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Dimensions of random affine code tree fractals

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Abstract. We study the dimension of code tree fractals, a class of fractals generated by a set of iterated function systems. We first consider deterministic affine code tree fractals, extending to the code tree fractal setting the classical result of Falconer and Solomyak on the Hausdorff dimension of self-affine fractals generated by a single iterated function system. We then calculate the almost sure Hausdorff, packing and box counting dimensions of a general class of random affine planar code tree fractals. The set of probability measures describing the randomness includes natural measures in random V-variable and homogeneous Markov constructions.

1. *Introduction*

The systematic study of dimensions of the attractors of iterated function systems was initiated by Hutchinson [**16**]. He proved the formula for Hausdorff dimensions of attractors of self-similar iterated function systems satisfying the open set condition. Since then, iterated function systems have been studied extensively and nowadays there exists a huge literature on them. Here we mention only a few results relevant to our purposes. In the case of affine maps, there is no counterpart to the open set condition guaranteeing that the dimension of the attractor is given by a concrete formula. However, in [**9**] Falconer proved that writing the affine maps as $F_i(x) = T_i(x) + a_i$, the dimension of the attractor is Lebesgue almost surely independent of the translation vectors a_i provided that the norms of the linear parts T_i are less than $\frac{1}{3}$. Furthermore, the dimension is given by the unique zero of a natural pressure. Solomyak [**24**] verified that $\frac{1}{3}$ may be replaced by $\frac{1}{2}$, and an

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example of Edgar [**6**] (see also [**7, 21, 23**]) shows that $\frac{1}{2}$ is the best possible upper bound in this setting.

In this paper we verify that the Hausdorff dimension is almost surely independent of the translation vectors for general affine code tree fractals, that is, for iterative constructions where the families of contractions may vary (see §2 for the definition). We also illustrate by examples that for general affine code tree fractals the dimension is not always given by the zero of the pressure.

A natural way to generalize deterministic iterated function systems is to add randomness to the construction. This can be done in various different ways. Jordan *et al* [**17**] studied a fixed affine iterated function system having a small independent random perturbation in translation vectors at each step of the construction. In this case no non-trivial upper bound for the contraction ratios is needed in the analogue of the result of Falconer [**9**] and Solomyak [**24**]. Gatzouras and Lalley [**15**] investigated random counterparts of self-affine Bedford–McMullen carpets. In their model the linear parts of the affine mappings are non-random, but the translation vectors and the number of maps are independently, randomly chosen at every step of the construction. In [**13**] Falconer and Miao studied random subsets of a fixed self-affine fractal. The randomness is introduced by choosing at each step of the construction a random subfamily of the original function system independently. They proved that the dimension of the random subset is almost surely given by the unique zero of the expected pressure.

In [**13, 15, 17**] the randomness is quite strong in the sense that there is total independence both in space, that is, between different nodes at a fixed construction level, and in scale or time, that is, once a node is chosen, its descendants are selected independently of the previous history. We will consider probability distributions which have certain independence only in time direction, more precisely, there exists almost surely an infinite sequence of neck levels N_n where all the sub code trees starting from level N_n are the same and the events depending only on the construction before a neck level N_k are independent of those depending only on the construction after N_k. The structure between neck levels can be chosen quite freely. In other words, our construction is locally random but globally almost homogeneous.

A special case of our setting (which in fact is the main motivation for our study) is the natural probability measure P of choosing random V-variable fractals considered by Barnsley *et al* [**1–3**]. A V-variable fractal is a code tree fractal where at each level of the code tree there are at most V different sub code trees. In [**3**] a formula for the P-almost sure Hausdorff dimension of random self-similar V-variable fractals satisfying a uniform open set condition is proved. We extend this result in two ways: by replacing similarities with affine maps and by considering a general class of probability measures including the measure P introduced in [**3**].

The paper is organized as follows: in §2 we give basic definitions and recall some well-known results. In §3 we extend the result of Falconer [**9**] and Solomyak [**24**] concerning the almost sure constancy of the Hausdorff dimension to deterministic code tree fractals (Theorem 3.2) and give examples illustrating reasons why the Hausdorff dimension is not given by the zero of the natural pressure function. Sections 4 and 5 are dedicated to random code tree fractals and contain our main results: in Theorem 4.3 we prove that,

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under quite weak assumptions on the probability measure on the set of code trees having a neck structure, the zero of the pressure exists and is independent of the code tree almost surely. In Theorem 5.1 we verify that, under certain additional assumptions, the zero of the pressure is equal to the typical Hausdorff, box counting and packing dimensions of the code tree fractal almost surely. Finally, in Proposition 5.7 it is observed that for self-similar code tree fractals the additional assumptions are not needed.

2. *Preliminaries*

We begin by defining a code tree fractal. Our definition is similar to the one in [**2**]. We denote by Λ an index set. Let $D \in \mathbb{N}$ and let $\mathbf{F} = \{F^\lambda \mid \lambda \in \Lambda\}$ be a family of iterated function systems such that $F^\lambda = \{f_1^\lambda, \ldots, f_{M_\lambda}^\lambda\}$. Here for all $i = 1, \ldots, M_\lambda$ the map $f_i^\lambda \colon \mathbb{R}^D \to \mathbb{R}^D$ is defined by $f_i^\lambda(x) = T_i^\lambda(x) + a_i^\lambda$, where $a_i^\lambda \in \mathbb{R}^D$ and T_i^λ is a non-singular linear mapping with $\sup_{\lambda\in\Lambda, i=1,\ldots,M_\lambda} \|T_i^\lambda\| < 1$ and $M = \sup_{\lambda\in\Lambda} M_\lambda < \infty$. Setting $I = \{1, \ldots, M\}$, the length of a word $\tau \in I^k$ is $|\tau| = k$. Consider a function $\omega \colon \bigcup_{k=0}^\infty I^k \to \Lambda$, where $I^0 = \emptyset$. Let $\Sigma_*^\omega \subset \bigcup_{k=0}^\infty I^k$ be the unique set satisfying the following conditions:

- $\emptyset \in \Sigma_*^\omega$;
- if $i_1 \cdots i_k \in \Sigma_*^\omega$ and $\omega(i_1 \cdots i_k) = \lambda$, then $i_1 \cdots i_k l \in \Sigma_*^\omega$ for all $l \le M_\lambda$ and $i_1 \cdots i_k l \notin \Sigma_*^\omega$ for any $l > M_\lambda$;
- if $i_1 \cdots i_k \notin \Sigma_*^\omega$, then $i_1 \cdots i_k l \notin \Sigma_*^\omega$ for any l.

The function ω restricted to Σ_*^ω is called an $\mathbf{F}$-valued code tree and the set of all $\mathbf{F}$-valued code trees is denoted by Ω. Equip $I^\mathbb{N}$ with the product topology. Let $\Sigma^\omega = \{\mathbf{i} = i_1 i_2 \cdots \in I^\mathbb{N} \mid i_1 \cdots i_n \in \Sigma_*^\omega \text{ for all } n \in \mathbb{N}\}$ be the compact set of infinite paths corresponding to a code tree $\omega \in \Omega$. For any $k \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^\omega \cup \bigcup_{j=k}^\infty I^j$, let $\mathbf{i}_k = i_1 \cdots i_k$ be the initial word of $\mathbf{i}$ with length k. We use the notations

$$f_{\mathbf{i}_k}^\omega = f_{i_1}^{\omega(\emptyset)} \circ f_{i_2}^{\omega(i_1)} \circ \cdots \circ f_{i_k}^{\omega(i_1 \cdots i_{k-1})}$$

and

$$T_{\mathbf{i}_k}^\omega = T_{i_1}^{\omega(\emptyset)} T_{i_2}^{\omega(i_1)} \cdots T_{i_k}^{\omega(i_1 \cdots i_{k-1})}.$$

For all $\mathbf{i} \in \Sigma^\omega$, define $Z^\omega(\mathbf{i}) = \lim_{k\to\infty} f_{\mathbf{i}_k}^\omega(0)$, where $0 \in \mathbb{R}^D$, and set $A^\omega = \{Z^\omega(\mathbf{i}) \mid \mathbf{i} \in \Sigma^\omega\}$. We refer to A^ω as the attractor or the code tree fractal corresponding to the code tree $\omega \in \Omega$. The attractor A^ω is well-defined since the maps f_i^λ are uniformly contracting. For $k \in \mathbb{N}$, $\omega \in \Omega$ and $\mathbf{i} \in \Sigma^\omega$, let

$$[\mathbf{i}_k] = \{\mathbf{j} \in \Sigma^\omega \mid j_l = i_l \text{ for all } l = 1, \ldots, k\}$$

be the cylinder starting with $i_1 \cdots i_k$. The length of the cylinder $[\mathbf{i}_k]$ is k.

Remark 2.1. (a) One could define code tree fractals for contractions on complete metric spaces, or more generally for maps such that the limit $Z^\omega(\mathbf{i})$ exists for any $\mathbf{i} \in \Sigma^\omega$. However, in this paper we consider only affine contractions on $\mathbb{R}^D$.

(b) Any compact subset of a fractal generated by a single iterated function system is a code tree fractal. Indeed, let A^W be the attractor of an iterated function system $W = \{w_1, \ldots, w_N\}$ on $\mathbb{R}^D$ and let $K \subset A^W$ be compact. Setting $I = \{1, \ldots, N\}$, we denote

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by $Z^W\colon I^{\mathbb{N}} \to \mathbb{R}^D$ the natural projection and define $\widetilde{K} = (Z^W)^{-1}(K)$. For $k \in \mathbb{N} \cup \{0\}$, let

$$\widetilde{K}_k = \{\mathbf{i}_k \in I^k \mid \mathbf{i}_k\mathbf{j} \in \widetilde{K} \text{ for some } \mathbf{j} \in I^{\mathbb{N}}\}.$$

Further, let $\Lambda = \{\lambda \subset I \mid \lambda \neq \emptyset\}$ and $\mathbf{F} = \{F^\lambda \mid \lambda \in \Lambda\}$, where $F^\lambda = \{w_i \mid i \in \lambda\}$. Defining the mapping $\omega\colon \bigcup_{k=0}^{\infty} \widetilde{K}_k \to \Lambda$ by

$$\omega(i_1 \cdots i_{k-1}) = \{i \mid i_1 \cdots i_{k-1} i \in \widetilde{K}_k\},$$

we have $A^\omega = K$. In particular, sub-self-affine sets are code tree fractals as compact subsets of self-affine sets. This in turn implies that attractors of graph directed Markov systems generated by affine maps are code tree fractals as well since they are sub-self-affine sets. See [**10, 11, 18, 19**].

It is also easy to see that a code tree fractal can be expressed as a subset of the attractor of a single possibly infinite iterated function system.

The way deterministic constructions are usually randomized depends heavily on the formalism used. It is useful to add some additional structure to our space in order to be able to compare results expressed in different formalisms.

Example 2.2. Suppose $\{w_1, w_2, w_3\}$ is an iterated function system, where $w_i = T_i + a_i$ for $i = 1, 2, 3$. Consider the code tree fractal constructed in Remark 2.1 corresponding to a compact $K \subset A^W$. Letting $\lambda = \{1, 2\}$ and $\lambda' = \{2, 3\}$, we have $F^\lambda = \{T_1^\lambda + a_1^\lambda, T_2^\lambda + a_2^\lambda\}$ and $F^{\lambda'} = \{T_1^{\lambda'} + a_1^{\lambda'}, T_2^{\lambda'} + a_2^{\lambda'}\}$, where $a_1^\lambda = a_1$, $a_2^\lambda = a_2$, $a_1^{\lambda'} = a_2$ and $a_2^{\lambda'} = a_3$. Thus, the translation vector a_2 is represented both by a_2^λ and $a_1^{\lambda'}$.

