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RESEARCH ARTICLE

Isometry groups of simply connected unimodular 4-dimensional Lie groups

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Abstract
We describe the full group of isometries of each left-invariant Riemannian metric on the simply connected unimodular nilpotent or solvable (R)-type Lie groups of dimension four. As an application of these results, we answer the following question: Does every nontrivial compact subgroup of the automorphism group arise as the isotropy subgroup of some left-invariant Riemannian metric?

1. Introduction

The main goal of this paper is to address the following problem.

Determine the full isometry group of each left-invariant Riemannian metric on the simply connected unimodular nilpotent or solvable (R)-type Lie groups of dimension four.

There are exactly seven such 4-dimensional Lie groups which are [21].

$$\mathrm{Nil}^3\times\mathbb{R} \qquad \mathrm{Nil}^4 \qquad \mathrm{Sol}^4_{m,n} \qquad \mathrm{Sol}^3\times\mathbb{R} \qquad \mathrm{Sol}^4_0 \qquad \mathrm{Sol}^{\prime 4}_0 \qquad \mathrm{Sol}^4_1$$

where the two Lie groups $\mathrm{Nil}^3\times\mathbb{R}$ and Nil^4 are nilpotent and the others are solvable (R)-type Lie groups. To understand why the Lie groups $\mathrm{Sol}^4_{m,n}$, $\mathrm{Sol}^3\times\mathbb{R}$, Sol^4_0, $\mathrm{Sol}^{\prime 4}_0$, Sol^4_1 are of type (R), see the abstract and Theorems 4.1, 4.2, 4.3, 4.4, 5.1 of the paper [21]. The classification of left-invariant Riemannian metrics on these Lie groups is due to Van Thuong [21]. There is a complete study of the two problems of classifying left-invariant Riemannian metrics, and determining their isometry groups, in the three-dimensional Lie groups given by [10, 11, 19] in the unimodular case and by [6] in the non-unimodular case. See also [3] for other general remarks.

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, and let g be a left-invariant Riemannian metric on G. The group of isometries of G with respect to the left-invariant Riemannian metric g denoted $\mathrm{Isom}(G,g)$ is the group of all diffeomorphisms θ of G such that the pullback of g by θ denoted θ^*g is equal to g. This means that

$$g(u,v)=g(\theta_*^{-1}u,\theta_*^{-1}v) \quad \forall u,v\in\mathfrak{g}, \text{(see [11])}$$

where θ_* is the differential of θ at the identity element e of G. This definition is used because to describe the group of isometries, it suffices to examine the isotropy subgroup at the identity element, as will be explained below. The isometry group $\mathrm{Isom}(G,g)$ is a Lie group under the compact-open topology and acts on G transitively [17]. The isotropy subgroup at the identity element e of G is denoted by $\mathrm{Isom}(G,g)_e$, it consists of isometries of G fixing the identity element e. Therefore, one simply gets that

$$\mathrm{Isom}(G,g)=L(G)\cdot\mathrm{Isom}(G,g)_e\cong G\cdot\mathrm{Isom}(G,g)_e$$

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where $L(G)$ is the subgroup of left translations which can be identified with G. In fact, if θ is an element of $\mathrm{Isom}(G, g)$ such that $\theta(e) = p \in G$, then θ decomposes as

$$\theta = L_p \circ (L_{p^{-1}} \circ \theta) \quad \text{where } L_{p^{-1}} \circ \theta \in \mathrm{Isom}(G, g)_e.$$

The product $L(G) \cdot \mathrm{Isom}(G, g)_e$ is not, in general, a semidirect product; it depends on whether $L(G)$ is normal in $\mathrm{Isom}(G, g)$. In [22, 23], it was proved that if G is nilpotent then

$$\mathrm{Isom}(G, g) \cong G \rtimes \mathrm{Aut}(G)_g. \tag{1}$$

where $\mathrm{Aut}(G)_g$ is the group of isometric automorphisms of G which is described below. The isometry groups of 4-dimensional nilpotent Lie groups with respect to a family of Lorentzian metrics are studied in [20].

Since G is simply connected, then $\mathrm{Aut}(G) \backsimeq \mathrm{Aut}(\mathfrak{g})$, and we have an action of this group on the set $\mathscr{L}$ of all left-invariant Riemannian metrics on G given by

$$\begin{aligned} \mathrm{Aut}(\mathfrak{g}) \times \mathscr{L} &\longrightarrow \mathscr{L} \\ (\theta, g) &\longmapsto \theta^* g = g_\theta \end{aligned}$$

where $g_\theta(u, v) = g(\theta^{-1}u, \theta^{-1}v)\ \forall u, v \in \mathfrak{g}$. See [11] for more detail.

We denote the isotropy subgroup of $\mathrm{Aut}(\mathfrak{g})$ at g by $\mathrm{Aut}(\mathfrak{g})_g = \{\theta \in \mathrm{Aut}(\mathfrak{g}) / \theta^* g = g\}$.

The Lie group G is said to be of type (R) (solvable of type (R) is also called completely solvable), if for every $x \in \mathfrak{g}$, the endomorphism $\mathrm{ad}\, x : \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues. Completely solvable Lie groups are in standard position inside their isometry group; consequently, a unimodular Lie group of this kind is normal in the isometry group of any left-invariant Riemannian metric g on it. In this case, the isometry group is given by [9, 11]

$$\mathrm{Isom}(G, g) \cong G \rtimes \mathrm{Aut}(G)_g. \tag{2}$$

The result in [11] is a consequence of two results given by Gordon with Wilson and Helgason in [9, 12]. In fact, Corollary (5.3) in [9], which is key to obtaining the result above, is itself a consequence of the work by Alekseevskii in [2]. In the case of the four-dimensional oscillator group, which is unimodular and solvable, but not of type (R), the problem of determining the group of isometries is resolved in the following paper [1]. To apply the previous two results, we need the automorphism group of our seven simply connected unimodular 4-dimensional Lie groups. The automorphism group of four-dimensional Lie groups has been studied by several authors, including [4, 5, 7, 21]. From the description of the automorphism group of the simply connected unimodular Lie groups of dimension four, given in [21], we provide the groups $\mathrm{Aut}(G)_g$ and then describe the full group of isometries of our four-dimensional Lie groups. In particular, we obtain a generalization of some of the results presented in [15].

2. Nilpotent and solvable (R)-type Lie groups of dimension 4

In this section, we describe the Lie bracket structures of the Lie algebras corresponding to simply connected nilpotent or solvable (R)-type Lie groups of dimension four. We also revisit the structures of such simply connected unimodular Lie groups.

The Lie group Nil^3 is referred to as the standard three-dimensional Heisenberg Lie group.

$\mathrm{Nil}^3 \times \mathbb{R}(U1[(2, 1)], \mu = 0$ in [16]): The Lie algebra of $\mathrm{Nil}^3 \times \mathbb{R}$ has a basis (e_i) such that [21]

$$\begin{aligned} [e_i, e_j] &= 0 \quad \text{for} \quad 1 \le i \le j \le 3 \\ [e_i, e_4] &= 0 \quad \text{for} \quad i = 1, 2 \\ [e_3, e_4] &= e_1. \end{aligned}$$

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Nil^4 ($U1[3]$ in [16]): The Lie algebra of Nil^4 has a basis (e_i) such that [21]

$$[e_i, e_j] = 0 \quad \text{for} \quad 1 \le i \le j \le 3$$

$$[e_1, e_4] = 0, \quad [e_2, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

The Lie algebras of the five solvable Lie groups of type (R) and of dimension four are described in [21] as follows

$$\begin{aligned}
\mathrm{Sol}^4_{m,n} \quad & (U1[1,1,1] \text{ in } [16]),\\
\mathrm{Sol}^3 \times \mathbb{R} \quad & (U1[1,1,1],\ \nu = 0 \text{ in } [16]),\\
\mathrm{Sol}^4_0 \quad & (U1[1,1,1],\ \lambda = \mu = 1,\ \nu = -2 \text{ in } [16]),\\
\mathrm{Sol}^{\prime 4}_0 \quad & (U1[2,1] \text{ in } [16]),\\
\mathrm{Sol}^4_1 \quad & (U3I0 \text{ in } [16]).
\end{aligned}$$

More precisely, in the case of Sol^4_1 ($U3I0$ in [16]), the basis (e_i) of the Lie algebra satisfies the following relations [21]

$$[e_2, e_3] = e_1, \quad [e_1, e_i] = 0 \quad \text{for all } i$$

$$[e_4, e_2] = e_2,$$

$$[e_4, e_3] = -e_3.$$

We now describe the matrix realizations of the simply connected unimodular Lie groups of dimension four. This problem was addressed by Biggs and Remsing in [4]. Our approach follows Table 3 in [21].

