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Dissipative conformal measures on locally compact spaces

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Abstract. The paper introduces a general method to construct conformal measures for a local homeomorphism on a locally compact non-compact Hausdorff space, subject to mild irreducibility-like conditions. Among other things, the method is used to give necessary and sufficient conditions for the existence of eigenmeasures for the dual Ruelle operator associated to a locally compact non-compact irreducible Markov shift equipped with a uniformly continuous potential function. As an application to operator algebras the results are used to determine for which β there are gauge invariant β-KMS weights on a simple graph C^*-algebra when the one-parameter automorphism group is given by a uniformly continuous real-valued function on the path space of the graph.

1. *Introduction*

A conformal measure for a discrete dynamical system made its first appearance in the work of Sullivan [**Sul83**], in connection with rational maps on the Riemann sphere, before the notion was coined and introduced in a more general setting by Denker and Urbanski [**DU91**]. Conformal measures corresponding to various potential functions now play important roles in the study of dynamical systems, for example in holomorphic dynamics where they are used as a tool to study the structure of the Julia set, among others. In the setting of topological Markov shifts conformal measures arise naturally via the thermodynamic formalism introduced by D. Ruelle, and they constitute a key ingredient in the study of topological Markov shifts with a countable number of states, [**Sar99**, **Sar09**]. During the last decade it has been realized that they also appear in connection with quantum statistical models that are based on C^*-algebras and one-parameter group actions arising from local homeomorphisms. This relation, which until recently only involved probability measures on the dynamical system side and KMS states on the operator algebra side, was extended in [**Tho14**] to a bijective correspondence between general (possibly infinite) conformal measures and KMS weights. It is this new connection between dynamical systems and C^*-algebras which motivates the present study. Conformal

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measures are often required to be probability measures, but for dynamical systems on non-compact spaces, such as countable state Markov shifts, it is crucial to allow the measures to be infinite. Nonetheless our knowledge of general, possibly infinite conformal measures is lacking when it comes to the problem of determining the KMS weights in the C^*-algebraic setting mentioned above, perhaps because of reluctance to consider measures that are not conservative.

The present paper seeks to improve on this by introducing a method to construct conformal measures for a local homeomorphism on a locally compact non-compact Hausdorff space which exploits the possibility of taking limits along sequences that go to infinity. The method works in a quite general non-compact setup and produces a non-zero fixed measure for the dual Ruelle operator when the pressure of the potential is non-positive, and not only when it is zero. More precisely, given a locally compact second countable Hausdorff space X and a local homeomorphism $\sigma : X \to X$, we shall say that (X, σ) is *cofinal* when the equality

$$X = \bigcup_{i,j\in\mathbb{N}} \sigma^{-i}(\sigma^j(U)) \tag{1.1}$$

holds for every open non-empty subset $U \subseteq X$, and that (X, σ) is *non-compact* when

$$\bigcup_{k=0}^{\infty} \sigma^k(U) \tag{1.2}$$

is not pre-compact (i.e. does not have compact closure) for any open non-empty subset $U \subseteq X$. Assuming that (X, σ) has these two properties and given any continuous real-valued function $\phi : X \to \mathbb{R}$, referred to in the following as the *potential*, there is a natural notion of pressure $\mathbb{P}(\phi)$, and the method to be described produces regular non-zero ϕ-conformal measures when $\mathbb{P}(-\phi) \leq 0$. The measures can be finite or infinite, but they are always dissipative when $\mathbb{P}(-\phi) < 0$.

This construction is an extension of the method by which positive eigenvectors of an infinite non-negative matrix can be produced, and has its roots in the way harmonic functions are constructed for Markov chains. See [**Tho14**] and [**Tho13**] for more details on this connection. The extension we present here is also inspired by the PhD thesis of Cyr [**Cyr10**], where a similar method was used to produce conformal measures for transient potentials with zero pressure on mixing countable state Markov shifts. In that setting our results are more complete because we obtain also a necessary condition for the existence of a ϕ-conformal measure: For a locally compact non-compact irreducible Markov shift and a uniformly continuous potential ϕ, a ϕ-conformal measure exists if and only if $\mathbb{P}(-\phi) \leq 0$.

Besides the countable state Markov shifts, a large and interesting class of examples come from transcendental entire maps. When there are no critical points in the Julia set the map is a local homeomorphism on its Julia set, giving rise to a dynamical system which is both cofinal and non-compact, possibly after the deletion of a single exceptional fixed point. A prominent example of this is given by the exponential family λe^z, and in §6 we consider the hyperbolic members of this family obtained by choosing λ real and between 0 and e^{-1}, in order to show that the method we introduce combines nicely with those developed by Mayer and Urbanski in [**MU08**, **MU10**], and gives rise to conformal

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measures for the geometric potential $\log|z|$, not only when the pressure function is zero, which by a result in [**MU08**] occurs exactly when the inverse temperature t is equal to the Hausdorff dimension HD of the radial Julia set, but for all $t \geq HD$. The additional conformal measures arise immediately from the general results once we have shown that the notion of pressure employed here agrees with that used by Mayer and Urbanski.

As indicated above, our interest in conformal measures stems from the bijective correspondence between $\beta\phi$-conformal measures on X and gauge invariant β-KMS weights for the one-parameter group arising from the triple (X, σ, ϕ) by a well-known canonical construction. In the final section we give a brief introduction to this connection and summarize the consequences of our results in the operator algebra setting. In particular, they allow us to extend the results from [**Tho14**] concerning the β-KMS weights of the gauge action on a simple graph algebra to actions coming from an arbitrary uniformly continuous potential.

2. *Conformal measures*

Throughout this paper X is a locally compact second countable Hausdorff space, $\sigma : X \to X$ is a local homeomorphism and $\phi : X \to \mathbb{R}$ is a continuous function.

Definition 2.1. A regular and non-zero Borel measure m on X is a *ϕ-conformal measure* when

$$m(\sigma(A)) = \int_A e^{\phi(x)}\, dm(x) \tag{2.1}$$

for every Borel subset $A \subseteq X$ with the property that $\sigma : A \to X$ is injective.

Let $C_c(X)$ be the space of continuous compactly supported functions on X. We study conformal measures via the *Ruelle operator* $L_\phi : C_c(X) \to C_c(X)$ defined such that

$$L_\phi(f)(x) = \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y). \tag{2.2}$$

Since $L_\phi : C_c(X) \to C_c(X)$ is linear and takes non-negative functions to non-negative functions, it follows from the Riesz representation theorem that L_ϕ defines a map $m \mapsto L_\phi^*(m)$ on the set of regular Borel measures m on X by the requirement that

$$\int_X f\, dL_\phi^*(m) = \int_X L_\phi(f)\, dm$$

for all $f \in C_c(X)$.

LEMMA 2.2. *Let m be a non-zero regular Borel measure on X. The following are equivalent.*

(1) *m is ϕ-conformal.*
(2) *$L_{-\phi}^*(m) = m$.*
(3) *When $U \subseteq X$ is an open subset such that $\sigma : U \to X$ is injective and $g \in C_c(U)$,*

$$\int_{\sigma(U)} g \circ (\sigma|_U)^{-1}\, dm = \int_U g e^{\phi}\, dm.$$

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(4) *There is a covering* $X = \bigcup_i U_i$ *of* X *by open subsets* U_i *such that, for every* i, $\sigma : U_i \to X$ *is injective and for all* $g \in C_c(U_i)$,

$$\int_{\sigma(U_i)} g \circ (\sigma|_{U_i})^{-1}\, dm = \int_{U_i} g e^{\phi}\, dm.$$

Proof. The equivalence between (1) and (2) was observed by Renault in [**Ren03**], although a version of it appeared, in a slightly different setting, in [**DU91**] where conformal measures were first introduced. Since the lemma is a crucial tool in the following, we present the proof here. We denote the characteristic function of a set S by 1_S.

(1) $\Rightarrow$ (3): Let U be as in (3). By linearity and continuity it suffices to establish the desired identity when $g = 1_A$ for some Borel subset $A \subseteq U$. In this case the identity is the same as (2.1).

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (2): By linearity it suffices to show that

$$\int_X L_{-\phi}(f)\, dm = \int_X f\, dm$$

when f is a supported in U_i. In that case the definition of $L_{-\phi}$ shows that

$$\int_X L_{-\phi}(f)\, dm = \int_{\sigma(U_i)} e^{-\phi((\sigma|_{U_i})^{-1}(x))} f((\sigma|_{U_i})^{-1}(x))\, dm,$$

which, thanks to (4), is equal to $\int_{U_i} f\, dm = \int_X f\, dm$.

