A CONSISTENCY CONDITION FOR ELECTRON WAVE FUNCTIONS

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1. *Introduction.* Dirac's wave equation of an electron enables one to solve for the wave function of an electron moving in an electromagnetic field. The wave function ψ has 4 components $\psi_1, \psi_2, \psi_3, \psi_4$, and the electromagnetic field is described by a four-vector A_μ, consisting of a scalar potential ϕ and a vector potential **A**.

In the usual problem, we solve for the wave function ψ when the potential A_μ is given. By doing so, we obtain the wave function of an electron which moves in a given electromagnetic field. It is of some interest to consider the reverse question: given the wave function ψ, what can we say about the electromagnetic potential A_μ, which is connected with ψ by Dirac's equation? Is A_μ uniquely determined, and if not, what is the extent to which it is arbitrary?

If it becomes possible to express A_μ as a function of $\psi_1, \psi_2, \psi_3, \psi_4$ and their derivatives, then it should be possible to express the electromagnetic field variables $f_{\mu\nu}$ $(= \partial A_\mu/\partial x^\nu - \partial A_\nu/\partial x^\mu)$ also in the same way. Thus, we would obtain a relationship between $f_{\mu\nu}$ and the ψ's. Such a relationship may have its uses, as the physical quantities that describe the electromagnetic field are the field variables $f_{\mu\nu}$ rather than the potentials which involve non-physical gauge variables. One would expect that the relationship between $f_{\mu\nu}$ and the ψ's would be a very complicated one. Nevertheless, it is of some interest to see if a useful relation may be obtained. In the course of looking for this relationship, it has come to my notice that any wave function ψ which describes the motion of an electron moving in any electromagnetic field should satisfy a consistency condition of the form

$$\psi' J \psi = 0, \tag{1}$$

where ψ is the column matrix $(\psi_1, \psi_2, \psi_3, \psi_4)$, ψ' is the transpose of ψ, that is, the 1×4 row matrix with elements $(\psi_1, \psi_2, \psi_3, \psi_4)$, and J is a 4×4 matrix involving the differential operator $\partial/\partial x^\mu$ and the Dirac matrices.

2. *Equations for the potentials.* The Dirac equations (1) are

$$[i\nabla_t + \phi + \boldsymbol{\alpha}.(-i\nabla + \mathbf{A}) + m\beta]\psi = 0, \tag{2}$$

where $\boldsymbol{\alpha}$ and β are the usual Dirac matrices, and for convenience of writing, we put $\hbar = c = e = 1$. Choosing the usual Dirac representation (1) for $\boldsymbol{\alpha}$ and β, and writing the equations as linear equations to determine ϕ, A_1, A_2, A_3, we obtain

$$\psi_1\phi + \psi_4 A_1 - i\psi_4 A_2 + \psi_3 A_3 = (-m - i\nabla_t)\psi_1 + (i\nabla_x + \nabla_y)\psi_4 + i\nabla_z\psi_3, \tag{3}$$

$$\psi_2\phi + \psi_3 A_1 + i\psi_3 A_2 - \psi_4 A_3 = (-m - i\nabla_t)\psi_2 + (i\nabla_x - \nabla_y)\psi_3 - i\nabla_z\psi_4, \tag{4}$$

$$\psi_3\phi + \psi_2 A_1 - i\psi_2 A_2 + \psi_1 A_3 = (m - i\nabla_t)\psi_3 + (i\nabla_x + \nabla_y)\psi_2 + i\nabla_z\psi_1, \tag{5}$$

$$\psi_4\phi + \psi_1 A_1 - i\psi_1 A_2 + \psi_2 A_3 = (m - i\nabla_t)\psi_4 + (i\nabla_x - \nabla_y)\psi_1 - i\nabla_z\psi_2. \tag{6}$$

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We may write this in the form

$$PX = Q, \tag{7}$$

where
$$P = \begin{pmatrix} \psi_1 & \psi_4 & -\psi_4 & \psi_3 \\ \psi_2 & \psi_3 & \psi_3 & -\psi_4 \\ \psi_3 & \psi_2 & -\psi_2 & \psi_1 \\ \psi_4 & \psi_1 & \psi_1 & -\psi_2 \end{pmatrix}, \quad X = \begin{pmatrix} \phi \\ A_1 \\ iA_2 \\ A_3 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}; \tag{8}$$

here Q_0, Q_1, Q_2, Q_3 are the expressions on the right-hand side of equations (3), (4), (5) and (6), respectively.

Normally we would expect to solve for the four unknowns ϕ, A_1, A_2, A_3 from the four equations (3), (4), (5) and (6). But we see that

$$\det P = 0. \tag{9}$$

Hence, the four equations are not linearly independent.

