion.

less obvious in algebraic judgments than in grounding geometric and arithmetical judgments as well. I remarked earlier that the symbols for operations that algebra shares with arithmetic denote the special kind of synthesis characteristic of mathematics, a synthesis of a strictly homogeneous manifold, and hence of *quanta*. Thus, the grounding of algebraic judgments also ultimately depends on the synthesis of *quanta* for their justification. Kant provides evidence for this when states the symbols belong to the mathematical treatment of magnitudes because algebra:

> … chooses a notation for all construction of magnitudes in general and the operations on them, designates the general concept of magnitudes in accordance with their different relations, and exhibits the procedures [*darstellt die Behandlung*] through which magnitude is generated and altered [*erzeugt und verändert*] in accordance with certain rules in intuition . . . (A717/B745).

This suggests that the spatial relations among the symbols for determinate and indeterminate *quantitas*, numerals, and the symbols for arithmetical operations allow them to be manipulated according to rules governing transformations of the symbols and their spatial relations that corresponds to the mathematical relationships among the *quantitas* of *quanta*. I will explain and defend this interpretation in the next book.

6.c Some additional roles for intuition

While I have been focusing on the synthesis of composition of *quanta* in intuition in grounding synthetic *a priori* mathematical judgments, it should be kept in mind that there are other roles for intuition as well. I will cite just three examples. I referred earlier to the representation of the composition of angles in in the proof of Euclid's Book I Proposition 32, which rests on the synthesis of composition of a homogeneous manifold. This judgment also depends on our ability to represent two angles as parts that are identical to the whole they compose, which is made possible by the fact that these determinate spaces are *extensive* magnitudes; that is, it is because the representation of the whole presupposes a representation of its parts that we can represent those parts as identical to the whole. The fact that the homogeneous manifold of space is extensive allows us to represent part/whole relations and ground judgments about them. In contrast, a light has an intensive magnitude, but its manifold is coalesced rather than aggregated by the synthesis of composition, so that the part-whole relations of the intensity are not represented; the intensity of a light is represented as a unity, not as composed of smaller intensities. As I discuss in *KMW* Section 4.1, we can only represent the intensity of the light as having a manifold at all, and hence as a magnitude (*quantum*) at all, by representing the variation of the intensity over time; moreover, our measurement of intensities (i.e., determining their *quantitas*) relies on and leverages the representation of extensive magnitudes.

The intuition of time also plays a crucial role in representing the successiveness of the synthesis of composition underlying number and hence arithmetic (A142-3/B182). Time first gives us the content of the concept of succession; it also gives the synthesis of composition of discrete *quanta* an ordering, which accounts for the ordinal properties of number.[29]

Yet another role for intuition is in our cognition of incongruent counterparts. Starting with the closing paragraphs of his 1768 *Concerning the Ultimate Ground of the Differentiation of Directions in Space*, but especially in his 1770 *Inaugural Dissertation*, Kant emphasizes the need for intuition to distinguish two completely similar figures that cannot be superimposed and to ground judgments about them.[30]

[29]See Sutherland (2017).

[30]See Kant's *Gesammelte Schriften,* Vol. 2: 375-383. I discuss Kant's 1768 argument and the role of intuition in it in some detail in Sutherland (2024).

6.d Concepts and judgments: complications

To bring more clarity to the role of intuition, I have been focusing on the difference between, on the one hand, the role of intuition in supplying sensible content to and establishing the objective reality of sensible mathematical concepts, and on the other hand, in grounding synthetic *a priori* judgments of mathematics, but the distinction between the two is not always easy to draw. The relation between exhibiting an object corresponding to an object and a synthetic *a priori* judgment based on the representation of the object—the relation between "intuition of" and "intuition that"—can be very tight.[31] As an example of something we immediately cognize, Kant refers to the fact that there are three angles in a figure enclosed by three straight lines (A303/B359). Kant also holds that axioms of geometry, such as that three points always lie in a plane, are cognized immediately. That is, mathematics is capable of axioms "because by means of the construction of concepts in the intuition of the object, it can connect the predicates of the later *a priori* and immediately" (A732/B760-1). I noted earlier that the *a priori* exhibition of an object corresponding to a mathematical concept is the *sine qua non* of mathematical cognition, but it is not just a necessary condition of the content of mathematical concepts; it is itself the immediate ground of the justification of at least some synthetic *a priori* judgments.

A judgment such as "7 + 5 = 12" is more complex. The concepts constructed would seem to be the concepts of 7 and of 5, but the construction of these concepts seems to depend in turn on the construction of the concept of unit, whose objects are pure units corresponding to fingers, for example, while the construction of the concepts of 7 and 5 also serve in the construction of the concept of the sum of 7 + 5. Finally, construction of this concept allows us to "see the number 12 arise" (B15-6); the judgment is that the sum of 7 + 5 and 12 are equal, which appears to be based on a recognition of the identity of, on the one hand, the magnitude produced by synthesizing (composing) 5 to 7, and on the other, the 12 that arises.[32] Perhaps Kant would allow a broadening of the notion of construction to allow that judgments can be constructed. While Kant almost exclusively refers to constructing a concept by exhibiting *a priori* an object or figure, in the A-edition of the *CPR* Transcendental Aesthetic, cut from the B-edition, Kant refers to the *a priori* construction of all geometrical principles [*aller geometrischen Grundsätze*] (A24). The distinction between the role of intuition in constructing a concept and grounding a judgment is not so easily discerned; the fact that Kant allows that we can immediately cognize the truth of some synthetic *a priori* judgments in our cognition of a mathematical object makes the role of intuition in constructing concepts and grounding judgments yet more difficult to disentangle. Regardless, I think there is sufficient evidence from Kant's theory of magnitudes for the claim that intuition plays a role in grounding arithmetical judgments by making it possible to represent the synthesis of composition of strictly homogeneous *quanta*. I believe that, in Kant's view, intuition plays a similar indirect role in grounding the judgments of algebra.

6.e Nonanalytic mathematical inferences

I turn now to whether, and if so how, intuition plays a role in mathematical inferences. This is a difficult issue and what I say here will have to be brief.[33] Kant holds that "all the inferences of

[31] I owe this formulation of the distinction to Parsons. See, for example, Parsons (2008), p. 139ff.

[32] This line of though raises a question: is the concept of the equality of 7 + 5 and 12 itself thereby constructed in grounding the judgment? That would connect construction of a concept directly to the construction of a judgment. My recognition of the equality is specific to this particular arithmetical proposition, so it would have to be a construction of the concept of this particular equality (not the concept of equality in general), that is, the equality-of-the-sum-of-7+5-and-12. I do not see any evidence that Kant makes this move.

[33] A fuller treatment would require consideration of logical inferences, and in particular the contrast between what we cognize immediately, immediate inferences, and syllogisms; it also of course requires taking into account the nature of demonstration and its relation to construction and how demonstrations differ from logical inferences. I will not be able to do that here and what I say starts with the texts that are the locus of debates concerning mathematical inference.

mathematicians proceed in accordance with the principle of contradiction [*nach dem Satze des Widerspruchs fortgehen*]," (A10/B14). Some have interpreted this to mean that every mathematical inference appeals to the principle of contradiction alone for its justification; others have interpreted it to merely mean that all inferences must be consistent with the principle of contradiction, or that mathematical proofs include no logical fallacies, because the principle of contradiction is a necessary though not sufficient criterion of all truth (A59/B84).[34] The latter reading allows that some inferences are synthetic and hence depend on intuition. Russell, Beth, Hintikka, and Friedman have all argued that intuition plays a logical role in those inferences, that is, that there are features of mathematical reasoning that can be accounted for in our logic today but which the limited logic of Kant's day could not. There are differing views about which logical roles are at issue and which deficiencies of traditional logic might have motivated Kant, but since Beth, the role of intuition in representing an individual object (corresponding in logical reasoning to existential instantiation) has received particular emphasis.[35] As Parsons has pointed out, these views are compatible with there also being other, nonlogical roles for intuition, which spurred a debate with Hintikka about whether there is also a phenomenological role for intuition.[36] The differences between geometry, arithmetic, and algebra are important for this issue; geometry provides the strongest case for the role of intuition in mathematical inferences, while its role in arithmetical and algebraic inferences is far less clear.