To allow identifications in our construction, we suppose that the set $\widehat{\Lambda} = \{(\lambda, i) \mid \lambda \in \Lambda \text{ and } i = 1, \ldots, M_\lambda\}$ is equipped with an equivalence relation $\sim$ satisfying the following two conditions:

- the cardinality $\mathcal{A}$ of the set of equivalence classes $\mathbf{a} := \widehat{\Lambda}/\sim$ is finite;
- for every $\lambda \in \Lambda$ we have $(\lambda, i) \sim (\lambda, j)$ if and only if $i = j$.

We use the notation $\mathbf{a}$ for the set of equivalence classes to emphasize that we will identify only the translation parts of the maps by the equivalence relation $\sim$. The assumption $\mathcal{A} < \infty$ guarantees that the Lebesgue measure, $\mathcal{L}^{D\mathcal{A}}$, can be used on the space $(\mathbb{R}^D)^{\mathcal{A}}$. By the second assumption, within any system $\{T_1^\lambda + a_1^\lambda, \ldots, T_{M_\lambda}^\lambda + a_{M_\lambda}^\lambda\}$ different translation vectors are never identified. Moreover, if $(\lambda, i) \sim (\lambda', j)$, then $a_i^\lambda = a_j^{\lambda'}$ as vectors in $\mathbb{R}^D$. Hence, we may consider $\mathbf{a}$ as an element of $\mathbb{R}^{D\mathcal{A}}$, and we denote the corresponding attractor by $A_{\mathbf{a}}^\omega$.

For a non-singular linear map $T\colon \mathbb{R}^D \to \mathbb{R}^D$, let $\sigma_i = \sigma_i(T)$, $1 \le i \le D$, be the singular values of T defined as the square roots of the eigenvalues of T^*T, where T^* is the adjoint of T. The singular values are enumerated in the decreasing order, that is,

$$0 < \sigma_D \le \sigma_{D-1} \le \cdots \le \sigma_2 \le \sigma_1 = \|T\|.$$

Recall that the singular values are the lengths of the semi-axes of the ellipsoid $T(B(0, 1))$, where $B(x, \rho) \subset \mathbb{R}^D$ is the closed ball with radius $\rho > 0$ centred at $x \in \mathbb{R}^D$. Define the singular value function as follows

$$\Phi^\alpha(T) = \begin{cases} \sigma_1\sigma_2 \cdots \sigma_{m-1}\sigma_m^{\alpha-m+1} & \text{if } 0 \le \alpha \le D, \\ \sigma_1\sigma_2 \cdots \sigma_{D-1}\sigma_D^{\alpha-D+1} & \text{if } \alpha > D, \end{cases}$$

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where m is the integer such that $m-1 \le \alpha < m$. The singular value function Φ^α is defined also for $\alpha > D$ to ensure the existence of the affinity dimension (see §3) for all affine code tree fractals. We have $\sigma_D(T)^\alpha \le \Phi^\alpha(T) \le \sigma_1(T)^\alpha$, and in particular, $\Phi^\alpha(T) = s^\alpha$ if T is a similitude with $\sigma_i = s$ for all $i = 1, \dots, D$. The singular value function is submultiplicative, that is,

$$\Phi^\alpha(TU) \le \Phi^\alpha(T)\Phi^\alpha(U)$$

for all linear maps T and U. For this and other properties of the singular value function see for example [**9**]. Throughout this paper we will make the assumption that there exist $\underline{\sigma}, \overline{\sigma} \in (0, 1)$ such that

$$0 < \underline{\sigma} \le \sigma_D(T_i^\lambda) \le \sigma_1(T_i^\lambda) \le \overline{\sigma} < 1$$

for all $\lambda \in \Lambda$ and for all $i = 1, \dots, M_\lambda$. Note that whilst the condition $\overline{\sigma} < 1$ follows from the uniform contractivity assumption, the existence of $\underline{\sigma} > 0$ is an extra assumption since Λ may be infinite even though $\mathcal{A} < \infty$. The lower bound $\underline{\sigma} > 0$ is not needed in §3.

3. *Deterministic code tree fractals*

In this section we show that the result of Falconer [**9**] (sharpened by Solomyak [**24**]) concerning self-affine iterated function systems can be extended to code tree fractals, that is, the Hausdorff dimension $\dim_{\mathrm{H}}(A_{\mathbf{a}}^\omega)$ of a code tree fractal $A_{\mathbf{a}}^\omega$ is typically independent of the translation vector $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. Theorem 3.2 is not only an important ingredient in the proof of our main theorem (Theorem 5.1) but is also of independent interest. The original proofs by Falconer and Solomyak in the case $|\Lambda| = 1$ generalize in a straightforward manner. For the convenience of the reader, we present the essential ideas.

We denote by M^α the α-dimensional natural measure defined for all Borel subsets E of Σ^ω by

$$M^\alpha(E) = \lim_{j\to\infty} M_j^\alpha(E),$$

where

$$M_j^\alpha(E) = \inf\left\{ \sum_{\mathbf{i}_k \in J} \Phi^\alpha(T_{\mathbf{i}_k}^\omega) \;\middle|\; J \subset \Sigma_*^\omega,\ E \subset \bigcup_{\mathbf{i}_k \in J} [\mathbf{i}_k] \text{ and } k \ge j \right\}.$$

The affinity dimension of Σ^ω is

$$d^\omega = \inf\{\alpha \mid M^\alpha(\Sigma^\omega) = 0\} = \sup\{\alpha \mid M^\alpha(\Sigma^\omega) = \infty\}.$$

The next lemma is the key tool in the proof of Theorem 3.2.

LEMMA 3.1. *Let $\rho > 0$ and assume that $\overline{\sigma} < \frac{1}{2}$. If α is non-integral with $0 < \alpha < D$ then there exists a constant $c_1 > 0$ such that for any $k \in \mathbb{N}$ and $\mathbf{i}, \mathbf{j} \in \Sigma^\omega$ with $\mathbf{i}_k = \mathbf{j}_k$ and $i_{k+1} \ne j_{k+1}$, we have*

$$\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathcal{L}^{D\mathcal{A}}(\mathbf{a})}{|Z_{\mathbf{a}}^\omega(\mathbf{i}) - Z_{\mathbf{a}}^\omega(\mathbf{j})|^\alpha} \le \frac{c_1}{\Phi^\alpha(T_{\mathbf{i}_k}^\omega)} \quad \textit{for all } \omega \in \Omega.$$

Proof. The points in $A_{\mathbf{a}}^\omega$ can be expressed as

$$Z_{\mathbf{a}}^\omega(\mathbf{i}) = a_{i_1}^{\omega(\emptyset)} + T_{i_1}^{\omega(\emptyset)} a_{i_2}^{\omega(i_1)} + T_{i_1}^{\omega(\emptyset)} T_{i_2}^{\omega(i_1)} a_{i_3}^{\omega(i_1 i_2)} + \cdots. \tag{3.1}$$

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If $\mathbf{i}, \mathbf{j} \in [\mathbf{i}_k]$, write $\mathbf{i} = \mathbf{i}_k\mathbf{i}'$ and $\mathbf{j} = \mathbf{i}_k\mathbf{j}'$ where $i'_l = i_{k+l}$ and $j'_l = j_{k+l}$ for $l \in \mathbb{N}$. Now

$$\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathcal{L}^{D\mathcal{A}}(\mathbf{a})}{|Z^{\omega}_{\mathbf{a}}(\mathbf{i}) - Z^{\omega}_{\mathbf{a}}(\mathbf{j})|^{\alpha}} = \int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathcal{L}^{D\mathcal{A}}(\mathbf{a})}{|T^{\omega}_{\mathbf{i}_k}(Z^{\omega'}_{\mathbf{a}}(\mathbf{i}') - Z^{\omega'}_{\mathbf{a}}(\mathbf{j}'))|^{\alpha}},$$

where the code tree ω' is defined by $\omega'(\mathbf{j}_l) = \omega(\mathbf{i}_k\mathbf{j}_l)$ for $l \in \mathbb{N}$.

Let

$$n_1 = \inf\{k \geq 2 \mid (\omega'(\mathbf{i}'_{k-1}), i'_k) \sim (\omega'(\mathbf{j}'_{k-1}), j'_k)\}.$$

We consider only the case $n_1 < \infty$; the remaining case $n_1 = \infty$ is easier (the second sum is not needed in (3.2)). Letting $\mathbf{a} = \{\beta_1, \ldots, \beta_{\mathcal{A}}\}$, we may assume without loss of generality that $[(\omega'(\emptyset), i'_1)] = \beta_1$, $[(\omega'(\emptyset), j'_1)] = \beta_2$ and $[(\omega'(\mathbf{i}'_{n_1-1}), i'_{n_1})] = [(\omega'(\mathbf{j}'_{n_1-1}), j'_{n_1})] \neq \beta_1$. Define a linear map $E\colon \mathbb{R}^{D\mathcal{A}} \to \mathbb{R}^{D\mathcal{A}}$ by

$$E(\beta_1, \ldots, \beta_{\mathcal{A}}) = (y_1, \beta_2, \ldots, \beta_{\mathcal{A}}),$$

where

$$y_1 = Z^{\omega'}_{\mathbf{a}}(\mathbf{i}') - Z^{\omega'}_{\mathbf{a}}(\mathbf{j}') = \beta_1 - \beta_2 + \sum_{i=1}^{\mathcal{A}} L_i(\beta_i)$$

and L_i is a linear transformation on $\mathbb{R}^D$ for all $i = 1, \ldots, \mathcal{A}$ (recall (3.1)). Further, since $\overline{\sigma} < \frac{1}{2}$, we have

$$\|L_1\| \leq \sum_{k=1}^{n_1-2} \overline{\sigma}^k + \sum_{k=n_1}^{\infty} 2\overline{\sigma}^k = \frac{\overline{\sigma} - \overline{\sigma}^{n_1-1}}{1-\overline{\sigma}} + \frac{2\overline{\sigma}^{n_1}}{1-\overline{\sigma}} < \frac{\overline{\sigma}}{1-\overline{\sigma}} < 1. \tag{3.2}$$

This implies that $\mathrm{Id} + L_1$ is invertible (where the identity transformation on $\mathbb{R}^D$ is denoted by Id), and therefore, E is invertible and $\det(E) = \det(\mathrm{Id} + L_1) \geq c_2 > 0$. By a change of variables, we obtain, using [**9**, Lemma 2.2] in the last inequality, that

$$\begin{aligned}
&\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathcal{L}^{D\mathcal{A}}(\mathbf{a})}{|T^{\omega}_{\mathbf{i}_k}(Z^{\omega'}_{\mathbf{a}}(\mathbf{i}') - Z^{\omega'}_{\mathbf{a}}(\mathbf{j}'))|^{\alpha}} \\
&\quad = |\det(E)|^{-1} \int_{y\in E(B(0,\rho))} \frac{d\mathcal{L}^{D\mathcal{A}}(y)}{|T^{\omega}_{\mathbf{i}_k}(y_1)|^{\alpha}} \\
&\quad \leq c_2^{-1} \int_{B^D(0,\rho)} \cdots \int_{B^D(0,\rho)} \int_{B^D(0,R)} \frac{d\mathcal{L}^D(y_1)\cdots d\mathcal{L}^D(y_{\mathcal{A}})}{|T^{\omega}_{\mathbf{i}_k}(y_1)|^{\alpha}} \leq \frac{c_1}{\Phi^{\alpha}(T^{\omega}_{\mathbf{i}_k})}
\end{aligned}$$

for some $0 < R < \infty$ and $c_1 < \infty$. Here the superscript D emphasizes that $B^D(0, r)$ is a closed ball in $\mathbb{R}^D$. Note that L_1 and thus E depend on $\mathbf{i}$, $\mathbf{j}$ and ω, but c_2, R and c_1 may be chosen to be independent of $\mathbf{i}$, $\mathbf{j}$ and ω. The assumption that α is non-integral with $0 < \alpha < D$ is needed when applying [**9**, Lemma 2.2]. □

Now we are ready to prove the main theorem of this section. The Hausdorff dimension is denoted by $\dim_{\mathrm{H}}$.