The universal covering group $\mathrm{Nil}^3 \times \mathbb{R}$, is the universal covering group for the Lie algebra $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ in [4]. It is given by

$$\mathrm{Nil}^3 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & x & w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} = p(w,x,y,z) : w,x,y,z \in \mathbb{R} \right\}.$$

The universal covering group Nil^4 is the universal covering group for the Lie algebra $\mathfrak{g}_{4.1}$ in [4]. It is given by

$$\mathrm{Nil}^4 = \left\{ \begin{bmatrix} 1 & z & \frac{z^2}{2} & w \\ 0 & 1 & z & -x+z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} = p(w,x,y,z) : w,x,y,z \in \mathbb{R} \right\}.$$

The universal covering group $\mathrm{Sol}^4_{\mathrm{m,n}}$ is the universal covering group for the Lie algebra $\mathfrak{g}^{\alpha,\beta}_{4.5}$ where $\alpha = -1 - \beta$ (see Table 3 in [21]) in [4]. It is given by

$$\mathrm{Sol}^4_{\mathrm{m,n}} = G^{\alpha,\beta}_{4.5} = \left\{ \begin{bmatrix} e^{-z} & 0 & 0 & w \\ 0 & e^{-\alpha z} & 0 & y \\ 0 & 0 & e^{(1+\alpha)z} & x \\ 0 & 0 & 0 & 1 \end{bmatrix} : w,x,y,z \in \mathbb{R} \right\}.$$

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The universal covering group $\mathrm{Sol}^3 \times \mathbb{R}$ is the universal covering group for the Lie algebra $\mathfrak{g}_{3.4}^0 \oplus \mathfrak{g}_1$ in [4]. It is isomorphic to $G_{3.4}^0 \times \mathbb{R}$ where

$$G_{3.4}^0 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh z & -\sinh z \\ y & -\sinh z & \cosh z \end{bmatrix} : x, y, z \in \mathbb{R} \right\} = \mathrm{SE}(1,1).$$

The universal covering group Sol_0^4 is the universal covering group for the Lie algebra $\mathfrak{g}_{4.5}^{\alpha,\beta}$ where $\alpha = \beta = \frac{-1}{2}$ (see Table 3 in [21]) in [4]. It is given by

$$\mathrm{Sol}_0^4 = G_{4.5}^{\alpha,\beta} = \left\{ \begin{bmatrix} e^{-z} & 0 & 0 & w \\ 0 & e^{\frac{1}{2}z} & 0 & y \\ 0 & 0 & e^{\frac{1}{2}z} & x \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}.$$

The universal covering group $\mathrm{Sol}_0^{'4}$ is the universal covering group for the Lie algebra $\mathfrak{g}_{4.2}^{\alpha}$ where $\alpha = -2$ (see Table 3 in [21]) in [4]. It is given by

$$\mathrm{Sol}_0^{'4} = G_{4.2}^{\alpha} = \left\{ \begin{bmatrix} e^{2z} & 0 & 0 & w \\ 0 & e^{-z} & 2ze^{-z} & -2x \\ 0 & 0 & e^{-z} & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}.$$

The universal covering group Sol_1^4 is the universal covering group for the Lie algebra $\mathfrak{g}_{4.8}^{-1}$ in [4]. It is given by

$$\mathrm{Sol}_1^4 = G_{4.8}^{-1} = \left\{ \begin{bmatrix} 1 & x & w \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix} = p(w, x, y, z) : w, x, y, z \in \mathbb{R} \right\}.$$

3. Metrics on 4-dimensional unimodular Lie groups

A left-invariant Riemannian metric on a Lie group G is simply an inner product $\langle \cdot, \cdot \rangle$ on the tangent space of G, which varies smoothly across G and such that the left translations are isometries.

It is well known that there is a bijective correspondence between left-invariant Riemannian metrics on a Lie group G and inner products on its Lie algebra $\mathfrak{g}$. Therefore, the classification of left-invariant Riemannian metrics on G is equivalent to the classification of inner products on $\mathfrak{g}$.

Proposition 3.1 [21]. *The set of inner products on $\mathfrak{g}$ is $(n^2+n)/2$-dimensional and can be identified with the set of upper triangular matrices with positive diagonal entries.*

Proof. This fact was already proven in [21]. We proceed with an alternative classical proof. The space of inner products on $\mathfrak{g}$ is the space of symmetric, positive definite matrices denoted S_n. We adopt the following notations.

$\mathrm{Tsup}_n :=$ the group consisting of upper triangular matrices with positive diagonal entries.

$\mathrm{Tinf}_n :=$ the group consisting of lower triangular matrices with positive diagonal entries.

$\mathrm{D}_n^+ :=$ the group consisting of diagonal matrices with positive elements.

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Consider the following map

$$\varphi : \mathrm{Tsup}_n \longrightarrow \mathrm{S}_n$$
$$B \longmapsto (B^{-1})^t(B^{-1})$$

It is clear that the map φ is well defined. For the injectivity, let $B, C \in \mathrm{Tsup}_n$ such that $\varphi(B) = \varphi(C)$. This is equivalent to the following equality $B^{-1}C = (C^{-1}B)^t$. Put $M = C^{-1}B$, then $B^{-1}C = (C^{-1}B)^t \Leftrightarrow M^{-1} = M^t$. Since Tsup_n is a group, then $M, M^{-1} \in \mathrm{Tsup}_n$. Since $M^{-1} = M^t \in \mathrm{Tinf}_n$, then $M \in \mathrm{Tsup}_n \cap \mathrm{Tinf}_n = \mathrm{D}_n^+$. Hence, we have $M^{-1} = M^t = M$, then $M = I_n$. This implies that $B = C$ and φ is injective.

For the surjectivity, let $A \in \mathrm{S}_n$ and consider the inner product on $\mathbb{R}^n$ defined by

$$\langle u, v\rangle = u^t A v, \forall u, v \in \mathbb{R}^n.$$

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. The Gram-Schmidt procedure produces an orthonormal basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ of the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ such that

$$[v_1, \ldots, v_n] = X \in Tsup_n.$$

This means that $Xe_i = v_i$. Since $\mathscr{B} = \{v_1, \ldots, v_n\}$ is an orthonormal basis of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, then

$$\mathrm{Mat}\,(\langle\cdot,\cdot\rangle, \mathscr{B}) = X^t A X = I_n.$$

Thus, $A = (X^{-1})^t(X^{-1}) = \varphi(X)$. Hence, φ is surjective.

Therefore, φ is a bijection. □

Remark 3.1 [21]. *The classification of inner products (symmetric positive definite matrices) on Lie algebras is equivalent to the classification of upper triangular matrices with positive diagonal entries.*

Now, following [21], we describe all left-invariant Riemannian metrics on simply connected unimodular nilpotent or solvable (R)-type 4-dimensional Lie groups, up to isometry. These metrics are represented by upper triangular matrices with positive diagonal entries. However, we will use the bijection φ, as introduced above, to describe the associated symmetric positive definite matrices.