I think there are good reasons to think that, in Kant's view, intuition plays a role in mathematical inferences. Kant distinguishes between apodictic proofs in philosophy and in demonstrations, where the latter are apodictic proofs that are intuitive. Demonstrations require the construction of concepts and hence giving an intuition *a priori* corresponding to them; since mathematics is characterized by its form, which is rational cognition from the construction of concepts, demonstrations are found in, and only found in, mathematics (A734/B762). Kant's emphasis on the role of intuition in this kind of proof strongly suggests that intuitions play a role in mathematical inferences themselves. Moreover, Kant echoes his *Prize Essay* in attributing to these proofs intuitive certainty, that is, self-evidence; he also states that algebra, by means of the signs exhibited in intuition "secures all inferences [*Schlüsse*] against mistakes by placing each of them before one's eyes [*vor Augen gestellt wird*]"; and about mathematics more generally, he states that with the aid of intuition "every false step becomes visible [*wobei jeder Fehltritt sichtbar wird*]" (A734-5/B761-2). These passages suggest not merely a logical role for intuition, but other possible roles for intuition in justifying mathematical inferences. Of course, seeing before one's eyes might be implicated in representing an individual object for a merely logical role of intuition, but the reference to what is made visible suggests more.[37] As I have stated, I believe that once Kant opened the door to the role of intuition in mathematical cognition, he opened it widely, finding multiple roles that it might play. Kant does not, however, give an inventory of those roles or explore them systematically. I believe it highly likely that Kant thought that, for example, intuition supported geometrical inferences that are based on the spatial representation of the relations among the parts of a figure, corresponding to what Ken Manders has called co-exact properties in his work on Euclid's geometry. Manders briefly suggests that Kant may have thought that intuition plays a role in our cognition of co-exact

[34]For the last "no logical fallacies" formulation and the reference to the principle of contradiction as a necessary criterion of all truth, see Friedman (1992), p. 83. Another reading of the phrase "in accordance with the principle of contradiction" has also been proposed, which I will not discuss here; see Hogan (2021).

[35]There is somereason for concern about how explicit Kant's recognition of the limits of the logic of his day was.

[36]See Parsons (2012), p. 102. There is also an underlying difficulty with distinguishing between a logical and a phenomenological role for intuition, since the distinction is defined relative to our notion of logical; for example, the singularity of intuitions playing the role of existential instantiation. Even if Kant was sensitive to the limits of the logic available to him, he might have thought that phenomenological features of intuition were necessary to play the logical role.

[37]See previous footnote.

properties, a view argued for by Lisa Shabel.[38] While I have not contributed to the discussion of diagrammatic reasoning in Kant, I believe that in Kant's view some, but not all, of the geometrical inferences involving co-exact properties depend on the synthesis of composition of a homogeneous manifold.

The role of intuition in arithmetical and algebraic inferences is significantly more challenging, and I will not be able to say much about that here. I will say that, while I am convinced that the intuitive representation of the synthesis of composition of a strictly homogeneous manifold is necessary to supply content to concepts of magnitudes, in grounding synthetic *a priori* judgments about them, and in many fundamental geometrical inferences, I am less sure about its role in supporting the inferences in arithmetic and algebra. This is in part because these subfields are, as I have noted, at a remove from *quanta* in a way that geometry is not. There is the further issue of the grounding of inferences in the spatial relations inherent in symbolic construction. Another, more basic reason I am less sure about the role of intuition in arithmetical and algebraic inferences is that Kant himself is unclear, an issue to which I now turn.

6.f The opacity of Kant's views

Kant does not distinguish and catalog all the ways that intuition contributes to mathematical cognition, for example, in representing a strictly homogeneous manifold, representing succession, and so forth. He also does not clearly distinguish between its contributions to supplying content to concepts, grounding judgments, and supporting inferences. When Kant describes the role of intuition in mathematics, he almost invariably highlights construction – even if not always by that name, then by description. Kant fairly consistently describes construction as representing *a priori* an object or a figure corresponding to a concept, which thereby supplies content for and establishes the objective reality of the concept.[39]

Kant also repeatedly appeals to the representation of objects or figures and often explicitly to construction when discussing the grounding of mathematical inferences. Kant characterizes demonstration in the Discipline of Pure Reason by its reliance on construction, states explicitly that it "derives its cognition from construction [*ihr Erkenntnis aus Konstruction ableitet*]," and discusses both geometrical and symbolic construction (A734/B762). Kant's lectures on logic reinforce the lack of clear distinction between constructions and demonstrations:

> If I do not cognize an object from mere concepts but rather *a priori* in an intuition that corresponds to the concept, then I construct a concept, and this is a demonstration. This proof has evidence in it, i.e., the proof is intuitive (*Wiener Logik*, 24:893, circa 1780.)

It is also noteworthy that in the Discipline of Pure Reason Kant states that the term "demonstration" already indicates that it proceeds through the intuition of the object (A734-5/B762-3). Kant's logic lectures make explicit what he has in mind: "'Demonstration' derives from *monstrare*, to display, to lay before the eyes." These passages echo the appeals to the visual representation of an object or figure from the *Prize Essay* to the *Critique of Pure Reason*.[40]

One might respond to Kant's claims by simply concluding that Kant did not carefully consider the distinction between the roles of intuition in supplying content to concepts, grounding judgments, or supporting inferences. Alternatively, on could broaden the notion of construction to include the construction of judgments, as suggested by the A-edition Transcendental Aesthetic

[38] See Manders (2008b) and Shabel (1998).

[39] In the previous section, however, I noted an exception that appears only in the A-edition of the Transcendental Analytic in which Kant refers to the construction of geometrical principles. I do not think it reflects his view of construction (except possibly by extending the notion to cover *immediate* judgments)

[40] I thank Chris Benzenberg and Clinton Tolley for conversations concerning the proper understanding of construction and demonstration as part of a larger discussion of the role of the faculty of reason (in contrast to the understanding) in mathematics.

passage I cited earlier, though I think at best that would cover *immediate* judgments. And one could expand the notion of demonstration beyond the inferential steps of a proof to include the *a priori* exhibition of an object corresponding to a concept. Even if we attempt to accommodate what Kant says in these ways, however, the distinction between the role of intuition in the three roles we are attempting to distinguish—concepts, judgments, and inferences—will be blurred. It is therefore unsurprising that it is difficult to clarify Kant's views more exactly. I believe this is a reflection of Kant's view that, although the justification of synthetic judgments and nonanalytic inferences in mathematics go beyond what is contained in the concepts involved, their justification nevertheless depends on the intuitive representation of the objects of the constructed mathematical concepts that are included in those judgments and inferences.

My attempt to bring the roles of intuition in Kant's philosophy of mathematics into sharper focus in this section is necessarily incomplete and leaves issues still to be addressed, but I hope it goes some way toward responding to Heis' request for more clarity in anticipation of a second book.

7. The Relation Between Empirical Cognition and Pure Mathematical Cognition

I now turn to a general issue raised by both Heis and Nunez, though in different ways: the relation between empirical cognition, such as cognition of a house, and pure mathematical cognition. Both think that I am, to use Nunez's formulation, "taking the conditions of possibility of experience and those of the exact sciences to be more intertwined than they are in Kant" (Nunez, p. 3 [**XX**]). There are two points at which they press this complaint; the role of strict homogeneity in empirical cognition and my account of construction. Heis objects to the claim that our representation of spatiotemporal magnitudes requires the representation of *pure* spatiotemporal regions. Both Heis and Nunez object to what I say about construction. I will address each point in turn. The upshot of my responses is that the conditions of the possibility of experience and those of the exact sciences are in fact very deeply intertwined.

7.a Empirical objects and strict homogeneity

To begin, a clarification. There are multiple ways to use the term "pure," but I would like to distinguish two. One can use it to mean with no admixture of anything empirical, or one can use it to mean strictly homogeneous, that is, without any qualitative differentiation.[41] The two senses of pure are closely tied in Kant, since the representation of strict homogeneity requires intuition, and intuition, abstracting from what is represented in it, is both a strictly homogeneous manifold and pure in the *a priori* sense. Heis objects to the claim that the representation of spatiotemporal magnitudes requires the representation of pure magnitudes in the sense of strictly homogeneous regions.

To set up Heis' criticism, recall that the principle of the A-edition Axioms of Intuition is that "All appearances are, as regards their intuition, extensive magnitudes," while in the B-edition the principle is that "All intuitions are extensive magnitudes." As I explain in Chapter 3 of *KMW*, the first two paragraphs of the B-edition argue for the claims that all appearances are magnitudes and that they are all extensive magnitudes, respectively. In the first paragraph, Kant defines a magnitude as a homogeneous manifold in intuition. Chapter 7 of *KMW*, a core chapter of the book, is devoted to arguing and providing textual support for two claims: that by homogeneity here, Kant means what I call "strict homogeneity," namely, that the manifold is qualitatively identical while numerically diverse; and that concepts on their own cannot represent a strictly homogeneous manifold, while intuition makes it possible to represent it.

