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ARTICLE

Reversed primes in arithmetic progressions for arbitrary bases

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Abstract
We prove analogs of the Dirichlet theorem on arithmetic progressions and the Siegel–Walfisz theorem for the digital reverses of primes in arbitrary bases, which we obtained only for large bases in a previous paper. The proof is based on a generalization of the result of Martin–Mauduit–Rivat (2014) on exponential sums over primes with so-called "digital" functions.

1. Introduction

In this article, we continue our study [1] of the distribution of reversed primes, that is, the digital reverses of prime numbers, in arithmetic progressions. In [1], the base g was restricted to $g \geq 31,699$. Recently, Chourasiya and Johnston [2] extended the admissible range of the base to $g \geq 26,000$. In the current article, as was announced in several places (see, e.g., [2, Section 1.2]), we shall remove the constraint on the base to include all bases greater than or equal to 2.

We recall the formal definition of the digital reverse. Take $g \in \mathbb{Z}_{\geq 2}$ as the base of the digital representation and write the base- g representation of $n \in \mathbb{Z}_{\geq 0}$ as

$$n = \sum_{i \geq 0} \varepsilon_i(n) g^i \quad \text{with} \quad \varepsilon_i(n) \in \{0, \dots, g-1\} \text{ for all } i \in \mathbb{Z}_{\geq 0}.$$

We define the length $\mathrm{len}(n)$ of n by

$$\mathrm{len}(n) := \min\{\ell \in \mathbb{Z}_{\geq 0} \mid \varepsilon_i(n) = 0 \text{ for all } i \geq \ell\},$$

that is, $\mathrm{len}(n)$ is the number of digits of n with the convention $\mathrm{len}(0) = 0$. For $n \in \mathbb{Z}_{\geq 0}$, we define

$$\mathrm{rev}(n) := \sum_{0 \leq i < \mathrm{len}(n)} \varepsilon_i(n) g^{\mathrm{len}(n)-i-1}, \tag{1.1}$$

which is the integer obtained by reading the base- g representation of n backward.

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We now state the infinitude of reversed primes in a given arithmetic progression for all bases.

Theorem 1.1 (Telhcirid's theorem on arithmetic progressions[1]). *For $g, a, q \in \mathbb{Z}$ with*

$$g \geq 2, \quad q \geq 1, \quad (a, q, g^2 - 1) = 1, \quad g \nmid (a, q),$$

there are infinitely many primes p such that $\mathrm{rev}(p) \equiv a \pmod{q}$.

Note that $(a, q, g^2 - 1) = 1$ and $g \nmid (a, q)$ are necessary conditions for the infinitude stated in Theorem 1.1 (see, e.g., Section 1 of [1]).

As in [1], Theorem 1.1 follows from a quantitative version. In order to avoid irregular behavior of the main term, we restrict the primes being reversed to have a fixed length $L \in \mathbb{N}$, that is, we study the asymptotic behavior of

$$\overleftarrow{\pi}_L(a, q) := \#\{g^{L-1} \leq p < g^L \mid \mathrm{rev}(p) \equiv a \pmod{q}\},$$

where, throughout this article, the letter p always denotes a prime number. Then the quantitative version of Theorem 1.1 is the following.

Theorem 1.2 (Zsiflaw–Legeis theorem[2]). *For $g, a, q \in \mathbb{Z}$ and $L \in \mathbb{N}$ with $g \geq 2$ and $q \geq 1$, we have*

$$\overleftarrow{\pi}_L(a, q) = \frac{\rho_g(a, q)}{q} \frac{g^L - g^{L-1}}{\log g^L} \left(1 + O\left(\frac{1}{L}\right)\right) + O(g^L \exp(-c\sqrt{L}))$$

provided that

$$q \leq \exp(c\sqrt{L})$$

with some constant $c > 0$, where $\rho_g(a, q)$ is given by

$$\rho_g(a, q) := \begin{cases} \dfrac{g - \mathbf{1}_{(q,g)|a}(q, g)}{g - 1} \cdot \dfrac{(q, g^2 - 1)}{\varphi((q, g^2 - 1))} & \text{if } (a, q, g^2 - 1) = 1 \text{ and } g \nmid (a, q), \\ 0 & \text{otherwise}, \end{cases}$$

and c and the implicit constants depend only on g and are effectively computable.

Remark. During the final write-up of this article, we discovered that Dartyge et al. [4, Theorem 1.4] have independently obtained the above results for all bases $g \geq 2$.

Our previous result [1] was proved by using the discrete circle method in the style of Maynard [10]. In order to improve the range of the base, we use a more direct method of Mauduit and Rivat [7] (see also Maynard [9]), which was used to study the distribution in arithmetic progressions of the sum of digits of primes. We shall reduce the problem to an estimate for an exponential sum of the form

$$\sum_{n \leq x} \Lambda(n) e(f(n)) \tag{1.2}$$

with a function f related to some digital property of integers. In Mauduit–Rivat [7], f is chosen to be the sum-of-digits function up to multiplication by a real parameter, whereas, in our case, the corresponding role will be played by the digital reverse with a certain modification. The method of Mauduit and Rivat to bound the exponential sum (1.2) has been generalized by Martin–Mauduit–Rivat [6] by replacing

[1] *Telhcirid* is *Dirichlet* read backward.
[2] Again *Zsiflaw–Legeis* is *Siegel–Walfisz* read backward.