THEOREM 3.2. *Let $\omega \in \Omega$ and assume that $\overline{\sigma} < \frac{1}{2}$. Then*

$$\dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) = \min\{D, d^{\omega}\}$$

for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$.

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Proof. As in [**9**, Proposition 5.1] it follows that

$$\dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) \le d^{\omega} \tag{3.3}$$

for every $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. Thus it suffices to prove that $\dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) \ge \min\{D, d^{\omega}\}$ for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. Let α be non-integral such that $0 < \alpha < \min\{D, d^{\omega}\}$. As in [**9**, Lemma 4.2] (see also [**22**, Proposition 2.8]), there exists a finite Borel measure μ^{ω} on Σ^{ω} and a constant $c(\omega)$ such that

$$\mu^{\omega}([\mathbf{i}_k]) \le c(\omega)\Phi^{\alpha}(T^{\omega}_{\mathbf{i}_k}) \tag{3.4}$$

for any cylinder $[\mathbf{i}_k]$.

Let $\rho > 0$ and $s < \alpha$. From Lemma 3.1 and (3.4) we get

$$\begin{aligned}
&\int_{\Sigma^{\omega}} \int_{\Sigma^{\omega}} \int_{\mathbf{a}\in B(0,\rho)} \frac{d\mathcal{L}^{D\mathcal{A}}(\mathbf{a})\, d\mu^{\omega}(\mathbf{i})\, d\mu^{\omega}(\mathbf{j})}{|Z^{\omega}_{\mathbf{a}}(\mathbf{i}) - Z^{\omega}_{\mathbf{a}}(\mathbf{j})|^{s}} \\
&\quad \le \sum_{k=0}^{\infty} \sum_{\mathbf{i}_k \in \Sigma^{\omega}_{*}} \left(\sup_{\mathbf{j},\mathbf{j}' \in [\mathbf{i}_k],\, j_{k+1} \ne j'_{k+1}} \int_{\mathbf{a}\in B(0,\rho)} \frac{\mu^{\omega}([\mathbf{j}_k])\mu^{\omega}([\mathbf{j}'_k])}{|Z^{\omega}_{\mathbf{a}}(\mathbf{j}) - Z^{\omega}_{\mathbf{a}}(\mathbf{j}')|^{s}} d\mathcal{L}^{D\mathcal{A}}(\mathbf{a}) \right) \\
&\quad \le c_1 \sum_{k=0}^{\infty} \sum_{\mathbf{i}_k \in \Sigma^{\omega}_{*}} \frac{\mu^{\omega}([\mathbf{i}_k])\mu^{\omega}([\mathbf{i}_k])}{\Phi^{s}(T^{\omega}_{\mathbf{i}_k})} \le c(\omega)c_1 \sum_{k=0}^{\infty} \sum_{\mathbf{i}_k \in \Sigma^{\omega}_{*}} \frac{\mu^{\omega}([\mathbf{i}_k])\Phi^{\alpha}(T^{\omega}_{\mathbf{i}_k})}{\Phi^{s}(T^{\omega}_{\mathbf{i}_k})} \\
&\quad \le c(\omega)c_1\mu^{\omega}(\Sigma^{\omega}) \sum_{k=0}^{\infty} \overline{\sigma}^{k(\alpha-s)} < \infty.
\end{aligned}$$

Finally, [**9**, Lemma 5.2] (essentially Fubini's theorem and the mass distribution principle) implies $\dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) \ge \alpha$ for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. The claim follows by choosing a sequence $\alpha_i \uparrow \min\{D, d^{\omega}\}$. □

Remark 3.3. In [**9**, Proposition 4.1 and Theorem 5.3], combined with [**24**], it is shown that if $\overline{\sigma} < \frac{1}{2}$ and $|\Lambda| = 1$, then the natural pressure exists and has a unique zero for self-affine iterated function systems, and that the Hausdorff dimension of the attractor is given by the zero of the pressure for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. It then follows that also the packing and box counting dimensions are equal to this number, see [**9**, Theorem 5.4] and [**12**, (3.17) and (3.29)]. However, in our setting this is not true, which is evident from Examples 3.5–3.7 below.

We will discuss the issue raised in Remark 3.3 in the remaining part of this section. We start by defining the natural pressure.

Definition 3.4. For every $k \in \mathbb{N}$ and $\alpha \ge 0$, let

$$S^{\omega}(k, \alpha) = \sum_{\mathbf{i}_k \in \Sigma^{\omega}_{*}} \Phi^{\alpha}(T^{\omega}_{\mathbf{i}_k}).$$

Define

$$p^{\omega}_{\inf}(\alpha) = \liminf_{k\to\infty} \frac{\log S^{\omega}(k, \alpha)}{k} \quad \text{and} \quad p^{\omega}_{\sup}(\alpha) = \limsup_{k\to\infty} \frac{\log S^{\omega}(k, \alpha)}{k}.$$

If $p^{\omega}_{\inf}(\alpha) = p^{\omega}_{\sup}(\alpha)$, the common value is called the (natural) pressure $p^{\omega}(\alpha)$.

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As in the next section, it is easy to see that $p^{\omega}_{\inf}$ and $p^{\omega}_{\sup}$ are decreasing functions in α with uniquely defined $\alpha_0 \le \alpha_1$ (depending on ω) such that $p^{\omega}_{\inf}(\alpha_0) = p^{\omega}_{\sup}(\alpha_1) = 0$.

Below we construct examples illustrating various reasons why the pressure might fail to give the Hausdorff dimension for general code tree fractals. Constructions with similar properties can be found in the literature, see for example Pesin and Weiss [**20**]. To highlight the features most important in our context, we give detailed descriptions here. All our examples are in $\mathbb{R}$ but they can easily be extended to $\mathbb{R}^D$ for any $D \in \mathbb{N}$.

Example 3.5. There exist code trees for which the pressure $p^{\omega}(\alpha)$ does not exist for any $\alpha > 0$.

For example, let $i = 0, 1$, and define $f_i, g_i : [0, 1] \to [0, 1]$ by $f_i(x) = x/8 + \frac{7}{8}i$ and $g_i(x) = x/4 + \frac{3}{4}i$ for all $x \in [0, 1]$. Let $F^1 = \{f_0, f_1\}$ and $F^2 = \{g_0, g_1\}$. The equivalence relation $\sim$ is trivial, that is, $\mathbf{a} \in \mathbb{R}^4$. Letting $1 = N_0 < N_1 < N_2 < \cdots$ be integers, set $\omega(\emptyset) = 2$ and for all $l = 0, 1, 2, \ldots$

$$\omega(\mathbf{i}_k) = \begin{cases} 1 & \text{if } N_{2l} \le k < N_{2l+1}, \\ 2 & \text{if } N_{2l+1} \le k < N_{2(l+1)}. \end{cases}$$

It is easy to see that for a sufficiently rapidly increasing sequence (N_i) we obtain

$$p^{\omega}_{\inf}(\alpha) = (1 - 3\alpha)\log 2 \quad \text{and} \quad p^{\omega}_{\sup}(\alpha) = (1 - 2\alpha)\log 2.$$

Observe that for an open set of translation vectors $\mathbf{a}$ we have

$$\overline{\dim}_{\mathrm{B}}(A^{\omega}_{\mathbf{a}}) = \dim_{\mathrm{p}}(A^{\omega}_{\mathbf{a}}) = \tfrac{1}{2} \quad \text{and} \quad \underline{\dim}_{\mathrm{B}}(A^{\omega}_{\mathbf{a}}) = \dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) = \tfrac{1}{3},$$

where $\underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$ are the lower and upper box counting dimensions, respectively, and $\dim_{\mathrm{p}}$ is the packing dimension.

The following example shows that the Hausdorff dimension of a code tree fractal may be strictly smaller than the unique zero of the pressure for all translation vectors.

Example 3.6. We construct a code tree ω for which the pressure $p^{\omega}(\alpha)$ exists for all $\alpha \ge 0$ and there exists a unique d such that $p^{\omega}(d) = 0$. However, $\dim_{\mathrm{H}}(A^{\omega}_{\mathbf{a}}) < d$ for all translation vectors $\mathbf{a}$.

Let $0 < r < R \le \frac{1}{3}$. We consider two systems $F = \{f_1, f_2, f_3\}$ and $G = \{g_1, g_2, g_3\}$ consisting of similarities on $\mathbb{R}$ such that $f_i(x) = rx + a^1_i$ and $g_i(x) = Rx + a^2_i$ for all $i = 1, 2, 3$ and $x \in \mathbb{R}$ with the trivial equivalence relation $\sim$, that is, $\mathbf{a} \in \mathbb{R}^6$. Taking a sequence of integers $1 = N_0 < N_1 < N_2 < \cdots$, define a code tree ω as follows: for each $n \in \mathbb{N} \cup \{0\}$, $m \in \{1, 2, 3\}$ and $\tau \in \bigcup_{k=0}^{\infty}\{1, 2, 3\}^k$ for which $N_{3n+(m-1)} \le |\tau| < N_{3n+m}$, set

$$\omega(\tau) = \begin{cases} F & \text{if } \tau_1 = m, \\ G & \text{otherwise.} \end{cases}$$

If $N_k \to \infty$ fast enough, we obtain for all $\alpha \ge 0$

$$p^{\omega}(\alpha) = \lim_{k\to\infty} \frac{\log S^{\omega}(k, \alpha)}{k} = \lim_{k\to\infty} \frac{\log(3^k R^{k\alpha})}{k} = \log 3 + \alpha \log R,$$

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giving $p^\omega(\alpha) = 0$ if and only if $\alpha = -\log 3/\log R$. On the other hand, it is straightforward to see that $M^\alpha(\Sigma^\omega) < \infty$ if $\alpha > -\log 3/\log r$. This implies by (3.3) that

$$\dim_{\mathrm{H}}(A^\omega_{\mathbf{a}}) \le d^\omega \le -\log 3/\log r < -\log 3/\log R.$$

Furthermore, $\dim_{\mathrm{p}}(A^\omega_{\mathbf{a}}) = -\log 3/\log R$ in an open set of translation vectors $\mathbf{a} \in \mathbb{R}^6$.

Observe that $p^\omega(\alpha)$ does not exist if we restrict Σ^ω to any of the branches $\Sigma^\omega_m = \{\mathbf{i} \in \Sigma^\omega \mid i_1 = m\}$ for $m \in \{1, 2, 3\}$. It is possible to modify the construction in such a way that the proportion of nodes where $\omega(\mathbf{i}_k) = F$ decreases to zero as k tends to infinity but for every $\mathbf{i}$ there are infinitely many l for which $\omega(\mathbf{i}_k) = F$ for all $N_l \le k < N_{l+1}$. Then for all $\alpha \ge 0$, we have $p^\omega(\alpha) = \log 3 + \alpha \log R$, and this remains true also for all branches of Σ^ω. The modification does not affect the Hausdorff dimension of the code tree fractal.

In the previous example the packing dimension of the code tree fractal is equal to the unique zero of the pressure in an open set of translation vectors. The last example of this section indicates that this is not always the case.

Example 3.7. There exists a code tree ω such that the pressure $p^\omega(\alpha)$ exists for all $\alpha \ge 0$ and for all translation vectors $\mathbf{a}$ neither $\dim_{\mathrm{H}}(A^\omega_{\mathbf{a}})$ nor $\dim_{\mathrm{p}}(A^\omega_{\mathbf{a}})$ agrees with the unique zero of the pressure.