(2) $\Rightarrow$ (1): To establish (2.1) we write $A = \bigsqcup_i A_i$ as a disjoint union of Borel sets A_i such that, for each i, there is an open and relatively compact subset $U_i \subseteq X$ where σ injective, and $A_i \subseteq U_i$. Then $m(\sigma(A)) = \sum_i m(\sigma(A_i))$ and

$$\int_A e^{\phi(x)}\, dm(x) = \sum_i \int_{A_i} e^{\phi(x)}\, dm(x).$$

It suffices therefore to establish (2.1) when $A \subseteq U$ for some open and relatively compact subset $U \subseteq X$ where σ is injective. Let $V \subseteq U$ be open. It follows from (2) that

$$\int_{\sigma(V)} e^{-\phi((\sigma|_U)^{-1}(x))} f((\sigma|_U)^{-1}(x))\, dm = \int_V f\, dm,$$

when $f \in C_c(V)$. Inserting for f an increasing sequence from $C_c(V)$ converging up to $e^{\phi} 1_V$, we find that (2.1) holds when $A = V$ is an open subset of U. Since both measures,

$$A \mapsto m(\sigma(A)) \quad \text{and} \quad A \mapsto \int_A e^{\phi}\, dm,$$

are finite Borel measures on U, they are also regular. It follows therefore that (2.1) holds for every Borel subset A of U. □

3. *Pressures associated with a cofinal local homeomorphism*

In this section we add first the assumption that σ is cofinal; that is, for every open non-empty subset U of X, identity (1.1) holds. We will use the notation $\|f\|$ for the supremum norm of a bounded function f on X, and we introduce the notation ϕ_n for the function

$$\phi_n(x) = \sum_{j=0}^{n-1} \phi(\sigma^j(x)).$$

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LEMMA 3.1. *Assume that (X, σ) is cofinal. Let K be a compact subset of X and $g \in C_c(X)$ a non-zero and non-negative function. There is an $N \in \mathbb{N}$ and a constant $C > 0$ such that*

$$|L_\phi^n(f)(x)| \le C\|f\| \sum_{1\le i,j\le N} L_\phi^{n+j-i}(g)(x) \tag{3.1}$$

for all $n > N$, all $x \in X$, and every function $f \in C_c(X)$ with support in K.

Proof. There is a $\delta > 0$ and an open non-empty subset $U \subseteq X$ such that $g(x) \ge \delta$ for all $x \in U$. Since σ is cofinal and K compact, there is an $N \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{1\le i,j\le N} \sigma^{-i}(\sigma^j(U)). \tag{3.2}$$

Set

$$C_1 = \sup\{e^{-\phi_i(z)} : 1 \le i \le N, z \in \operatorname{supp} g\},$$
$$C_2 = \sup\{e^{\phi_i(z)} : 1 \le i \le N,\ z \in K\}.$$

Consider an element $f \in C_c(X)$ with support in K and let $x \in X$, $n \in \mathbb{N}$, $n > N$. Since $\sigma^{-n}(x) \cap K$ is a finite set, it follows from (3.2) that we can write $\sigma^{-n}(x) \cap K$ as a disjoint union

$$\sigma^{-n}(x) \cap K = \bigsqcup_{1\le i,j\le N} B_{i,j},$$

where $B_{i,j} \subseteq \sigma^{-i}(\sigma^j(U))$ are finite sets. For $y \in B_{i,j}$ we choose $a_y \in U$ such that $\sigma^i(y) = \sigma^j(a_y)$. Then

$$L_\phi^n(f)(x) = \sum_{y\in\sigma^{-n}(x)\cap K} e^{\phi_n(y)} f(y) = \sum_{1\le i,j\le N} \sum_{y\in B_{i,j}} e^{\phi_n(y)} f(y). \tag{3.3}$$

For $y \in B_{i,j}$,

$$e^{\phi_n(y)}|f(y)| = e^{\phi_i(y)} e^{\phi_{n-i}(\sigma^i(y))}|f(y)| = e^{\phi_i(y)} e^{\phi_{n-i}(\sigma^j(a_y))}|f(y)|.$$

Since $|f(y)| \le \delta^{-1}\|f\| g(a_y)$, we find that

$$\begin{aligned} e^{\phi_n(y)}|f(y)| &\le \delta^{-1}\|f\| e^{\phi_i(y)} e^{\phi_{n-i}(\sigma^j(a_y))} g(a_y) \\ &= \delta^{-1}\|f\| e^{\phi_i(y)} e^{-\phi_j(a_y)} e^{\phi_{n-i+j}(a_y)} g(a_y) \le \|f\|\delta^{-1} C_1 C_2 e^{\phi_{n-i+j}(a_y)} g(a_y). \end{aligned} \tag{3.4}$$

To control the ambiguity of the association $y \mapsto a_y$, note that

$$M_i = \sup_{y\in K} \#\{z \in K : \sigma^i(z) = \sigma^i(y)\}$$

is finite for every $i \in \mathbb{N}$. Set $M = \max_{1\le i\le N} M_i$. Then

$$\sum_{y\in B_{i,j}} e^{\phi_{n-i+j}(a_y)} g(a_y) \le M \sum_{z\in\sigma^{-n-j+i}(x)} e^{\phi_{n-i+j}(z)} g(z) = M L_\phi^{n-i+j}(g)(x). \tag{3.5}$$

By combining (3.3), (3.4) and (3.5) we obtain (3.1) if we set $C = \delta^{-1} C_1 C_2 M$. □

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COROLLARY 3.2. *Assume that σ is cofinal. Let $f, g \in C_c(X)$ be non-zero and non-negative. It follows that*

$$\limsup_n (L_\phi^n(f)(x))^{1/n} = \limsup_n (L_\phi^n(g)(x))^{1/n}$$

for all $x \in X$.

Proof. Let K be a compact set containing the support of f and let C and N be the numbers from Lemma 3.1. If $\lambda > 0$ and $L_\phi^n(g)(x) \leq \lambda^n$ for all $n \geq k$, it follows from (3.1) that, for all $n \geq k + N$, there is a $j \in [n - N, n + N]$ such that

$$(C\|f\|N^2)^{-1} L_\phi^n(f)(x) \leq \lambda^j.$$

Thus

$$(C\|f\|N^2)^{-1/n} (L_\phi^n(f)(x))^{1/n} \leq \alpha_n \lambda,$$

for all large n, where $\alpha_n = \max\{\lambda^{(j/n)-1} : n - N \leq j \leq n + N\}$. Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (C\|f\|N^2)^{-1/n} = 1$ we conclude first that $\limsup_n (L_\phi^n(f)(x))^{1/n} \leq \lambda$ and then that

$$\limsup_n (L_\phi^n(f)(x))^{1/n} \leq \limsup_n (L_\phi^n(g)(x))^{1/n}.$$

This argument shows that if one of the two supremum limits is finite then so is the other, and by symmetry that they agree. Hence if one is infinite, so is the other. □

In the following we denote by $C_c(X)^+$ the set of non-negative and non-zero elements of $C_c(X)$. Using the convention that $\log 0 = -\infty$ and $\log \infty = \infty$, it follows from Corollary 3.2 that when σ is cofinal, we can define

$$\mathbb{P}_x(\phi) = \log\left(\limsup_n (L_\phi^n(f)(x))^{1/n}\right) \in [-\infty, \infty],$$

independently of which element $f \in C_c(X)^+$ we use. We subsequently define the *pressure* $\mathbb{P}(\phi)$ of ϕ to be

$$\mathbb{P}(\phi) = \sup_{x\in X} \mathbb{P}_x(\phi),$$

which is again an extended real number, that is, $\mathbb{P}(\phi) \in [-\infty, \infty]$.

3.1. *The pressures associated to a cofinal Markov shift.* In this section we relate the pressure defined above to the Gurevich pressure known from topological Markov shifts. Because we can, and since it is important for the applications to graph C^*-algebras, we work in the same generality as in [**Tho14**] and [**Tho13**], rather than restricting attention to irreducible or mixing Markov shifts.

Let G be a directed graph with vertex set V_G and edge set E_G. We assume that both V_G and E_G are countable, and that G is 'row-finite', in the sense that the number of edges emitted from any vertex is finite. Furthermore, we assume that there are no sinks, that is, that every vertex emits an edge.

