which you start, and you must go straight from town to town – never turning off at crossroads."

The answer on page 369 simply said: "The number of different routes is 264. It is quite a difficult puzzle, and consideration of space does not admit of my showing the best method of making the count."

This sounded like a challenge. Since the graph is a complete graph on five vertices I decided to start by counting all the paths whose first visits to each town occur in the order *ABCDE*. Any of the 23 non-trivial permutations of *B*, *C*, *D* and *E* should then give a different valid path. Now the only possible ways to start are *ABCAD*, *ABCDA*, *ABCDB* and *ABCDE*. In each case, on removal of the used arcs and untangling the remainder to remove crossings it is easy to list and count the possible continuations, getting 6, 4, 6 and 6, respectively. This gives a total of 22, and multiplying by 24 gives 528. I have no idea why Dudeney got his answer. The number of different Eulerian cycles is indeed 264, but each should be counted twice, since it passes through A on two occasions. It is also interesting that (the late) Martin Gardner, although he corrected many of the errors in the original, has missed this one.

It was certainly a lot easier than the proof of Fermat's Last Theorem!

Reference

1. Henry-Ernest-Dudeney-536-Puzzles-and-Curious-Problems-Scribner-1970.pdf (xn--webducation-dbb.com)

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Feedback

On 105.28: Peter Giblin writes: In his interesting Note Clive Johnson establishes conditions for a sequence of the form $x_{m+1} = \dfrac{ax_m + b}{cx_m + d}$, $m = 0, 1, 2, \ldots$, where $ad \neq bc$ and x_0 is a given real initial term, to be periodic with a given period n. (Traditionally if $cx_m + d = 0$ then x_{m+1} is declared to be ∞ and the next term x_{m+2} is a/c. That is the real line is 'compactified with a single point at infinity'.) In what follows I shall assume the sequence is real, that is a, b, c, d and x_0 are real numbers, but this is not strictly necessary. Summarising his statements:

> The above sequence has period $n > 1$, that is $x_n = x_0$ for all (real) choices of initial term x_0 and this does not hold for smaller values of n, if, and only if, the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the property that M_n is a scalar matrix (the identity matrix multiplied by a real number), and this is not true for any smaller value of n. Furthermore this holds if,

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and only if, the eigenvalues λ_1, λ_2 of M satisfy $(\lambda_1/\lambda_2)^n = 1$ and λ_1/λ_2 is a primitive nth root of 1. (The (necessarily complex) eigenvalues of M are the roots of

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

and λ_1^n, λ_2^n are the eigenvalues of M^n.)

As the author remarks, it is hard to find explicit references to these results, though they are in some sense 'well known'! It is very useful to have a reference for them, especially as the detailed implications are laid bare. I have two supplementary remarks, the first of which I owe to a conversation with Bronisław Wajnryb of the Rzeszów University of Technology in Poland some years ago.

Let $\lambda_1/\lambda_2 = r$ say so that $a + d = \lambda_1 + \lambda_2 = \lambda_2(1 + r)$ and $ad - bc = \lambda_1\lambda_2 = r\lambda_2^2$. Hence

$$ad - bc = r\left(\frac{a+d}{1+r}\right)^2 \text{ so that } \frac{(a+d)^2}{ad-bc} = \frac{1}{r} + 2 + r = 2\big(\mathrm{Re}(r) + 1\big),$$

where $\mathrm{Re}(r)$ is the real part of r, namely $\cos\big(2\pi k/n\big)$ for some k coprime to n. The point of this is that it shows that all primitive nth roots of 1 with distinct real parts give *quadratic* conditions on the coefficients a, b, c, d determining the sequence. We can take $c = 1$ without loss of generality and then each quadratic condition is of the form $(a+d)^2 = 2(ad-b)\big(\mathrm{Re}(r)+1\big)$ which is a surface in a, b, d space given as a graph, with b a quadratic function of a, c, d. It is not hard to check that for $n > 2$ this surface is a paraboloid lying in the half-space $b \leqslant 0$ (for $n = 2$ it is the plane $a + d = 0$) and that all the surfaces have as limiting points the constant sequences where $ad - bc = 0$, that is $b = ad$ here. The result is particularly striking in the case $n = 5$ where there are explicit formulas for $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\cos\frac{4\pi}{5} = -\frac{1+\sqrt{5}}{4}$ which are the distinct real parts of two primitive fifth roots of 1. These give an explicit factorization of the condition for a Möbius sequence of period 5, namely

