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On a separation theorem of Rogers

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In (1) (where references and terminology may be found), C. A. Rogers has proved the following surprising theorem.

Let $\mathscr{H}$ be a class of sets, all Souslin-$\mathscr{F}$ sets, in a topological space X. If A is a Souslin-$\mathscr{H}$ set and B is an analytic set that does not meet A, then there is a Borelian-$\mathscr{H}$ set C with $A \subset C$ and $C \cap B = \varnothing$.

A careful analysis of his reasoning, in the light of Sierpiński's proof ((2), (3)) of an abstract form of the Souslin-Lusin separation theorem, led me to the direct proof to be presented here. No use will be made of transfinite numbers or the theory of sieves, and this may make it easier to decide to what extent further generalization is possible.

We may and shall suppose that $\mathscr{H}$ is closed under the operations of countable union and countable intersection. All point sets referred to will be understood to be in X. Given sets V, W, we write $V \rhd W$ if there exists a set $C \in \mathscr{H}$ with $V \subset C \subset X \backslash W$; if not, we write $V \not\rhd W$. In this notation, we are to prove that $A \rhd B$. We use three simple lemmas.

Lemma 1. *If $V_r \rhd W$ for $r = 1, 2, \ldots$ then $\bigcup_{r=1}^{\infty} V_r \rhd W$.*

Proof. Let $V_r \subset C_r \subset X \backslash W$, where $C_r \in \mathscr{H}$. Then

$$\bigcup_{r=1}^{\infty} V_r \subset \bigcup_{r=1}^{\infty} C_r \subset X \backslash W \quad \text{and} \quad \bigcup_{r=1}^{\infty} C_r \in \mathscr{H}.$$

Lemma 2. *If $V \rhd W \cap D$, where $V \subset D \in \mathscr{H}$, then $V \rhd W$.*

Proof. Let $V \subset C \subset X \backslash (W \cap D)$, where $C \in \mathscr{H}$. Then $V \subset C \cap D \subset X \backslash W$ and $C \cap D \in \mathscr{H}$.

Lemma 3. *If $V \rhd W_r$ for $r = 1, 2, \ldots$ then $V \rhd \bigcup_{r=1}^{\infty} W_r$.*

Proof. Let $V \subset C_r \subset X \backslash W_r$, where $C_r \in \mathscr{H}$. Then

$$V \subset \bigcap_{r=1}^{\infty} C_r \subset X \backslash \left(\bigcup_{r=1}^{\infty} W_r \right) \quad \text{and} \quad \bigcap_{r=1}^{\infty} C_r \in \mathscr{H}.$$

Now we introduce some notation. By hypothesis, we can write

$$A = \bigcup_{\mathbf{i} \in} \bigcap_{n=1}^{\infty} A(\mathbf{i}|n)$$

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where $A(\mathbf{i}|n) \in \mathscr{H}$, and

$$A(\mathbf{i}|n) = \bigcup_{\mathbf{j} \in \mathbf{I}} \bigcap_{m=1}^{\infty} F(\mathbf{i}|n; \mathbf{j}|m),$$

where $F(\mathbf{i}|n; \mathbf{j}|m) \in \mathscr{F}$. Moreover

$$B = \bigcup_{\mathbf{k} \in \mathbf{I}} B(\mathbf{k}),$$

where $B\colon \mathbf{I} \to \mathscr{K}$ is semicontinuous. We define

$$A^*(\mathbf{i}|n) = \bigcup_{\mathbf{l}|n = \mathbf{i}|n} \bigcap_{r=1}^{\infty} A(\mathbf{l}|r),$$

$$F^*(\mathbf{i}|n; \mathbf{j}|m) = \bigcup_{\mathbf{l}|m = \mathbf{j}|m} \bigcap_{r=1}^{\infty} F(\mathbf{i}|n; \mathbf{l}|r),$$

$$B^*(\mathbf{k}|n) = \bigcup_{\mathbf{l}|n = \mathbf{k}|n} B(\mathbf{l}).$$

Denoting the empty sequence by $\square$, we make the conventions that

$$A^*(\square) = A, \quad F^*(\mathbf{i}|n; \square) = A(\mathbf{i}|n), \quad B^*(\square) = B.$$

We observe that $A^*(\mathbf{i}|n)$ depends only on $i_1, \ldots, i_n$, and similarly for the sets F^*, B^*; in such cases we may regard $\mathbf{i}|n$ as standing for the finite sequence $(i_1, \ldots, i_n)$. We have

$$A^*(\mathbf{i}|n) = \bigcup_{\mathbf{l}|n = \mathbf{i}|n} A^*(\mathbf{l}|n+1), \tag{1}$$

the union on the right being regarded as a countable union, and similarly

$$F^*(\mathbf{i}|n; \mathbf{j}|m) = \bigcup_{\mathbf{l}|m = \mathbf{j}|m} F^*(\mathbf{i}|n; \mathbf{l}|m+1), \tag{2}$$

$$B^*(\mathbf{k}|n) = \bigcup_{\mathbf{l}|n = \mathbf{k}|n} B^*(\mathbf{l}|n+1). \tag{3}$$