[41] I discuss the two notions of purity in the representation of the pure units of arithmetic in Sutherland (2008).

Heis, however, points out an apparent problem. According to the principle of the Axioms of Intuition, all appearances are magnitudes. But appearances are not qualitatively identical manifolds. So, it seems that "homogeneous" cannot mean qualitatively identical after all. Heis suggests that perhaps on my interpretation Kant should have qualified the principle to state that "all appearances *with respect to their form* are magnitudes," or another similar qualification, such as "all appearances *contain* a homogeneous manifold" (Heis, p. 9 [**XX**]). That is precisely what Kant does in his A-edition formulation of the principle, with his qualification that all appearance are, *as regards their intuition* [*ihre Anschauung nach*] extensive magnitudes. My close reconstruction of Kant's argument in Chapter 3 of *KMW* shows that Kant begins with the claim that all appearances *contain*, *as regards their form* [*enthalten, der Form nach*] an intuition in space and time, which grounds all of them *a priori* (B202). (Note the reference to the *a priori* grounding in space and time here.) He next states that the apprehension of an appearance depends on the "synthesis of the manifold through which the representation of a determinate space and time are generated." Now, the determination of our representation of a house, say, is sensitive to the way in which sensibility is affected (I will return to this point in a moment) , but it is a determination of the manifolds of space and time, which are themselves strictly homogeneous and *a priori*, a determination carried out by means of a synthesis of composition governed by pure *a priori* concepts, namely, the categories of quantity. And Kant closes the argument with the claim that appearances are magnitudes since "as intuitions in space or time they must be represented through the same synthesis as that through which space and time in general are determined" (B203).[42] In short, an appearance, such as a house, with its all its qualities—periwinkle doors and shutters, say—is a magnitude in virtue of the homogeneous manifold of the determinate space and time it contains, that is, in virtue of its shape and duration, abstracting from its qualities.[43] Kant suggests such an understanding of the pure shape of an object in the Transcendental Aesthetic, where he separates from the representation of a body all that understanding thinks about it and all that belongs to sensation to arrive at extension and shape [*Gestalt*], which he says belong to pure intuition (A20-1/B35).[44] And in Section 26 of the B-Deduction, Kant states that that "in making an empirical intuition of a house into perception through apprehension of the manifold … I as it were draw its shape [*Gestalt*]…" (B162), which suggests a close connection to the drawing of a line that Kant repeatedly states is fundamental to pure geometry. Kant also states that motion as a *description* [*Beschreibung*] of space belongs not only to geometry but even to transcendental philosophy (B155).[45] As I understand Kant, our cognition of an empirical object includes a pure component, that is, the strictly homogeneous *a priori* manifolds of space and time, determined in accordance with the pure *a priori* categories of quantity by means of the synthesis of composition, which generates the representation of the determinate space the object occupies for a determinate duration, a synthesis carried out in response to the way in which sensibility is affected. The last is the key point—despite the appearance containing a determination of the pure manifolds of space and time, their determination is responsive to the way in which sensibility is affected, and that is what distinguishes the pure *a priori* construction of mathematical concepts from our apprehension and perception of appearances.[46] And it is insofar as appearances *contain* these pure determinate spaces and times that they *are* magnitudes. In a footnote, Heis adds a further objection to the move from *containing* a

[42]Anticipating the next paragraph, Kant adds to the claim that appearances are all magnitudes the phrase "and indeed extensive magnitudes." For an explanation of this segue, see *KMW* Section 3.2.

[43]As does Kant, I focus on space and shape over time and duration in what follows.

[44]Guyer and Wood translate "*Gestalt*" as "form," which in this context is misleading.

[45]A point repeatedly emphasized by Michael Friedman.

[46]Note that I say that it is responsive to the way in which sensibility is affected; I am avoiding the phrase "what is empirically given in sensibility" in an attempt to skirt entanglement in debates about the Myth of the Given. I think that Kant is clearly committed to the view that sensibility is affected by the thing in itself and that our cognition is responsive to that affection. I return to this issue in Section 10 below.

determinate space and time to *being* a magnitude; after all, he points out, gelato contains eggs but is not egg. True enough for gelato and what it contains. But a careful reading of Kant's argument makes it clear that Kant is making exactly this move; that is, appearances *are* magnitudes *as regards to their intuition* in virtue of the determinate intuitions they *contain.* Not all relations of containment exclude direct predication of this sort; I in some sense "contain" animality, and in virtue of that fact I *am* an animal, regardless of my other properties. If it seems like a stretch for Kant to make this move for extensive magnitudes in general, I think it is because in Kant's view, being an extensive magnitude is the core essential property of appearances.[47]

Nevertheless, Heis is correct that Kant's claim that appearances as regards their form *are* magnitudes can lead to the apparent contradiction he indicates. By magnitudes in the argument of the Axioms of Intuition, Kant means *quantum* in particular. Heis quotes me summarizing a passage from notes on Kant's lectures where I say *quanta* exhibit strict homogeneity. Kant says in that passage that "homogeneity is specific identity with numerical difference, and a *quantum* consists of homogeneous parts," so I was closely following Kant. To make sense of such a claim, we need to understand "*quantum*" to refer inclusively to either a pure homogeneous manifold with nothing empirical mixed in, as in pure geometry for example, or an appearance *insofar as* it is a *quantum*, that is, insofar as it *contains*, *as regards its intuition*, a determinate space and time that is a strictly homogeneous manifold. That is the way I understand Kant.

In sum, our empirical cognition of, for example, a house, empirical though it maybe, is generated through the synthesis of the strictly homogeneous manifolds of pure *a priori* space and time, a synthesis guided by the pure *a priori* categories of quantity, carried out in response to the way in which sensibility is affected. Thus, the conditions of the possibility of experience and those of the exact sciences are indeed thoroughly intertwined, and indeed at the deepest level.

In *KMW* Section 7.9, I address the worry that red and blue blocks of the same size cannot be combined to form a larger volume of blocks because they are not strictly homogeneous, and strict homogeneity is a condition of the synthesis of composition characteristic of mathematical combination. Heis refers to the red and blue blocks as an example in his criticism. I hope what I have said here helps to clarify what I said in Section 7.9 Each of the red and blue blocks contains a strictly homogeneous manifold of space, namely the determinate space each occupies. In combining the blocks to constitute a larger volume, they are combined with respect to the strictly homogeneous spatial extent of the blocks—the volumes of the red and blue blocks combine mathematically to constitute a larger volume *insofar as* they occupy space, while their redness and blueness play no part in their mathematical combination with respect to volume.

Heis worries that allowing that appearances are magnitudes (and more particularly extensive magnitudes), which contain a strictly homogeneous manifold, does away with the need for pure mathematical objects in the science of quantity. I take it he means that we would no longer need to rely on the *a priori* construction of mathematical objects in intuition if we could simply mathematically combine empirical objects. But that is not the case. First, because the determinate spaces contained in an appearance are synthesized in response to the way in which sensibility is affected, they are inexact: a block that was 2 inches on a side even to a very high degree of approximation would still only approximate 8 cubic inches, and a set of five such blocks would only approximate 40 cubic inches. Moreover, the justification of claims about the size of a block or a set of blocks would depend on what is presented in empirical intuition and would only be empirically justified; such cognitions would fail to have the status of pure *a priori* mathematics. The fact that the blocks

[47]It is for this reason fitting that the Axioms are first in the order of principles. Note as well that Kant holds that sensations and the real properties of objects corresponding to them *have* an intensive magnitude. Kant does not make a parallel claim that appearances *are* intensive magnitudes; the real properties corresponding to sensations are not as central to what an appearance is as its being an extensive magnitude.

are combined with respect to the strictly homogeneous *a priori* extensions of space they occupy does not give the combination the status of pure *a priori* mathematics.[48]

7.b The apprehension of appearances and construction

On my interpretation, appearances count as magnitudes insofar as they contain a strictly homogeneous manifold in space and time, and since that manifold is devoid of qualitative difference, the manifold is supplied by the pure manifolds of space and time, which are *a priori.* Moreover, the synthesis of composition of the strictly homogeneous manifold in space and time is the same synthesis upon which pure *a priori* mathematics is grounded. Finally, the synthesis in both cases is governed by the categories of quantity, which are pure *a priori* concepts (A163/B204). It is in this way that the conditions of possibility of experience and those of the exact sciences are in fact at the most fundamental level intertwined in Kant.[49] But pure mathematics is rational cognition from the *construction* of concepts, which is an *a priori* exhibition of an intuition corresponding to a concept. Thus, the role of *a priori* intuition, pure *a priori* concepts of the understanding, and the synthesis common to both t could be taken to suggest that the synthesis of composition underlying appearances is also an instance of construction.