$$\big(2a^2 + \big(3+\sqrt{5}\big)bc + \big(1-\sqrt{5}\big)ad + 2d^2\big)\big(2a^2 + \big(3-\sqrt{5}\big)bc + \big(1+\sqrt{5}\big)ad + 2d^2\big) = 0.$$

Attractive plots can be made of the nested paraboloids arising from this construction (taking $c = 1$ as above) and these have been incorporated into a number of projects by Nuffield Research Placement sixth formers, for example Matthew Temple, whose work is written up in [1].

The second remark is that it is easily proved by induction that $2\cos nx$ is a *monic* polynomial in $2\cos x$ with integer coefficients, that is with leading coefficient 1. For example, writing C for $2\cos x$, we have

$$2\cos 3x = C^3 - 3C$$

and

$$2\cos 8x = C^8 - 8C^6 + 20C^4 - 16C^2 + 2.$$

It follows from this that if $2\cos x$ is rational, say p/q in lowest terms with $q > 1$, then x cannot be of the form $2k\pi/n$. Since $\cos 2k\pi = 1$ the latter

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leads to the equation

$$2q^N = p^N + a_1qp^{N-1} + a_2q^2p^{N-2} + \dots,$$

for some N and integers $a_1, a_2, \dots$. This is impossible as q divides every term except p^N. From this we deduce for example that the Möbius sequence with $a = 1, b = 1, c = -2, d = 1$ cannot give an n-cycle for any n, since the ratio of eigenvalues comes to $\frac{1}{3}(-1 + 2\sqrt{2}i)$ with real part $-\frac{1}{3}$, that is $2\cos x = -\frac{2}{3}$ in the notation above. So the ratio of eigenvalues cannot be an nth root of 1 for any n. This Möbius sequence is 'chaotic'.

References

1. Matthew Temple, *Oscillating Möbius sequences*, Nuffield Bursary Project 2012, https://www.liv.ac.uk/~pjgiblin/papers/Matthew.pdf

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On 105.49: Owen Toller writes: Des MacHale starts with the apparently uncontentious statement that "If $P = Q$, then $Q = P$". But anyone who has tried to teach elementary algebra knows that actual usage of the equals sign does not always work like this. No learner would be happy with giving the last line of an equation as "$3 = x$" instead of "$x = 3$". In such a case, of course "=" means "the value of x is 3", which is not at all the same as an abbreviation for ".aeq." in Robert Recorde's usage. And this is not a trivial point – learners really struggle with it! Suppose a learner is faced with the equation $\frac{12}{x} = 18$. Many want to start by dividing both sides by 12, and obtain, wrongly, $x = 18 \div 12$. So a teacher will often advise learners to clear fractions first, going to $12 = 18x$, after which they are less likely to go wrong. (The situation is even more clear-cut with an inequality such as $12 - 5x < 2$.) But learners find it counterintuitive to put x on the right of an expression, as they are used from a very early stage to give their answers with x on the left-hand side of an equals sign. Of course there is no practical prospect of outlawing the equals sign in "assign" statements such as "$x = 3$" at this stage of mathematical history (though there might have been something to be said for it in the brief period when programming was taught more widely). Readers may care to ask themselves which equals signs in Professor MacHale's article are non-symmetric in the sense I have indicated.

On transitivity, Professor MacHale gives a good example in his Example 9; but many teachers and examiners frown upon learners who attempt to prove trigonometric identities of the form '$A \equiv B$' by showing that $A \equiv C$ and $B \equiv C$. Of course here the issue is usually claimed to be the irreversibility of the implications involved, but in most examples encountered in school the implications are all two-way anyway.

So perhaps the conclusion ought to be that some equals signs are more symmetric (or perhaps even more transitive) than others?

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