An *infinite path* in G is an element $p = (p_i)_{i=1}^\infty \in (E_G)^{\mathbb{N}}$ such that $r(p_i) = s(p_{i+1})$ for all i, where we have used the notation $r(e)$ and $s(e)$ for the range and source of an edge $e \in E_G$, respectively. A finite path $\mu = e_1 e_2 \cdots e_n$ is defined similarly, and we extend

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the range and source maps to finite paths such that $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. The number of edges in μ is its *length* and we denote it by $|\mu|$. We let $\mathcal{P}(G)$ denote the set of infinite paths in G and extend the source map to $\mathcal{P}(G)$ such that $s(p) = s(p_1)$ when $p = (p_i)_{i=1}^{\infty}$. To describe the topology of $\mathcal{P}(G)$, let $\mu = e_1 e_2 \cdots e_n$ be a finite path in G. We can then consider the *cylinder*

$$Z(\mu) = \{p \in \mathcal{P}(G) : p_i = e_i,\ i = 1, 2, \ldots, n\}.$$

$\mathcal{P}(G)$ is a totally disconnected second countable locally compact Hausdorff space in the topology for which the collection of cylinders is a base [**KPRR97**]. The left shift $\sigma : \mathcal{P}(G) \to \mathcal{P}(G)$ is the map defined such that

$$\sigma(e_0 e_1 e_2 e_3 \cdots) = e_1 e_2 e_3 \cdots .$$

Note that σ is a local homeomorphism. It is not difficult to see that $(\mathcal{P}(G), \sigma)$ is cofinal if and only if G is cofinal in the sense introduced in [**KPRR97**]. If $v \in V_G$ and $p \in \mathcal{P}(G)$, there is a finite path μ in G and an $i \in \mathbb{N}$ such that $s(\mu) = v$ and $r(\mu) = s(p_i)$. In the following we assume that G is a countable graph such that:

- G is row-finite;
- G has no sinks; and
- G is cofinal.

For simplicity we summarize these conditions by saying that G is *cofinal* when they hold. We denote by NW_G the set of vertices v which are contained in a loop, meaning that there is a finite path μ such that $s(\mu) = r(\mu) = v$. These vertices are called *non-wandering*, and together with the edges they emit they constitute an irreducible (or strongly connected) subgraph of G which we also denote by NW_G.

For a continuous real-valued function $\phi : \mathcal{P}(G) \to \mathbb{R}$ the *Gurevich pressure* $P_{NW_G}(\phi)$ of the restriction of ϕ to $\mathcal{P}(NW_G)$ is defined to be

$$\limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[v]}(y), \tag{3.6}$$

where v is a vertex in NW_G and $[v] = \{x \in \mathcal{P}(G) : s(x) = v\}$; see, for example, [**Sar09**]. It has been shown in increasing generality by Sarig [**Sar99**, **Sar09**] that the limsup above is a limit which is independent of the choice of v when G is mixing (or primitive), and ϕ satisfies some condition on its variation. In the general irreducible case, the sequence involved in (3.6) will not converge, but the independence of the choice of vertex is a general fact, at least when ϕ is uniformly continuous in an appropriate metric. To make this precise, set

$$\mathrm{var}_k(\phi) = \sup\{|\phi(x) - \phi(y)| : x_i = y_i,\ i = 1, 2, \ldots, k\}.$$

We shall work with the assumption that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. Note that this condition is implied by the uniform continuity of ϕ with respect to any metric d on $\mathcal{P}(G)$ with the property that for any $\delta > 0$ there is a $k \in \mathbb{N}$ such that $x_i = y_i,\ i = 1, 2, \ldots, k \Rightarrow d(x, y) \leq \delta$.

The next lemma follows from [**Sar09**, Proposition 3.2] when G is primitive and ϕ has the Walters property.

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LEMMA 3.3. *Assume that G is cofinal and that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. The value of* (3.6) *does not depend on the choice of $v \in NW_G$, and for every finite path μ in NW_G,*

$$P_{NW_G}(\phi) = \limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{Z(\mu)}(y).$$

Proof. Let $v = s(\mu)$. Then

$$\sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{Z(\mu)}(y) \le \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[v]}(y)$$

for all n, and hence

$$\limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{Z(\mu)}(y) \le \limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[v]}(y). \tag{3.7}$$

Let w be any vertex in NW_G and let $\epsilon > 0$. Since NW_G is irreducible, there is a finite path p in G such that $s(p) = r(\mu)$, $r(p) = w$. By assumption there is a $k \in \mathbb{N}$ such that $\mathrm{var}_k(\phi) \le \epsilon$. Let $w_1, w_2, \ldots, w_l$ be the vertices that can be reached from w by a path of length k. For each w_i we choose a finite path q_i such that $s(q_i) = w_i$, $r(q_i) = s(\mu)$. Let $n > k$ and set

$$M_i = \{y \in [w] : \sigma^n(y) = y,\ r(y_k) = w_i\},$$

and define $\chi_i : M_i \to \{y \in Z(\mu) : \sigma^{n+L_i}(y) = y\}$, where $L_i = |\mu p| + |q_i| + k$, such that $\chi_i(y)$ is the infinite path which repeats the loop starting with μ, followed by p, the first $n + k$ edges in y and ends with q_i. In symbols,

$$\chi_i(y) = (\mu p y_{[1,n+k]} q_i)^\infty.$$

By using the fact that $\mathrm{var}_k(\phi) \le \epsilon$ we find that

$$e^{\phi_n(y)} \le e^{n\epsilon} e^{\phi_n(\sigma^{|\mu p|}(\chi_i(y)))}.$$

By comparing $\phi_n(\sigma^{|\mu p|}(\chi_i(y)))$ to $\phi_{n+L_i}(\chi_i(y))$ one sees that

$$e^{\phi_n(\sigma^{|\mu p|}(\chi_i(y)))} \le C_i e^{\phi_{n+L_i}(\chi_i(y))},$$

where

$$C_i = \sup\{e^{-\phi_{|q_i|+k}(z)} : z \in [w]\} \cdot \sup\{e^{-\phi_{|\mu p|}(z)} : z \in Z(\mu)\}.$$

Since $\{y \in [w] : \sigma^n(y) = y\} = \bigsqcup_{i=1}^l M_i$ and each χ_i is injective, this leads to the estimate

$$\sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[w]}(y) \le C e^{n\epsilon} \sum_{i=1}^{l} \sum_{\sigma^{n+L_i}(y)=y} e^{\phi_{n+L_i}(y)} 1_{Z(\mu)}(y),$$

where $C = \max_{1\le i\le l} C_i$, and we conclude therefore that

$$\limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[w]}(y) \le \epsilon + \limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{Z(\mu)}(y).$$

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Since $\epsilon > 0$ was arbitrary we can combine with (3.7) to get

$$\limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[w]}(y) \le \limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{Z(\mu)}(y)$$
$$\le \limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{\phi_n(y)} 1_{[v]}(y).$$

The desired conclusion follows from this by using the freedom in the choice of μ, v and w. □

PROPOSITION 3.4. *Let $X = \mathcal{P}(G)$ be the space of infinite paths in a cofinal graph G, and let σ be the left shift on $\mathcal{P}(G)$. Let $\phi : \mathcal{P}(G) \to \mathbb{R}$ be a continuous function such that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. Then:*

(a) $\mathbb{P}(\phi) = -\infty$ *when* $NW_G = \emptyset$; *and*

(b) $\mathbb{P}(\phi)$ *is the Gurevich pressure* $P_{NW_G}(\phi)$ *of* $\phi|_{\mathcal{P}(NW_G)}$ *when* $NW_G \neq \emptyset$.

Proof. Let $x \in \mathcal{P}(G)$ and let $\epsilon > 0$ be arbitrary. Choose $k \in \mathbb{N}$ such that $\mathrm{var}_k(\phi) \le \epsilon$ and let μ be the path of length k with $x \in Z(\mu)$. If $x \notin \mathcal{P}(NW_G)$, the set $\sigma^{-n}(x) \cap Z(\mu)$ is empty for all n and hence $L^n_\phi(1_{Z(\mu)})(x) = 0$ for all n, leading to the conclusion that $\mathbb{P}_x(\phi) = -\infty$. Assume then that $x \in \mathcal{P}(NW_G)$. Let $n > k$. There is an obvious bijection $\chi : \sigma^{-n}(x) \cap Z(\mu) \to \{y \in Z(\mu) : \sigma^n(y) = y\}$ such that $\chi(z)_i = z_i$, $i = 1, 2, \ldots, n+k$. Since $\mathrm{var}_k(\phi) \le \epsilon$ this leads first to the estimates

$$\sum_{\sigma^n(y)=y} e^{\phi_n(y)-n\epsilon} 1_{Z(\mu)}(y) \le L^n_\phi(1_{Z(\mu)})(x) \le \sum_{\sigma^n(y)=y} e^{\phi_n(y)+n\epsilon} 1_{Z(\mu)}(y), \tag{3.8}$$

and then by Lemma 3.3 to the conclusion that $P_{NW_G}(\phi) - \epsilon \le \mathbb{P}_x(\phi) \le P_{NW_G}(\phi) + \epsilon$. Hence

$$\mathbb{P}_x(\phi) = \begin{cases} -\infty, & x \notin \mathcal{P}(NW_G), \\ P_{NW_G}(\phi), & x \in \mathcal{P}(NW_G). \end{cases}$$

□

4. *Constructing conformal measures*

In this section we return to the setting where X is a second countable locally compact Hausdorff space, $\sigma : X \to X$ is a cofinal local homeomorphism and $\phi : X \to \mathbb{R}$ is continuous. We add the assumption that (X, σ) is *non-compact*, in the sense that there is no open non-empty subset $U \subseteq X$ such that $\bigcup_{n=0}^\infty \sigma^n(U)$ is pre-compact. Since (X, σ) is assumed cofinal, this additional condition is satisfied when there is just a single point x whose orbit closure $\overline{\bigcup_{k=1}^\infty \phi^k(x)}$ is not compact.