This would be a mistake. Nunez and Heis rightly call me out for identifying the synthesis involved in the apprehension of an appearance with construction in a few footnotes. Both Nunez and Heis point out that I claim that the cognition of a determinate space or time requires construction in intuition (*KMW* 45n34), though the cognition of an appearance, which includes the cognition of a determinate space and time, does not allow of construction. For the most part, I am more careful. For example, in §26 of the B-Deduction, Kant states that in the perception of a house through apprehension of its manifold "I as it were [*gleichsam*]" draw its shape in agreement with this synthetic unity of the manifold in space, strongly suggesting a close connection to the construction of figures in geometry; he then ties the synthetic unity to "the category of **magnitude**, with which the synthesis of apprehension, i.e., the perception, must therefore be in agreement" (B162). As Nunez points out, I emphasize that Kant only refers to an "as it were" drawing of the house (p. 44n30). Yet in a subsequent footnote I erroneously refer to the construction of the house (p.47n36). *Mea culpa.* While these are only a few slips, both Nunez and Heis are right to take them as revealing. I think of the synthesis of apprehension as containing an "as it were" construction, corresponding to an "as it were" drawing of the shape of a house, which is responsive to the way in which sensibility is affected by the thing in itself. But I was not clear enough about the distinction between "as it were" construction and construction proper, and I thank both Nunez and Heis for the correction.

In pressing me on this issue, Heis and Nunez each identify a feature of construction that is absent in the case of the apprehension of an appearance. Heis does not focus on Kant's characterization of construction as the *a priori* exhibition of an object corresponding to a concept. Instead, he says that "[c]onstruction. … "is the kind of cognitive procedure … in a demonstration. … that makes it possible to draw a general conclusion from the properties of a concrete but representative individual" (Heis, p. 12 **[XX]**). Heis thus broadens the requirements of construction to include what would be required to serve this crucial role in demonstrations. Citing Kant's discussion of definitions in the Discipline of Pure Reason (A727-31/B755-9), Heis claims that a "concept can be constructed only if it possesses a (real) mathematical definition" (Heis, p. 12 **[XX]**). This isn't quite what Kant says; Kant's point is that the only concepts that can be defined are those that are "electively thought" [*willkürlich gedacht*], by which he means that they are deliberately made up

[48]This is not the end of the story, for there are further issues about the nature of applied mathematics in light of this interpretation. The calculation of the volume of a block or group of blocks counts as mathematical in virtue of the pure mathematics underlying the calculation.

[49]And it is for these reasons that mathematics applies "in all its precision" to appearances (A165/B206).

[*vorsätzlich gemacht*].[50] That is not yet sufficient, however; a defined concept must also define a true object [*wahren Gegenstand definieren*], which requires that the object and its possibility be given through the elective concept, and the only way to do that is to exhibit *a priori* the object corresponding the concept, that is, to construct the concept (A729-30/B757-8). Thus, Kant's point here is that mathematical definitions require construction, not the other way around. Definitions properly so called are crucial to mathematical demonstrations; because I make the concept, I know what I wanted to think in the concept; it contains just that which the definition would think through it. (A729/B757; A731/B759). Kant here is echoing a theme he already sounded in 1763 in the *Prize Essay* (2:290ff). Knowing what you think in the defined concept is essential for drawing a general conclusion from a concrete representative individual in a mathematical proof; otherwise, you might surreptitiously draw general conclusions from features of the individual that do not follow from the defined concept, for example, drawing a conclusion about triangles that only holds for isosceles triangles. But Kant does not argue that a concept can be constructed only if it possesses a real mathematical definition. That is why in the *Metaphysical Foundations of Natural Science* Kant can construct the concept of matter, which is a concept given in experience.[51] And it is well that Kant does not hold that only concepts with real mathematical definitions can be constructed, since Kant claims that "only the concept of magnitudes can be constructed, i.e., exhibited *a priori* in intuition" (A714/B743), and the concepts of magnitude are grounded in the categories of quantity, which are *a priori* given concepts that do not admit of definition, only exposition (A728/B757).[52] Heis rejects the claim that the concept of *quantum* can be constructed for the reason that it has no real, mathematical definition (see Heis, fn. 12 [**XX**]), but this is based on the mistaken claim that only defined mathematical concepts can be constructed and flies in the face of Kant's explicit assertion that the concept of magnitudes and only the concept of magnitudes can be constructed.[53]

Nevertheless, Heis is correct that the perception of an empirical object, such as a dog, does not involve construction. When I perceive a dog, I determine the space it occupies, thereby generating a determinate space; I "as it were" draw its shape, that is, I "as it were" construct it. As I described in the previous section and alluded to in this section, the generation of a determinate space is in response to the way in which sensibility is affected—and that is what makes it an only "as it were" construction. That is, and this is the crucial point, the exhibition of the object is not *a priori*, because it is responsive to what is empirically given—despite the fact that the determination of the space is effected by the same synthesis employed in mathematical cognition, the strictly homogeneous spatial manifold that is synthesized is *a priori*, and the synthesis is guided by pure *a priori* concepts. I described the responsiveness of apprehension to what is empirically given and contrasted it with the elective construction of concepts in mathematical cognition in *KMW*, but it was embedded in a discussion of nonconceptualism and did not make explicit reference to construction.[54] I was not nearly perspicuous enough in *KMW* about the responsiveness of apprehension to the way in which sensibility is affected.

Nunez highlights a different feature of construction that is absent in the apprehension of appearances. He distinguishes between mathematical construction and the empirical apprehension of, say, a house by appealing to the fact that the former is elective [*willkürlich*] and hence a matter of

[50]I will say more about the term "elective" shortly.

[51]One can do so by analyzing the concept of matter and making use only of what is found in it that relates to the pure intuitions of space and time (4:472).

[52]See also A240-1 and A241n, where Kant states that the categories, in contrast to mathematical concepts, cannot be defined. (The significance of these passages could be questioned, however, since they were dropped in the B-edition.)

[53]I am not arguing that the construction of concepts that are not, in Kant's strict sense, defined is completely unproblematic; there are puzzles raised by Kant's views that date back to 1763. Also, Nunez (2014) argues that the categories are in fact capable of real definitions. Unfortunately, I cannot pursue these issues further here.

[54]I address the difference between empirical and mathematical cognition in *KMW* Sections 4.5.2, and also in 5.5.2 in a discussion of the Kantian conceptualist/nonconceptualist debate. I will return to that debate next.

the will in a way that the latter is not (Nunez, p. 5 [**XX**]). It has always been clear to me that construction *in the proper sense* is restricted to pure mathematical cognition, is thoroughly *a priori*, and is elective; in fact, I have emphasized the elective nature of construction, arguing for the use of the term "elective" over "arbitrary."[55] Thus, Nunez and I are in full agreement. My view is that mathematical construction is elective *because* it is *completely* pure, in the sense that it includes no influence of the way in which sensibility is affected by the thing in itself, and without such influence the synthesis could only be the result of my free choice.[56] In contrast, Nunez foregrounds the role of the will in construction, rightly noting that the fact that mathematical cognition and apprehension employ the same synthesis should not obscure the difference in what drives the synthesis. We agree, however, that describing construction as elective distinguishes the complete purity of mathematical construction from the exercise of the same synthesis in the apprehension and perception of empirical objects. When that synthesis is exercised in apprehension and perception, it responds to the way in which sensibility is affected by the thing in itself.[57]

8. Strict Homogeneity: Further Clarifications

I responded earlier to Heis's concern that qualitatively distinguishable magnitudes such as red and blue blocks do not require the representation of strict homogeneity as a condition of their composition. Nunez raises a few additional worries one might have about the claim that mathematical cognition requires strict homogeneity, which gives me a chance to further clarify its role. The first concerns how the account can accommodate the qualitative features of mathematical objects, for example, that the diagonal of a cube is the cube root of the sides, or that 12 has the distinctive property of being the sole sum of 5 + 7 (Nunez, p. 2 [**XX**]). As Nunez puts it, these objects have specific differences and hence qualities and hence are also conceptualizable. As a result, mathematical objects can be distinguished conceptually by their qualitative differences—instances of cubes can be distinguished from other kinds of solids and 12 can be distinguished from other numbers, for example. But if concepts can be used to distinguish mathematical objects, we have no need for intuition to make it possible to distinguish them.