For $i = 0, 1$, define $f_i : [0, 1] \to [0, 1]$ by $f_i(x) = x/2 + a_i$ for all $x \in [0, 1]$, where $a_0 = 0$ and $a_1 = \frac{1}{2}$. Set $F^1 = \{f_0, f_1\}$, $F^2 = \{f_0\}$ and $F^3 = \{f_1\}$. We identify a_0 in F^2 with a_0 in F^1 and a_1 in F^3 with a_1 in F^1. Let ω be such that the corresponding attractor is $A^\omega_{\mathbf{a}} = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ (see Remark 2.1). Clearly, $\dim_{\mathrm{H}}(A^\omega_{\mathbf{a}}) = \dim_{\mathrm{p}}(A^\omega_{\mathbf{a}}) = 0$ and $\dim_{\mathrm{B}}(A^\omega_{\mathbf{a}}) = \frac{1}{2}$. Moreover, denoting by $N(k)$ the number of dyadic intervals of length 2^{-k} that meet $A^\omega_{\mathbf{a}}$, we obtain for all $\alpha \ge 0$

$$p^\omega(\alpha) = -\alpha \log 2 + \lim_{k\to\infty} \frac{\log N(k)}{k},$$

implying $p^\omega(\alpha) = 0$ if and only if $\alpha = \dim_{\mathrm{B}}(A^\omega_{\mathbf{a}}) = \frac{1}{2}$. Let us now consider translations of this system. Since the attractor $A^\omega_{\mathbf{a}}$ is a countable set for all $\mathbf{a} \in \mathbb{R}^2$, we have $\dim_{\mathrm{p}}(A^\omega_{\mathbf{a}}) = 0$ for all $\mathbf{a} \in \mathbb{R}^2$. Note that the zero of the pressure p^ω is independent of $\mathbf{a} \in \mathbb{R}^2$.

We finish this section with a proposition concerning 1-variable code trees (for the definition see Example 4.5).

PROPOSITION 3.8. *Suppose that ω is a 1-variable (homogeneous) code tree and* $\mathbf{F}$ *contains only similarities. Then d^ω equals the unique α_0 such that $p^\omega_{\inf}(\alpha_0) = 0$.*

Proof. Letting $\beta > \alpha_0$, there is $c < 1$ satisfying

$$\liminf_{k\to\infty} (S^\omega(k, \beta))^{1/k} < c.$$

Consequently, for arbitrarily large values of k, we get $\sum_{\mathbf{i}_k \in \Sigma^\omega_*} \Phi^\beta(T^\omega_{\mathbf{i}_k}) < c^k$ which implies that $M^\beta(\Sigma^\omega) = 0$. Letting $\beta \downarrow \alpha_0$, we get $d^\omega \le \alpha_0$.

To prove the opposite inequality, consider $\beta > d^\omega$. Then $M^\beta(\Sigma^\omega) = 0$. Given $\varepsilon > 0$, we find an index set $J \subset \Sigma^\omega_*$ such that $\Sigma^\omega \subset \bigcup_{\tau\in J} [\tau]$ and $\sum_{\tau\in J} \Phi^\beta(T^\omega_\tau) < \varepsilon$. Since Σ^ω is compact, we may assume that J is finite. Defining

$$n_{\min} = \min\{|\tau| \mid \tau \in J\} \quad \text{and} \quad n_{\max} = \max\{|\tau| \mid \tau \in J\},$$

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we may assume that J minimizes

$$\sum_{\tau\in K}\Phi^{\beta}(T_{\tau}^{\omega}) \tag{3.5}$$

among all covering sets K that satisfy $n_{\min}\le|\tau|\le n_{\max}$ for all $\tau\in K$. Consider $\zeta\in\Sigma_{n_{\min}}^{\omega}$ and let $J_{\zeta}=\{\tau\in J\mid[\tau]\subset[\zeta]\}$. Since J minimizes (3.5) and each map f_i^{λ} is a similarity, it follows from the homogeneity of ω that $\sum_{\tau\in J_{\zeta}}\Phi^{\beta}(T_{\tau}^{\omega})=\Phi^{\beta}(T_{\zeta}^{\omega})$. Consequently, we obtain

$$S^{\omega}(n_{\min},\beta)=\sum_{\tau\in J}\Phi^{\beta}(T_{\tau}^{\omega})<\varepsilon.$$

Choosing sufficiently large $n_{\min}\in\mathbb{N}$ and sufficiently small $\varepsilon>0$, it follows that $\liminf_{k\to\infty}S^{\omega}(k,\beta)=0$. This yields $p_{\inf}^{\omega}(\beta)\le 0$, and therefore, $\alpha_0\le\beta$. Letting $\beta\downarrow d^{\omega}$, we get $\alpha_0\le d^{\omega}$. □

Remark 3.9. There is not much to be said about the dimension of a code tree fractal for a fixed translation parameter $\mathbf{a}\in\mathbb{R}^{D\mathcal{A}}$. The following can be gleaned from the proof of [**8**, Theorem 7.3]. Let $\omega\in\Omega$. Assume that all the maps are similarities and for some fixed $\mathbf{a}\in\mathbb{R}^{D\mathcal{A}}$ the following uniform open set condition is satisfied: there is a non-empty open set O such that for each $\lambda\in\Lambda$ we have

$$\bigcup_{m=1}^{M_{\lambda}}f_m^{\lambda}(O)\subset O\quad\text{and}\quad f_m^{\lambda}(O)\cap f_n^{\lambda}(O)=\emptyset\quad\text{if } m\ne n. \tag{3.6}$$

Then $\dim_{\mathrm{H}}(A_{\mathbf{a}}^{\omega})=d^{\omega}$.

4. *Random code tree fractals and pressure*

As illustrated in the previous section, in general, the dimension of a code tree fractal is not necessarily given by the zero of the natural pressure. In this section we consider a general class of random affine code tree fractals for which the pressure almost surely exists, is independent of the code tree and has a unique zero. When constructing the random sets in [**13, 17**], independent choices are made at different steps of the construction and the limiting sets have a stochastically self-repeating structure both in space and in scale whilst in our model the random sets are spatially nearly homogeneous. More precisely, suppose that $(\mu_k)_{k=0}^{\infty}$ is a family of probability measures, where each μ_k is a measure on the set of finite code trees of length k. Suppose also that ν is a probability measure on $\mathbb{N}$ with finite first moment. This generates a probability measure P on Ω in the following way: choose a random number N_1 according to ν. Select labels of nodes from Λ from level 0 up to level N_1-1 at random according to μ_{N_1-1}. This specifies a randomly chosen code tree up to level N_1-1. Repeat this procedure, that is, generate a realization of N_2-N_1 according to ν. For some fixed subtree rooted at level N_1 choose labels of nodes of this subtree at random according to the probability measure $\mu_{N_2-N_1-1}$. For all other subtrees rooted at level N_1 select the same labels as in the generated one. This specifies the randomly chosen code tree up to level N_2-1. Continuing in this manner will uniquely define P according to Kolmogorov's extension theorem.

Next we present a convenient formalism to study constructions described above.

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Definition 4.1. Let $\widetilde{\Omega}$ be the set of $(\omega, N) \in \Omega \times \mathbb{N}^{\mathbb{N}}$ such that:

- $N_m < N_{m+1}$ for all $m \in \mathbb{N}$;
- if $\mathbf{i}_{N_m}\mathbf{j}_l, \mathbf{i}'_{N_m} \in \Sigma_*^\omega$, then $\mathbf{i}'_{N_m}\mathbf{j}_l \in \Sigma_*^\omega$ and $\omega(\mathbf{i}_{N_m}\mathbf{j}_l) = \omega(\mathbf{i}'_{N_m}\mathbf{j}_l)$,

where for each $(\omega, N) \in \widetilde{\Omega}$ the sequence $(N_m)_{m\in\mathbb{N}}$ is the list of *neck levels*, that is, all the sub code trees rooted at a neck level N_m are identical. Define $\Xi\colon \widetilde{\Omega} \to \widetilde{\Omega}$ by $\Xi(\omega, N) = (\hat{\omega}, \hat{N})$, where $\hat{N}_m = N_{m+1} - N_1$ and $\hat{\omega}(\mathbf{j}_l) = \omega(\mathbf{i}_{N_1}\mathbf{j}_l)$ for all $m, l \in \mathbb{N}$. The elements of $\widetilde{\Omega}$ are denoted by $\tilde{\omega}$. We equip $\widetilde{\Omega}$ with the topology generated by the cylinders

$$[(\omega, N)_m] = \{(\hat{\omega}, \hat{N}) \in \widetilde{\Omega} \mid \hat{N}_i = N_i \text{ for all } i \le m \text{ and } \hat{\omega}(\tau) = \omega(\tau) \text{ for all } \tau \text{ with } |\tau| < N_m\}$$

and use the Borel σ-algebra on $\widetilde{\Omega}$.

Remark 4.2. (a) Observe that since N_1 is a neck level, the definition of Ξ is independent of the choice of $\mathbf{i}_{N_1}$. With the chosen topology, Ξ is continuous. Since there is no uniform upper bound for N_1, the space $\widetilde{\Omega}$ is not compact.

(b) The functions p^ω, $\Phi^\alpha(T^\omega)$ etc. defined in the previous sections have natural extensions to $\widetilde{\Omega}$, that is, $p^{(\omega,N)} = p^\omega$, $\Phi^\alpha(T^{(\omega,N)}) = \Phi^\alpha(T^\omega)$, etc. We denote them by $p^{\tilde{\omega}}$, $\Phi^\alpha(T^{\tilde{\omega}})$, etc.

We complete this section by proving the existence of the zero of the pressure.

THEOREM 4.3. *Let $0 < \underline{\sigma} \le \overline{\sigma} < 1$ be as in §2. Assume that P is an ergodic Ξ-invariant probability measure on $\widetilde{\Omega}$ such that $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$. Then for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the pressure $p^{\tilde{\omega}}(\alpha)$ exists for all $\alpha \in [0, \infty[$ and is independent of $\tilde{\omega} \in \widetilde{\Omega}$. Furthermore, $p^{\tilde{\omega}}$ is strictly decreasing and there exists a unique α_0 such that $p^{\tilde{\omega}}(\alpha_0) = 0$ for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$.*

Proof. Consider $\alpha \in [0, \infty[$. For $n < m \in \mathbb{N} \cup \{0\}$, let

$$\Sigma_*^{\tilde{\omega}}(n, m) = \{i_{N_n+1} \cdots i_{N_m} \mid \mathbf{i}_{N_n} i_{N_n+1} \cdots i_{N_m} \in \Sigma_*^{\tilde{\omega}}\},$$

where $N_0 = 0$. The fact that N_n is a neck level implies that the definition is independent of the choice of $\mathbf{i}_{N_n}$. Setting

$$X_{n,m}(\tilde{\omega}) = \log\Bigg(\sum_{i_{N_n+1}\cdots i_{N_m} \in \Sigma_*^{\tilde{\omega}}(n,m)} \Phi^\alpha(T_{i_{N_n+1}}^{\tilde{\omega}(i_1\cdots i_{N_n})} \cdots T_{i_{N_m}}^{\tilde{\omega}(i_1\cdots i_{N_m-1})})\Bigg),$$

we have

$$X_{n+1,m+1} = X_{n,m} \circ \Xi$$

by the definition of Ξ. Note that $X_{0,n}(\tilde{\omega}) = \log(S^{\tilde{\omega}}(N_n, \alpha))$, and, moreover, submultiplicativity of Φ^α gives

$$X_{0,m} \le X_{0,n} + X_{n,m} \tag{4.1}$$

for any $0 < n < m$. Since

$$(N_{n+1} - N_n)\log(\underline{\sigma}^\alpha) \le X_{n,n+1} \le (N_{n+1} - N_n)\log(\overline{\sigma}^\alpha M)$$

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and $N_1 \circ \Xi^n = N_{n+1} - N_n$, combining the Ξ-invariance of P and the assumption $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$ gives that $X_{n,m}$ is P-integrable for all $n, m \in \mathbb{N} \cup \{0\}$ with $n < m$. From Kingman's subadditive ergodic theorem (see for example Durrett [**5**]) it follows that the limit

$$\lim_{n\to\infty} \frac{\log(S^{\tilde{\omega}}(N_n, \alpha))}{n} =: \tilde{p}^{\tilde{\omega}}(\alpha)$$

exists for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. Furthermore, $\tilde{p}^{\tilde{\omega}}$ is P-almost surely independent of $\tilde{\omega}$ by ergodicity of P. Observe that in the definition of $\tilde{p}^{\tilde{\omega}}$ there is a neck level N_n in the numerator whilst in the denominator we have n. Birkhoff's ergodic theorem implies that the finite non-random limit

$$\lim_{n\to\infty} \frac{N_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} N_1 \circ \Xi^k = b \geq 1$$

exists for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. In particular,

$$\lim_{n\to\infty} \frac{N_{n+1} - N_n}{n} = 0 = \lim_{n\to\infty} \frac{N_{n+1} - N_n}{N_n}, \tag{4.2}$$

and therefore, the limit

$$\lim_{n\to\infty} \frac{\log(S^{\tilde{\omega}}(N_n, \alpha))}{N_n} = \frac{\tilde{p}^{\tilde{\omega}}(\alpha)}{b}$$

exists for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$.