Moreover

$$A^*(\mathbf{i}|n) \subset A(\mathbf{i}|n), \tag{4}$$

$$F^*(\mathbf{i}|n; \mathbf{j}|m) \subset \bigcap_{\mu=1}^{m} F(\mathbf{i}|n; \mathbf{j}|\mu). \tag{5}$$

The proof of the theorem will be based upon the following auxiliary proposition: *if* $A \not\Vdash B$ *then in* $\mathbf{I}$ *there are sequences* $\mathbf{i}$, $\mathbf{k}$, *and* $\mathbf{j}_\sigma$ $(\sigma = 1, 2, \ldots)$ *such that for all* s *we have*

$$A^*(\mathbf{i}|s) \not\Vdash B^*(\mathbf{k}|s) \cap \bigcap_{\sigma=1}^{s} F^*(\mathbf{i}|\sigma; \mathbf{j}_\sigma|s). \tag{6}$$

The hypothesis $A \not\Vdash B$ can be regarded as the case $s = 0$ of (6). Therefore, in order to prove the proposition by an inductive construction of the required sequences, it will be sufficient to show that if (6) holds for a given s and for given sequences $\mathbf{i}|s$, $\mathbf{k}|s$, and $\mathbf{j}_\sigma|s$ $(\sigma = 1, \ldots, s)$, all of length s, then we can extend these to sequences $\mathbf{i}|s+1$, $\mathbf{k}|s+1$, $\mathbf{j}_\sigma|s+1$ $(\sigma = 1, \ldots, s)$ of length $s+1$, and find a sequence $\mathbf{j}_{s+1}|s+1$, such that

$$A^*(\mathbf{i}|s+1) \not\Vdash B^*(\mathbf{k}|s+1) \cap \bigcap_{\sigma=1}^{s+1} F^*(\mathbf{i}|\sigma; \mathbf{j}_\sigma|s+1). \tag{7}$$

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First, it follows from (6), (1), and Lemma 1 that for some extension $\mathbf{i}|s+1$ of $\mathbf{i}|s$ we have

$$A^*(\mathbf{i}|s+1) \not\Vdash R_s, \tag{8}$$

where R_s denotes the right-hand side of (6). Secondly, it follows from (8), (4), and Lemma 2 that $A^*(\mathbf{i}|s+1) \not\Vdash R_s \cap A(\mathbf{i}|s+1)$, that is,

$$A^*(\mathbf{i}|s+1) \not\Vdash B^*(\mathbf{k}|s) \cap \bigcap_{\sigma=1}^{s} F^*(\mathbf{i}|\sigma; \mathbf{j}_\sigma|s) \cap F^*(\mathbf{i}|s+1; \square). \tag{9}$$

Thirdly, it follows from (9), (2), (3), and Lemma 3 that for some extensions $\mathbf{k}|s+1$ of $\mathbf{k}|s$ and $\mathbf{j}_\sigma|s+1$ of $\mathbf{j}_\sigma|s$ $(\sigma = 1, \ldots, s)$ and some sequence $\mathbf{j}_{s+1}|s+1$ we have (7). Our proposition is thereby proved.

Now suppose if possible that $A \cap B = \varnothing$ but $A \not\Vdash B$. Then

$$B(\mathbf{k}) \cap \bigcap_{\sigma=1}^{\infty} \bigcap_{\mu=1}^{\infty} F(\mathbf{i}|\sigma; \mathbf{j}_\sigma|\mu) \subset B \cap \bigcap_{\sigma=1}^{\infty} A(\mathbf{i}|\sigma) \subset B \cap A = \varnothing,$$

the various sequences being those whose existence is asserted by our auxiliary proposition, and since $B(\mathbf{k})$ is compact and the sets F are closed, it follows that for some n we have

$$B(\mathbf{k}) \cap \bigcap_{\sigma=1}^{n} \bigcap_{\mu=1}^{n} F(\mathbf{i}|\sigma; \mathbf{j}_\sigma|\mu) = \varnothing.$$

The semicontinuity of B implies that for some value of m we have

$$B^*(\mathbf{k}|m) \cap \bigcap_{\sigma=1}^{n} \bigcap_{\mu=1}^{n} F(\mathbf{i}|\sigma; \mathbf{j}_\sigma|\mu) = \varnothing.$$

Let $s = \max(m, n)$; then *a fortiori*

$$B^*(\mathbf{k}|s) \cap \bigcap_{\sigma=1}^{s} \bigcap_{\mu=1}^{s} F(\mathbf{i}|\sigma; \mathbf{j}_\sigma|\mu) = \varnothing.$$

But by (5) this set contains the right-hand side of (6), and thus $A^*(\mathbf{i}|s) \not\Vdash \varnothing$, which is false by (4).

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