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the sum-of-digits function by a class of more general arithmetic functions called "digital functions." An arithmetic function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ defined with a map $\alpha\colon \{0,\dots,g-1\} \to \mathbb{R}$ by

$$f(n) := \sum_{0\leq i<\mathrm{len}(n)} \alpha(\varepsilon_i(n)) \tag{1.3}$$

is said to be "digital" by Martin–Mauduit–Rivat. However, our digital reverse rev (or its modification) is not a digital function in the above sense since any arithmetic function f satisfying (1.3) admits the bound

$$|f(n)| \leq \Big(\max_{0\leq m<g} |\alpha(m)|\Big)\mathrm{len}(n) \ll \log(n+1)$$

while rev does not. We thus generalize the notion of digital functions one step further by introducing the following "weakly" digital functions.

Definition 1.1. Let $g \in \mathbb{Z}_{\geq 2}$ and let $\mathscr{A}_g$ be the set of sequences of maps

$$\boldsymbol{\alpha} = (\alpha_i\colon \{0,\dots,g-1\} \to \mathbb{R})_{i=0}^{\infty}.$$

We call an element of $\mathscr{A}_g$ a *seed*. For $\boldsymbol{\alpha} \in \mathscr{A}_g$, we define the *weakly digital function* $f_{\bullet,\boldsymbol{\alpha}}$ *generated by the seed* $\boldsymbol{\alpha}$ as the sequence of functions $(f_{\lambda,\boldsymbol{\alpha}}\colon \mathbb{Z}_{\geq 0} \to \mathbb{R})_{\lambda=0}^{\infty}$ given by

$$f_{\lambda,\boldsymbol{\alpha}}(n) := \sum_{0\leq i<\lambda} \alpha_i(\varepsilon_i(n)).$$

Note that $f_{\lambda,\boldsymbol{\alpha}}$ has a period g^{λ}. We also write $f_{\lambda} = f_{\lambda,\boldsymbol{\alpha}}$ for simplicity.

In other words, $f_{\lambda,\boldsymbol{\alpha}}(n)$ is defined by looking at the λ least significant digits $\varepsilon_0(n),\dots,\varepsilon_{\lambda-1}(n)$, regardless of whether λ is larger or smaller than the length of n, and then assigning to each digit $\varepsilon_i(n)$ the value $\alpha_i(\varepsilon_i(n))$ and finally summing those values. The rules α_i assigning values to digits now depend on the position i, unlike in (1.3). Also, in (1.3), the number of digits involved is fixed to be the length of n, whereas here the number of digits is an additional parameter.

The resulting estimate for the exponential sum will be stated as Theorem 7.1 below (see also (4.3) and (4.5) for some definitions). For the proof of Theorem 7.1, we follow the argument of Martin–Mauduit–Rivat [6], but we need to keep track of the shifts of $\boldsymbol{\alpha} \in \mathscr{A}_g$ and to modify the auxiliary function Ψ given by formula (19) of [6] as in (4.7) below. A related general estimate was obtained by Mauduit and Rivat [8, Theorem 1] for functions satisfying the carry property and a uniform Fourier-decay condition. However, the uniformity with respect to the seed in our exponential sum estimateis required for the direct application to the digital reverse.

This article is organized as follows. We first prove some basic properties of weakly digital functions in Section 3. We then introduce some lemmas on exponential sums with weakly digital functions in Section 4, especially the L^1, L^{∞} and hybrid bounds. In Sections 5 and 6, we prepare estimates for the Type I and Type II sums. In Section 7, we prove Theorem 7.1. Finally, we deduce Theorems 1.1, 1.2, and 8.2 in Section 8. We expect to develop further results on the distribution of weakly digital functions over primes in a later paper.

2. Notation

Besides those introduced elsewhere in the article, we use the following notation.

We use the convention $\mathbb{N} := \{1,2,3,\dots\}$ so $0 \notin \mathbb{N}$. The letter p is reserved for prime numbers. The letter c is used for positive constants, not necessarily the same at each occurrence. The arithmetic function $\varphi(n)$ is the Euler totient function, $\mu(n)$ is the Möbius function, $\tau(n)$ is the divisor function, that is, the number of positive divisors of n, and $\Lambda(n)$ is the von Mangoldt function. For integers $n_1,\dots,n_r$, we write $(n_1,\dots,n_r)$ for the greatest common divisor of $n_1,\dots,n_r$.

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For a real number x, we let $e(x) := \exp(2\pi i x)$. We also write $[x]$ for the integer part of x, that is, the greatest integer $\le x$, $\lceil x \rceil$ denotes the least integer $\ge x$, and $\|x\| := \min_{n\in\mathbb{Z}}|x-n|$ is the distance of x from the nearest integer.

For a logical formula P, we write $\mathbf{1}_P$ for the indicator function of P.

If a theorem or a lemma is stated with the phrase "where the implicit constant depends on $a, b, c, \ldots$," then every implicit constant in the corresponding proof may also depend on $a, b, c, \ldots$ without being specifically mentioned.