For any $m \in \mathbb{N}$, let $l(m)$ be a random integer such that $N_{l(m)} \leq m < N_{l(m)+1}$. Since $l(m) \leq m$ and

$$(\underline{\sigma}^{\alpha})^{N_{l(m)+1}-N_{l(m)}} S^{\tilde{\omega}}(N_{l(m)}, \alpha) \leq S^{\tilde{\omega}}(m, \alpha) \leq (\overline{\sigma}^{\alpha} M)^{N_{l(m)+1}-N_{l(m)}} S^{\tilde{\omega}}(N_{l(m)}, \alpha),$$

equation (4.2) implies that the pressure

$$\lim_{m\to\infty} \frac{\log(S^{\tilde{\omega}}(m, \alpha))}{m} = p^{\tilde{\omega}}(\alpha) = \frac{\tilde{p}^{\tilde{\omega}}(\alpha)}{b} \tag{4.3}$$

exists for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. By ergodicity, $p^{\tilde{\omega}}(\alpha)$ is P-almost surely independent of $\tilde{\omega} \in \widetilde{\Omega}$.

We continue by verifying that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the pressure $p^{\tilde{\omega}}(\alpha)$ exists for all $\alpha \in [0, \infty[$, is strictly decreasing in α and has a unique zero at α_0. It follows from Fubini's theorem that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the pressure $p^{\tilde{\omega}}(\alpha)$ exists for $\mathcal{L}$-almost all $\alpha \in [0, \infty[$. Consider such $\tilde{\omega} \in \widetilde{\Omega}$. Using the properties of the singular value function, we obtain for all $\alpha \in [0, \infty[$, for all $\delta > 0$ and for all $m \in \mathbb{N}$

$$\underline{\sigma}^{m\delta} \leq \frac{S^{\tilde{\omega}}(m, \alpha + \delta)}{S^{\tilde{\omega}}(m, \alpha)} \leq \overline{\sigma}^{m\delta}.$$

Hence, both

$$\liminf_{n\to\infty} \frac{\log S^{\tilde{\omega}}(n, \alpha)}{n} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log S^{\tilde{\omega}}(n, \alpha)}{n}$$

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are strictly decreasing and continuous in α. Since for $\mathcal{L}$-almost all $\alpha \in [0, \infty[$ the pressure $p^{\tilde{\omega}}(\alpha)$ exists, it follows that the limit exists for all $\alpha \in [0, \infty[$ and is continuous and strictly decreasing in α. Finally, the fact that for all $n \in \mathbb{N}$

$$\frac{\log S^{\tilde{\omega}}(n, 0)}{n} \geq 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \lim_{n\to\infty} \frac{\log S^{\tilde{\omega}}(n, \alpha)}{n} = -\infty$$

yields the existence of a unique α_0 such that $p^{\tilde{\omega}}(\alpha_0) = 0$. This completes the proof. □

Remark 4.4. By (4.3), the functions $\tilde{p}^{\tilde{\omega}}$ and $p^{\tilde{\omega}}$ have the same unique zero for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$.

Example 4.5. (a) The definition of $\widetilde{\Omega}$ was inspired by the construction of random V-variable fractals introduced by Barnsley *et al* (see [**1–3**]). Let $V \geq 1$ be an integer. A code tree is said to be V-variable if there are at most V distinct sub code trees at each level, and a code tree fractal is V-variable if the code tree is V-variable. The attractors of ordinary iterated function systems are 1-variable fractals also known as homogeneous fractals. In random V-variable code trees the distribution of the spacing between necks has exponentially decreasing tails implying that N_1 has a finite expectation. In particular, the probability measure related to random V-variable fractals introduced by Barnsley *et al* satisfies the assumptions of Theorem 4.3.

It was shown in [**3**] that, under the assumption that all maps are similarities, $S^{\tilde{\omega}}(n, \alpha)$ can be expressed in terms of products of random $V \times V$-matrices and, using the law of large numbers, one obtains

$$p^{\tilde{\omega}}(\alpha) = \int_{\widetilde{\Omega}} \log\Bigg(\sum_{\mathbf{i}_{N_1} \in \Sigma^{\tilde{\omega}}_*(0,1)} (r_{i_1}^{\tilde{\omega}(\emptyset)} r_{i_2}^{\tilde{\omega}(i_1)} \cdots r_{i_{N_1}}^{\tilde{\omega}(i_1 \cdots i_{N_1-1})})^{\alpha} \Bigg) \, dP(\tilde{\omega}),$$

where r_i^{λ} is the similarity ratio of f_i^{λ}. This property is useful when estimating the pressure and the dimension numerically. In general, it is hard to give good numerical estimates for the pressure when $\int_{\widetilde{\Omega}} N_1(\tilde{\omega}) \, dP(\tilde{\omega})$ is large.

(b) Another example of systems satisfying the assumptions of Theorem 4.3 are random Markov homogeneous fractals. Suppose that $\Lambda = \{1, \ldots, \ell\}$ is finite and Q is an ergodic Markov transition matrix on Λ, that is, Q is an $\ell \times \ell$-matrix with non-negative elements and row-sums equal to 1. Let P_0 be a probability measure on Λ.

A probability measure on the set of code trees Ω is generated in the following way: suppose $\omega(\emptyset) = \omega_0$ is chosen according to P_0. Next choose ω_1 according to the probability measure defined by the row ω_0 in Q, that is, $\omega_1 = j$ with probability $Q_{\omega_0 j}$. Further, let $\omega(i) = \omega_1$ for all $i = 1, \ldots, M_{\omega(\emptyset)}$. This defines a measure on Ω up to level 1. We continue inductively in the same manner, that is, $\omega_{k+1} = j$ with probability $Q_{\omega_k j}$ and $\omega(i_1 \cdots i_k l) = \omega_{k+1}$ for all $l = 1, \ldots, M_{\omega(i_1 \ldots i_k)}$. This defines a probability measure P on Ω.

Every realization of the above construction is spatially homogeneous since at every level all the sub code trees are identical. By ergodicity P has a self-repeating time structure in the following sense: let $N_i(\omega)$ be the smallest level $k > N_{i-1}(\omega)$ such that $\omega_{N_i(\omega)} = \omega_0$, where $N_0(\omega) = 0$. The sequence $(N_i)_{i\in\mathbb{N}}$ will define regenerative time points in the sense that the probability measure P on Ω will have self-repeating structure at these random times.

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5. *Dimensions of random code tree fractals*

In this section we show that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the zero of the pressure is equal to the Hausdorff dimension of $A_{\mathbf{a}}^{\tilde{\omega}}$ for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$. For this purpose, we restrict our consideration to the case $D = 2$ and impose additional assumptions on the probability measure P. For a discussion concerning the assumptions of the following theorem, see Remark 5.6.

THEOREM 5.1. *Let $D = 2$ and $0 < \underline{\sigma} \le \overline{\sigma} < \frac{1}{2}$. Assume that P is an ergodic Ξ-invariant probability measure on $\widetilde{\Omega}$. Further, suppose that*

$$P\big(\{\tilde{\omega} \in \widetilde{\Omega} \mid \text{ there exists } v \in \mathbb{R}^2 \backslash \{0\} \text{ such that } T^{\tilde{\omega}}_{\mathbf{i}_{N_1}}(v) \text{ are parallel for all } \mathbf{i}_{N_1} \in \Sigma^{\tilde{\omega}}_*(0,1)\}\big) < 1, \tag{5.1}$$

$$\sum_{l=1}^{\infty} P\left(\left\{\tilde{\omega} \in \widetilde{\Omega} \mid N_1(\tilde{\omega}) \ge \frac{c_0 l}{\log l}\right\}\right) < \infty \quad \text{for all } c_0 > 0 \tag{5.2}$$

and the σ-algebras generated by $\{\tilde{\omega}(\tau) \mid |\tau| < N_m\}$ and $\{\tilde{\omega}(\tau) \mid |\tau| \ge N_m\}$ are independent for all $m \in \mathbb{N}$. Then for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$

$$\dim_{\mathrm{H}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \dim_{\mathrm{p}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \dim_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \min\{\alpha_0, 2\}$$

for $\mathcal{L}^{2\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{2\mathcal{A}}$, where α_0 is the zero of the pressure given in Theorem 4.3.

Proof. Theorem 5.1 is proved as a consequence of a sequence of lemmas. We will first verify that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$, $\min\{\alpha_0, 2\} \le \dim_{\mathrm{H}}(A_{\mathbf{a}}^{\tilde{\omega}})$ for $\mathcal{L}^{2\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{2\mathcal{A}}$. Then we will prove the inequality $\overline{\dim}_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) \le \min\{\alpha_0, 2\}$ for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ and for all $\mathbf{a} \in \mathbb{R}^{2\mathcal{A}}$. The claim then follows since the packing dimension and the lower box counting dimension are between the Hausdorff and upper box counting dimensions (see [**12**, (3.17) and (3.29)]).

To obtain the lower estimate we construct for all $\alpha < \alpha_0$ and for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ a probability measure $\mu^{\tilde{\omega}}$ on $\Sigma^{\tilde{\omega}}$ such that for some constant $c(\tilde{\omega})$

$$\mu^{\tilde{\omega}}([\mathbf{i}_l]) \le c(\tilde{\omega}) \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_l}) \tag{5.3}$$

for all $l \in \mathbb{N}$ (recall (3.4) and the proof of Theorem 3.2). Observe that inequality (3.4) is valid only for $\alpha < d^{\tilde{\omega}}$. For $\tilde{\omega} \in \widetilde{\Omega}$ and $\alpha < \alpha_0$ the measure $\mu^{\tilde{\omega}}$ on $\Sigma^{\tilde{\omega}}$ is defined in the following way: let $m \in \mathbb{N}$ and set

$$\mu^{\tilde{\omega}}_m = \frac{\sum_{\mathbf{i}_{N_m} \in \Sigma^{\tilde{\omega}}_*(0,m)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_m}}) \delta_{\mathbf{i}_{N_m}}}{\sum_{\mathbf{i}_{N_m} \in \Sigma^{\tilde{\omega}}_*(0,m)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_m}})}, \tag{5.4}$$

where $\delta_{\mathbf{i}_{N_m}}$ is the Dirac measure at a fixed point of the cylinder $[\mathbf{i}_{N_m}]$. The choice of the cylinder point plays no role in what follows. Since $\Sigma^{\tilde{\omega}}$ is compact, the sequence $(\mu^{\tilde{\omega}}_m)_{m \in \mathbb{N}}$ has a converging subsequence with a limit measure $\mu^{\tilde{\omega}}$. Observe that the converging subsequence may depend on $\tilde{\omega} \in \widetilde{\Omega}$.