In the following we write $\lim_{k\to\infty} x_k = \infty$ when $\{x_k\}$ is a sequence in X with the property that for any compact subset $K \subseteq X$ there is an $N \in \mathbb{N}$ such that $x_k \notin K$ for all $k \ge N$.

LEMMA 4.1. *Assume that (X, σ) is cofinal and non-compact. Let $h \in C_c(X)^+$. There is a sequence $\{x_k\}$ in X such that*

$$\sum_{n=0}^\infty L^n_\phi(h)(x_k) > 0$$

for all k and $\lim_{x\to\infty} x_k = \infty$.

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Proof. Set $U = \{x \in X : h(x) > 0\}$ and let $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ be a sequence of open pre-compact subsets in X such that $X = \bigcup_k V_k$. Since (X, σ) is non-compact there is for every $k \in \mathbb{N}$ an element

$$x_k \in \left(\bigcup_{n=0}^{\infty} \sigma^n(U) \backslash V_k\right).$$

The sequence $\{x_k\}$ has the stated properties. □

A sequence $\{x_k\}$ with the properties stipulated in Lemma 4.1 will be called *h-diverging*.

LEMMA 4.2. *Assume that (X, σ) is cofinal and non-compact, and that*

$$\sum_{n=0}^{\infty} |L_\phi^n(f)(x)| < \infty$$

for all $x \in X$ and all $f \in C_c(X)$. Let $h \in C_c(X)^+$. For every h-diverging sequence $\{x_k\} \subseteq X$ there is a subsequence $\{x_{k_i}\}$ such that the limit

$$\lim_{i\to\infty} \frac{\sum_{n=0}^{\infty} L_\phi^n(f)(x_{k_i})}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_{k_i})} \tag{4.1}$$

exists for all $f \in C_c(X)$.

Proof. Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ be a sequence of open relatively compact sets such that $X = \bigcup_i V_i$. Then

$$C_c(X) = \bigcup_i C_0(V_i), \tag{4.2}$$

where $C_0(V_i)$ denotes the Banach space of continuous functions on V_i that vanish at infinity. Since $C_0(V_i)$ is separable there is a sequence $\{g_n\} \subseteq C_c(X)$ such that $\{g_n\} \cap C_0(V_i)$ is dense in $C_0(V_i)$ for all i. It follows from Lemma 3.1 that for each i there are $N_i \in \mathbb{N}$ and $C_i > 0$ such that

$$\left|\sum_{n=0}^{\infty} L_\phi^n(f)(x)\right| \leq C_i N_i^2 \|f\| \sum_{n=0}^{\infty} L_\phi^n(h)(x)$$

for all $f \in C_0(V_i)$ and all $x \notin \bigcup_{j=0}^{N_i} \sigma^j(\overline{V_i})$. Since $\lim_{k\to\infty} x_k = \infty$ this implies that

$$\left|\frac{\sum_{n=0}^{\infty} L_\phi^n(f)(x_k)}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_k)}\right| \leq C_i N_i^2 \|f\| \tag{4.3}$$

for all $f \in C_0(V_i)$ and all sufficiently large $k \in \mathbb{N}$. A diagonal sequence argument shows that there is a subsequence $\{x_{k_i}\}$ such that

$$\lim_{i\to\infty} \frac{\sum_{n=0}^{\infty} L_\phi^n(g_j)(x_{k_i})}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_{k_i})}$$

exists for all j. Let $f \in C_0(V_i)$. It follows from (4.3) that

$$\begin{aligned}\left|\frac{\sum_{n=0}^{\infty} L_\phi^n(g_j)(x_{k_l})}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_{k_l})} - \frac{\sum_{n=0}^{\infty} L_\phi^n(f)(x_{k_l})}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_{k_l})}\right| &= \left|\frac{\sum_{n=0}^{\infty} L_\phi^n(g_j - f)(x_{k_l})}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_{k_l})}\right| \\ &\leq C_i N_i^2 \|g_j - f\|\end{aligned}$$

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for all large l when $g_j \in C_0(V_i)$. Since $\{g_n\} \cap C_0(V_i)$ is dense in $C_0(V_i)$ it follows from this, in combination with (4.2), that the limit (4.1) exists for all $f \in C_c(X)$. □

LEMMA 4.3. *Assume that (X, σ) is cofinal and non-compact, and that $\mathbb{P}(\phi) < 0$. Let $h \in C_c(X)^+$. There is a regular Borel measure m on X such that $L_\phi^*(m) = m$ and $\int_X h\, dm = 1$.*

Proof. It follows from the definition of $\mathbb{P}(\phi)$ that $\sum_{n=0}^{\infty} |L_\phi^n(f)(x)| < \infty$ for all $x \in X$ and all $f \in C_c(X)$. By Lemmas 4.2 and 4.1 there is a sequence $\{x_k\}$ in X such that $\lim_{k\to\infty} x_k = \infty$ and the limit

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} L_\phi^n(f)(x_k)}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_k)}$$

exists for all $f \in C_c(X)$. Riesz's representation theorem provides us therefore with a regular Borel measure m on X such that

$$\int_X f\, dm = \lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} L_\phi^n(f)(x_k)}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_k)}$$

for all $f \in C_c(X)$. Note that

$$\int_X f\, dm - \int_X L_\phi(f)\, dm = \lim_{k\to\infty} \frac{f(x_k)}{\sum_{n=0}^{\infty} L_\phi^n(h)(x_k)} = 0$$

for all $f \in C_c(X)$ since $\lim_{k\to\infty} x_k = \infty$. □

THEOREM 4.4. *Assume that (X, σ) is cofinal and non-compact, and that $\mathbb{P}(-\phi) \le 0$. Let $h \in C_c(X)$ be non-negative and non-zero. There is a ϕ-conformal measure m such that $\int_X h\, dm = 1$.*

Proof. Let $n \in \mathbb{N}$ and note that $\mathbb{P}(-\phi - (1/n)) = \mathbb{P}(-\phi) - (1/n) < 0$. Hence Lemmas 2.1 and 4.3 give us a $\phi + (1/n)$-conformal measure m_n for each $n \in \mathbb{N}$, with the additional property that $\int_X h\, dm_n = 1$. Let $V_1 \subseteq V_2 \subseteq \cdots$ be the sets from the proof of Lemma 4.2. It follows from the way the m_n were constructed, in particular from (4.3), that there are numbers $M_k > 0$, not depending of n, such that

$$\left| \int_X f\, dm_n \right| \le M_k \|f\| \tag{4.4}$$

for all $f \in C_0(V_k)$ and all n. (It is necessary here to check that the constant C in Lemma 3.1 can be chosen independently of n.) Thus the sequence of linear functionals on $C_0(V_k)$ arising from integration with respect to the m_n is contained in the ball of radius M_k in the dual space of $C_0(V_k)$. By compactness of this ball in the weak* topology we deduce that the sequence has a convergent subsequence for each k. Combining (4.2) with a diagonal sequence argument, this leads to the conclusion that there is a subsequence $\{m_{n_i}\}$ such that the limit $\lim_{i\to\infty} \int_X f\, dm_{n_i}$ exists for all $f \in C_c(X)$. By the Riesz representation theorem this gives us a regular Borel measure m on X such that

$$\lim_{i\to\infty} \int_X f\, dm_{n_i} = \int_X f\, dm$$

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for all $f \in C_c(X)$. In particular, $\int_X h\, dm = 1$. To check that m is ϕ-conformal, let U be an open subset of X such that σ is injective on U. For each i and each $g \in C_c(U)$,

$$\int_{\sigma(U)} g \circ (\sigma|_U)^{-1}(x)\, dm_{n_i}(x) = \int_U g(x) e^{\phi(x)+n_i^{-1}}\, dm_{n_i}(x)$$

by Lemma 2.2. Since $g \in C_c(V_k)$ for some sufficiently large k and since $ge^{\phi+n_i^{-1}}$ converges uniformly to ge^{ϕ}, it follows from (4.4) that we can take the limit $i \to \infty$ to find that

$$\int_{\sigma(U)} g \circ (\sigma|_U)^{-1}(x)\, dm(x) = \int_U g(x) e^{\phi}\, dm(x).$$

Hence m is ϕ-conformal, again by Lemma 2.2. □

4.1. *Conformal measures for cofinal Markov shifts.* We return now to the setting of §3.1.