3. Basic properties of weakly digital functions

As mentioned in the introduction, we need to keep track of the shifts of a seed $\boldsymbol{\alpha} \in \mathscr{A}_g$. Therefore, for $j \in \mathbb{Z}_{\ge 0}$ and $\boldsymbol{\alpha} \in \mathscr{A}_g$, we define the j-shift $\boldsymbol{\alpha}^{[j]} = (\alpha_i^{[j]})_{i=0}^{\infty} \in \mathscr{A}_g$ of $\boldsymbol{\alpha}$ by

$$\alpha_i^{[j]} := \alpha_{i+j}.$$

Note that we then have $\alpha_{i+k}^{[j]} = \alpha_i^{[j+k]}$ for $i, j, k \in \mathbb{Z}_{\ge 0}$. We also write

$$f_\lambda^{[j]} = f_{\lambda,\boldsymbol{\alpha}}^{[j]} := f_{\lambda,\boldsymbol{\alpha}^{[j]}}.$$

In this section, we first prove a certain "g-additivity" of weakly digital functions $f_\lambda^{[j]}$.

As in Martin–Mauduit–Rivat [6], we say that a function $f\colon \mathbb{Z}_{\ge 0} \to \mathbb{C}$ is *g-additive* if

$$f(g^k m + n) = f(g^k m) + f(n)$$

for all $k, m, n \in \mathbb{Z}_{\ge 0}$ with $0 \le n < g^k$. We also say that $f\colon \mathbb{Z}_{\ge 0} \to \mathbb{C}$ is *strongly g-additive* if

$$f(g^k m + n) = f(m) + f(n)$$

for all $k, m, n \in \mathbb{Z}_{\ge 0}$ with $0 \le n < g^k$. Our weakly digital functions are, in general, neither strongly g-additive nor g-additive in the strict sense, but they satisfy the following similar identity.

Lemma 3.1. *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, and $\lambda, j, k, m, n \in \mathbb{Z}_{\ge 0}$ with $0 \le k \le \lambda$ and $n < g^k$, we have*

$$f_\lambda^{[j]}(g^k m + n) = f_{\lambda-k}^{[j+k]}(m) + f_k^{[j]}(n).$$

Proof. Since $n < g^k$, we have

$$g^k m + n = \sum_{i\ge 0} \varepsilon_i(m) g^{i+k} + \sum_{0\le i<k} \varepsilon_i(n) g^i = \sum_{i\ge k} \varepsilon_{i-k}(m) g^i + \sum_{0\le i<k} \varepsilon_i(n) g^i,$$

and so

$$\varepsilon_i(g^k m + n) = \begin{cases} \varepsilon_i(n) & \text{for } 0 \le i < k, \\ \varepsilon_{i-k}(m) & \text{for } i \ge k. \end{cases}$$

Using $0 \le k \le \lambda$, we thus have

$$\begin{aligned} f_\lambda^{[j]}(g^k m + n) &= \sum_{k\le i<\lambda} \alpha_i^{[j]}(\varepsilon_{i-k}(m)) + \sum_{0\le i<k} \alpha_i^{[j]}(\varepsilon_i(n)) \\ &= \sum_{0\le i<\lambda-k} \alpha_{i+k}^{[j]}(\varepsilon_i(m)) + f_k^{[j]}(n) \\ &= \sum_{0\le i<\lambda-k} \alpha_i^{[j+k]}(\varepsilon_i(m)) + f_k^{[j]}(n) = f_{\lambda-k}^{[j+k]}(m) + f_k^{[j]}(n). \end{aligned}$$

This completes the proof. □

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Using the above variant of g-additivity, we prove the next lemma, which reduces exponential sums with an arbitrary upper endpoint to those with upper endpoints given by powers of g.

Lemma 3.2. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $L, j \in \mathbb{Z}_{\geq 0}$, $x \in \mathbb{R}_{\geq 1}$, and $\beta \in \mathbb{R}$ with $1 \leq x \leq g^L$, we have*

$$\left| \sum_{0 \leq n < x} e(f_L^{[j]}(n) - \beta n) \right| \leq (g-1) \sum_{0 \leq \lambda \leq \frac{\log\lfloor x \rfloor}{\log g}} \left| \sum_{0 \leq n < g^\lambda} e(f_\lambda^{[j]}(n) - \beta n) \right|.$$

Proof. Let $N := \lceil x \rceil \geq 1$. We then have

$$\sum_{0 \leq n < x} e(f_L^{[j]}(n) - \beta n) = \sum_{0 \leq n < N} e(f_L^{[j]}(n) - \beta n). \tag{3.1}$$

Let $k := [\frac{\log N}{\log g}]$ so that $g^k \leq N < g^{k+1}$. Since $x \leq g^L$, we have $k \leq L$. We then write

$$N = g^k y + z \quad \text{with} \quad y \in [1, g) \cap \mathbb{Z} \text{ and } z \in [0, g^k) \cap \mathbb{Z}. \tag{3.2}$$

We prove the assertion by induction on $k \in \{0, \dots, L\}$.

Initial case k=0. In this case, we have $N \leq g - 1$, and so the trivial bound gives

$$\left| \sum_{0 \leq n < x} e(f_L^{[j]}(n) - \beta n) \right| = \left| \sum_{0 \leq n < N} e(f_L^{[j]}(n) - \beta n) \right| \leq N \leq g - 1.$$

Hence, the assertion holds.