By submultiplicativity of Φ^{α}, we have for all $n, m \in \mathbb{N}$ and $N_{n-1} \le l < N_n$ that

$$\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) \le \frac{\Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_l}) \sum_{\mathbf{j}, \mathbf{i}_l \mathbf{j} \in \Sigma^{\tilde{\omega}}_*(0,n)} \sum_{\mathbf{k}_{N_m} \in \Sigma^{\tilde{\omega}}_*(n,n+m)} \Phi^{\alpha}(T^{\tilde{\omega}}_{(\mathbf{i}_l)\mathbf{j}}) \Phi^{\alpha}(T^{\Xi^n(\tilde{\omega})}_{\mathbf{k}_{N_m}})}{\sum_{\mathbf{i}_{N_n} \in \Sigma^{\tilde{\omega}}_*(0,n)} \sum_{\mathbf{k}_{N_m} \in \Sigma^{\tilde{\omega}}_*(n,n+m)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}} T^{\Xi^n(\tilde{\omega})}_{\mathbf{k}_{N_m}})}, \tag{5.5}$$

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where the last $|\mathbf{j}|$ maps of $T^{\tilde\omega}_{\mathbf{i}\mathbf{j}}$ are denoted by $T^{\tilde\omega}_{(\mathbf{i}_l)\mathbf{j}}$. In order to prove (5.3), we need to estimate the denominator of (5.5) from below by

$$\tilde c(\tilde\omega) \sum_{\mathbf{i}_{N_n}\in\Sigma^{\tilde\omega}_*(0,n)} \sum_{\mathbf{k}_{N_m}\in\Sigma^{\tilde\omega}_*(n,n+m)} \Phi^\alpha(T^{\tilde\omega}_{\mathbf{i}_{N_n}})\Phi^\alpha(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}}) \tag{5.6}$$

for some $\tilde c(\tilde\omega) > 0$.

The verification of the lower bound (5.6) is divided into three different steps. The first two steps are Lemmas 5.4 and 5.5 in which the assumptions (5.1), (5.2) and the independence are not needed. However, the assumptions of Theorem 4.3 have to be valid.

Since $D = 2$, it is enough to consider $0 \le \alpha \le 2$. Letting $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ be linear maps, set

$$\underline{\Phi}^\alpha(T) = \begin{cases} \sigma_2(T)^\alpha & \text{if } \alpha \le 1, \\ \sigma_2(T)\sigma_1(T)^{\alpha-1} & \text{if } \alpha > 1. \end{cases}$$ □

LEMMA 5.2. *Let $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ be linear maps and $0 \le \alpha \le 2$. Then*

$$\Phi^\alpha(TU) \ge \underline{\Phi}^\alpha(T)\Phi^\alpha(U).$$

Proof. The fact that $\sigma_1(TU) \ge \sigma_2(T)\sigma_1(U)$ gives the claim for $\alpha \le 1$. If $\alpha > 1$, write $\beta = 2 - \alpha$. Since $\det(T) = \sigma_1(T)\sigma_2(T)$ and $\sigma_1(T^{-1}) = \sigma_2(T)^{-1}$, we obtain

$$\Phi^\alpha(T) = \sigma_1(T^{-1})^\beta \det(T) \quad \text{and} \quad \underline{\Phi}^\alpha(T) = \sigma_2(T^{-1})^\beta \det(T), \tag{5.7}$$

implying

$$\begin{aligned} \Phi^\alpha(TU) &= \sigma_1(U^{-1}T^{-1})^\beta \det(TU) \ge \sigma_2(T^{-1})^\beta \det(T)\sigma_1(U^{-1})^\beta \det(U) \\ &= \underline{\Phi}^\alpha(T)\Phi^\alpha(U). \end{aligned}$$ □

Remark 5.3. Observation (5.7) implies that $\underline{\Phi}^\alpha$ is supermultiplicative, that is, $\underline{\Phi}^\alpha(TU) \ge \underline{\Phi}^\alpha(T)\underline{\Phi}^\alpha(U)$ for all linear maps $T, U : \mathbb{R}^2 \to \mathbb{R}^2$. Similarly as in the proof of Theorem 4.3 this in turn implies that for P-almost all $\tilde\omega \in \widetilde\Omega$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}_{N_n}\in\Sigma^{\tilde\omega}_*(0,n)} \underline{\Phi}^\alpha(T^{\tilde\omega}_{\mathbf{i}_{N_n}}) =: \underline{\tilde p}^{\tilde\omega}(\alpha) \le \tilde p^{\tilde\omega}(\alpha)$$

exists for all $\alpha \in [0, 2]$.

LEMMA 5.4. *Inequality (5.3) holds for P-almost all $\tilde\omega \in \widetilde\Omega$ for which $\underline{\tilde p}^{\tilde\omega}(\alpha) > 0$.*

Proof. Let $\tilde\omega \in \widetilde\Omega$ satisfy $\underline{\tilde p}^{\tilde\omega}(\alpha) > 0$ and (4.2). There exists $\zeta > 1$ such that for all sufficiently large $n \in \mathbb{N}$

$$\sum_{\mathbf{i}_{N_n}\in\Sigma^{\tilde\omega}_*(0,n)} \underline{\Phi}^\alpha(T^{\tilde\omega}_{\mathbf{i}_{N_n}}) \ge \zeta^n.$$

Hence, inequality (5.5), Lemma 5.2 and the fact $\overline{\sigma} < 1$ yield

$$\mu^{\tilde\omega}_{n+m}([\mathbf{i}_l]) \le M^{N_n - N_{n-1}} \zeta^{-n} \Phi^\alpha(T^{\tilde\omega}_{\mathbf{i}_l})$$

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for all sufficiently large $n \in \mathbb{N}$, for all $m \in \mathbb{N}$ and for all $N_{n-1} \le l < N_n$. By (4.2), for every $\varepsilon > 0$ there exists n_ε such that $N_n - N_{n-1} \le \varepsilon n$ for all $n \ge n_\varepsilon$. This implies the existence of a constant $c(\tilde{\omega}) < \infty$ such that

$$\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) \le c(\tilde{\omega})\Phi^\alpha(T^{\tilde{\omega}}_{\mathbf{i}_l})$$

for all $n, m \in \mathbb{N}$ and $N_{n-1} \le l < N_n$. Since $[\mathbf{i}_l]$ is open, we have $\mu^{\tilde{\omega}}([\mathbf{i}_l]) \le \liminf_{k\to\infty} \mu^{\tilde{\omega}}_{n+m_k}([\mathbf{i}_l])$, where (μ_{n+m_k}) is the subsequence converging to $\mu^{\tilde{\omega}}$. This completes the proof. □

For the remaining two cases of the proof of inequality (5.3), we need the following notation. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. If $\sigma_1(T) > \sigma_2(T)$ there exist unique (up to sign) unit vectors $v_1(T)$ and $v_2(T)$ such that $|T(v_i)| = \sigma_i$ for $i = 1, 2$. Set $w_i(T) = T(v_i)$ for $i = 1, 2$. Observe that $v_1(T)$ and $v_2(T)$ are perpendicular to each other since they are eigenvectors of the self-adjoint map T^*T, and $w_1(T)$ and $w_2(T)$ are also perpendicular as the semi-axes of an ellipse. If $\sigma_1(T) = \sigma_2(T)$, let $v_1(T)$ and $v_2(T)$ be any pair of perpendicular unit vectors. Given two linear maps $T, U : \mathbb{R}^2 \to \mathbb{R}^2$, write

$$\widehat{w}_1(U) := \frac{w_1(U)}{|w_1(U)|} = a v_1(T) + b v_2(T),$$

where a and b depend on T and U. Then

$$\sigma_1(TU) \ge |T(w_1(U))| \ge |a|\sigma_1(T)\sigma_1(U)$$

and the fact $\sigma_2(TU) \ge \sigma_2(T)\sigma_2(U)$ gives

$$\Phi^\alpha(TU) \ge |a|^\alpha \Phi^\alpha(T)\Phi^\alpha(U). \tag{5.8}$$

Since $\alpha < \alpha_0$, we have $\tilde{p}^{\tilde{\omega}}(\alpha) > 0$, and by Lemma 5.4, it remains to consider the case $\underline{\tilde{p}}^{\tilde{\omega}}(\alpha) < \tilde{p}^{\tilde{\omega}}(\alpha)$. For every such $\tilde{\omega} \in \widetilde{\Omega}$, there exist $\lambda > 1$, $0 < \rho < \lambda$, $0 < \gamma < 1$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{\mathbf{i}_{N_n} \in \Sigma^{\tilde{\omega}}_*(0,n)} \Phi^\alpha(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) > \lambda^n \quad \text{and} \quad \sum_{\mathbf{i}_{N_n} \in \Sigma^{\tilde{\omega}}_*(0,n)} \underline{\Phi}^\alpha(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) < (\gamma\rho)^n \tag{5.9}$$

for all $n \ge n_0$. Since the set

$$\widetilde{\Omega}^< = \{\tilde{\omega} \in \widetilde{\Omega} \mid \underline{\tilde{p}}^{\tilde{\omega}}(\alpha) < \tilde{p}^{\tilde{\omega}}(\alpha)\}$$

can be represented as a countable union of sets consisting of points which satisfy (5.9) with rational λ, ρ and γ, we may fix $\lambda > 1$, $0 < \rho < \lambda$ and $0 < \gamma < 1$ for the rest of the proof. We denote the corresponding set by $\widetilde{\Omega}^<_{\lambda,\rho,\gamma}$, and let

$$E_{n_0} = \{\tilde{\omega} \in \widetilde{\Omega}^<_{\lambda,\rho,\gamma} \mid (5.9) \text{ is valid for all } n \ge n_0\}.$$

Since $\bigcup_{n_0=1}^\infty E_{n_0} = \widetilde{\Omega}^<_{\lambda,\rho,\gamma}$, we may also fix $n_0 \in \mathbb{N}$ for the rest of the proof.

Let $1 < \lambda_1 < \lambda$ and $\lambda_1/\lambda < \gamma_1 < 1$. For all $n, m \in \mathbb{N}$, $\mathbf{i}_{N_n} \in \Sigma^{\tilde{\omega}}_*(0, n)$ and $\mathbf{k}_{N_m} \in \Sigma^{\tilde{\omega}}_*(n, n+m)$, let

$$\widehat{w}_1(T^{\Xi^n(\tilde{\omega})}_{\mathbf{k}_{N_m}}) = a v_1(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) + b v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}), \tag{5.10}$$

where a and b are functions of $\mathbf{i}_{N_n}$ and $\mathbf{k}_{N_m}$, and define

$$C^{\tilde{\omega}}_n(\mathbf{k}_{N_m}) = \{\mathbf{i}_{N_n} \in \Sigma^{\tilde{\omega}}_*(0, n) \mid |a| > \lambda_1^{-(n/\alpha)}\}.$$

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LEMMA 5.5. *Let $\underline{\tilde{p}}^{\tilde{\omega}}(\alpha) < \tilde{p}^{\tilde{\omega}}(\alpha)$. Assuming that for all large n, $m \in \mathbb{N}$ we have*

$$\sum_{\mathbf{i}_{N_n} \in C_n^{\tilde{\omega}}(\mathbf{k}_{N_m})} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) > \gamma_1^n \sum_{\mathbf{i}_{N_n} \in \Sigma_*^{\tilde{\omega}}(0,n)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) \tag{5.11}$$

for all $\mathbf{k}_{N_m} \in \Sigma_*^{\tilde{\omega}}(n, n+m)$, inequality (5.3) is valid.