LEMMA 4.5. *Let $X = \mathcal{P}(G)$ be the space of infinite paths in a cofinal graph G, and let σ be the left shift on $\mathcal{P}(G)$. Let $\phi : \mathcal{P}(G) \to \mathbb{R}$ be a function such that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. Assume that there is a ϕ-conformal measure m. Then $\mathbb{P}(-\phi) \le 0$.*

Proof. Thanks to Proposition 3.4 we can assume that $NW_G \neq \emptyset$. Let $v \in NW_G$ and $\epsilon > 0$ be arbitrary. There is a $k \in \mathbb{N}$ such that $\mathrm{var}_k(\phi) \le \epsilon$. Consider a finite path μ in NW_G of length k with $s(\mu) = v$. Then

$$\sum_{\sigma^n(y)=y} e^{-\phi_n(y)} 1_{Z(\mu)}(y) \le \sum_{y\in\sigma^{-n}(x)} e^{-\phi_n(y)+n\epsilon} 1_{Z(\mu)}(y)$$

for all $x \in Z(\mu)$ and all $n > k$; cf. (3.8). It follows that

$$\begin{aligned} m(Z(\mu)) \sum_{\sigma^n(y)=y} e^{-\phi_n(y)} 1_{Z(\mu)}(y) &\le e^{n\epsilon} \int_{Z(\mu)} L^n_{-\phi}(1_{Z(\mu)})\, dm \\ &\le e^{n\epsilon} \int_{\mathcal{P}(G)} L^n_{-\phi}(1_{Z(\mu)})\, dm = e^{n\epsilon} m(Z(\mu)) \quad (4.5) \end{aligned}$$

for all large n. Note that $m(Z(\mu)) > 0$ since m is ϕ-conformal and (X, σ) is cofinal. It follows therefore from (4.5) that

$$\limsup_n \frac{1}{n} \log \sum_{\sigma^n(y)=y} e^{-\phi_n(y)} 1_{Z(\mu)}(y) \le \epsilon.$$

Thanks to Lemma 3.3 and Proposition 3.4, this completes the proof. □

To formulate the next theorem we need a stronger condition on ϕ when NW_G is non-empty and finite. Following Walters [**Wal01**], we say that ϕ *satisfies Bowen's condition* on NW_G when there is a $C > 0$ such that

$$\left| \sum_{i=0}^{n-1} [\phi(\sigma^i(x)) - \phi(\sigma^i(y))] \right| \le C$$

for all $x = (x_i)_{i=0}^{\infty}$, $y = (y_i)_{i=0}^{\infty} \in \mathcal{P}(NW_G)$ such that $x_i = y_i$, $i = 0, 1, 2, \ldots, n-1$, and all $n \ge 1$.

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THEOREM 4.6. *Assume that G is a cofinal graph and that $\phi : \mathcal{P}(G) \to \mathbb{R}$ is a function such that* $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$.

(1) *Assume that $NW_G = \emptyset$. There is a ϕ-conformal measure for the left shift on $\mathcal{P}(G)$.*
(2) *Assume that NW_G is non-empty and finite. Assume that ϕ satisfies Bowen's condition on NW_G. There is a ϕ-conformal measure for the left shift on $\mathcal{P}(G)$ if and only if $\mathbb{P}(-\phi) = 0$, and it is then unique up multiplication by a scalar.*
(3) *Assume that NW_G is infinite. There is a ϕ-conformal measure for the left shift if and only if $\mathbb{P}(-\phi) \leq 0$.*

Proof. It is easy to see that in cases (1) and (3) there is an element $x \in \mathcal{P}(G)$ such that $\lim_{j\to\infty} \sigma^j(x) = \infty$. Hence (X, σ) is non-compact in these cases and the stated conclusions follow from (a) of Proposition 3.4, Lemma 4.5 and Theorem 4.4.

Consider case (2). Let p be the global period of NW_G. Then $\mathcal{P}(NW_G)$ is the disjoint union of compact and open sets X_i, $i = 1, 2, \ldots, p$, such that $\sigma(X_i) = X_{i+1}$, mod p. Furthermore, the restriction of σ^p to X_p is a mixing subshift of finite type. Since ϕ satisfies Bowen's condition on NW_G by assumption, it follows that ϕ_p satisfies Bowen's condition on X_p with respect to σ^p. It follows therefore from [**Wal01**, Theorems 1.3 and 2.16] that there is a ϕ_p-conformal measure for σ^p on X_p if and only if the pressure of the restriction of $-\phi_p$ to X_p (with respect to σ^p) is zero. It follows from (b) of Proposition 3.4 that this pressure is $p\mathbb{P}(-\phi)$. Since a ϕ-conformal measure for σ on $\mathcal{P}(G)$ will restrict to a ϕ_p-conformal measure for σ^p on X_p, we deduce that there can only be a ϕ-conformal measure for σ on $\mathcal{P}(G)$ if $\mathbb{P}(-\phi) = 0$. Assume then that $\mathbb{P}(-\phi) = 0$. It follows from the theorems of Walters quoted above that there is a ϕ_p-conformal measure for σ^p on X_p, and that it is unique up to multiplication by a scalar.

It suffices now to show that a ϕ_p-conformal measure ν for σ^p on X_p extends uniquely to a ϕ-conformal measure μ for σ on $\mathcal{P}(G)$. For each $i < p$ there is a partition $X_i = \bigsqcup_j B_{i,j}$ of X_i into compact and open sets such that $\sigma^{p-i} : B_{i,j} \to X_p$ is injective for each j. To extend ν to a ϕ-conformal measure μ on $\mathcal{P}(NW_G)$ we must therefore define μ on X_i by the requirement that

$$\int_{X_i} g \, d\mu = \sum_j \int_{\sigma^{p-i}(B_{i,j})} g((\sigma^{p-i}|_{B_{i,j}})^{-1}(x)) e^{-\phi_{p-i}((\sigma^{p-i}|_{B_{i,j}})^{-1}(x))} \, d\nu(x) \tag{4.6}$$

when $g \in C(X_i)$. In particular, we see that an extension of ν to a ϕ-conformal measure for σ on $\mathcal{P}(NW_G)$ is unique. To see that such an extension exists it is possible to show that the recipe (4.6) provides the required extension by using the fact that ν is ϕ_p-conformal on X_p. Alternatively, one can first extend ν to a measure ν on $\mathcal{P}(NW_G)$ such that $\nu(X_i) = 0$, $i \neq p$, and then take

$$\mu = \sum_{i=0}^{p-1} (\mathcal{L}^*_{-\phi})^i(\nu),$$

where $\mathcal{L}_{-\phi}$ denotes the compression of $L_{-\phi}$ to $C(\mathcal{P}(NW_G))$, that is,

$$\mathcal{L}_{-\phi}(g)(x) = \sum_{y \in \sigma^{-1}(x) \cap \mathcal{P}(NW_G)} e^{-\phi(y)} g(y)$$

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when $g \in C(\mathcal{P}(NW_G))$. To extend μ from $\mathcal{P}(NW_G)$ to $\mathcal{P}(G)$ note that

$$\int_{\sigma([v])} g \circ (\sigma|_{[v]})^{-1}\, d\mu = \int_{[v]} g e^{\phi}\, d\mu \tag{4.7}$$

when $v \in NW_G$ and $g \in C([v])$. Set $H_0 = NW_G$ and

$$H_n = \{w \in V_G : r(s^{-1}(w)) \subseteq H_{n-1}\}$$

for $n \geq 1$. Then $H_n \subseteq H_{n+1}$ for all n and $\bigcup_n H_n = V_G$ because G is cofinal; see the proof of [**Tho13**, Lemma 2.4]. When $w \in H_1$ we can define a regular Borel measure μ_w on $[w]$ by the requirement that

$$\int_{[w]} f\, d\mu_w = \int_{\sigma([w])} e^{-\phi((\sigma|_{[w]})^{-1}(x))} f \circ (\sigma|_{[w]})^{-1}(x)\, d\mu(x)$$

for all $f \in C([w])$. We extend μ to a regular Borel measure on $\{x \in \mathcal{P}(G) : s(x) \in H_1\}$, by setting

$$\mu(B) = \mu(B \cap \mathcal{P}(NW_G)) + \sum_{w \in H_1 \setminus H_0} \mu_w(B \cap [w]).$$

Then (4.7) holds for all $v \in H_1$. Continuing by induction we get a regular Borel measure on $\mathcal{P}(G)$ such that (4.7) holds for $v \in V_G$. It follows then from Lemma 2.1 that μ is ϕ-conformal, and it is clear from the construction that it is the only ϕ-conformal measure extending ν. □

Remark 4.7. In cases (1) and (3) of Theorem 4.6 a ϕ-conformal measure is generally not unique.