Induction step. Assume that the assertion holds for the case $0 \leq k < K$ with $K \in \{1, \dots, L\}$. We shall prove the case $k = K$. By (3.2), we have

$$\begin{aligned} \sum_{0 \leq n < N} e(f_L^{[j]}(n) - \beta n) &= \sum_{0 \leq m < y} \sum_{g^K m \leq n < g^K (m+1)} e(f_L^{[j]}(n) - \beta n) \\ &\quad + \sum_{g^K y \leq n < N} e(f_L^{[j]}(n) - \beta n) \\ &= \sum_{0 \leq m < y} \sum_{0 \leq n < g^K} e(f_L^{[j]}(g^K m + n) - \beta(g^K m + n)) \\ &\quad + \sum_{0 \leq n < z} e(f_L^{[j]}(g^K y + n) - \beta(g^K y + n)) \\ &=: \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.} \end{aligned} \tag{3.3}$$

For the sum $\sum_1$, by using Lemma 3.1 and $y \in [1, g) \cap \mathbb{Z}$, we have

$$\begin{aligned} \left| \sum\nolimits_1 \right| &= \left| \sum_{0 \leq m < y} e(f_{L-K}^{[j+K]}(m) - \beta g^K m) \sum_{0 \leq n < g^K} e(f_K^{[j]}(n) - \beta n) \right| \\ &\leq (g-1) \left| \sum_{0 \leq n < g^K} e(f_K^{[j]}(n) - \beta n) \right|. \end{aligned} \tag{3.4}$$

For the sum $\sum_2$, using Lemma 3.1, we obtain

$$\left| \sum\nolimits_2 \right| = \left| e(f_{L-K}^{[j+K]}(y) - \beta g^K y) \sum_{0 \leq n < z} e(f_K^{[j]}(n) - \beta n) \right| \leq \left| \sum_{0 \leq n < z} e(f_K^{[j]}(n) - \beta n) \right|.$$

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When $z \geq 1$, we have $0 \leq [\frac{\log\lceil z\rceil}{\log g}] = [\frac{\log z}{\log g}] < K$ and so the induction hypothesis implies that

$$\left|\sum\nolimits_2\right| \leq (g-1) \sum_{0\leq\lambda<K} \left| \sum_{0\leq n<g^\lambda} e(f_\lambda^{[j]}(n) - \beta n)\right|. \tag{3.5}$$

This holds clearly if $z = 0$. On inserting (3.4) and (3.5) into (3.3) and using (3.1), we obtain

$$\left| \sum_{0\leq n<x} e(f_L^{[j]}(n) - \beta n)\right| \leq (g-1) \sum_{0\leq\lambda\leq K} \left| \sum_{0\leq n<g^\lambda} e(f_\lambda^{[j]}(n) - \beta n)\right|,$$

that is, the assertion for the case $k = K$. □

4. Exponential sums for weakly digital functions

We study the distribution of weakly digital functions via discrete Fourier analysis. For a given $\boldsymbol{\alpha} \in \mathscr{A}_g$, $\lambda, j \in \mathbb{Z}_{\geq 0}$, and $\beta \in \mathbb{R}$, we introduce a normalized exponential sum

$$F_\lambda^{[j]}(\beta) := \frac{1}{g^\lambda} \sum_{0\leq n<g^\lambda} e(f_\lambda^{[j]}(n) - \beta n) \quad \text{and} \quad F_\lambda(\beta) := F_\lambda^{[0]}(\beta). \tag{4.1}$$

Note that the function $\beta \mapsto F_\lambda^{[j]}(\beta)$ has a period 1. For $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i, j \in \mathbb{Z}_{\geq 0}$, and $\beta \in \mathbb{R}$, we also use

$$\varphi_i^{[j]}(\beta) := \left| \sum_{0\leq n<g} e(\alpha_i^{[j]}(n) - \beta n)\right| \quad \text{and} \quad \varphi_i(\beta) := \varphi_i^{[0]}(\beta). \tag{4.2}$$

In this section, we prepare several results on the exponential sum $F_\lambda^{[j]}(\beta)$.

4.1. Product formula

A key tool for studying digital properties of integers is, as in the previous studies, the following product formula for the exponential sum $F_\lambda^{[j]}(\beta)$.

Lemma 4.1. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $\lambda, j \in \mathbb{Z}_{\geq 0}$, and $\beta \in \mathbb{R}$, we have*

$$|F_\lambda^{[j]}(\beta)| = \frac{1}{g^\lambda} \prod_{0\leq i<\lambda} \varphi_i^{[j]}(\beta g^i).$$

Proof. By considering the base-g representation of n in (4.1), we have

$$\begin{aligned} |F_\lambda^{[j]}(\beta)| &= \frac{1}{g^\lambda} \left| \sum_{0\leq n_0,\dots,n_{\lambda-1}<g} e\left(\sum_{0\leq i<\lambda} \alpha_i^{[j]}(n_i) - \sum_{0\leq i<\lambda} \beta n_i g^i\right)\right| \\ &= \frac{1}{g^\lambda} \prod_{0\leq i<\lambda} \left| \sum_{0\leq n<g} e(\alpha_i^{[j]}(n) - \beta g^i n)\right|. \end{aligned}$$

By recalling the definition of $\varphi_i^{[j]}(\beta)$, we obtain the assertion. □

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We then deduce a recurrence formula which will be convenient in some arguments below.