Proof. Let $n \geq n_0$ and $m \in \mathbb{N}$ be so large that inequality (5.11) holds. Consider $N_{n-1} \leq l < N_n$. From (5.5), (5.8), assumption (5.11) and the first inequality in (5.9) we get

$$\mu^{\tilde{\omega}}_{n+m}([\mathbf{i}_l]) \leq \left(\frac{\lambda_1}{\lambda}\right)^n M^{N_n - N_{n-1}} \gamma_1^{-n} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_l}).$$

Proceeding as in the proof of Lemma 5.4, it follows that (5.3) holds. □

Below we will prove that the assumption in (5.11) holds for P-almost all $\tilde{\omega} \in \widetilde{\Omega}^{<}$. The heuristic idea of the proof is as follows: if the assumption in (5.11) is false, then the longer semiaxes of most of the ellipses $T^{\Xi^n(\tilde{\omega})}_{\mathbf{k}_{N_m}}(B(0,1))$ are almost parallel to $v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}})$ for most $\mathbf{i}_{N_n}$. In particular, most of $v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}})$ are nearly parallel to each other. By assumption (5.2), the distance $N_n - N_{n-1}$ is small compared to n, and therefore, $v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-1}}})$ is roughly parallel to the image of $v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}})$ under $T^{\Xi^{n-1}(\tilde{\omega})}_{\mathbf{i}_{N_1}}$. In this manner one finds a vector which is mapped in the same way by all the maps between the levels N_{n-1} and N_n, and by assumption (5.1), the probability of this event is smaller than $\eta < 1$. Selecting an integer h in a suitable manner and repeating this argument in the blocks $[N_{n-h}, N_{n-h+1}], \ldots, [N_{n-1}, N_n]$ one may attach a vector v_t to each $[N_{n-t}, N_{n-t+1}]$ such that v_t is mapped in the same manner by all the maps between the levels N_{n-t} and N_{n-t+1}. By independence, this happens with probability less than η^h, and the Borel–Cantelli lemma implies that this may happen only for finitely many $n \in \mathbb{N}$.

We will now make this heuristic idea precise. Consider $c_0 > 0$. For $L \in \mathbb{N}$, denote by $\widetilde{\Omega}(L)$ the set of $\tilde{\omega} \in \widetilde{\Omega}$ for which

$$N_{l+1} - N_l \leq \frac{c_0 l}{\log l} \tag{5.12}$$

for all $l \geq L$. Since $N_1 \circ \Xi^n = N_{n+1} - N_n$ and P is Ξ-invariant, assumption (5.2) combined with the Borel–Cantelli lemma implies that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ there is an $L \in \mathbb{N}$ such that $\tilde{\omega} \in \widetilde{\Omega}(L)$. Fix $L \in \mathbb{N}$ for the rest of the proof.

By considering a decreasing sequence of events, it follows from assumption (5.1) that there exist $\delta > 0$ and $\eta < 1$ such that

$$\begin{aligned} P(\{\tilde{\omega} \in \widetilde{\Omega} \mid \text{ there exists } v \in \mathbb{R}^2 \backslash \{0\} \text{ such that } \sphericalangle(T^{\tilde{\omega}}_{\mathbf{i}_{N_1}}(v),\, T^{\tilde{\omega}}_{\mathbf{i}'_{N_1}}(v)) \leq \delta \\ \text{for all } \mathbf{i}_{N_1}, \mathbf{i}'_{N_1} \in \Sigma_*^{\tilde{\omega}}(0,1)\}) < \eta, \end{aligned} \tag{5.13}$$

where $\sphericalangle(v, u)$ is the angle between $u, v \in \mathbb{R}^2 \backslash \{0\}$. For $n \in \mathbb{N}$, let

$$\begin{aligned} H_n = \{\tilde{\omega} \in \widetilde{\Omega} \mid \text{ there exists } v \in \mathbb{R}^2 \backslash \{0\} \text{ such that } \sphericalangle(T^{\Xi^{n-1}(\tilde{\omega})}_{\mathbf{i}_{N_1}}(v),\, T^{\Xi^{n-1}(\tilde{\omega})}_{\mathbf{i}'_{N_1}}(v)) \\ \leq \delta \text{ for all } \mathbf{i}_{N_1}, \mathbf{i}'_{N_1} \in \Sigma_*^{\tilde{\omega}}(n-1, n)\}. \end{aligned}$$

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Since $H_n = \Xi^{-(n-1)}(H_1)$, we have $P(H_n) < \eta$ for all $n \in \mathbb{N}$. Furthermore, by assumption, the events H_n and H_m are independent for $n \neq m$. Observe that H_n is independent of γ, γ_1, λ, λ_1, ρ, n_0 and L.

For all $n, m \in \mathbb{N}$ with $n \geq n_0$, set

$$G_{n,m} = \{\tilde{\omega} \in E_{n_0} \mid \text{ there exists } \mathbf{k}_{N_m} \in \Sigma_*^{\tilde{\omega}}(n, n+m) \text{ for which inequality (5.11) is not valid}\},$$

and define $G_n = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} G_{n,m}$. Let

$$B_n^{\tilde{\omega}} = \left\{ \mathbf{i}_{N_n} \in \Sigma_*^{\tilde{\omega}}(0, n) \;\middle|\; \frac{\sigma_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}})}{\sigma_1(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}})} \geq \left(\frac{\rho}{\lambda}\right)^{n/\beta} \right\},$$

where $\beta = \alpha$ for $\alpha \leq 1$ and $\beta = 2 - \alpha$ for $\alpha > 1$. From (5.9) one obtains

$$\sum_{\mathbf{i}_{N_n} \in B_n^{\tilde{\omega}}} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) < \gamma^n \sum_{\mathbf{i}_{N_n} \in \Sigma_*^{\tilde{\omega}}(0,n)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) \tag{5.14}$$

for all $n \geq n_0$.

By Lemmas 5.4 and 5.5 it is enough to prove that for P-almost all $\tilde{\omega} \in E_{n_0}$ the number of those $n \geq n_0$, for which $\tilde{\omega} \in G_n$, is finite, that is, $P(\bigcap_{k=n_0}^{\infty} \bigcup_{n=k}^{\infty} G_n) = 0$. Let $n \geq n_0$ and $\tilde{\omega} \in G_n$. Fix $m \in \mathbb{N}$ and $\mathbf{k}_{N_m} \in \Sigma_*^{\tilde{\omega}}(n, n+m)$ for which inequality (5.11) is not valid. Choose $h \in \mathbb{N}$ with $h \leq h_n$ where $h_n = \lfloor C \log n \rfloor$ is the integer part of $C \log n$ for some large C to be fixed later. We claim that for large enough $n \in \mathbb{N}$ there is $\mathbf{i}_{N_{n-h}} \in \Sigma_*^{\tilde{\omega}}(0, n-h) \backslash B_{n-h}^{\tilde{\omega}}$ such that $\mathbf{i}_{N_{n-h}} \mathbf{j}_{N_h} \in \Sigma_*^{\tilde{\omega}}(0, n) \backslash C_n^{\tilde{\omega}}(\mathbf{k}_{N_m})$ for all $\mathbf{j}_{N_h} \in \Sigma_*^{\tilde{\omega}}(n-h, n)$. Indeed, if for every $\mathbf{i}_{N_{n-h}} \in \Sigma_*^{\tilde{\omega}}(0, n-h) \backslash B_{n-h}^{\tilde{\omega}}$ there exists $\mathbf{j}_{N_h} \in \Sigma_*^{\tilde{\omega}}(n-h, n)$ such that $\mathbf{i}_{N_{n-h}} \mathbf{j}_{N_h} \in C_n^{\tilde{\omega}}(\mathbf{k}_{N_m})$, inequalities (5.14) and (5.12) yield

$$\begin{aligned}
\sum_{\mathbf{i}_{N_n} \in C_n^{\tilde{\omega}}(\mathbf{k}_{N_m})} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}) &\geq \sum_{\mathbf{i}_{N_{n-h}} \in \Sigma_*^{\tilde{\omega}}(0,n-h) \backslash B_{n-h}^{\tilde{\omega}}} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}} T^{\Xi^{n-h}(\tilde{\omega})}_{\mathbf{j}_{N_h}}) \\
&\geq (\underline{\sigma}^{\alpha})^{N_n - N_{n-h}} (1 - \gamma^{n-h}) \sum_{\mathbf{i}_{N_{n-h}} \in \Sigma_*^{\tilde{\omega}}(0,n-h)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}}) \\
&\geq \left(\frac{\underline{\sigma}^{\alpha}}{\overline{\sigma}^{\alpha} M}\right)^{hc_0(n/\log n)} (1 - \gamma^{n-h}) \sum_{\mathbf{i}_{N_n} \in \Sigma_*^{\tilde{\omega}}(0,n)} \Phi^{\alpha}(T^{\tilde{\omega}}_{\mathbf{i}_{N_n}}).
\end{aligned} \tag{5.15}$$

Choosing sufficiently small $c_0 > 0$ and sufficiently large $n \in \mathbb{N}$ (recall $h \leq C \log n$), this contradicts the assumption that inequality (5.11) is not valid.

Consider $\mathbf{i}_{N_{n-h}} \in \Sigma_*^{\tilde{\omega}}(0, n-h) \backslash B_{n-h}^{\tilde{\omega}}$ and $\mathbf{j}_{N_h} \in \Sigma_*^{\tilde{\omega}}(n-h, n)$ such that $\mathbf{i}_{N_{n-h}} \mathbf{j}_{N_h} \in \Sigma_*^{\tilde{\omega}}(0, n) \backslash C_n^{\tilde{\omega}}(\mathbf{k}_{N_m})$. For any unit vector $v \in S^1 \subset \mathbb{R}^2$, write

$$\frac{T^{\Xi^{n-h}(\tilde{\omega})}_{\mathbf{j}_{N_h}}(v)}{|T^{\Xi^{n-h}(\tilde{\omega})}_{\mathbf{j}_{N_h}}(v)|} = c(v) v_1(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}}) + d(v) v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}}). \tag{5.16}$$

Defining

$$v_2^{-1} = \frac{(T^{\Xi^{n-h}(\tilde{\omega})}_{\mathbf{j}_{N_h}})^{-1}(v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}}))}{|(T^{\Xi^{n-h}(\tilde{\omega})}_{\mathbf{j}_{N_h}})^{-1}(v_2(T^{\tilde{\omega}}_{\mathbf{i}_{N_{n-h}}}))|} \in S^1,$$

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we observe that

$$\begin{aligned}|c(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))|\ \sigma_1(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})\ |T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))|\\ \le |T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))| \le |T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}(v_2^{-1})|\\ = \sigma_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})|T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_2^{-1})|.\end{aligned}$$

Since $\mathbf{i}_{N_{n-h}} \notin B^{\tilde\omega}_{n-h}$, we have for all $n \ge L-h$

$$\begin{aligned}|c(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))| &\le \frac{\sigma_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})}{\sigma_1(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})}\frac{|T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_2^{-1})|}{|T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))|}\\ &< \left(\frac{\rho}{\lambda}\right)^{(n-h)/\beta}\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{hc_0(n/\log n)}. \end{aligned} \tag{5.17}$$