The reason for the introduction of Bowen's condition in case (2) is that we do not know (more precisely, the author does not know) if there can be a ϕ-conformal measure on a mixing one-sided subshift of finite type when $\mathbb{P}(-\phi) < 0$ and the potential ϕ is continuous, but does not satisfy Bowen's condition. By [**Wal01**, Theorem 2.16] it is possible to use a condition slightly weaker than Bowen's, but beyond that nothing seems to be known. When $\mathbb{P}(-\phi) = 0$ there *is* a ϕ-conformal measure, also when ϕ is only assumed to be continuous. This follows from [**Th011**, Theorem 6.9] since the spectral radius of the Ruelle operator is 1 when $\mathbb{P}(\phi) = 0$ by [**Wal01**, Theorem 1.3]. It is, however, not clear if the measure is unique in general. Therefore, without assuming Bowen's condition or the slightly weaker condition used by Walters in [**Wal01**, Theorem 2.16], the only thing we can say in case (2) is that there is a ϕ-conformal measure if $\mathbb{P}(-\phi) = 0$, and none if $\mathbb{P}(-\phi) > 0$.

For a mixing topological Markov shift Sarig has shown the existence of an $e^{\mathbb{P}(\phi)}$-eigenmeasure for the dual Ruelle operator when the potential has summable variation and is recurrent [**Sar99**, **Sar09**]. Van Cyr extended this to transient potentials in his thesis [**Cyr10**]. We can now supplement their results as follows.

THEOREM 4.8. *Let G be a countable connected directed graph with finite out-degree at each vertex. Assume that G is not finite. Let $\phi : \mathcal{P}(G) \to \mathbb{R}$ be a function such that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$, and let $t \in \mathbb{R}$. There is a non-zero regular Borel measure m on $\mathcal{P}(G)$ such that*

$$L_\phi^*(m) = e^t m$$

if and only if $t \geq \mathbb{P}(\phi)$.

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Proof. $\mathcal{P}(G)$ is locally compact because G has finite out-degree at each vertex, and second countable because G is countable. The dynamical system $(\mathcal{P}(G), \sigma)$ is cofinal because G is connected, and non-compact because G is also infinite. Therefore the theorem follows from Lemma 2.2 and Theorem 4.6(3) after the observation that $\mathbb{P}(\phi - t) = \mathbb{P}(\phi) - t$. □

It follows from Sarig's results that there is a unique regular conservative e^t-eigenmeasure for the dual Ruelle operator when $t = \mathbb{P}(\phi)$, provided ϕ is recurrent, G is aperiodic, and $\sum_{k=2}^{\infty} \mathrm{var}_k(\phi) < \infty$, cf. [**Sar09**]. His results do not require G to have finite out-degree at the vertices. As will be shown in the next section, at least in the locally compact setting, e^t-eigenmeasures must be dissipative when $t > \mathbb{P}(\phi)$, and when $t = \mathbb{P}(\phi)$ and ϕ is transient.

5. *Dissipativity*

In this section we assume only that X is a second countable locally compact Hausdorff space, $\sigma : X \to X$ is a local homeomorphism and $\phi : X \to \mathbb{R}$ is continuous.

LEMMA 5.1. *Assume that m is ϕ-conformal. Then $m \circ \sigma^{-1}$ is absolutely continuous with respect to m.*

Proof. Write $X = \bigsqcup_{i \in \mathbb{N}} A_i$ as a disjoint union where the A_i are Borel subsets of X such that σ is injective on each A_i, and let $B \subseteq X$ be a Borel set. Since the conformality assumption implies that

$$m(B \cap \sigma(A_i)) = m(\sigma(\sigma^{-1}(B) \cap A_i)) = \int_{\sigma^{-1}(B) \cap A_i} e^{\phi(x)}\, dm(x),$$

it follows that $m(B) = 0 \Rightarrow m(\sigma^{-1}(B) \cap A_i) = 0$ for all $i \Rightarrow m(\sigma^{-1}(B)) = 0$. □

In general $m \circ \sigma^{-1}$ is not equivalent to m; it is if and only if $m(X \backslash \sigma(X)) = 0$. It follows from Lemma 5.1 that a ϕ-conformal measure m gives rise to a *Hopf decomposition* of X. That is,

$$X = C \sqcup D,$$

where C and D are disjoint Borel sets with the following properties (see [**Kre85**, §1.3]):

(1) $\sigma(C) \subseteq C$;

(2) for every Borel subset $A \subseteq C$,

$$m\left(A \setminus \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \sigma^{-k}(A)\right) = 0;$$

(3) $D = \bigcup_{n=1}^{\infty} D_n$, where each D_n is wandering in the sense that

$$D_n \cap \sigma^{-k}(D_n) = \emptyset, \quad k = 1, 2, 3, \ldots.$$

The set C is the *conservative part* of σ.

For a given ϕ-conformal measure m, the Hopf decomposition is unique modulo m-null sets, and we say that m is *dissipative* when the conservative part is an m-null set. Thus m is dissipative if and only if X is the union of a countable collection of wandering Borel sets, up to an m-null set.

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LEMMA 5.2. *Let m be a ϕ-conformal measure. Assume that $\sum_{n=1}^{\infty} L_{-\phi}^n(f)(x) < \infty$ for m-almost every x when $f \in C_c(X)$ is non-negative. It follows that m is dissipative.*

Proof. Let C be the conservative part of σ. Assume for a contradiction that $m(C) > 0$. Since m is regular there is a non-negative $f \in C_c(X)$ such that

$$C_1 = \{x \in C : f(x) \geq 1\}$$

has positive m-measure. Note that $m(C_1) < \infty$. Since $\sum_{k=1}^{\infty} L_{-\phi}^k(f)(x) < \infty$ for m-almost all x by assumption, there is an $N \in \mathbb{N}$ such that

$$C_2 = \left\{x \in C_1 : \sum_{k=1}^{\infty} L_{-\phi}^k(f)(x) \leq N\right\}$$

has positive m-measure. Note that

$$Nm(C_2) \geq \int_{C_2} \sum_{k=1}^{\infty} L_{-\phi}^k(f)(x)\, dm(x) = \sum_{k=1}^{\infty} \int_X 1_{C_2} L_{-\phi}^k(f)\, dm. \tag{5.1}$$

Let $\{g_k\}$ be a uniformly bounded sequence from $C_c(X)$ which converges m-almost everywhere to 1_{C_2}. Then

$$\begin{aligned}\int_X 1_{C_2} L_{-\phi}^k(f)\, dm &= \lim_{n\to\infty} \int_X g_n L_{-\phi}^k(f)\, dm = \lim_{n\to\infty} \int_X L_{-\phi}^k(f g_n \circ \sigma^k)\, dm \\ &= \lim_{n\to\infty} \int_X f g_n \circ \sigma^k\, dm = \int_X f 1_{C_2} \circ \sigma^k\, dm \geq \int_{C_2} 1_{C_2} \circ \sigma^k\, dm.\end{aligned}$$

Substituting into (5.1), we get that

$$\int_{C_2} \sum_{k=1}^{\infty} 1_{C_2} \circ \sigma^k\, dm \leq Nm(C_2) < \infty. \tag{5.2}$$

However, since σ is infinitely recurrent on C by (2) above, we know that

$$\sum_{k=1}^{\infty} 1_{C_2} \circ \sigma^k(x) = \infty$$

for m-almost every $x \in C_2$. This contradicts (5.2) since $m(C_2) > 0$. □

The following proposition is an immediate consequence of Lemma 5.2.

PROPOSITION 5.3. *Assume that (X, σ) is cofinal and that $\mathbb{P}(-\phi) < 0$. It follows that every ϕ-conformal measure for σ is dissipative.*

We remark that Lemma 5.2 can also be used to show that for a transient potential, as defined by Sarig in [**Sar99**], any conformal measure is dissipative.

Although the conformal measures for a potential ϕ with $\mathbb{P}(-\phi)$ negative must be dissipative, they may very well be finite. This occurs already for constant potentials on certain locally compact mixing Markov shifts.

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6. *Conformal measures in the exponential family*

Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic map and $J(h)$ the Julia set of h. We assume that h is transcendental, that is, not a polynomial, and then $J(h)$ is closed, unbounded and totally invariant under h, namely, $h^{-1}(J(h)) = J(h)$. If we assume that $h'(z) \neq 0$ for all $z \in J(h)$, it follows that h is locally injective on $J(h)$ and hence that $h : J(h) \to J(h)$ is a local homeomorphism. By Montel's theorem, [**Mil06**, Theorem 3.7], there is a set $\mathcal{E}(h) \subseteq \mathbb{C}$, consisting of at most one point x, which must be totally h-invariant in the sense that $h^{-1}(x) = \{x\}$, such that for any open subset U of $\mathbb{C}$ with $U \cap J(h) \neq \emptyset$, $\bigcup_{i,j\in\mathbb{N}}^{\infty} h^{-j}(h^i(U)) = \mathbb{C}\backslash\mathcal{E}(h)$. It follows that $J(h)\backslash\mathcal{E}(h)$ is totally h-invariant, locally compact in the relative topology and that $h : J(h)\backslash\mathcal{E}(h) \to J(h)\backslash\mathcal{E}(h)$ is cofinal. Another application of Montel's theorem shows that $(J(h)\backslash\mathcal{E}(h), h)$ is also non-compact. Hence the results of the previous sections apply to this dynamical system.