Lemma 4.2. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $\lambda \in \mathbb{N}$, $j \in \mathbb{Z}_{\geq 0}$, and $\beta \in \mathbb{R}$, we have*

$$|F_\lambda^{[j]}(\beta)| = |F_{\lambda-1}^{[j+1]}(\beta g)| \cdot \frac{1}{g}\varphi_j(\beta).$$

Proof. By Lemma 4.1, we have

$$|F_\lambda^{[j]}(\beta)| = \frac{1}{g^{\lambda-1}} \prod_{1 \leq i < \lambda} \varphi_i^{[j]}(\beta g^i) \cdot \frac{1}{g}\varphi_0^{[j]}(\beta).$$

By shifting the variable i by 1 and using $\varphi_{i+1}^{[j]} = \varphi_i^{[j+1]}$ and $\varphi_0^{[j]} = \varphi_j^{[0]} = \varphi_j$, we obtain

$$|F_\lambda^{[j]}(\beta)| = \frac{1}{g^{\lambda-1}} \prod_{0 \leq i < \lambda-1} \varphi_i^{[j+1]}(\beta g \cdot g^i) \cdot \frac{1}{g}\varphi_j(\beta) = |F_{\lambda-1}^{[j+1]}(\beta g)| \cdot \frac{1}{g}\varphi_j(\beta).$$

This completes the proof. □

4.2. The L^∞-bound

We next prepare a pointwise bound for $F_\lambda^{[j]}(\beta)$. We first extract a small cancellation from each of the factors in the product formula (Lemma 4.1) by using the terms with indices m and n in the definition (4.2) of $\varphi_i^{[j]}(\beta)$.

Lemma 4.3. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i, j, m, n \in \mathbb{Z}_{\geq 0}$ with $0 \leq m < n < g$ and $\beta \in \mathbb{R}$, we have*

$$\varphi_i^{[j]}(\beta) \leq g \exp\left(-\frac{8}{g}\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n)\|^2 \right).$$

Proof. Since $m \neq n$, we have

$$\begin{aligned}\varphi_i^{[j]}(\beta) &\leq |e(\alpha_i^{[j]}(m) - \beta m) + e(\alpha_i^{[j]}(n) - \beta n)| + \Bigg| \sum_{\substack{0 \leq k < g \\ k \notin \{m,n\}}} e(\alpha_i^{[j]}(k) - \beta k) \Bigg| \\ &\leq |1 + e(\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n))| + g - 2.\end{aligned}$$

We now use the inequality

$$|1 + e(x)| = 2|\cos \pi x| = 2 \cos \pi \|x\| \leq 2(1 - 4\|x\|^2),$$

which is valid for all $x \in \mathbb{R}$. This gives

$$\begin{aligned}\varphi_i^{[j]}(\beta) &\leq 2(1 - 4\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n)\|^2) + g - 2 \\ &= g\left(1 - \frac{8}{g}\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n)\|^2\right) \\ &\leq g \exp\left(-\frac{8}{g}\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n)\|^2 \right)\end{aligned}$$

since $1 - x \leq e^{-x}$ for any $x \in \mathbb{R}$. This completes the proof. □

In order to obtain an L^∞-bound for $F_\lambda^{[j]}(\beta)$ that is uniform in β, we combine the cancellations obtained from Lemma 4.3 for pairs of consecutive factors in the product formula (Lemma 4.1), thereby

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eliminating the dependence on β. To this end, we introduce the following notation. For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, and $i, j \in \mathbb{Z}_{\geq 0}$, define

$$\gamma_i^{[j]}(\boldsymbol{\alpha}) := \frac{2\log 2}{(g-1)g^4(\log g)^2} \sum_{0 \leq m < n < g} \left\| \left(g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)\right) - \left(g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n)\right) \right\|^2. \tag{4.3}$$

We also write $\gamma_i(\boldsymbol{\alpha}) := \gamma_i^{[0]}(\boldsymbol{\alpha})$. Since $\|x\| \leq \frac{1}{2}$ for all $x \in \mathbb{R}$, we have

$$\gamma_i^{[j]}(\boldsymbol{\alpha}) \in \left[0, \frac{\log 2}{4g^3(\log g)^2}\right] \subset \left[0, \frac{1}{20}\right) \tag{4.4}$$

since

$$\gamma_i^{[j]}(\boldsymbol{\alpha}) \leq \frac{\log 2}{2(g-1)g^4(\log g)^2} \sum_{0 \leq m < n < g} 1 = \frac{\log 2}{4g^3(\log g)^2} \leq \frac{1}{32\log 2} < \frac{1}{20}.$$

Note that we defined $\gamma_i^{[j]}(\boldsymbol{\alpha})$ to be smaller than what is strictly necessary for the proof of the next lemma. This is only to simplify the subsequent arguments.

Lemma 4.4. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i, j \in \mathbb{Z}_{\geq 0}$, and $\beta \in \mathbb{R}$, we have*

$$\left(\varphi_i^{[j]}(\beta g^i)\varphi_{i+1}^{[j]}(\beta g^{i+1})\right)^{\frac{1}{2}} \leq g^{1-\gamma_i^{[j]}(\boldsymbol{\alpha})}.$$

Proof. For $0 \leq m < n < g$, we have

$$\begin{aligned} &g\left(\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\right) - \left(\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\right) \\ &= \left(g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)\right) - \left(g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n)\right). \end{aligned}$$

By using $\|x+y\| \leq \|x\| + \|y\|$ and $\|gx\| \leq g\|x\|$, we thus get

$$\begin{aligned} &\frac{1}{g+1}\left\|\left(g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)\right) - \left(g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n)\right)\right\| \\ &= \frac{1}{g+1}\left\|g\left(\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\right) - \left(\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\right)\right\| \\ &\leq \frac{1}{g+1}\left(g\left\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\right\| + \left\|\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\right\|\right) \\ &\leq \max\left(\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\|, \|\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\|\right). \end{aligned}$$