We continue by showing that $T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}}))$ and $T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}'_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}}))$ are nearly parallel to each other for all $\mathbf{j}_{N_h}, \mathbf{j}'_{N_h} \in \Sigma^{\tilde\omega}_*(n-h,n)$ when $n \in \mathbb{N}$ is sufficiently large. Indeed, recalling (5.10) and (5.16), we have

$$\begin{aligned}T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})) &= a\ T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_1(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))\\ &\quad + b\ |T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))|(c(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))v_1(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})\\ &\quad + d(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})),\end{aligned}$$

and thus

$$\begin{aligned}&|\tan(\sphericalangle(T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})),\ v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})))|\\ &\le \left|\frac{c(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))}{d(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))}\right| + \left|\frac{ea}{bd(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))}\right|\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{N_h},\end{aligned}$$

where b, $d(v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))$ and e are close to one. The factor e appears since $T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(v_1(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}\mathbf{j}_{N_h}}))$ need not to be perpendicular to $v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})$. By using the definition of $C^{\tilde\omega}_n(\mathbf{k}_{N_m})$ and inequality (5.17) we deduce that for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned}&\sphericalangle(T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})),\ T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}'_{N_h}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})))\\ &\quad\le 2\max_{\mathbf{j}=\mathbf{j}_{N_h},\mathbf{j}'_{N_h}}\sphericalangle(T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}}(\widehat{w}_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})),\ v_2(T^{\tilde\omega}_{\mathbf{i}_{N_{n-h}}})) \le 5\gamma_2^{n-h}\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{hc_0(n/\log n)}\end{aligned} \tag{5.18}$$

for all $\mathbf{j}_{N_h}, \mathbf{j}'_{N_h} \in \Sigma^{\tilde\omega}_*(n-h,n)$, where $\gamma_2 = \max\{\lambda_1^{-1/\alpha}, (\rho/\lambda)^{1/\beta}\}$.

Writing $\mathbf{j}_{N_h} = \mathbf{j}_{N_1}\mathbf{j}_{N_{h-1}}$, we get

$$T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_h}}(w_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}})) = T^{\Xi^{n-h}(\tilde\omega)}_{\mathbf{j}_{N_1}}(T^{\Xi^{n-h+1}(\tilde\omega)}_{\mathbf{j}_{N_{h-1}}}(w_1(T^{\Xi^n(\tilde\omega)}_{\mathbf{k}_{N_m}}))).$$

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Now, for any fixed $\mathbf{j}_{N_{h-1}} \in \Sigma_*^{\tilde{\omega}}(n-h+1, n)$ inequality (5.18) implies

$$\begin{aligned} &\sphericalangle(T_{\mathbf{j}_{N_1}}^{\Xi^{n-h}(\tilde{\omega})}(T_{\mathbf{j}_{N_{h-1}}}^{\Xi^{n-h+1}(\tilde{\omega})}(w_1(T_{\mathbf{k}_{N_m}}^{\Xi^n(\tilde{\omega})}))),\ T_{\mathbf{j}'_{N_1}}^{\Xi^{n-h}(\tilde{\omega})}(T_{\mathbf{j}_{N_{h-1}}}^{\Xi^{n-h+1}(\tilde{\omega})}(w_1(T_{\mathbf{k}_{N_m}}^{\Xi^n(\tilde{\omega})})))) \\ &\quad \le 5\gamma_2^{n-h}\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{hc_0(n/\log n)} \end{aligned} \tag{5.19}$$

for all $\mathbf{j}_{N_1}, \mathbf{j}'_{N_1} \in \Sigma_*^{\tilde{\omega}}(n-h, n-h+1)$. Let $n_\delta \in \mathbb{N}$ be such that

$$5\gamma_2^{n_\delta - h_{n_\delta}}\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{h_{n_\delta}c_0(n_\delta/\log n_\delta)} < \delta.$$

From (5.19) we see that for large enough $n \ge n_\delta$ we have $\tilde{\omega} \in H_{n-h+1}$ for all $h \le h_n$. By independence, inequality (5.13) implies that for all such $n \ge n_\delta$

$$P(G_n) \le P\left(\bigcap_{l-h_n+1}^{n} H_l\right) < \eta^{h_n}.$$

Select sufficiently large C and sufficiently small c_0 such that

$$\eta^C < e^{-1}, \quad \sqrt{\gamma_2}\left(\frac{\overline{\sigma}}{\underline{\sigma}}\right)^{Cc_0} < 1 \quad \text{and} \quad \gamma_1 < \left(\frac{\underline{\sigma}^\alpha}{\overline{\sigma}^\alpha M}\right)^{Cc_0}.$$

Moreover, let $n \in \mathbb{N}$ be so large that

$$n - C\log n > \frac{n}{2}, \quad n \ge \max\{2L, n_\delta, 2n_0\} \quad \text{and} \quad \left(\gamma_1\left(\frac{\overline{\sigma}^\alpha M}{\underline{\sigma}^\alpha}\right)^{Cc_0}\right)^n < 1 - \sqrt{\gamma}.$$

Observing that for such n all the arguments above (see (5.15), (5.17) and (5.19)) hold, gives $P(G_n) \le \eta^{\lfloor C\log n\rfloor}$. Since $\sum_{n=1}^\infty \eta^{C\log n} < \infty$, the Borel–Cantelli lemma implies that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}(L) \cap E_{n_0}$ we have $\tilde{\omega} \in G_n$ only for finitely many $n \in \mathbb{N}$. Since this is true for all $L, n_0 \in \mathbb{N}$ (and for all rational $1 < \lambda_1 < \lambda$, $0 < \rho < \lambda$, $0 < \gamma < 1$ and $\lambda_1/\lambda < \gamma_1 < 1$), inequality (5.3) is valid for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$. Hence the inequality $\dim_{\mathrm{H}} A_{\mathbf{a}}^{\tilde{\omega}} \ge \min\{\alpha_0, 2\}$ follows for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$, for $\mathcal{L}^{2\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{2\mathcal{A}}$.

It remains to prove that for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$

$$\overline{\dim}_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) \le \min\{\alpha_0, 2\} \tag{5.20}$$

for all $\mathbf{a} \in \mathbb{R}^{2\mathcal{A}}$. When $\alpha_0 \ge 2$, this certainly holds. Notice that by Theorem 4.3, for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the pressure exists for all $\alpha \in [0, \infty[$ and further, $p^{\tilde{\omega}}(\alpha) < 0$ for $2 > \alpha > \alpha_0$. Fix $2 > \alpha > \alpha_0$. Then for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$, for some $0 < \gamma < 1$, and for all large k, we have $S^{\tilde{\omega}}(k, \alpha) \le \gamma^k$. Summing up over k gives

$$\sum_{k=0}^{\infty} \sum_{\mathbf{i}_k \in \Sigma_*^{\tilde{\omega}}} \Phi^\alpha(T_{\mathbf{i}_k}^{\tilde{\omega}}) \le C_1 < \infty.$$

The fact $\overline{\dim}_{\mathrm{B}}(A_{\mathbf{a}}^{\tilde{\omega}}) \le \alpha$ now follows as in [**13**, Theorem 4.1]. This implies (5.20) since for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$ the argument can be carried out for all $\alpha_0 < \alpha < 2$, completing the proof of Theorem 5.1. □

We conclude this section by discussing the assumptions of Theorem 5.1.

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Remark 5.6. We say that a family $F^\lambda = \{f_1, \ldots, f_n\}$ is parallel if there exists v such that $f_1(v), \ldots, f_n(v)$ are parallel. If a family F^λ is not parallel, it is irreducible in the sense of [**4**], see also [**14**]. Note that the complement of the set of parallel families is open in any reasonable metric. The set of non-parallel families is also dense in the following sense: suppose $F^\lambda = \{f_1, \ldots, f_n\}$ is parallel. Then the family $F^{\lambda'} = \{f_1, \ldots, f_n, R_\varepsilon \circ f_1\}$, where R_ε is a rotation by angle ε, is not parallel. If we view F^λ as a degenerate family $\{f_1, \ldots, f_n, f_1\}$, then $F^{\lambda'}$ is close to $\{f_1, \ldots, f_n, f_1\}$. Assumption (5.1) is thus weak in the sense that if it is not satisfied then after a small perturbation in the family **F** and in the measure P it is satisfied. In particular, if all the families in **F** are non-parallel, then (5.1) is valid for any P for which $P(\{\tilde{\omega} \in \widetilde{\Omega} \mid N_1(\tilde{\omega}) = 1\}) > 0$. Further, the validity of (5.1) is not destroyed by small perturbations. Of course, it should be noted that ergodicity is not necessarily preserved under perturbations.

Assumption (5.1) is not necessary for the validity of Theorem 5.1. Indeed, it is not difficult to show that Theorem 5.1 still holds if assumption (5.1) is replaced by the assumption that all the maps in **F** fix two given directions. Therefore, the deterministic iterated function system considered in [**6**, Example 2] (see also [**7, 21, 23**]) is a special case of our theorem. Hence, in Theorem 5.1, the assumption $\overline{\sigma} < \frac{1}{2}$ is necessary.

By (4.2), the condition $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$ implies that for any $\varepsilon > 0$ we have $N_{l+1} - N_l < \varepsilon l$ for large enough $l \in \mathbb{N}$. Thus, by (5.12) assumption (5.2) is only slightly stronger than the condition $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$.

The strongest assumptions in Theorem 5.1 are the independence between neck levels and the condition $D = 2$. In the proof the role of independence is quite clear whilst the restriction $D = 2$ is more hidden. The latter assumption is used in the definition of the lower pressure $\underline{\tilde{p}}^{\tilde{\omega}}$. Clearly, the definition could be modified in the higher dimensional case. However, for $D \geq 3$ the main problem is that ellipsoids have more than two semiaxes and, for example, the counterpart of inequality (5.17) is not obvious.

Examples of measures satisfying assumption (5.2) and the independence between neck levels are discussed in Example 4.5. For example, assuming exponential decay for $P(\{\tilde{\omega} \in \widetilde{\Omega} \mid N_1(\tilde{\omega}) \geq t\})$ implies (5.2).

According to the following proposition for self-similar code tree fractals the assumptions of Theorem 4.3 (with $\overline{\sigma} < 1$ replaced by $\overline{\sigma} < \frac{1}{2}$) are sufficient for the validity of the dimension formula.

PROPOSITION 5.7. *Let P be an ergodic Ξ-invariant probability measure on $\widetilde{\Omega}$ such that $\int_{\widetilde{\Omega}} N_1(\tilde{\omega})\, dP(\tilde{\omega}) < \infty$. Assume that* **F** *is a family of iterated function systems consisting of similarities on $\mathbb{R}^D$ and $0 < \underline{\sigma} \leq \overline{\sigma} < \frac{1}{2}$. Then for P-almost all $\tilde{\omega} \in \widetilde{\Omega}$*

$$\dim_{\mathrm{H}}(A_{\mathbf{a}}^{\tilde{\omega}}) = \min\{\alpha_0, D\}$$

for $\mathcal{L}^{D\mathcal{A}}$-almost all $\mathbf{a} \in \mathbb{R}^{D\mathcal{A}}$.

Proof. Since the mappings are similarities, Φ^α is multiplicative. We may proceed as in the proof of Lemma 5.4 to verify that the measure constructed in (5.4) satisfies (5.3). The conclusion of Proposition 5.7 then follows as in the proof of Theorem 3.2 as described in the beginning of the the proof of Theorem 5.1. □

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Remark 5.8. Combining (5.3) with the methods of [**3**], we obtain a generalization of the main theorem in [**3**], that is, under the assumptions of Theorem 4.3 and assuming that the maps are similarities, the dimension formula is valid P-almost surely for any $\mathbf{a} \in \mathbb{R}^{D,\mathcal{A}}$ for which the uniform open set condition (3.6) is satisfied.

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