The claim is not, however, that the only way to distinguish any already conceptualized mathematical object is by means of intuition, that is, by distinguishing distinct parts or locations of a qualitatively indistinguishable manifold. Nor is the claim that you cannot conceptualize mathematical properties that make essential reference to strict homogeneity. With the aid of intuition, we can certainly form concepts, such as the concept of the diagonal of a cube, that depend on intuition for their content; the role of intuition in representing a strictly homogeneous magnitude is at a deeper level than this concept formation. The claim is that concepts on their own, without the aid of intuition in representing strict homogeneity, cannot represent *quanta* such as a cube at all, and hence cannot represent the synthesis of composition which is characteristic of them and which allows both their representation and their mathematical treatment. Nunez finishes this line of argument with the claim that "it looks like each in-itself identical part of space and time will

[55]See Sutherland (2010), where I first proposed that 'elective' was a better translation of "*willkürlich*" than "arbitrary," which to that point had been the standard translation (and often still is). Although "arbitrary" means decided by a person's choice, it has also come to connote being chosen seemingly at random or on a whim rather than in a reasoned or methodical way (see the Oxford English Dictionary Online and the Merriam-Webster Dictionary online entries for "arbitrary," both consulted March 23, 2026). Thus, to my ear "elective" better captures the connotation that the synthesis is a matter of the will and free choice without suggesting that it is random or haphazard.

[56]A similar point holds for the electively synthesized marks in a mathematical definition, discussed earlier.

[57]In a recent review of my book, Katherine Dunlop gives a fine-grained and penetrating analysis of my interpretation of the sameness of synthesis of mathematical and empirical cognition (see Dunlop, 2025). I think her core concerns parallel those of Heis and Nunez; that is, that I have drawn mathematical cognition and empirical cognition too closely together. She worries, for example, about what counts as figurative synthesis. I believe further clarifications in line with my reply to Nunez and Heis would assuage those worries, though I cannot give them here.

have properties that are specific to it." This is missing the heart of the view. Each of the parts of space are self-identical, but the point is that they are all *qualitatively* identical, and that concepts cannot represent qualitatively identical manifoldness without the aid of intuition. With the aid of intuition, however, we can represent the manifold by means of concepts. And the possibility of cognizing the measure of the diagonal of a cube (the *quantitas* of this line segment relative to one of the sides) already depends on the fact that the line is composed of a strictly homogeneous *quantum*. Mathematical concepts are ultimately concepts of magnitude, which are sensible concepts, and depend on intuition for their content.

Consider the most general case I considered in *KMW* Section 5.2. One might think that you could form a concept of a strict homogeneity—a concept of a manifold (a manyness or a muchness thought through the category of plurality) that is numerically different without being qualitatively different—by unifying concepts that do not essentially refer to intuition into one concept, for example, by employing concepts of plurality, difference, quality, and negation. Let us suppose for the sake of argument that were possible. Note first, however, that the concept is defined with a negative condition, namely, "a manifold without qualitative difference" and that without appeal to intuition, there is nothing that answers to this concept. More importantly, however, it would at best have a logical content; unless it is given content by its reference to intuition, we cannot grasp what it would be to be a strictly homogeneous manifold, because concepts on their own are incapable of representing such a manifold. Yet intuiting a strictly homogeneous manifold allows you to reflect a concept from the content of your intuiting to acquire content for that concept. All mathematical concepts are ultimately concepts of magnitude, and all concepts of magnitude are sensible concepts whose content makes essential reference to intuition, and in particular, to their property of being a homogeneous manifold.

Similarly, the concept of the ratio of a diagonal of a cube to its side is a pure sensible concept that makes essential reference to a strictly homogeneous manifold. The ratio of a diagonal of a cube to its side is conceptualizable, but only because we can represent the cube and the diagonal in intuition by means of representing a strictly homogeneous manifold, which is required to represent a line at all. And the same applies to the concept of 12 and its relation to 7 and 5 . On Kant's understanding, the concept of 12 ultimately makes essential reference to a manifold of units as such, that is, units that are not qualitatively distinguished. Moreover, the concept of addition makes essential reference to a kind of composition that concepts can acquire through reflection on the synthesis of composition of a homogeneous manifold. Finally, the concept of equality makes essential reference to identity of quantity without qualitative difference. Again, these are pure sensible concepts whose content includes essential reference to strictly homogeneous manifolds.

Nunez reformulates his question in another way, but I take him to be raising a related but distinct question. He asks why the fact about cubes that the diagonal is root three of one of its sides is a synthetic truth. Why is it not a part of the concept cube? Similarly, why is the unique fact that 12 = 5 + 7 contained in the concept of 12 synthetic? Why is not this contained in the concept 12? (Nunez, p. 3 [**XX**]).

The answer is that these are propositions about the diagonal of cubes and the number 12 that make claims about *quantitas*, which concern how big something is. As I discussed earlier in Section 6, in the arithmetical case, the bare *concept* of unifying 7 + 5 is not enough to ground the proposition concerning its relation to 12; you need to *compose* 5 to 7 and that requires a synthesis of composition of a strict homogeneity. Concepts on their own cannot represent the special sort of composition grounding this proposition; all we could think through concepts alone is the unification of concepts with each other, which is the result of a quite different sort of synthesis. Of course, once we compose magnitudes, we can form a concept of the synthesis of composition of a homogeneous manifold, but this concept is subsequent to the synthesis; its sensible content depends on synthesizing such a manifold.

This is not to say that Kant's understanding of the quality/quantity distinction and the qualities of *quanta* and *quantitas* are perspicuous. In fact, I think that Kant uses the notion of quality in two

ways, one that contrasts with the notion of *quantum* and the other that contrasts with *quantitas*, the latter of which is closely related to the Leibnizian–Wolffian contrast between quality and quantity. Kant's views are hard to untangle, which might be an artifact of his attempting to reform the Leibnizian–Wolffian theory of quantity while also reworking the Euclidean theory of magnitudes. Sections 8.3 and 8.4 of *KMW* were my best attempt to address these issues, while Section 8.5 directly considered the qualities of magnitudes, such as why Kant considers the straightness or curvature of a line to be a quality, which bear directly on Nunez's worries.

9. Equality

Anderson asks me to say more about the relation of equality and the role that intuition plays in allowing its representation. To begin, there are two sorts of equality in the broad sense: the equality of continuous magnitudes, such as volumes, and the equality of collections of things regarded as units, that is, equimultiplicity. Kant treats such collections as discrete magnitudes, so equality in the broad sense covers both continuous and discrete magnitudes.

As I discuss in Chapter 9, the relation of equality is crucial to bridging the gap between our cognition of the mereology of magnitudes grounded in the categories of quantity (unity, plurality, and allness), and a full account of the mathematics of magnitudes. The part-whole relation allows a strict partial ordering of magnitudes of a particular kind, such as volumes, with respect to size: one magnitude is greater than another if the latter is a fully contained proper part of the former; it is partial because it does not order disjoint or merely overlapping magnitudes of the same kind. Kant, however, states that one magnitude is greater than another if the latter is *equal* to a proper part of the former, which can be used to induce a total ordering of magnitudes of a particular kind with respect to size. Moreover, as I mentioned in the *Précis*, the relation of equality is the linchpin of what I call the "general theory of concrete measurement," which I argue in Chapter 9 is a tacitly presupposed theory supporting both Euclidean geometry and Kant's philosophy of mathematics. The relation of equality plays three roles in that theory: first, in the composition of *equal* units in measuring; second, in the *equality* between the totality of composed equal units and what is measured; and third, in adding equal units of one magnitude an *equimultiple* number of times that one adds equal units of another magnitude. This last role makes not just simple measurement but the Eudoxian theory of proportions possible, which in turn makes the mathematical treatment of magnitudes possible.