Thus, by Lemma 4.3, we have

$$\begin{aligned} &\varphi_i^{[j]}(\beta g^i)\varphi_{i+1}^{[j]}(\beta g^{i+1}) \\ &\leq g^2 \exp\left(-\frac{8}{g}\left(\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\|^2 + \|\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\|^2\right)\right) \\ &\leq g^2 \exp\left(-\frac{8}{g}\max\left(\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)\|, \|\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n)\|\right)^2\right) \\ &\leq g^2 \exp\left(-\frac{8}{g(g+1)^2}\left\|\left(g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)\right) - \left(g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n)\right)\right\|^2\right). \end{aligned}$$

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By using the inequality

$$\frac{8}{g(g+1)^2}\max_{0\le m<n<g}\left\|\left(g\alpha_i^{[j]}(m)-\alpha_{i+1}^{[j]}(m)\right)-\left(g\alpha_i^{[j]}(n)-\alpha_{i+1}^{[j]}(n)\right)\right\|^2$$
$$\ge\frac{16}{(g-1)g^2(g+1)^2}\sum_{0\le m<n<g}\left\|\left(g\alpha_i^{[j]}(m)-\alpha_{i+1}^{[j]}(m)\right)-\left(g\alpha_i^{[j]}(n)-\alpha_{i+1}^{[j]}(n)\right)\right\|^2$$

and

$$\frac{16}{(g-1)g^2(g+1)^2}\ge\frac{4\log 2}{(g-1)g^4\log g},$$

we obtain the result. □

We can now prove an L^∞-bound for $F_\lambda^{[j]}(\beta)$. For $g\in\mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha}\in\mathscr{A}_g$, $\lambda, j\in\mathbb{Z}_{\ge 0}$, we now introduce

$$\sigma_\lambda^{[j]}(\boldsymbol{\alpha}):=\sum_{0\le i<\lambda}\gamma_i^{[j]}(\boldsymbol{\alpha})\quad\text{and}\quad\sigma_\lambda(\boldsymbol{\alpha}):=\sigma_\lambda^{[0]}(\boldsymbol{\alpha}).\tag{4.5}$$

Note that (4.4) gives

$$\sigma_\lambda^{[j]}(\boldsymbol{\alpha})\in\left[0,\frac{\log 2}{4g^3(\log g)^2}\lambda\right]\subset\left[0,\frac{1}{4g^3\log g}\lambda\right]\subset\left[0,\frac{\lambda}{20}\right]\tag{4.6}$$

and we trivially have $\sigma_0^{[j]}(\boldsymbol{\alpha})=0$. Further, $\sigma_\lambda^{[j]}(\boldsymbol{\alpha})$ is nondecreasing with respect to $\lambda\in\mathbb{Z}_{\ge 0}$.

Lemma 4.5 (L^∞-bound). *For $g\in\mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha}\in\mathscr{A}_g$, $\lambda, j\in\mathbb{Z}_{\ge 0}$, and $\beta\in\mathbb{R}$, we have*

$$|F_\lambda^{[j]}(\beta)|\ll g^{-\sigma_\lambda^{[j]}(\boldsymbol{\alpha})},$$

where the implicit constant depends only on g.

Proof. We may assume $\lambda\ge 1$. By Lemma 4.1 and the trivial bound $\varphi_i^{[j]}(\beta)\le g$, we have

$$|F_\lambda^{[j]}(\beta)|\le\frac{1}{g^{\lambda-1}}\prod_{0\le i<\lambda-1}\varphi_i^{[j]}(\beta g^i)^{\frac12}\prod_{1\le i<\lambda}\varphi_i^{[j]}(\beta g^i)^{\frac12}=\frac{1}{g^{\lambda-1}}\prod_{0\le i<\lambda-1}\left(\varphi_i^{[j]}(\beta g^i)\varphi_{i+1}^{[j]}(\beta g^{i+1})\right)^{\frac12}.$$

Then, by Lemma 4.4, we obtain the assertion since

$$\sum_{0\le i<\lambda-1}\gamma_i^{[j]}(\boldsymbol{\alpha})=\sum_{0\le i<\lambda}\gamma_i^{[j]}(\boldsymbol{\alpha})-\gamma_{\lambda-1}^{[j]}(\boldsymbol{\alpha})>\sigma_\lambda^{[j]}(\boldsymbol{\alpha})-\frac{1}{20}$$

by (4.4). □

4.3. The L^1-bound

We next bound the discrete L^1-moment with a congruence condition

$$\sum_{\substack{0\le h<g^\lambda\\ h\equiv a\pmod{kg^\delta}}}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|,$$

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where $kg^{\delta} \mid g^{\lambda}$ and $g \nmid k$. To this end, for $i \in \mathbb{Z}_{\geq 0}$, $R \mid g$, $S \mid g$, and $t \in \mathbb{R}$, we introduce

$$\Psi_i(t,R,S) := \frac{1}{g^2} \sum_{0 \leq r < R} \varphi_{i+1}\left(\frac{g(t+r)}{RS}\right) \sum_{0 \leq s < S} \varphi_i\left(\frac{t+r}{RS} + \frac{s}{S}\right) \tag{4.7}$$

and prepare an estimate for $\Psi_i(t,R,S)$.