Remark 3.2. *In each unimodular Lie group of dimension four, we will investigate different metrics that yield different isometry groups. This means that for any other metric, its isometry group is isomorphic to that of one of the investigated metrics.*

3.1. Left-invariant Riemannian metrics on $\mathbf{Nil}^3 \times \mathbb{R}$

By Theorem 3.1 in [21], any metric on $\mathrm{Nil}^3 \times \mathbb{R}$ is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \alpha > 0.$$

According to the bijection φ, the left-invariant Riemannian metric (symmetric positive definite matrix) associated with M is

$$\langle\cdot,\cdot\rangle = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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3.2. Left-invariant Riemannian metrics on $\mathbf{Nil^4}$

By Theorem 3.2 in [21], any metric on Nil^4 is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \alpha, \gamma > 0 \quad \beta \geq 0.$$

We distinguish two cases associated to this metric

1. If $\beta = 0$, then the left-invariant Riemannian metric associated with M is

$$\langle \cdot, \cdot \rangle_1 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. If $\beta > 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_2 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & \frac{-\beta}{\alpha^2\gamma} & 0 & 0 \\ \frac{-\beta}{\alpha^2\gamma} & \frac{1}{\gamma^2} + \frac{\beta^2}{\alpha^2\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.3. Left-invariant Riemannian metrics on $\mathbf{Sol^4_{m,n}}$

By Theorem 4.3 in [21], any metric on $\mathrm{Sol}^4_{\mathrm{m,n}}$ is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} 1 & \alpha & \beta & 0 \\ 0 & 1 & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad \mu > 0 \quad \alpha, \gamma \geq 0 \quad \beta \in \mathbb{R}.$$

We distinguish three cases associated with this metric

1. If $\alpha = \beta = \gamma = 0$, then the left-invariant Riemannian metric associated with M is

$$\langle \cdot, \cdot \rangle_1 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix}$$

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2. If $\alpha > 0$ and $\beta = \gamma = 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_2 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

3. If $\alpha > 0$, $\beta \neq 0$ and $\gamma \geq 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_3 = \begin{bmatrix} 1 & -\alpha & \alpha\gamma - \beta & 0 \\ -\alpha & 1+\alpha^2 & -\alpha(\alpha\gamma - \beta) - \gamma & 0 \\ \alpha\gamma - \beta & -\alpha(\alpha\gamma - \beta) - \gamma & (\alpha\gamma - \beta)^2 + \gamma^2 + 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}.$$

The remaining cases for the parameters $(\alpha, \beta, \gamma, \mu)$ do not give rise to additional isometry groups.

3.4. Left-invariant Riemannian metrics on $\mathbf{Sol^3 \times \mathbb{R}}$

By Theorem 4.4 in [21], any metric on $\mathrm{Sol}^3 \times \mathbb{R}$ is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} 1 & \alpha & \beta & 0 \\ 0 & 1 & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad \mu > 0 \quad \alpha, \gamma \geq 0 \quad \beta \in \mathbb{R}.$$

Hence, this case is similar to the precedent one, we distinguish the following left-invariant Riemannian metrics

$$\langle \cdot, \cdot \rangle_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix} \qquad \langle \cdot, \cdot \rangle_2 = \begin{bmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

$$\langle \cdot, \cdot \rangle_3 = \begin{bmatrix} 1 & -\alpha & \alpha\gamma - \beta & 0 \\ -\alpha & 1+\alpha^2 & -\alpha(\alpha\gamma - \beta) - \gamma & 0 \\ \alpha\gamma - \beta & -\alpha(\alpha\gamma - \beta) - \gamma & (\alpha\gamma - \beta)^2 + \gamma^2 + 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}.$$

The remaining cases for the parameters $(\alpha, \beta, \gamma, \mu)$ do not give rise to additional isometry groups.

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3.5. Left-invariant Riemannian metrics on $\mathbf{Sol}_0^4$

By Theorem 4.1 in [21], any metric on Sol_0^4 is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} \quad \alpha \geq 0 \quad \beta > 0.$$

We distinguish two cases associated with this metric

1. If $\alpha = 0$, then the left-invariant Riemannian metric associated with M is

$$\langle \cdot, \cdot \rangle_1 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\beta^2} \end{bmatrix}$$

2. If $\alpha > 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_2 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & 0 & -\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\beta^2} \end{bmatrix}.$$

3.6. Left-invariant Riemannian metrics on $\mathbf{Sol}_0^{'4}$

By Theorem 4.2 in [21], any metric on $\mathrm{Sol}_0^{'4}$ is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} 1 & 0 & \beta & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad \alpha, \mu > 0 \quad \beta \geq 0 \quad \gamma \in \mathbb{R}.$$

We distinguish two cases associated with this metric

1. If $\beta = \gamma = 0$, then the left-invariant Riemannian metric associated with M is

$$\langle \cdot, \cdot \rangle_1 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\alpha^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

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2. If $\beta \neq 0$ and $\gamma = 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_2 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} 1 & 0 & -\beta & 0 \\ 0 & \frac{1}{\alpha^2} & 0 & 0 \\ -\beta & 0 & 1+\beta^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix}.$$

The remaining cases for the parameters $(\alpha, \beta, \gamma, \mu)$ do not give rise to additional isometry groups.

3.7. Left-invariant Riemannian metrics on $\mathbf{Sol}_1^4$

By Theorem 5.1 in [21], any metric on Sol_1^4 is equivalent, up to isometry, to the following metric

$$M = \begin{bmatrix} \alpha & \beta & \gamma & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix} \quad \alpha, \nu > 0 \quad \beta, \mu \geq 0 \quad \gamma \in \mathbb{R}.$$

We distinguish three cases associated with this metric

1. If $\beta = \gamma = \mu = 0$, then the left-invariant Riemannian metric associated with M is

$$\langle \cdot, \cdot \rangle_1 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

2. If $\beta > 0$ and $\gamma = \mu = 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_2 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & \frac{-\beta}{\alpha^2} & 0 & 0 \\ \frac{-\beta}{\alpha^2} & 1+\left(\frac{\beta}{\alpha}\right)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

3. If $\beta, \mu > 0$ and $\gamma = 0$, then our left-invariant Riemannian metric is

$$\langle \cdot, \cdot \rangle_3 = \left(M^{-1}\right)^t \left(M^{-1}\right) = \begin{bmatrix} \frac{1}{\alpha^2} & \frac{-\beta}{\alpha^2} & \frac{\beta\mu}{\alpha^2} & 0 \\ \frac{-\beta}{\alpha^2} & 1+\left(\frac{\beta}{\alpha}\right)^2 & \frac{-\beta^2\mu}{\alpha^2}-\mu & 0 \\ \frac{\beta\mu}{\alpha^2} & \frac{-\beta^2\mu}{\alpha^2}-\mu & \left(\frac{\beta\mu}{\alpha}\right)^2+\mu^2+1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}.$$

The remaining cases for the parameters $(\alpha, \beta, \gamma, \mu, \nu)$ do not give rise to additional isometry groups.

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4. The isometry groups

Let us recall some well-known and important results about the isometry groups of left-invariant Riemannian metrics on Lie groups.

Assume that our Lie group G is compact, then for any left-invariant Riemannian metric on G. The isometry group $\mathrm{Isom}(G, g)$ is also compact [17].

If G is compact simple and if g is a left-invariant Riemannian metric on G. Then $\mathrm{Isom}_0(G, g) \subset L(G)R(G)$ [18]. This means that any isometry in the connected component of the identity $\mathrm{Isom}_0(G, g)$ decomposes into a left translation and a right translation.

Let θ be an automorphism of G and let g be a left-invariant Riemannian metric on G. The pullback $\theta^* g$ of g by θ is defined by: $\theta^* g(u, v) = g\left(\theta_*^{-1}(u), \theta_*^{-1}(v)\right)\ \forall u, v \in \mathfrak{g}$ where θ_* is the differential of θ at the identity element e of G. This equation is equivalent to the following matrix equation $[g] = [\theta_*]^t[\theta^* g][\theta_*]$. Hence, in term of matrix calculation, the group of isometric automorphisms of G is given by [11]

$$\mathrm{Aut}(G)_g = \left\{\theta \in \mathrm{Aut}(G) \mid [g] = [\theta_*]^t[g][\theta_*]\right\}. \tag{3}$$

Remark 4.1. *It is a standard fact that if two left-invariant Riemannian metrics on a Lie group G are equivalent up to automorphism, then their isometry groups are conjugate.*

According to this remark, one only needs to describe the isometry groups of representative metrics of the non-equivalent classes of left-invariant Riemannian metrics on the Lie group.

Remark 4.2. *The computations of the groups of isometric automorphisms determined by the matrix equation*

$$A^t \langle \cdot, \cdot \rangle_i A = \langle \cdot, \cdot \rangle_i$$

were carried out using MAPLE. The detailed calculations are presented in each case.