In Chapter 8, I argue that Kant accepts the traditional philosophical definitions, found in his rationalist predecessors of equality as identity of quantity and similarity as identity of quality, but that he radically revises it by introducing his distinction between *quanta* and *quantitas* and arguing that the former is a strictly homogeneous manifold that requires intuition for its representation. Given the importance of the relation of equality, I consider, in Section 9.8, where the relation of equality might best fit into Kant's architectonic scheme, speculating that equality defined as the identity of quantity might belong to the concepts of reflection, in particular to the concepts of identity and difference, as described in the Amphiboly of the *CPR*.

But there is also something more substantive to be gleaned from the Amphiboly with respect to equality. I have placed great weight on the Amphiboly discussion of identity and difference, which I discuss in *KMW* Section 7.5; Kant argues in the Amphiboly against Leibniz's principle of the identity of indiscernibles, and on the basis of what Kant says there, I have argued that because intuition is a qualitatively identical manifold, intuition allows us to represent bare numerical diversity. As a consequence, intuition makes it possible to represent two *quanta*, insofar as they are homogeneous manifolds represented intuition, as numerically diverse despite having the same *quantitas*, that is, being quantitatively identical, and hence equal. As mentioned earlier, all appearances *as regards their intuition* are extensive magnitudes; they may be discernible with regard to some qualities or other, but intuition is what allows us to represent them, merely insofar as they are *quanta*, as numerically diverse while also being equal. The representation of equality

among magnitudes is, of course, fundamental to their mathematical treatment, and as I indicated earlier, plays three roles in the Euxodian theory of proportions.

Whether arithmetical or algebraic equations such as $7 + 5 = 12$ or $a = \sqrt{b}$, which also concern *quantitas*, are statements of identity or equality is a further issue; it depends on the relation between symbolic and ostensive construction and the level of abstraction expressed in these equations, an issue I plan to address in my next book. Regardless, the representation of equality is fundamental to mathematical cognition.

10. Kant's Theory of Magnitudes and Kantian Nonconceptualism

In Chapter 4 of *KMW*, I give an interpretation of extensive magnitudes that avoids a threat of an infinite regress in our representation of them. I point out that my account undermines an argument nonconceptualists have used to argue for their position, but I studiously avoided saying more about the conceptualist/nonconceptualist debate. Despite my efforts, Anderson wants to draw me in and presses me to say more about the implications of my interpretation for the debate.

There has been a remarkable amount of work on the specifically Kantian version of the conceptualist/nonconceptualist debate in the last decades, which has included disagreements about what conceptualism and nonconceptualism means or should mean in thinking about Kant's philosophy. It has connections to and was influenced by, while differing from, the version found in the contemporary philosophy of mind (see Allais, 2016). Even the specifically Kantian version of the debate has also transformed from its origins in Wilfred Sellars' articulation of the Myth of the Given. While all agree that concepts are required for cognition, there were unclarities or differences of opinion about whether the locus of the debate concerns perception or intuition; the consensus appears now to be focused on intuition, though Kant's claims concerning apprehension and perception still play a role in claims about intuition. There have been different notions of object in play, as well as different notions of content. The debate has also included several ways of drawing a distinction between strong and moderate conceptualism (Land, 2015b, 2016) or between conceptualism and intellectualism (McClear, 2016). One could defend a moderate conceptualism according to which intuitions require a sensible synthesis which in turn requires spontaneity and the activity of the understanding, whether or not that involves concepts. And a nonconceptualist could argue for the claim that intuition does not constitutively require concepts, even if in fact all human intuitions are transformed by concepts (see Allais, 2016). Both the conceptualists and the nonconceptualists have refined and moderated their claims and the positions are not as far apart as they once were, though room for substantial disagreement remains. One might shun an attempt to enter the fray, and Nunez praises me for avoiding it in my book. Yet, I agree with Allais that though the debate has not been definitively settled, it has occasioned a more nuanced understanding of Kant's views, and I believe that connecting the controversy with Kant's theory of magnitudes would be mutually beneficial. Unfortunately, there was no way for me to do justice to the lively debates in my book, much less for me to do so here, but I will venture to mention four ways in which Kant's theory of magnitude bears on it.

The first concerns the resolution to the threat of infinite regress in Kant's account of extensive magnitudes, which I addressed in Chapter 4 of my book. In a nutshell, Kant states that an extensive magnitude is one in which the representation of the parts makes possible the representation of a whole and therefore necessarily precedes the latter, and those parts are synthesized through a successive synthesis of productive imagination. But if each of these parts is in turn an extensive magnitude, then they require a representation of their parts, and so on. The resolution draws on the distinction between determinately and indeterminately represented intuitions. An extensive magnitude is a determinate intuition, which is the result of the successive synthesis, while the synthesized parts are only indeterminately represented.

There are, however, different understandings of these indeterminate intuitions. According to Lucy Allais and Sasha Golob, the need for indeterminately represented parts provides an argument

for a kind of intuitive atomism and hence for nonconceptualism, because in their view the parts must be represented particulars prior to being synthesized in order to serve their function.[58] I argue, however, that this misconstrues the nature of these indeterminately represented parts and their synthesis. The latter is, in the first instance, a continuous synthesis of a continuous homogeneous manifold starting from a point, in which no particular parts are represented or presented.[59] I argue that Kant's discussion of the continuity of magnitudes strongly supports this reading and in fact explains the applicability of the calculus to appearances, which is crucial for the foundations of natural science. A proper understanding of Kant's theory of magnitudes and their mathematical character therefore undercuts one argument for nonconceptualism, but I was careful to state in the book that doing so did not settle the conceptualist/nonconceptualist debate, and I refrained from saying more; what follows is the more for which Anderson asks.

The second consequence of Kant's theory of magnitudes is a broader point: it introduces a complication that alters the framework of the debate. Kant thinks that a fundamental role of intuition is to allow us to represent a continuous homogeneous manifold whose parts are only indeterminately represented, but the role of intuition in the representation of such a manifold appears to be at odds with the singularity of intuition, which is widely construed as requiring a representation of a singular object, that is, a particular individual. Taking the case of an outer representation, it requires a representation that differentiates a spatial object from the surrounding space and from other objects. This understanding of the singularity of intuition is quite natural given Kant's focus in the *Critique of Pure Reason* on the conditions for the cognition of objects of experience, a robust notion of object the cognition of which requires a great deal in Kant's account. I argue in Chapter 5, however, that the singularity of intuition is not incompatible with the representation of a homogenous manifold because the singularity of intuition is best understood as a way of representing singularly, which allows for the representation of a continuous homogeneous manifold—similar to the way in which a map can represent a continuous expanse of space without a determinate representation of the parts of the expanse. This representation is more basic than, and in fact underlies and makes possible, the singular representation of a singular object; in fact, it is the ultimate source of the singularity of the representation of the singular object.

This understanding of the place of the representation of a homogenous manifold obviously has direct relevance for understanding Kant's account of the conditions for cognizing singular objects. It is also directly relevant to the conceptualist/nonconceptualist debate, which has almost entirely focused on the conditions for the singular representation of singular objects, where singularity is usually construed as the representation of an object distinguished from its surround and other objects. Since there is a role for intuition in representing a homogeneous manifold that is at a more basic level than the representation of singular objects, the conceptualist and nonconceptualist alike must address whether and how concepts might be required at each level. It thus adds further complexity to the debate. One could be a nonconceptualist about intuition's role in representing a continuous indeterminate homogeneous manifold, but a conceptualist about the representation of singular objects in it, or one could either be a nonconceptualist or a conceptualist about both. A few participants in the conceptualist/nonconceptualist debate have taken this further complexity into account, in particular, Emily Carson and Thomas Land; I will return to their work below.

The third way in which Kant's theory of magnitudes bears on the debate concerns its origins in Wilfred Sellars. While I think the recent versions and refinements of Kantian conceptualism and nonconceptualism are of value in their own right, the history and origins of the debate, tracing back

[58]See Allais (2015), pp. 149–150, who cites Golob (2011).