In this setting probability conformal measures have been constructed by Mayer and Urbanski [**MU08**, **MU10**] for a large class of entire functions when the potential ϕ is chosen carefully. When h comes from the exponential family $h(z) = \lambda e^z$, the potential considered by Mayer and Urbanski is

$$\phi(z) = \log|z|, \tag{6.1}$$

or some 'tame' perturbation of ϕ. The inverse temperature β for which a finite $\beta\phi$-conformal measure exists is invariably unique, but we can now show that at least for the hyperbolic members of the exponential family λe^z the situation is very different when infinite conformal measures are also considered. To substantiate this we assume that $h = E_\lambda$ where $E_\lambda(z) = \lambda e^z$ for some $0 < \lambda < e^{-1}$. In this case $\mathcal{E}(E_\lambda) = \emptyset$. It follows from [**MU08**, Proposition 4.5] that

$$A_\beta := \sup_{x\in J(E_\lambda)} \sum_{y\in E_\lambda^{-1}(x)} |y|^{-\beta} < \infty$$

for all $\beta > 1$, a fact which is also easy to verify directly in the present case. This implies, in particular, that we can define $L_{-\beta\phi}$ by the same formula as above,

$$L_{-\beta\phi}(g)(x) = \sum_{y\in E_\lambda^{-1}(x)} |y|^{-\beta} g(y),$$

as a positive linear operator on the vector space of bounded functions on $J(E_\lambda)$ for all $\beta > 1$. In order to combine the methods and results of this paper with those of [**MU08**], we only have to prove the following lemma.

LEMMA 6.1. *When $E_\lambda : J(E_\lambda) \to J(E_\lambda)$ for some $\lambda \in]0, e^{-1}[$ and ϕ is the potential* (6.1)*,*

$$\mathbb{P}(-\beta\phi) = \limsup_n \frac{1}{n} \log L^n_{-\beta\phi}(1)(x)$$

for all $x \in J(E_\lambda)$ and all $\beta > 1$.

Proof. It is shown in [**MU08**] that $\limsup_n (1/n) \log L^n_{-\beta\phi}(1)(x)$ is independent of $x \in J(E_\lambda)$. Since we clearly have the inequality

$$\limsup_n \frac{1}{n} \log L^n_{-\beta\phi}(1)(x) \geq \limsup_n \frac{1}{n} \log L^n_{-\beta\phi}(f)(x)$$

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for any non-negative $f \in C_c(J(E_\lambda))$ with $\|f\| \leq 1$, it suffices to find a single element $x_0 \in J(E_\lambda)$ and a non-negative function $f \in C_c(J(E_\lambda))$ such that

$$\limsup_n \frac{1}{n} \log L^n_{-\beta\phi}(f)(x_0) \geq \limsup_n \frac{1}{n} \log L^n_{-\beta\phi}(1)(x_0). \tag{6.2}$$

For this purpose we need the following observation which follows from [**MU08**, Lemma 5.3]:

Observation 6.2. Set $B_R = \{z \in \mathbb{C} : |z| \leq R\}$. Then

$$\lim_{R\to\infty} \sup_{x\in J(E_\lambda)} \sum_{y\in E_\lambda^{-1}(x)\setminus B_R} |y|^{-\beta} = 0$$

when $\beta > 1$.

It is well known that $J(E_\lambda)$ contains a fixed point x_0 for E_λ. It follows from Observation 6.2 that there is an $R > 0$ such that

$$m := \sup_{x\in J(E_\lambda)} \sum_{y\in E_\lambda^{-1}(x)\setminus B_R} |y|^{-\beta} < |x_0|^{-\beta}. \tag{6.3}$$

Let $f \in C_c(J(E_\lambda))$ be a non-negative function such that $f(z) = 1$ when $z \in B_R \cap J(E_\lambda)$. Then $f + 1_{J(E_\lambda)\setminus B_R} \geq 1$ on $J(E_\lambda)$ and hence

$$L_{-\beta\phi}(f)(x) + L_{-\beta\phi}(1_{J(E_\lambda)\setminus B_R})(x) \geq L_{-\beta\phi}(1)(x) \tag{6.4}$$

for all $x \in J(E_\lambda)$. It follows from (6.4) first that

$$L_{-\beta\phi}(f)(x) \geq L_{-\beta\phi}(1)(x) - m,$$

and then that

$$L^n_{-\beta\phi}(f)(x) \geq L^n_{-\beta\phi}(1)(x) - mL^{n-1}_{-\beta\phi}(1)(x),$$

for all n and x. By using the fact that

$$L^n_{-\beta\phi}(1)(x_0) = \sum_{y\in E_\lambda^{-1}(x_0)} |y|^{-\beta} L^{n-1}_{-\beta\phi}(1)(y) \geq |x_0|^{-\beta} L^{n-1}_{-\beta\phi}(1)(x_0),$$

we obtain the estimate

$$L^n_{-\beta\phi}(f)(x_0) \geq (|x_0|^{-\beta} - m)L^{n-1}_{-\beta\phi}(1)(x_0),$$

for all n. Thanks to (6.3), this proves (6.2). □

It follows from Lemma 6.1 that $\mathbb{P}(-\beta\phi) = P(\beta)$, where P is the pressure function considered in [**MU08**, Proposition 7.2]. We can therefore deduce from [**MU08**, Proposition 7.2] that there is a unique $\beta_0 > 1$ such that $\mathbb{P}(-\beta_0\phi) = 0$, and that $\mathbb{P}(-\beta\phi) < 0$ for all $\beta > \beta_0$. As shown in [**MU08**], the number β_0 is the Hausdorff dimension $HD(J_r(E_\lambda))$ of the radial Julia set

$$J_r(E_\lambda) = \{z \in J(E_\lambda) : \liminf_n |E^n_\lambda(z)| < \infty\}.$$

In particular, $\beta_0 < 2$ by [**MU08**, Corollary 1.4]. A main tool in [**MU08**] for the study of $J_r(E_\lambda)$ is a $\beta_0\phi$-conformal measure. It follows now from Theorem 4.4 that they exist for all $\beta \geq HD(J_r(E_\lambda))$, and from Proposition 5.3 that they are dissipative when $\beta > HD(J_r(E_\lambda))$, that is, we have the following proposition.

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PROPOSITION 6.3. *For each* $\lambda \in]0, e^{-1}[$ *and each* $\beta \geq HD(J_r(E_\lambda))$ *there is a* $\beta\phi$*-conformal measure for* $E_\lambda : J(E_\lambda) \to J(E_\lambda)$. *For* $\beta > HD(J_r(E_\lambda))$ *the measures are dissipative.*

As shown in [**MU08**] there is a Borel probability measure on $J(E_\lambda)$ which is a $\beta\phi$-conformal measure when $\beta = HD(J_r(E_\lambda))$, and in [**MU10**] it is shown that it is unique. The paper [**MU10**] contains much more information on this Borel probability measure.

If we consider the potential

$$\psi(z) = \log |E_\lambda(z)|,$$

there is a bijective correspondence between the $\beta\phi$-conformal and the $\beta\psi$-conformal measures given by sending the $\beta\phi$-conformal measure m to the $\beta\psi$-conformal measure $|z|^\beta dm(z)$. It follows from the argument in [**Tho12**, Remark 5.7] that there are no finite $\beta\psi$-conformal measures for any β, but Proposition 6.3 shows that infinite conformal measures exist for all $\beta \geq HD(J_r(E_\lambda))$:

COROLLARY 6.4. *For each* $\lambda \in]0, e^{-1}[$ *and each* $\beta \geq HD(J_r(E_\lambda))$ *there is a* $\beta\psi$*-conformal measure for* $E_\lambda : J(E_\lambda) \to J(E_\lambda)$. *For* $\beta > HD(J_r(E_\lambda))$ *the measures are dissipative.*

7. *KMS weights from conformal measures*

A local homeomorphism σ on a locally compact Hausdorff space X gives rise to an étale groupoid Γ_σ and hence a C^*-algebra $C^*(\Gamma_\sigma)$ by a construction introduced in increasing generality by Renault, Deaconu and Anantharaman-Delaroche [**Ren80**, **Dea95**, **Ana97**]. To describe the construction, set

$$\Gamma_\sigma = \{(x, k, y) \in X \times \mathbb{Z} \times X : \exists n, m \in \mathbb{N}, k = n - m, \sigma^n(x) = \sigma^m(y)\}.$$

This is a groupoid with the set of composable pairs being

$$\Gamma_\sigma^{(2)} = \{((x, k, y), (x', k', y')) \in \Gamma_\sigma \times \Gamma_\sigma : y = x'\}.$$

Multiplication and inversion are given by

$$(x, k, y)(y, k', y') = (x, k + k', y') \quad \text{and} \quad (x, k, y)^{-1} = (y, -k, x).$$