Note that determining the isometry groups of each Lie group enables us to answer Question 4.3.

Question 4.3. *Does every nontrivial compact subgroup of the automorphism group arise as the isotropy subgroup of some left-invariant Riemannian metric?*

4.1. Nilpotent case

Using the result (1), we give the isometry group of our two nilpotent four-dimensional Lie groups.

4.1.1. Isometry group of $\mathrm{Nil}^3 \times \mathbb{R}$

Theorem 4.4. *The isometry group of* $\mathrm{Nil}^3 \times \mathbb{R}$ *is given by*

$$\mathrm{Isom}\left(\mathrm{Nil}^3 \times \mathbb{R}, \langle \cdot, \cdot \rangle\right) \cong \left(\mathrm{Nil}^3 \times \mathbb{R}\right) \rtimes (\mathrm{diag}\{\pm 1, \pm 1, \mathrm{O}(2)\})$$

Proof. The automorphism group $\mathrm{Aut}\left(\mathrm{Nil}^3 \times \mathbb{R}\right)$ consists of elements of the form [21]

$$\begin{bmatrix} ad - bc & x & y & u \\ 0 & e & z & v \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}) \quad e \in \mathbb{R}^* \quad x, y, z, u, v \in \mathbb{R}.$$

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$$\text{Let } A = \begin{bmatrix} ad-bc & x & y & u \\ 0 & e & z & v \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \in \mathrm{Aut}\left(\mathrm{Nil}^3 \times \mathbb{R}\right), \text{ then by (3)}$$

$$A \in \mathrm{Aut}\left(\mathrm{Nil}^3 \times \mathbb{R}\right)_{\langle\cdot,\cdot\rangle}$$

$$\Leftrightarrow \quad A^t \langle\cdot,\cdot\rangle A = \langle\cdot,\cdot\rangle$$

$$\Leftrightarrow \quad \begin{bmatrix} ad-bc & 0 & 0 & 0 \\ x & e & 0 & 0 \\ y & z & a & c \\ u & v & b & d \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ad-bc & x & y & u \\ 0 & e & z & v \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} \frac{(ad-bc)^2}{\alpha^2} & \frac{(ad-bc)x}{\alpha^2} & \frac{(ad-bc)y}{\alpha^2} & \frac{(ad-bc)u}{\alpha^2} \\ \frac{(ad-bc)x}{\alpha^2} & \frac{x^2}{\alpha^2}+e^2 & \frac{xy}{\alpha^2}+ez & \frac{xu}{\alpha^2}+ev \\ \frac{(ad-bc)y}{\alpha^2} & \frac{xy}{\alpha^2}+ez & \frac{y^2}{\alpha^2}+z^2+a^2+c^2 & \frac{yu}{\alpha^2}+zv+ab+cd \\ \frac{(ad-bc)u}{\alpha^2} & \frac{xu}{\alpha^2}+ev & \frac{yu}{\alpha^2}+zv+ab+cd & \frac{u^2}{\alpha^2}+v^2+b^2+d^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By equating the $(1,2)$, $(1,3)$, and $(1,4)$ entries of the two matrices above, we obtain $x = y = u = 0$. Then, equating the $(2,3)$ and $(2,4)$ entries yields $z = v = 0$. Hence, the matrix equality reduces to

$$\begin{bmatrix} \frac{(ad-bc)^2}{\alpha^2} & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & a^2+c^2 & ab+cd \\ 0 & 0 & ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2, \quad e = \pm 1, \quad ad - bc = \pm 1$$

$$\Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{O}(2), \quad e = \pm 1.$$

Hence,

$$\mathrm{Isom}\left(\mathrm{Nil}^3 \times \mathbb{R}, \langle\cdot,\cdot\rangle\right) \cong \left(\mathrm{Nil}^3 \times \mathbb{R}\right) \rtimes \left(\mathrm{diag}\left\{\pm 1, \pm 1, \mathrm{O}(2)\right\}\right).$$

□

Answer. The answer to Question 4.3 is negative. For example, the compact subgroup $(\mathbb{Z}_2)^2$ of $\mathrm{Aut}\left(\mathrm{Nil}^3 \times \mathbb{R}\right)$ cannot occur as the isotropy subgroup of any left-invariant Riemannian metric.

4.1.2. Isometry group of Nil^4

Theorem 4.5. *The isometry group of* Nil^4 *is given by*

$$\mathrm{Isom}\left(\mathrm{Nil}^4, \langle\cdot,\cdot\rangle_1\right) \cong \mathrm{Nil}^4 \rtimes (\mathbb{Z}_2)^2$$

$$\mathrm{Isom}\left(\mathrm{Nil}^4, \langle\cdot,\cdot\rangle_2\right) \cong \mathrm{Nil}^4 \rtimes (\mathbb{Z}_2).$$

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Proof. The automorphism group $\mathrm{Aut}(\mathrm{Nil}^4)$ consists of elements of the form [21]

$$\begin{bmatrix} ad^2 & ed & x & y \\ 0 & ad & e & z \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{bmatrix} \quad a, d \in \mathbb{R}^* \quad b, e, x, y, z \in \mathbb{R}.$$

Let $A = \begin{bmatrix} ad^2 & ed & x & y \\ 0 & ad & e & z \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{bmatrix} \in \mathrm{Aut}(\mathrm{Nil}^4)$, then by (3)

$$\begin{aligned} & A \in \mathrm{Aut}(\mathrm{Nil}^4)_{\langle\cdot,\cdot\rangle_1} \\ \Leftrightarrow \quad & A^t \langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1 \\ \Leftrightarrow \quad & \begin{bmatrix} ad^2 & 0 & 0 & 0 \\ ed & ad & 0 & 0 \\ x & e & a & 0 \\ y & z & b & d \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ad^2 & ed & x & y \\ 0 & ad & e & z \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \Leftrightarrow \quad & \begin{bmatrix} \frac{a^2d^4}{\alpha^2} & \frac{a\,d^3 e}{\alpha^2} & \frac{a\,d^2 x}{\alpha^2} & \frac{a\,d^2 y}{\alpha^2} \\ \frac{a\,d^3 e}{\alpha^2} & \frac{e^2 d^2}{\alpha^2} + \frac{a^2 d^2}{\gamma^2} & \frac{edx}{\alpha^2} + \frac{ade}{\gamma^2} & \frac{edy}{\alpha^2} + \frac{adz}{\gamma^2} \\ \frac{a\,d^2 x}{\alpha^2} & \frac{edx}{\alpha^2} + \frac{ade}{\gamma^2} & \frac{x^2}{\alpha^2} + \frac{e^2}{\gamma^2} + a^2 & \frac{xy}{\alpha^2} + \frac{ez}{\gamma^2} + ab \\ \frac{a\,d^2 y}{\alpha^2} & \frac{edy}{\alpha^2} + \frac{adz}{\gamma^2} & \frac{xy}{\alpha^2} + \frac{ez}{\gamma^2} + ab & \frac{y^2}{\alpha^2} + \frac{z^2}{\gamma^2} + b^2 + d^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

By equating the $(1,2)$, $(1,3)$, and $(1,4)$ entries of the two matrices above, we obtain $e = x = y = 0$. Then, equating the $(2,4)$ and $(3,4)$ entries yields $z = b = 0$. Hence, the matrix equality reduces to

$$\begin{bmatrix} \frac{a^2d^4}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{a^2d^2}{\gamma^2} & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & d^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Leftrightarrow a = \pm 1, \quad d = \pm 1.$$

Therefore,

$$\begin{aligned} \mathrm{Aut}(\mathrm{Nil}^4)_{\langle\cdot,\cdot\rangle_1} &= \left\{ I_4, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\} \\ &\cong (\mathbb{Z}_2)^2 . \end{aligned}$$

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Hence,

$$\mathrm{Isom}\left(\mathrm{Nil}^4, \langle\cdot,\cdot\rangle_1\right) \cong \mathrm{Nil}^4 \rtimes (\mathbb{Z}_2)^2 .$$