[59]I say "in the first instance" because the representation of a determinate *quantum* is more basic than and does not require determination of the *quantitas* of the *quanta*, that is, how big it is according to some measure. Moreover, not all synthesis of the composition of magnitudes is continuous; Kant also includes a synthesis of discrete magnitudes into collections. In fact, the synthesis of discrete magnitudes in collections underlies measurement and hence the determination of the *quantitas* of both continuous and discrete *quanta*.

through many recent writers and futher back through the work of Robert Hanna, John McDowell, and Wilfred Sellars, is also of philosophical interest. In Chapter 5, I claim that the singularity of intuition is best understood as a way of representing singularly; in a recent article, I have further defended that claim and criticized Sellars' understanding of Kant.[60] My analysis relies on the distinction between intuiting and intuited in a manner similar to Sellars, but the position I argue for conflicts with Sellar's criticism of Kant's account of intuition and the form of intuition. Sellars believes that Kant uses "intuition" ambiguously for a this-such intuition, on the one hand, and an impression–intuition, on the other.[61] Sellars construes the former as representing singular objects, such as a "this-cube," which requires intuition for reference to a "this" and a sortal concept for the "such." Impression–intuition, in contrast, is a more basic notion of intuition that corresponds to a manifold of sense impressions; it belongs to sheer receptivity and is nonconceptual and supplies the brute fact or constraining element of perceptual experience. Those familiar with the Myth of the Given will see where this is going—Sellars thinks Kant errs in countenancing the notion of an impression-intuition.

I argue that Sellars is simply mistaken in identifying a more basic notion of intuition with a manifold of sense impressions. But what is important for present purposes is Sellars' further argument. Sellars thinks that Kant's ambiguity in the notion of intuition is reflected in two corresponding notions of *form* of intuition. Sellars argues that construing space as the form of intuition in the basic sense is incoherent and nonsense. His argument ultimately rests on the view that the more basic notion of intuition is non-representational, so that "intuition" in the basic sense can only refer to mental states as such (as opposed to representational contents); since mental states are not spatial, space cannot be their form. Sellars' argument depends on the idea that having objective purport is only achieved with this-such intuitions of singular objects, which involve concepts. There is no room in Sellars for the idea that intuition allows us to represent a homogeneous manifold singularly, which is what in turn allows us to represent singular objects.

More is required to fill out this critique of Sellar's way of reading Kant, but the upshot is that Sellars thinks that this-such intuition is the only coherent notion of intuition in Kant, and because he assumes that the singularity of intuition consists in the representation of a singular object, and hence a this-such, it already involves concepts. Sellars overlooks the more basic role of intuition in singularly representing a homogeneous manifold, whose form is to represent singularly. Sellars may wish to argue that the possibility of singularly representing a homogeneous manifold, insofar as this is representational and has objective purport, requires concepts. Kant's manner of speaking about intuitions at least strongly suggests that they have objective purport, that is, that intuitions as such have intentional content. If one thought that having objective purport itself requires concepts, then conceptualism would entail that the role of intuition in representing a homogeneous manifold already in some way involves concepts. But that argument depends on the conditions for having representational content at all, not on the conditions for the representation of a singular object as a this-such.

The fourth way in which Kant's theory of magnitudes bears on the debate concerns whether the role of the categories in mathematical cognition provides an argument for conceptualism. In pure mathematics, the representation of a mathematical object such as a triangle requires construction of a concept of magnitude, that is, exhibiting *a priori* an intuition corresponding to the concept. So, concepts are required for any intuition of a mathematical object, and conceptualism holds for the objects of pure mathematics.

At an earlier point in my work, I thought that the thorough intertwining of the conditions of possibility of experience and those of the exact sciences provided a possible argument for conceptualism about the intuition of empirical objects as well. I was further influenced in this direction by

[60]See Sutherland (forthcoming).

[61]The labels "this-such intuition" and "impression–intuition" are mine.

the thoughtful work of Emily Carson and Thomas Land, both of whom were arguing for a form of conceptualism based in part in reaction to my work on Kant's theory of magnitudes.[62]

While writing Chapter 5 and reflecting on Allais' nonconceptualism, however, I became less sanguine about this line of thought; I was also influenced at that time by exchanges with Tyler Burge and Matt Boyle, who convinced me that I was reading too much into empirical cognition under the influence of my focus on Kant's philosophy of mathematics. I did not leave room for the possibility that in the case of empirical cognition, my representation of a determinate intuition is responsive to an indeterminate intuition that presents not merely a continuous indeterminate homogeneous manifold, as in the case of pure *a priori* mathematical cognition, but *presents a singular object* in that manifold in virtue of the affection of sensibility by the thing in itself. The continuous synthesis of the manifold in generating a determinate empirical intuition would, in the first instance, still be a continuous synthesis of the homogeneous manifold, but it would be responsive to the singular *object* presented in it.[63] The presentation of this singular object would be indeterminate insofar as it has not yet been brought under the synthesis of composition under the categories of quantity that generates a determinate intuition of the object. This opens the door for a nonconceptualist interpretation of empirical intuition, and makes room for different notions of objects here, whether "perceptual particulars," or particular property instances located in particular regions of space.[64] The synthesis of the continuous indeterminate homogeneous manifold would in either case be carried out in response to the indeterminate intuition of a singular object presented in it. I discussed this possible nonconceptualist position in *KMW* Sections 4.5.2 and 5.5.2; I prescinded from endorsing it myself in an effort to stay out of the fray, but it is a position that I think is reasonable to defend and in fact one to which I am sympathetic.

Conceptualists of all sorts will balk, of course. They might argue that intuition of any sort of singular object or perceptual particular is a representation of a this-such and hence requires concepts, in line with Sellar's view. Or they might argue that any objective purport whatsoever requires concepts—or, adopting a weaker version of conceptualism, at least a spontaneous synthesis of the understanding. The real upshot of all this is that both the conceptualist and nonconceptualist need to take into account that the most basic role of intuition is to represent a homogeneous manifold rather than a singular object, but at least as far as I can see right now, I do not think that Kant's theory of magnitudes settles these questions one way or another.[65]

11. Determinate and Indeterminate Magnitudes

Nunez calls into question my interpretation of indeterminate magnitudes and the distinction between them and determinate magnitudes (Nunez, p. 5 [**XX**]). I briefly review my account before addressing Nunez's specific complaint. Although in a few instances Kant refers to intuitions as determining, in virtually all contexts determination is an act of the spontaneity of the understanding, and moreover it almost always employs concepts.[66] In Chapter 2 of *KMW*, I make much of the distinction between indeterminate and determinate intuitions and magnitudes, using it to explain

[62]Emily Carson's paper presented in Tel Aviv in 2009 and subsequently published as Carson (2020), as well as Thomas Land (2014), shaped my views. See also Land (2015b). I was also influenced at that time by conceptualist readings of the strategy of the B-edition Transcendental Deduction.

[63]Thus, the responsiveness to what is empirically presented in intuition I described in Section 7 is here construed as a responsiveness of a singular *object* presented in intuition.

[64]These suggestions correspond to those mentioned in Allais (2016) and McLear (2016).

[65]I presented my interpretation of the role of intuition in representing strictly homogeneous manifolds at the University of Pittsburgh in 2012. At least as I presented my views at the time, John McDowell stated that he did not think that they entailed either conceptualism or nonconceptualism. I disagreed then, but now I agree.

[66]An exception is in the famously fraught footnote at B160-1, where Kant suggests that the understanding determines sensibility in a manner that precedes all concepts. Kant also a few times refers to intuitions as determining, but these are exceptional instances.

the different treatments of both space and time and of mathematics in the Transcendental Aesthetic and the Transcendental Analytic. I argue that the discussion of space, time, and mathematics in the Transcendental Aesthetic abstracts as far as possible from the contributions of the understanding, and hence from the determining activity of the understanding under concepts, while it is precisely the task of the Transcendental Analytic to take those contributions of the understanding into account. Thus, when Kant states in the Transcendental Aesthetic that space and time are magnitudes, he has in mind indeterminate magnitudes, while the Axioms of Intuition concerns determinate magnitudes. I argue in Chapter 2 of *KMW* that in the B-edition of the formulation of the Axioms principle, it would have been more precise for Kant to have said that all *determinate* intuitions are extensive magnitudes.

The determination of intuitions in the representation of a determinate magnitude (*quantum*) is effected through a synthesis of composition of the homogeneous manifold of intuition in accordance with the categories of quantity. In the first instance, the composition is a continuous synthesis of a continuous and homogeneous manifold, paradigmatically exemplified in the continuous generation of the representation of a line carried out in the continuous homogeneous manifold of space.[67] The line is a determinate magnitude because it is determined through this synthesis and because the line is contained within boundaries—the starting point and endpoint of the synthesis—and these boundaries are themselves determined through the synthesis. But, and this is crucial, it is not having boundaries which alone constitute its being a determinate magnitude; it is the fact that the understanding runs through the manifold in an act of continuous synthesis. At the same time, no particular potentially representable parts of the line are determinately represented; that is, none of its potentially representable parts are themselves determinate magnitudes. This is what avoids the infinite regress in the representation of extensive magnitudes, as I discussed earlier.