3. *Consistency condition.* The rank of the matrix P is three, as there are 3×3 minors which are non-vanishing. If we take as critical minor

$$\begin{vmatrix} \psi_1 & \psi_4 & -\psi_4 \\ \psi_2 & \psi_3 & \psi_3 \\ \psi_3 & \psi_2 & -\psi_2 \end{vmatrix} \tag{10}$$

then there is a consistency condition which has to be satisfied by the Q's, namely

$$\begin{vmatrix} \psi_1 & \psi_4 & -\psi_4 & Q_0 \\ \psi_2 & \psi_3 & \psi_3 & Q_1 \\ \psi_3 & \psi_2 & -\psi_2 & Q_2 \\ \psi_4 & \psi_1 & \psi_1 & Q_3 \end{vmatrix} = 0, \tag{11}$$

which when simplified gives

$$(\psi_1\psi_2 - \psi_3\psi_4)(-\psi_2 Q_0 + \psi_1 Q_1 + \psi_4 Q_2 - \psi_3 Q_3) = 0. \tag{12}$$

Hence, either
$$-\psi_2 Q_0 + \psi_1 Q_1 + \psi_4 Q_2 - \psi_3 Q_3 = 0, \tag{13}$$

or
$$\psi_1\psi_2 - \psi_3\psi_4 = 0. \tag{14}$$

On substituting for Q_0, Q_1, Q_2, Q_3 in equation (14) we find that the terms in m cancel out, giving the equation

$$\begin{aligned} &\psi_1[-i\nabla_t\psi_2 + (i\nabla_x - \nabla_y)\psi_3 - i\nabla_z\psi_4] + \psi_2[i\nabla_t\psi_1 + (-i\nabla_x - \nabla_y)\psi_4 - i\nabla_z\psi_3] \\ &+ \psi_3[i\nabla_t\psi_4 + (-i\nabla_x + \nabla_y)\psi_1 + i\nabla_z\psi_4] + \psi_4[-i\nabla_t\psi_3 + (i\nabla_x + \nabla_y)\psi_2 + i\nabla_z\psi_1] = 0. \end{aligned} \tag{15}$$

Dividing throughout by i, this equation may be written in the form

$$\psi' J \psi = 0, \tag{1}$$

where
$$J \equiv \begin{pmatrix} 0 & -\nabla_t & \nabla_x + i\nabla_y & -\nabla_z \\ \nabla_t & 0 & -\nabla_z & -\nabla_x + i\nabla_y \\ -\nabla_x - i\nabla_y & \nabla_z & 0 & \nabla_t \\ \nabla_z & \nabla_x - i\nabla_y & -\nabla_t & 0 \end{pmatrix} \tag{16}$$

$$= \beta\alpha_x\alpha_z(\nabla_t - \alpha_x\nabla_x - \alpha_y\nabla_y - \alpha_z\nabla_z). \tag{17}$$

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The equation (14) may be written in the form

$$\psi' K \psi = 0, \tag{18}$$

where

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -\beta\alpha_y\alpha_z. \tag{19}$$

The condition (19) is a special condition, which will not be satisfied generally by ψ, but only in some specially simple case. Leaving this case out, we obtain the more general condition (1).

We note that J is independent of the mass m of the electron. This implies that condition (1) should be satisfied by the wave function of any charged particle of spin $\frac{1}{2}$ (and not merely of electrons) and moving in any electromagnetic field.

One notes that J is an anti-symmetric matrix operator and that

$$\det J = \Box^2, \tag{20}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

It is of interest to check whether the electron wave functions of those cases where the solution is explicitly known do in fact satisfy the condition (1). The author has verified this result for the wave function of a free electron, and Kathirgamanathan in an unpublished paper has verified it for the wave function of an electron in a central field of force.

4. *Alternative derivation.* It has been indicated to the author by Prof. Dirac that the condition (1) may also be derived in a simple way as follows:

If γ is an anti-symmetric 4×4 matrix,

$$\psi' \gamma \psi = 0. \tag{21}$$

If γ is such that $\gamma\alpha_x$, $\gamma\alpha_y$, $\gamma\alpha_z$, $\gamma\beta$ are all antisymmetric, then

$$\psi' \gamma (\phi + \alpha_x A_x + \alpha_y A_y + \alpha_z A_z + m\beta)\psi = 0. \tag{22}$$

So if γ satisfies the wave equation, we have

$$\psi' \gamma (i\nabla_t - i\alpha_x \nabla_x - i\alpha_y \nabla_y - i\alpha_z \nabla_z)\psi = 0 \tag{23}$$

which is the same as condition (1) when $\gamma = \beta\alpha_x\alpha_z$.

We can see that γ is uniquely determined except for a multiplying constant.

5. *The potentials.* The four equations (3), (4), (5) and (6), being linearly dependent equations, do not determine ϕ, A_1, A_2, A_3 completely. If we take ϕ as a parameter, we may express A_1, A_2, A_3 in terms of ϕ. Omitting details of calculation, the result we obtain is

$$A_s = \frac{\psi' \alpha_s R \psi}{\psi' a \psi}, \quad s = x, y, z, \tag{24}$$

where

$$R = \alpha_y \beta [m + i\phi - i(\nabla_t - \alpha_x \nabla_x - \alpha_y \nabla_y - \alpha_z \nabla_z)], \tag{25}$$

$$a = \alpha_z \beta. \tag{26}$$

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6. *Discussion.* The fact that any electron wave function, whatever the electromagnetic field, satisfies a simple condition of the form (1) is of some interest. There is added interest owing to the circumstance that the condition does not depend on the charge or mass of the particle. Hence, the result holds for the wave function of any particle of spin $\frac{1}{2}$ moving in any electromagnetic field.

Two questions appear to deserve further investigation. One is whether the condition (1) has any physical meaning. The author has not succeeded in discovering a reasonable physical interpretation. The second question is that as one notes that the expression (24) for A as a function of ψ is a comparatively simple one, whether the relationship between ψ and the field quantities $f_{\mu\nu}$ would also be comparatively simple. The author hopes to deal with these equations in a later paper.

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