For the metric $\langle\cdot,\cdot\rangle_2$, we have by (3):

$$\begin{aligned}
& A \in \mathrm{Aut}\left(\mathrm{Nil}^4\right)_{\langle\cdot,\cdot\rangle_2} \\
\Leftrightarrow \quad & A^t\langle\cdot,\cdot\rangle_2 A = \langle\cdot,\cdot\rangle_2 \\
\Leftrightarrow \quad & \begin{bmatrix} ad^2 & 0 & 0 & 0 \\ ed & ad & 0 & 0 \\ x & e & a & 0 \\ y & z & b & d \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2} & \frac{-\beta}{\alpha^2\gamma} & 0 & 0 \\ \frac{-\beta}{\alpha^2\gamma} & \frac{1}{\gamma^2}+\frac{\beta^2}{\alpha^2\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ad^2 & ed & x & y \\ 0 & ad & e & z \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & \frac{-\beta}{\alpha^2\gamma} & 0 & 0 \\ \frac{-\beta}{\alpha^2\gamma} & \frac{1}{\gamma^2}+\frac{\beta^2}{\alpha^2\gamma^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

We first note that $\mathrm{Aut}\left(\mathrm{Nil}^4\right)_{\langle\cdot,\cdot\rangle_2} \subset \mathrm{O}(4)$ and:

$$\begin{bmatrix} ad^2 & 0 & 0 & 0 \\ ed & ad & 0 & 0 \\ x & e & a & 0 \\ y & z & b & d \end{bmatrix} \begin{bmatrix} ad^2 & ed & x & y \\ 0 & ad & e & z \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\Leftrightarrow \quad & \begin{bmatrix} a^2d^4 & a\,d^3e & a\,d^2x & a\,d^2y \\ a\,d^3e & a^2d^2+e^2d^2 & ade+edx & adz+edy \\ a\,d^2x & ade+edx & a^2+e^2+x^2 & ab+ez+xy \\ a\,d^2y & adz+edy & ab+ez+xy & b^2+d^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\Leftrightarrow \quad & e = x = y = z = b = 0, \quad a = \pm 1, \quad d = \pm 1.
\end{aligned}$$

Now since the form of A is restricted, one simply gets that (it suffices to check which elements of $(\mathbb{Z}_2)^2$ preserve the metric $\langle\cdot,\cdot\rangle_2$)

$$\begin{aligned}
& A \in \mathrm{Aut}\left(\mathrm{Nil}^4\right)_{\langle\cdot,\cdot\rangle_2} \Leftrightarrow A^t\langle\cdot,\cdot\rangle_2 A = \langle\cdot,\cdot\rangle_2 \\
& \Leftrightarrow A \in \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \cong \mathbb{Z}_2.
\end{aligned}$$

Hence, $\mathrm{Aut}\left(\mathrm{Nil}^4\right)_{\langle\cdot,\cdot\rangle_2} \cong \mathbb{Z}_2$. Therefore, $\mathrm{Isom}\left(\mathrm{Nil}^4, \langle\cdot,\cdot\rangle_2\right) \cong \mathrm{Nil}^4 \rtimes \mathbb{Z}_2$. □

Answer. The answer to Question 4.3 is affirmative. Note that $(\mathbb{Z}_2)^2$ has three distinct subgroups of order 2; however, these are different subgroups inside $(\mathbb{Z}_2)^2$, even though they are all isomorphic to $\mathbb{Z}_2$. We emphasize that it is these abstract group types that occur as isotropy groups of left-invariant Riemannian metrics.

Remark 4.6. *Note that when we find the group of isometric automorphisms of the first class of metrics, the problem becomes easy, because the group of isometric automorphisms of the first class of metrics is the maximal one. This fact has a significant impact on the rest of the article.*

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A major development in the study of left-invariant metrics on solvable Lie groups was the discovery that distinguished metrics have maximal isometry groups. More precisely, Jablonski and Jablonski–Gordon established this property for Einstein and Ricci soliton metrics in [8, 13, 14].

4.2. Solvable (R)-type case

Using the result (2), we give the isomerty groups of our five solvable Lie groups of type (*R*).

4.2.1. Isometry group of $\mathrm{Sol}^4_{\mathrm{m,n}}$

The automorphism group of $\mathrm{Sol}^4_{m,n}$ consists of elements of the form [21]

$$\begin{bmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad a, b, c \in \mathbb{R}^*, \quad x, y, z \in \mathbb{R}.$$

Theorem 4.7. *The group of isometric automorphisms of* $\mathrm{Sol}^4_{m,n}$ *is given by*

$$\mathrm{Aut}\left(\mathrm{Sol}^4_{m,n}\right)_{\langle\cdot,\cdot\rangle_1} \cong (\mathbb{Z}_2)^3$$
$$\mathrm{Aut}\left(\mathrm{Sol}^4_{m,n}\right)_{\langle\cdot,\cdot\rangle_2} \cong (\mathbb{Z}_2)^2$$
$$\mathrm{Aut}\left(\mathrm{Sol}^4_{m,n}\right)_{\langle\cdot,\cdot\rangle_3} \cong \mathbb{Z}_2.$$

Proof. Let $A = \begin{bmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}^4_{\mathrm{m,n}})$. Then by (3)

$$A \in \mathrm{Aut}\left(\mathrm{Sol}^4_{\mathrm{m,n}}\right)_{\langle\cdot,\cdot\rangle_1}$$
$$\Leftrightarrow \quad A^t\langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1$$
$$\Leftrightarrow \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \begin{bmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} a^2 & 0 & 0 & ax \\ 0 & b^2 & 0 & by \\ 0 & 0 & c^2 & cz \\ ax & by & cz & x^2+y^2+z^2+\frac{1}{\mu^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix}$$
$$\Leftrightarrow \quad x = y = z = 0, \quad a = \pm 1, \quad b = \pm 1, \quad c = \pm 1.$$

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Therefore,

$$\mathrm{Aut}(\mathrm{Sol}^4_{\mathrm{m,n}})_{\langle\cdot,\cdot\rangle_1} = \left\{ \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \cong (\mathbb{Z}_2)^3.$$

For the metric $\langle\cdot,\cdot\rangle_2$, by using (3), one gets that

$$\begin{aligned}
& A \in \mathrm{Aut}\big(\mathrm{Sol}^4_{\mathrm{m,n}}\big)_{\langle\cdot,\cdot\rangle_2} \\
\Leftrightarrow \quad & A^t\langle\cdot,\cdot\rangle_2 A = \langle\cdot,\cdot\rangle_2 \\
\Leftrightarrow \quad & \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \begin{bmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \\
\Leftrightarrow \quad & \begin{bmatrix} a^2 & -a\alpha b & 0 & -a\alpha y + ax \\ -a\alpha b & b^2(\alpha^2+1) & 0 & -b\alpha x + b(\alpha^2+1)y \\ 0 & 0 & c^2 & cz \\ (-y\alpha+x)a & (-x\alpha+y(\alpha^2+1))b & cz & (-y\alpha+x)x + (-x\alpha+y(\alpha^2+1))y + z^2 + \frac{1}{\mu^2} \end{bmatrix} \\
& \qquad = \begin{bmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \\
\Leftrightarrow \quad & z=0, \quad a=\pm1, \quad b=\pm1, \quad ab=1, \quad c=\pm1 \\
\Leftrightarrow \quad & z=0, \quad a=b=\pm1, \quad c=\pm1.
\end{aligned}$$

By equating the (1, 4) and (2, 4) entries of the two matrices above, we see that x, y verify the following system

$$\begin{cases} x - \alpha y = 0 \\ -\alpha x + \left(1+\alpha^2\right) y = 0 \end{cases}$$

This implies that $x = y = 0$. Hence,

$$\begin{aligned}
\mathrm{Aut}\big(\mathrm{Sol}^4_{\mathrm{m,n}}\big)_{\langle\cdot,\cdot\rangle_2} &= \left\{ I_4, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \\
&\cong (\mathbb{Z}_2)^2.
\end{aligned}$$