In contrast to the representation of determinate continuous *quanta*, the representations of space and time as wholes discussed in the Transcendental Aesthetic are not represented through a continuous synthesis of their manifolds, nor are they represented as bounded. They are indeterminate magnitudes.

I have just described the determination of continuous *quanta*, but there is another level of determination of *quantitas*, which is the "how big" of a *quantum.* The representation of a determinate *quantum* is more basic and does not require the determination of its *quantitas.* The determination of *quantitas* only occurs when we measure something in accordance with a unit of measure, where the unit, the plurality of units, and the totality of that plurality fall under the categories of unity, plurality, and allness. Significantly it also presupposes the notion of equality, which is required for measurement. It is one thing to simply represent a determinate continuous *quantum*, for example, a line, and quite another to determine its *quantitas*, for example, how many shorter line segments are equal to it, or, for an empirical example, how many feet long a piece of lumber is. As I argue in my book, one of Kant's important innovations is to draw this distinction. It certainly seems to be the case that we can represent something continuous and extended in space without measuring how many equal units of some specified unit it contains. Moreover, as I argue in Chapter 9 of *KMW*, Kant is quite sensitive to the further requirements of measurement, in particular, the need for equality. Kant's account of our representation of *quanta*—its grounding in the categories of quantity and the nature of the synthesis of composition—do not require the relation of equality.

[67]By "line" here and in what follows I mean finite line segment. The continuity of this synthesis is necessary for his understanding of space and time and for the continuity of the spatial and temporal objects which appear in it. The continuity of this synthesis is also crucial to his understanding of the applicability of the calculus to the material world. See *KMW* Section 4.6. For further discussion of Kant's understanding of the calculus, see Sutherland (2021).

I say "in the first instance" because there is also a synthesis of discrete magnitudes in the representation of collections, which I address next.

In addition to continuous *quanta* and their *quantitas*, there are also discrete *quanta*, that is collections of things insofar as they are units, and their *quantitas*, that is, their measure in answer to the question "how big?" Here, too it is one thing to represent a determinate collection, for example, a collection of shoes in my closet, and another to determine its *quantitas* according to some unit, for example, how many shoes or how many pairs of shoes there are. So, in addition to the distinction between indeterminate and determinate *quanta*, we also have to keep in mind the distinction between *quantum* and *quantitas* and the distinction between continuous and discrete *quanta* and the *quantitas* of each.

Distinguishing between the different senses of magnitude is further complicated by the fact that, in Kant's view, the representation of the units of discrete *quanta* ultimately depends on the representation of continuous *quanta*, and because the determination of the *quantitas* of a continuous *quanta* requires regarding the continuous *quanta* as composed of equal units, and hence presupposes the representation of discrete *quanta* and their measurement. These entanglements reflect a long history of grappling with the relationship between continuous magnitudes and number. There are also further complications with Kant's account, which make his views difficult to decipher; two salient examples include his conception of number, which includes both cardinal and ordinal elements, and his doctrine of schemata.

What is most important for the issue at hand are the distinctions between the representation of indeterminate continuous *quanta*, determinate continuous *quanta*, and the *quantitas* of these *quanta*. Nunez objects to my interpretation of indeterminate *quanta* in the proof of the thesis of the First Antinomy (A462n/B454n) and offers his own interpretation. Kant argues there against the possibility of representing the world as an infinite given whole of simultaneously existing things. Kant interprets this possibility as requiring the representation of the magnitude [*Größe*] of the world as a *quantum*, by which he means the *quantitas* of this *quantum*, that is, with the representation of the measure of its size. I will not discuss the proof as a whole and will focus on what Kant says about representing an indeterminate *quantum.*

Kant claims that unless a magnitude of a *quantum* is given within certain boundaries of any intuition, then it can only be thought through the synthesis of its parts, and the totality of the *quantum* only through the completed synthesis, or through the repeated addition of units to each other. What is of interest is the exception to his claim expressed in the unless clause—the case in which a magnitude of a *quantum* is given within certain boundaries of any intuition. A footnote clarifies that he means the case in which we intuit an indeterminate *quantum* as whole when it is enclosed within boundaries; in that case, we do not need to construct its totality through measurement, because the boundaries determine its completeness. How does Kant understand our thinking an indeterminate *quantum* as enclosed within boundaries?

As I just noted earlier, the representation of a determinate *quantum* requires a synthesis of composition that runs through a homogeneous manifold and also determines its limits through its starting and ending, as exemplified in the drawing of a line. Since space has multiple dimensions, it is possible to use the representation of determinate lines to delimit an area, such a square. The area is enclosed within boundaries and is thereby intuited as a whole, but the homogeneous manifold of space contained within those boundaries has not been determined through a successive synthesis of its manifold and hence is not itself a determinate *quantum.* Similarly, we can represent six square planes to delimit a volume, namely, a cube. The volume is enclosed within boundaries and intuited as whole, but the space contained within those boundaries has not been determined through a continuous successive synthesis of its manifold. It follows that representing indeterminate *quanta* as a whole in this way precludes directly measuring it by specifying a unit and synthesizing those units in a totality and thereby determining its *quantitas.* Its *quantitas* can nevertheless be indirectly determined if the *quantitas* of the delimiting sides are determined; if I know the length of the sides of a square the *quantitas* of the enclosed area is indirectly determined, and similarly for the *quantitas* of the sides of a cube and the enclosed volume. What is important is that Kant is allowing for two sorts of indeterminate *quanta*; space and time as infinite given wholes discussed in the

Transcendental Aesthetic, and indeterminate *quanta* whose manifolds are not synthesized and yet are represented as a whole within boundaries.

Nunez construes the distinction between indeterminate and determinate magnitudes in a quite different way. In his view, indeterminate magnitudes are represented in empirical apprehension and determinate magnitudes are represented through *a priori* construction, where he understands *a priori* construction as requiring measurement (Nunez, p. 5ff [**XX**]). In short, a determinate magnitude is one that has been measured. Note first that this understanding of *a priori* construction ignores the distinction between the representation of a *quantum* and the representation of its *quantitas* in *pure a priori* cognition; on Nunez's view, one cannot represent a *quantum* such as a line without measuring it. As I noted earlier, I do not think it is plausible, nor do I think it is Kant's view, that whenever we represent a line by drawing it, we must measure it, determining how many equal units of some chosen unit of measure are contained in it.

According to Nunez's reading of the First Antinomy, we can empirically apprehend an indeterminate magnitude within boundaries, where the apprehension involves the deployment of the categories of quantity and allows us to empirically apprehend it as a *quantum* prior to determining its *quantitas* through an *a priori* construction. This gives Nunez an account of the conditions of the possibility of experience that are less intertwined with the conditions for mathematical cognition and the exact sciences than the account I give. Empirical cognition does not require measurement.

Unfortunately, I do not think this can work. Note that Nunez preserves the distinction between *quanta* and *quantitas* in the case of empirical apprehension while rejecting it for the construction of geometrical figures in imagination, which makes his rejection of it in the latter case even less plausible. Moreover, and I think this is decisive, the argument of the Axioms of Intuition turns on Kant's claim that appearances can only be apprehended and taken up into empirical consciousness through the same synthesis as that through which the representations of a determinate space or time are generated, and Kant states explicitly that this is the synthesis required for the representation of a *quantum*, so it is at the deeper level of *quanta* that empirical apprehension and mathematical cognition are intertwined, not just when the *quantitas* of an empirical intuition is determined. Moreover, everything Kant says in the Axioms about the determination of spaces and times, the synthesis of composition of a homogeneous manifold, and the concept of magnitude (*Quanti*) applies to mathematical objects as well, while nothing he says or argues presupposes the relation of equality or measurement; there is nothing beyond part/whole relations. It is the fact that apprehension of appearances and mathematical cognition are already fused at the level of *quanta* that makes appearances susceptible to measurement. It is true that mathematical cognition *really* gets going once we introduce measurement and hence the determination of *quantitas*. But our apprehension of appearances and mathematical cognition are already fused at the level of *quanta*, even though only the synthesis of *pure a priori* mathematical cognition is elective and counts as construction in the proper sense.

Once again, I am grateful for the thoughtfulness and depth of the comments provided by Anderson, Heis, and Nunez, which range from broadly programmatic to detailed issues of textual interpretation. I hope my responses have clarified my interpretation and hence Kant's philosophy of mathematics and its relation to human experience.

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