Γ_σ is a locally compact Hausdorff space in the topology where open sets U, V in X with the property that σ^n is injective on U and σ^m is injective on V give rise to the open set

$$\{(x, k, y) \in U \times (n - m) \times V : \sigma^n(x) = \sigma^m(y)\}$$

in Γ_σ, and where sets of this form constitute a base for the topology. The set $C_c(\Gamma_\sigma)$ of compactly supported functions on Γ_σ is a $*$-algebra with product

$$fg(x, k, y) = \sum_{z, n-m=k} f(x, n, z)g(z, m, y)$$

and involution $f^*(x, k, y) = \overline{f(y, -k, x)}$. The C^*-algebra $C^*(\Gamma_\sigma)$ is the completion of $C_c(\Gamma_\sigma)$ with respect to a natural norm; see, for example, [**Ana97**] or [**Tho10**]. Since we have been working with the condition that ϕ is cofinal it is worth pointing out that cofinality

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of ϕ is almost equivalent to the simplicity of $C^*(\Gamma_\sigma)$. In fact, by [**Tho10**, Theorem 4.16], $C^*(\Gamma_\sigma)$ is simple if and only if σ is cofinal (a condition which was called 'irreducibility' in [**Tho10**]) and $\{x \in X : \sigma^k(x) = x\}$ has non-empty interior for all $k \in \mathbb{N}$. In particular, if (X, σ) is non-compact, $C^*(\Gamma_\sigma)$ is simple if and only if σ is cofinal.

A continuous function $\phi : X \to \mathbb{R}$ gives rise to a continuous one-parameter automorphism group α^ϕ_t, $t \in \mathbb{R}$, on $C^*(\Gamma_\sigma)$ defined such that

$$\alpha^\phi_t(f)(x, k, y) = \lim_{n\to\infty} e^{it(\phi_{k+n}(x)-\phi_n(y))} f(x, k, y)$$

when $f \in C_c(\Gamma_\phi)$. The case where ϕ is the constant function 1 yields the so-called *gauge action*.

Any regular Borel measure m on X gives rise to a densely defined lower semi-continuous weight φ_m on $C^*(\Gamma_\sigma)$ such that

$$\varphi_m(f) = \int_X f(x, 0, x)\, dm(x)$$

when $f \in C_c(\Gamma_\phi)$. This construction is the link between conformal measures and KMS weights because it turns out that for any $\beta \in \mathbb{R}$, a regular Borel measure m is $\beta\phi$-conformal if and only if the weight φ_m is a β-KMS weight for the one-parameter group α^ϕ; see [**Tho14**, Proposition 2.1 and Lemma 3.2]. Furthermore, all gauge-invariant β-KMS weights arise in this way by [**Tho14**, Proposition 3.1]. Thanks to this relation between KMS weights and conformal measures the preceding methods and results have consequences for KMS weights, some of which we now summarize. For example, we get the following from Theorem 4.4.

THEOREM 7.1. *Assume that (X, σ) is non-compact and cofinal, and that $\phi : X \to \mathbb{R}$ is a continuous function. Let $\beta \in \mathbb{R}$ be a real number such that $\mathbb{P}(-\beta\phi) \leq 0$. Then there is a gauge invariant β-KMS weight for the one-parameter group α^ϕ on $C^*(\Gamma_\sigma)$.*

When G is a cofinal graph, like those considered in §§3.1 and 4.1, the C^*-algebra $C^*(\Gamma_\sigma)$ coming from the shift σ on $\mathcal{P}(G)$ is known as the graph C^*-algebra associated with G (cf. [**KPRR97**]), and it is usually denoted by $C^*(G)$. From the results above regarding more general potentials we easily get the following consequences.

THEOREM 7.2. *Let G be a cofinal graph and $\phi : \mathcal{P}(G) \to \mathbb{R}$ a function such that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. Consider the one-parameter group α^ϕ on $C^*(G)$.*

(1) *Assume that $NW_G = \emptyset$. There is a gauge-invariant β-KMS weight for α^ϕ for all $\beta \in \mathbb{R}$.*
(2) *Assume that NW_G is non-empty and finite. Assume that ϕ satisfies Bowen's condition on NW_G. There is a gauge-invariant β-KMS weight for α^ϕ if and only if $\mathbb{P}(-\beta\phi) = 0$, and if it exists this β-KMS weight is unique up to multiplication by a scalar.*
(3) *Assume that NW_G is infinite. There is a gauge-invariant β-KMS weight for α^ϕ if and only if $\mathbb{P}(-\beta\phi) \leq 0$.*

Proof. Combine Theorem 4.6 above with [**Tho14**, Proposition 3.1]. □

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It should be noted that in case (2) of Theorem 7.2 the function $\beta \mapsto \mathbb{P}(-\beta\phi)$ may not have any zeroes, and hence KMS weights (or states if $G = NW_G$) may not exist; see [**KR06**, Example 3.7].

COROLLARY 7.3. *Let G be a cofinal graph and $\phi : \mathcal{P}(G) \to]0, \infty[$ a function such that $\lim_{k\to\infty} \mathrm{var}_k(\phi) = 0$. Consider the one-parameter group α^ϕ on $C^*(G)$.*

(1) *Assume that $NW_G = \emptyset$. There is a gauge-invariant β-KMS weight for α^ϕ for all $\beta \in \mathbb{R}$.*
(2) *Assume that NW_G is non-empty and finite. Assume that ϕ satisfies Bowen's condition on NW_G. There is a $\beta_0 \in]0, \infty[$ such that there is a gauge invariant β-KMS weight for α^ϕ if and only if $\beta = \beta_0$, and it is then unique up to multiplication by a scalar.*
(3) *Assume that NW_G is infinite. Assume also that ϕ is bounded away from 0 and ∞, that is, there are $0 < a \le b < \infty$ such that $\phi(y) \in [a, b]$ for all $y \in \mathcal{P}(G)$. There is a $\beta_0 \in]0, \infty]$ such that a gauge-invariant β-KMS weight for α^ϕ exists if and only if $\beta \ge \beta_0$.*

Proof. Only (2) and (3) require proof. Consider first case (3) and assume that the Gurewich entropy $P_{NW_G}(0) = \mathbb{P}(0)$ of NW_G is infinite. For any $\epsilon > 0$, any $x \in \mathcal{P}(NW_G)$ and any $\beta \in \mathbb{R}$, there is a finite path μ in NW_G such that

$$\min\{e^{-\beta a}, e^{-\beta b}\}^n e^{-n\epsilon} \sum_{\sigma^n(y)=y} 1_{Z(\mu)}(y) \le L^n_{-\beta\phi}(1_{Z(\mu)})(x)$$

for all large n; cf. (3.8). Since $e^{\mathbb{P}(0)} = \limsup_n (\sum_{\sigma^n(y)=y} 1_{Z(\mu)}(y))^{1/n} = \infty$, it follows that $\mathbb{P}_x(-\beta\phi) = \infty$. By Lemma 4.5 there are therefore no $\beta\phi$ conformal measures and we have to set $\beta_0 = \infty$ in this case.

Case (2) and case (3) with $\mathbb{P}(0)$ finite can be handled together. In both cases the proof depends on the finiteness and continuity of the function $\beta \mapsto \mathbb{P}(-\beta\phi)$. To establish these properties, note that there are constants $0 < a \le b < \infty$ such that $a \le \phi(x) \le b$ for all $x \in \mathcal{P}(NW_G)$. In case (3) this is an assumption, and in case (2) it follows from compactness of $\mathcal{P}(NW_G)$. Let $\beta, \beta' \in \mathbb{R}$, $\beta \le \beta'$, and consider a non-negative $f \in C_c(\mathcal{P}(G))$. Then

$$\begin{aligned}\sum_{y\in\sigma^{-n}(x)} e^{(\beta-\beta')nb} e^{-\beta\phi_n(y)} f(y) &\le \sum_{y\in\sigma^{-n}(x)} e^{-\beta'\phi_n(y)} f(y) \\ &\le \sum_{y\in\sigma^{-n}(x)} e^{(\beta-\beta')na} e^{-\beta\phi_n(y)} f(y),\end{aligned}$$

leading to the estimates

$$(\beta - \beta')b + \mathbb{P}(-\beta\phi) \le \mathbb{P}(-\beta'\phi) \le (\beta - \beta')a + \mathbb{P}(-\beta\phi).$$

It follows first that $\mathbb{P}(-\beta\phi) \in \mathbb{R}$ for all $\beta \in \mathbb{R}$ since $\mathbb{P}(0)$ is finite; in case (2) because NW_G is and in case (3) by assumption. Once this is established the estimates above show that $\beta \mapsto \mathbb{P}(-\beta\phi)$ is continuous, strictly decreasing and converges to $-\infty$ when $\beta \to \infty$. In this way (2) and (3) follow from the corresponding cases of Theorem 7.2. □

If we take ϕ to be a unit constant in Corollary 7.3, we recover Theorem 4.3 in [**Tho14**].

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