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Finally, using the fact that $(\mathbb{Z}_2)^3$ as above is the maximal group of isometric automorphisms of $\mathrm{Sol}^4_{\mathrm{m,n}}$, a straightforward verification shows that

$$\begin{aligned} A \in \mathrm{Aut}\big(\mathrm{Sol}^4_{\mathrm{m,n}}\big)_{\langle\cdot,\cdot\rangle_3} &\Leftrightarrow A^t\langle\cdot,\cdot\rangle_3 A = \langle\cdot,\cdot\rangle_3 \\ &\Leftrightarrow A \in \{I_4, \mathrm{diag}\{-1,-1,-1,1\}\} \cong \mathbb{Z}_2. \end{aligned}$$

□

Corollary 4.8. *The isometry group of* $\mathrm{Sol}^4_{m,n}$ *is given by*

$$\begin{aligned} \mathrm{Isom}\big(\mathrm{Sol}^4_{m,n}, \langle\cdot,\cdot\rangle_1\big) &\cong \mathrm{Sol}^4_{m,n} \rtimes (\mathbb{Z}_2)^3 \\ \mathrm{Isom}\big(\mathrm{Sol}^4_{m,n}, \langle\cdot,\cdot\rangle_2\big) &\cong \mathrm{Sol}^4_{m,n} \rtimes (\mathbb{Z}_2)^2 \\ \mathrm{Isom}\big(\mathrm{Sol}^4_{m,n}, \langle\cdot,\cdot\rangle_3\big) &\cong \mathrm{Sol}^4_{m,n} \rtimes \mathbb{Z}_2. \end{aligned}$$

Answer. The answer to Question 4.3 is affirmative. Note that $(\mathbb{Z}_2)^3$ has seven distinct subgroups of order 2 and seven distinct subgroups of order 4; however, these are different subgroups inside $(\mathbb{Z}_2)^3$, even though they are all isomorphic to $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$, respectively. We emphasize that it is these abstract group types that occur as isotropy groups of left-invariant Riemannian metrics.

4.2.2. Isometry group of $\mathrm{Sol}^3 \times\mathbb{R}$

We denote by $\mathbf{D}(4)$ the dihedral group of order 8.

Theorem 4.9. *The group of isometric automorphisms of* $\mathrm{Sol}^3 \times\mathbb{R}$ *is given by*

$$\begin{aligned} \mathrm{Aut}\big(\mathrm{Sol}^3 \times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_1} &\cong \mathbf{D}(4) \times \mathbb{Z}_2 \\ \mathrm{Aut}\big(\mathrm{Sol}^3 \times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_2} &\cong (\mathbb{Z}_2)^2 \\ \mathrm{Aut}\big(\mathrm{Sol}^3 \times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_3} &\cong \mathbb{Z}_2. \end{aligned}$$

Proof. The group of automorphisms of $\mathrm{Sol}^3 \times\mathbb{R}$ consists of all elements of $\mathrm{GL}(4,\mathbb{R})$ of the form [21]

$$\begin{bmatrix} e & 0 & 0 & x \\ 0 & a & 0 & y \\ 0 & 0 & d & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e & 0 & 0 & x \\ 0 & 0 & b & y \\ 0 & c & 0 & z \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad a,b,c,d,e \in \mathbb{R}^*, x,y,z \in \mathbb{R}.$$

Let $A = \begin{bmatrix} e & 0 & 0 & x \\ 0 & a & 0 & y \\ 0 & 0 & d & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}^3 \times\mathbb{R})$. Then by (3)

$$\begin{aligned} & A \in \mathrm{Aut}(\mathrm{Sol}^3 \times\mathbb{R})_{\langle\cdot,\cdot\rangle_1} \\ \Leftrightarrow \quad & A^t\langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1 \\ \Leftrightarrow \quad & \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & d & 0 \\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \begin{bmatrix} e & 0 & 0 & x \\ 0 & a & 0 & y \\ 0 & 0 & d & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \end{aligned}$$

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$$\Leftrightarrow \quad \begin{bmatrix} e^2 & 0 & 0 & ex \\ 0 & a^2 & 0 & ay \\ 0 & 0 & d^2 & dz \\ ex & ay & dz & x^2+y^2+z^2+\dfrac{1}{\mu^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad x=y=z=0, \quad e=\pm 1, \quad a=\pm 1, \quad d=\pm 1$$

$$\Leftrightarrow \quad A \in \left\{ \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \cong (\mathbb{Z}_2)^3.$$

Let $A = \begin{bmatrix} e & 0 & 0 & x \\ 0 & 0 & b & y \\ 0 & c & 0 & z \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$. Then by (3)

$$A \in \mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})_{\langle \cdot,\cdot \rangle_1}$$

$$\Leftrightarrow \quad A^t \langle \cdot,\cdot \rangle_1 A = \langle \cdot,\cdot \rangle_1$$

$$\Leftrightarrow \quad \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & b & 0 & 0 \\ x & y & z & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix} \begin{bmatrix} e & 0 & 0 & x \\ 0 & 0 & b & y \\ 0 & c & 0 & z \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} e^2 & 0 & 0 & ex \\ 0 & c^2 & 0 & cz \\ 0 & 0 & b^2 & by \\ ex & cz & by & x^2+y^2+z^2+\dfrac{1}{\mu^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad x=y=z=0, \quad e=\pm 1, \quad b=\pm 1, \quad c=\pm 1$$

$$\Leftrightarrow \quad A \in \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}$$

$$\cup \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}$$

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Hence,

$$\mathrm{Aut}\big(\mathrm{Sol}^3\times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_1}$$
$$=(\mathbb{Z}_2)^3\cup\left\{\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}\right\}$$
$$\cup\left\{\begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix},\begin{bmatrix}-1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}\right\}$$

Therefore, $\mathrm{Aut}\big(\mathrm{Sol}^3\times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_1}$ is a group of order 16 such that:
The following 11 elements are of order 2

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}.$$

The following 4 elements are of order 4

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}.$$

In the classification of groups of order 16, the group that contains 11 elements of order 2 and 4 elements of order 4 is exactly $\mathbf{D}_4\times\mathbb{Z}_2$. Thus,

$$\mathrm{Aut}\big(\mathrm{Sol}^3\times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_1}\cong\mathbf{D}(4)\times\mathbb{Z}_2.$$

The maximal group of isometric automorphisms of $\mathrm{Sol}^3\times\mathbb{R}$ is $\mathbf{D}(4)\times\mathbb{Z}_2$, whose elements are listed above. To determine the group of isometric automorphisms corresponding to the metric $\langle\cdot,\cdot\rangle_2$, it suffices to check which elements of $\mathbf{D}(4)\times\mathbb{Z}_2$ preserve this metric. A straightforward verification shows that

$$\mathrm{Aut}\big(\mathrm{Sol}^3\times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_2}=\mathrm{Aut}(\mathrm{Sol}^4_{m,n})_{\langle\cdot,\cdot\rangle_2}\cong(\mathbb{Z}_2)^2.$$

The same for the third metric, we have

$$\mathrm{Aut}\big(\mathrm{Sol}^3\times\mathbb{R}\big)_{\langle\cdot,\cdot\rangle_3}=\mathrm{Aut}(\mathrm{Sol}^4_{m,n})_{\langle\cdot,\cdot\rangle_3}\cong\mathbb{Z}_2.$$

□

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Corollary 4.10. *The isometry group of* $\mathrm{Sol}^3 \times \mathbb{R}$ *is given by*

$$\begin{aligned}
\mathrm{Isom}\big(\mathrm{Sol}^3 \times \mathbb{R}, \langle\cdot,\cdot\rangle_1\big) &\cong \big(\mathrm{Sol}^3 \times \mathbb{R}\big) \rtimes (\mathbf{D}(4) \times \mathbb{Z}_2)\\
\mathrm{Isom}\big(\mathrm{Sol}^3 \times \mathbb{R}, \langle\cdot,\cdot\rangle_2\big) &\cong \big(\mathrm{Sol}^3 \times \mathbb{R}\big) \rtimes (\mathbb{Z}_2)^2\\
\mathrm{Isom}\big(\mathrm{Sol}^3 \times \mathbb{R}, \langle\cdot,\cdot\rangle_3\big) &\cong \big(\mathrm{Sol}^3 \times \mathbb{R}\big) \rtimes \mathbb{Z}_2.
\end{aligned}$$

Answer. The answer to Question 4.3 is negative. For example, the compact subgroup $(\mathbb{Z}_2)^3$ of $\mathrm{Aut}\big(\mathrm{Sol}^3 \times \mathbb{R}\big)$ cannot occur as the isotropy subgroup of any left-invariant Riemannian metric.