We first prove some basic lemmas on the moments of φ_i.

Lemma 4.6. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i \in \mathbb{Z}_{\geq 0}$, $R \in \mathbb{N}$, and $a \in \mathbb{Z}$ with $R \mid g$ and $(R,a) = 1$, we have*

$$\max_{\beta \in \mathbb{R}} \sum_{0 \leq r < R} \varphi_i\left(\beta + \frac{ar}{R}\right)^2 \leq g^2.$$

Proof. By expanding the square and using the orthogonality with $(R,a) = 1$, we have

$$\begin{aligned} \sum_{0 \leq r < R} \varphi_i\left(\beta + \frac{ar}{R}\right)^2 &= \sum_{0 \leq m,n < g} e(\alpha_i(m) - \alpha_i(n) - \beta(m-n)) \sum_{0 \leq r < R} e\left(-\frac{ar}{R}(m-n)\right) \\ &\leq R \sum_{\substack{0 \leq m,n < g \\ m \equiv n \pmod R}} 1 = g^2. \end{aligned}$$

This completes the proof. □

Lemma 4.7. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i \in \mathbb{Z}_{\geq 0}$, $U \in \mathbb{N}$, and $\beta \in \mathbb{R}$ with $U \geq 2g-1$, we have*

$$\sum_{0 \leq u < U} \varphi_i\left(\beta + \frac{u}{U}\right)^4 = U \sum_{|h| < g} \left| \sum_{0 \leq n, n+h < g} e(\alpha_i(n+h) - \alpha_i(n)) \right|^2.$$

Proof. By expanding the square, we have

$$\varphi_i\left(\beta + \frac{u}{U}\right)^2 = \sum_{0 \leq m,n < g} e\left(\alpha_i(m) - \alpha_i(n) - \left(\beta + \frac{u}{U}\right)(m-n)\right).$$

By writing $h = m - n$, we can rewrite the right-hand side as

$$\varphi_i\left(\beta + \frac{u}{U}\right)^2 = \sum_{|h| < g} c(h) e\left(-\left(\beta + \frac{u}{U}\right)h\right) \quad \text{with} \quad c(h) := \sum_{\substack{0 \leq m,n < g \\ h = m-n}} e(\alpha_i(m) - \alpha_i(n)).$$

By the orthogonality property, we thus have

$$\begin{aligned} \sum_{0 \leq u < U} \varphi_i\left(\beta + \frac{u}{U}\right)^4 &= \sum_{|h_1|, |h_2| < g} c(h_1)\overline{c(h_2)} e(-\beta(h_1 - h_2)) \sum_{0 \leq u < U} e\left(-\frac{u}{U}(h_1 - h_2)\right) \\ &= U \sum_{\substack{|h_1|, |h_2| < g \\ h_1 \equiv h_2 \pmod U}} c(h_1)\overline{c(h_2)} e(-\beta(h_1 - h_2)). \end{aligned} \tag{4.8}$$

By the assumption $U \geq 2g-1$, we have

$$|h_1|, |h_2| < g \text{ and } h_1 \equiv h_2 \pmod U \implies h_1 = h_2.$$

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By using this fact in (4.8) and noting that

$$c(h) = \sum_{0 \le n, n+h < g} e(\alpha_i(n+h) - \alpha_i(n)),$$

we obtain the assertion. □

We bound $\Psi_i(t, R, S)$ by considering the two cases $S \neq g$ and $S = g$ separately.

Lemma 4.8. *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i \in \mathbb{Z}_{\ge 0}$, $t \in \mathbb{R}$, and $R, S \in \mathbb{N}$ with $R \mid g$, $S \mid g$ and $(R, g/S) = 1$, we have*

$$\Psi_i(t, R, S)^2 \le \frac{1}{g/S} \left\lceil \frac{g}{RS} \right\rceil \cdot RS.$$

In particular, if $R \ge 2$ and $S \neq g$, we have

$$\Psi_i(t, R, S)^2 \le \frac{2}{3} RS.$$

Proof. By the Cauchy–Schwarz inequality, we have

$$\Psi_i(t, R, S)^2 \le \frac{1}{g^4} \sum_{0 \le r < R} \varphi_{i+1}\left(\frac{(g/S)t + (g/S)r}{R} \right)^2 \sum_{0 \le r < R} \left(\sum_{0 \le s < S} \varphi_i\left(\frac{t+r}{RS} + \frac{s}{S} \right) \right)^2.$$

By Lemma 4.6 with the assumption $(R, g/S) = 1$, we have

$$\Psi_i(t, R, S)^2 \le \frac{1}{g^2} \sum_{0 \le r < R} \left(\sum_{0 \le s < S} \varphi_i\left(\frac{t+r}{RS} + \frac{s}{S} \right) \right)^2.$$

By applying the Cauchy–Schwarz inequality once more, we get

$$\Psi_i(t, R, S)^2 \le \frac{S}{g^2} \sum_{0 \le r < R} \sum_{0 \le s < S} \varphi_i\left(\frac{t+r}{RS} + \frac{s}{S} \right)^2. \tag{4.9}$$