4.2.3. Isometry group of Sol_0^4

Theorem 4.11. *The isometry group of* Sol_0^4 *is given by*

$$\begin{aligned}
\mathrm{Isom}\big(\mathrm{Sol}_0^4, \langle\cdot,\cdot\rangle_1\big) &\cong \mathrm{Sol}_0^4 \rtimes (\mathrm{diag}\,\{\mathrm{O}(2), \pm 1, 1\})\\
\mathrm{Isom}\big(\mathrm{Sol}_0^4, \langle\cdot,\cdot\rangle_2\big) &\cong \mathrm{Sol}_0^4 \rtimes (\mathbb{Z}_2)^2.
\end{aligned}$$

Proof. The automorphism group $\mathrm{Aut}(\mathrm{Sol}_0^4)$ consists of elements of the form [21]

$$\begin{bmatrix} a & b & 0 & x\\ c & d & 0 & y\\ 0 & 0 & e & z\\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a & b\\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{R}) \quad e \in \mathbb{R}^* \quad x, y, z \in \mathbb{R}.$$

Let $A = \begin{bmatrix} a & b & 0 & x\\ c & d & 0 & y\\ 0 & 0 & e & z\\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_0^4)$, then by (3)

$$\begin{aligned}
& A \in \mathrm{Aut}\big(\mathrm{Sol}_0^4\big)_{\langle\cdot,\cdot\rangle_1}\\
\Leftrightarrow \quad & A^t \langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1\\
\Leftrightarrow \quad & \begin{bmatrix} a & c & 0 & 0\\ b & d & 0 & 0\\ 0 & 0 & e & 0\\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix} \begin{bmatrix} a & b & 0 & x\\ c & d & 0 & y\\ 0 & 0 & e & z\\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix}\\
\Leftrightarrow \quad & \begin{bmatrix} a^2+c^2 & ab+cd & 0 & ax+cy\\ ab+cd & b^2+d^2 & 0 & bx+dy\\ 0 & 0 & e^2 & ez\\ ax+cy & bx+dy & ez & x^2+y^2+z^2+\frac{1}{\beta^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix}.
\end{aligned}$$

By equating the $(3,4)$ and $(4,4)$ entries of the two matrices above, we obtain

$$z = x = y = 0.$$

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Hence, the matrix equality reduces to

$$\begin{bmatrix} a^2+c^2 & ab+cd & 0 & 0 \\ ab+cd & b^2+d^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2, \quad e = \pm 1$$

$$\Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{O}(2), \quad e = \pm 1.$$

Hence

$$\mathrm{Isom}\left(\mathrm{Sol}_0^4, \langle\cdot,\cdot\rangle_1\right) \cong \mathrm{Sol}_0^4 \rtimes (\mathrm{diag}\{\mathrm{O}(2), \pm 1, 1\}) .$$

For the metric $\langle\cdot,\cdot\rangle_2$, using (3), one gets that

$$A \in \mathrm{Aut}(\mathrm{Sol}_0^4)_{\langle\cdot,\cdot\rangle_2}$$

$$\Leftrightarrow \quad A^t \langle\cdot,\cdot\rangle_2 A = \langle\cdot,\cdot\rangle_2$$

$$\Leftrightarrow \quad \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & e & 0 \\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix} \begin{bmatrix} a & b & 0 & x \\ c & d & 0 & y \\ 0 & 0 & e & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a^2+c^2 & ab+cd & -a\alpha e & -a\alpha z + ax + cy \\ ab+cd & b^2+d^2 & -b\alpha e & -b\alpha z + bx + dy \\ -a\alpha e & -b\alpha e & e^2(\alpha^2+1) & -e\alpha x + e(\alpha^2+1)z \\ (-z\alpha+x)a+cy & (-z\alpha+x)b+dy & (-x\alpha+z(\alpha^2+1))e & (-z\alpha+x)x + y^2 + (-x\alpha+z(\alpha^2+1))z + \frac{1}{\beta^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta^2} \end{bmatrix}$$

By equating the (3, 3) and (3, 1) entries of the two matrices above, we obtain $e = \pm 1$ and $ae = 1$. Since $\alpha > 0$ and $e \neq 0$, equating the (3, 2) entries yields $b = 0$. Next, the (2, 2) entries imply that $d = \pm 1$. The (2, 1) entries then imply that $c = 0$, while the (4, 2) entries give $y = 0$. Finally, the (4, 1) and (4, 3) entries show that x and z satisfy the following system:

$$\begin{cases} x - \alpha z = 0 \\ -\alpha x + \left(1+\alpha^2\right) z = 0 \end{cases}$$

This implies that $x = z = 0$. Hence,

$$\mathrm{Aut}(\mathrm{Sol}_0^4)_{\langle\cdot,\cdot\rangle_2} = \left\{ I_4, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\cong (\mathbb{Z}_2)^2.$$

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Consequently,

$$\mathrm{Isom}\big(\mathrm{Sol}_0^4, \langle\cdot,\cdot\rangle_2\big) \cong \mathrm{Sol}_0^4 \rtimes (\mathbb{Z}_2)^2.$$

□

Answer. The answer to Question 4.3 is negative. For example, the compact subgroup $\mathbb{Z}_2$ of $\mathrm{Aut}(\mathrm{Sol}_0^4)$ cannot occur as the isotropy subgroup of any left-invariant Riemannian metric.

4.2.4. Isometry group of $\mathrm{Sol}_0^{'4}$

Theorem 4.12. *The isometry group of* $\mathrm{Sol}_0^{'4}$ *is given by*

$$\mathrm{Isom}\big(\mathrm{Sol}_0^{'4}, \langle\cdot,\cdot\rangle_1\big) \cong \mathrm{Sol}_0^{'4} \rtimes (\mathbb{Z}_2)^2$$
$$\mathrm{Isom}\big(\mathrm{Sol}_0^{'4}, \langle\cdot,\cdot\rangle_2\big) \cong \mathrm{Sol}_0^{'4} \rtimes \mathbb{Z}_2.$$

Proof. Let $A = \begin{bmatrix} a & b & 0 & x \\ 0 & a & 0 & y \\ 0 & 0 & e & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_0^{'4})$ (see [21]). Then by (3)

$$\begin{aligned}
& A \in \mathrm{Aut}\big(\mathrm{Sol}_0^{'4}\big)_{\langle\cdot,\cdot\rangle_1} \\
\Leftrightarrow \quad & A^t \langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1 \\
\Leftrightarrow \quad & \begin{bmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & e & 0 \\ x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \begin{bmatrix} a & b & 0 & x \\ 0 & a & 0 & y \\ 0 & 0 & e & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \\
\Leftrightarrow \quad & \begin{bmatrix} a^2 & ab & 0 & ax \\ ab & b^2 + \frac{a^2}{\alpha^2} & 0 & bx + \frac{ay}{\alpha^2} \\ 0 & 0 & e^2 & ez \\ ax & bx + \frac{ay}{\alpha^2} & ez & x^2 + \frac{y^2}{\alpha^2} + z^2 + \frac{1}{\mu^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu^2} \end{bmatrix} \\
\Leftrightarrow \quad & x = y = z = b = 0, \quad e = \pm 1, \quad a = \pm 1.
\end{aligned}$$

Thus,

$$\mathrm{Aut}(\mathrm{Sol}_0^{'4})_{\langle\cdot,\cdot\rangle_1} = \left\{ I_4, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

This is exactly the Klein 4-group, thus $\mathrm{Aut}(\mathrm{Sol}_0^{'4})_{\langle\cdot,\cdot\rangle_1} \cong (\mathbb{Z}_2)^2$. Hence,

$$\mathrm{Isom}\big(\mathrm{Sol}_0^{'4}, \langle\cdot,\cdot\rangle_1\big) \cong \mathrm{Sol}_0^{'4} \rtimes (\mathbb{Z}_2)^2.$$

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For the metric $\langle\cdot,\cdot\rangle_2$, it suffices to determine which of the elements of $(\mathbb{Z}_2)^2$ listed above preserve this metric. We obtain that

$$A \in \mathrm{Aut}\big(\mathrm{Sol}_0'^4\big)_{\langle\cdot,\cdot\rangle_2} \Leftrightarrow A^t\langle\cdot,\cdot\rangle_2 A = \langle\cdot,\cdot\rangle_2$$

$$\Leftrightarrow A \in \left\{ I_4, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \cong \mathbb{Z}_2.$$

Therefore,

$$\mathrm{Isom}\big(\mathrm{Sol}_0'^4, \langle\cdot,\cdot\rangle_2\big) \cong \mathrm{Sol}_0'^4 \rtimes \mathbb{Z}_2.$$

□

Answer. The answer to Question 4.3 is affirmative. Note that $(\mathbb{Z}_2)^2$ has three distinct subgroups of order 2; however, these are different subgroups inside $(\mathbb{Z}_2)^2$, even though they are all isomorphic to $\mathbb{Z}_2$. We emphasize that it is these abstract group types that occur as isotropy groups of left-invariant Riemannian metrics.

4.2.5. Isometry group of Sol_1^4

Theorem 4.13. *The group of isometric automorphisms of* Sol_1^4 *is given by*

$$\mathrm{Aut}\big(\mathrm{Sol}_1^4\big)_{\langle\cdot,\cdot\rangle_1} \cong \mathbf{D}(4)$$

$$\mathrm{Aut}\big(\mathrm{Sol}_1^4\big)_{\langle\cdot,\cdot\rangle_2} \cong \mathbb{Z}_2$$

$$\mathrm{Aut}\big(\mathrm{Sol}_1^4\big)_{\langle\cdot,\cdot\rangle_3} = \{I_4\}.$$

Proof. The group of automorphisms of Sol_1^4 consists of all elements of $\mathrm{GL}(4,\mathbb{R})$ of the form [21]

$$\begin{bmatrix} ad & -aq & -dp & x \\ 0 & a & 0 & p \\ 0 & 0 & d & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -bc & cp & bq & x \\ 0 & 0 & b & p \\ 0 & c & 0 & q \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad a,b,c,d \in \mathbb{R}^*, x,p,q \in \mathbb{R}.$$

Let $A = \begin{bmatrix} ad & -aq & -dp & x \\ 0 & a & 0 & p \\ 0 & 0 & d & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_1^4)$. Then by (3)

$$A \in \mathrm{Aut}(\mathrm{Sol}_1^4)_{\langle\cdot,\cdot\rangle_1}$$

$$\Leftrightarrow \quad A^t\langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1$$

$$\Leftrightarrow \begin{bmatrix} ad & 0 & 0 & 0 \\ -aq & a & 0 & 0 \\ -dp & 0 & d & 0 \\ x & p & q & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix} \begin{bmatrix} ad & -aq & -dp & x \\ 0 & a & 0 & p \\ 0 & 0 & d & q \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

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$$\Leftrightarrow \begin{bmatrix} \frac{a^2d^2}{\alpha^2} & -\frac{a^2dq}{\alpha^2} & -\frac{a\,d^2p}{\alpha^2} & \frac{adx}{\alpha^2} \\ -\frac{a^2dq}{\alpha^2} & \frac{a^2q^2}{\alpha^2}+a^2 & \frac{aqdp}{\alpha^2} & -\frac{aqx}{\alpha^2}+ap \\ -\frac{a\,d^2p}{\alpha^2} & \frac{aqdp}{\alpha^2} & \frac{d^2p^2}{\alpha^2}+d^2 & -\frac{dpx}{\alpha^2}+dq \\ \frac{adx}{\alpha^2} & -\frac{aqx}{\alpha^2}+ap & -\frac{dpx}{\alpha^2}+dq & \frac{x^2}{\alpha^2}+p^2+q^2+\frac{1}{\nu^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad x=p=q=0, \quad a=\pm 1, \quad d=\pm 1$$

$$\Leftrightarrow \quad A \in \left\{ I_4, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

Let $A = \begin{bmatrix} -bc & cp & bq & x \\ 0 & 0 & b & p \\ 0 & c & 0 & q \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_1^4)$. Then by (3)

$$A \in \mathrm{Aut}(\mathrm{Sol}_1^4)_{\langle\cdot,\cdot\rangle_1}$$

$$\Leftrightarrow \quad A^t \langle\cdot,\cdot\rangle_1 A = \langle\cdot,\cdot\rangle_1$$

$$\Leftrightarrow \quad \begin{bmatrix} -bc & 0 & 0 & 0 \\ cp & 0 & c & 0 \\ bq & b & 0 & 0 \\ x & p & q & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix} \begin{bmatrix} -bc & cp & bq & x \\ 0 & 0 & b & p \\ 0 & c & 0 & q \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} \frac{b^2c^2}{\alpha^2} & -\frac{b\,c^2p}{\alpha^2} & -\frac{b^2cq}{\alpha^2} & -\frac{bcx}{\alpha^2} \\ -\frac{b\,c^2p}{\alpha^2} & \frac{c^2p^2}{\alpha^2}+c^2 & \frac{cpbq}{\alpha^2} & \frac{cpx}{\alpha^2}+cq \\ -\frac{b^2cq}{\alpha^2} & \frac{cpbq}{\alpha^2} & \frac{b^2q^2}{\alpha^2}+b^2 & \frac{bqx}{\alpha^2}+bp \\ -\frac{bcx}{\alpha^2} & \frac{cpx}{\alpha^2}+cq & \frac{bqx}{\alpha^2}+bp & \frac{x^2}{\alpha^2}+p^2+q^2+\frac{1}{\nu^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu^2} \end{bmatrix}$$

$$\Leftrightarrow \quad x=p=q=0, \quad b=\pm 1, \quad c=\pm 1$$

$$\Leftrightarrow A \in \left\{ \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}.$$

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Consequently:

$$\begin{aligned}
&\mathrm{Aut}\big(\mathrm{Sol}_1^4\big)_{\langle\cdot,\cdot\rangle_1} \\
&= \left\{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\right\} \\
&\cup \left\{\begin{bmatrix}-1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}\right\} \\
&= \left\langle \begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&-1\end{bmatrix}\right\rangle \cong \mathbf{D}(4).
\end{aligned}$$

Therefore, $\mathrm{Aut}(\mathrm{Sol}_1^4)_{\langle\cdot,\cdot\rangle_1} \cong \mathbf{D}(4)$.

The maximal group of isometric automorphisms of Sol_1^4 is $\mathbf{D}(4)$, whose elements are listed above. To determine the group of isometric automorphisms corresponding to the metric $\langle\cdot,\cdot\rangle_2$, it suffices to check which elements of $\mathbf{D}(4)$ preserve this metric. A straightforward verification shows that

$$\mathrm{Aut}\big(\mathrm{Sol}_1^4\big)_{\langle\cdot,\cdot\rangle_2} = \{I_4, \mathrm{diag}\{-1,-1,1,1\}\} \cong \mathbb{Z}_2.$$

The same for the third metric, we obtain $\mathrm{Aut}(\mathrm{Sol}_1^4)_{\langle\cdot,\cdot\rangle_3} = \{I_4\}$. □

Corollary 4.14. *The isometry group of* Sol_1^4 *is given by*

$$\begin{aligned}
\mathrm{Isom}\big(\mathrm{Sol}_1^4, \langle\cdot,\cdot\rangle_1\big) &\cong \mathrm{Sol}_1^4 \rtimes \mathbf{D}(4) \\
\mathrm{Isom}\big(\mathrm{Sol}_1^4, \langle\cdot,\cdot\rangle_2\big) &\cong \mathrm{Sol}_1^4 \rtimes \mathbb{Z}_2 \\
\mathrm{Isom}\big(\mathrm{Sol}_1^4, \langle\cdot,\cdot\rangle_3\big) &\cong \mathrm{Sol}_1^4 .
\end{aligned}$$

Answer. The answer to Question 4.3 is negative. For example, the compact subgroup $(\mathbb{Z}_2)^2$ of $\mathrm{Aut}(\mathrm{Sol}_1^4)$ cannot occur as the isotropy subgroup of any left-invariant Riemannian metric.

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