By expanding the square and using the orthogonality, we have

$$\begin{aligned} \sum_{0 \le s < S} \varphi_i\left(\frac{t+r}{RS} + \frac{s}{S} \right)^2 &= \sum_{0 \le m, n < g} e\left(\alpha_i(m) - \alpha_i(n) - \frac{t+r}{RS}(m-n) \right) \sum_{0 \le s < S} e\left(-\frac{s}{S}(m-n) \right) \\ &= S \sum_{\substack{0 \le m, n < g \\ m \equiv n \pmod{S}}} e\left(\alpha_i(m) - \alpha_i(n) - \frac{t+r}{RS}(m-n) \right). \end{aligned}$$

By (4.9), we thus have

$$\Psi_i(t, R, S)^2 \le \frac{S^2}{g^2} \sum_{0 \le r < R} \sum_{\substack{0 \le m, n < g \\ m \equiv n \pmod{S}}} e\left(\alpha_i(m) - \alpha_i(n) - \frac{t+r}{RS}(m-n) \right).$$

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By swapping the summation, we obtain

$$\Psi_i(t,R,S)^2 \le \frac{S^2}{g^2} \sum_{\substack{0\le m,n<g\\ m\equiv n \pmod S}} \left| \sum_{0\le r<R} e\left(-\frac{r}{RS}(m-n)\right)\right|.$$

By writing $m = a+\mu S$ and $n = a+\nu S$, we get

$$\begin{aligned}\Psi_i(t,R,S)^2 &\le \frac{S^2}{g^2} \sum_{0\le a<S} \sum_{0\le \mu,\nu<\frac{g}{S}} \left| \sum_{0\le r<R} e\left(-\frac{r}{RS}((a+\mu S)-(a+\nu S))\right)\right| \\ &= \frac{S^3}{g^2} \sum_{0\le \mu,\nu<\frac{g}{S}} \left| \sum_{0\le r<R} e\left(-\frac{r}{R}(\mu-\nu)\right)\right| = \frac{RS^3}{g^2} \sum_{\substack{0\le \mu,\nu<\frac{g}{S}\\ \mu\equiv\nu \pmod R}} 1.\end{aligned}$$

In the last sum, we can write $\ell R = \mu-\nu$ by the condition $\mu\equiv\nu \pmod R$ to obtain

$$\sum_{\substack{0\le \mu,\nu<\frac{g}{S}\\ \mu\equiv\nu \pmod R}} 1 = \sum_{|\ell|<\frac{g}{RS}} \sum_{\substack{0\le \mu,\nu<\frac{g}{S}\\ \mu-\nu=\ell R}} 1 = \frac{g}{S} \sum_{|\ell|<\frac{g}{RS}} \left(1-\frac{|\ell|}{(\frac{g}{RS})}\right).$$

We thus have

$$\Psi_i(t,R,S)^2 \le \frac{RS^2}{g} \sum_{|\ell|<\frac{g}{RS}} \left(1-\frac{|\ell|}{(\frac{g}{RS})}\right). \tag{4.10}$$

By noting that $\lceil \frac{g}{RS}\rceil \ge \frac{g}{RS} > 0$ and so $\lceil \frac{g}{RS}\rceil \ge 1$, we can estimate the last sum as

$$\begin{aligned}\sum_{|\ell|<\frac{g}{RS}} \left(1-\frac{|\ell|}{(\frac{g}{RS})}\right) &= 1+2 \sum_{1\le \ell\le \lceil \frac{g}{RS}\rceil -1} \left(1-\frac{\ell}{(\frac{g}{RS})}\right) \\ &= 1+2\left(\left\lceil \frac{g}{RS}\right\rceil -1\right) - \frac{1}{\frac{g}{RS}}\left\lceil \frac{g}{RS}\right\rceil \left(\left\lceil \frac{g}{RS}\right\rceil -1\right) \\ &\le 1+\left(\left\lceil \frac{g}{RS}\right\rceil -1\right) = \left\lceil \frac{g}{RS}\right\rceil .\end{aligned}$$

On inserting this estimate into (4.10), we obtain the first assertion.

For the latter assertion, by the assumption $S \mid g$, we have

$$\frac{1}{g/S}\left\lceil \frac{g}{RS}\right\rceil \le \frac{1}{g/S}\left(\frac{g}{RS}+1-\frac{1}{R}\right).$$

By assuming $R\ge 2$ and $S\ne g$, we have $R, g/S \ge 2$ since $S \mid g$. Also, since $(R, g/S) = 1$, we do not have $R = g/S = 2$. If $R\ge 3$ and $g/S\ge 2$, then we have

$$\frac{1}{g/S}\left\lceil \frac{g}{RS}\right\rceil \le \frac{1}{R}+\frac{1}{g/S}\left(1-\frac{1}{R}\right) \le \frac{1}{R}+\frac{1}{2}\left(1-\frac{1}{R}\right) = \frac{1}{2R}+\frac{1}{2} \le \frac{2}{3}.$$

Otherwise, we have $R = 2$ and $g/S\ge 3$ and so

$$\frac{1}{g/S}\left\lceil \frac{g}{RS}\right\rceil \le \frac{1}{R}+\frac{1}{g/S}\left(1-\frac{1}{R}\right) \le \frac{1}{R}+\frac{1}{3}\left(1-\frac{1}{R}\right) = \frac{2}{3R}+\frac{1}{3} = \frac{2}{3}.$$

On inserting these estimates into the first assertion, we obtain the lemma. □

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Lemma 4.9. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $i \in \mathbb{Z}_{\geq 0}$, $t \in \mathbb{R}$, and $R \in \mathbb{N}$ with $R \mid g$ and $R \geq 2$, we have*

$$\Psi_i(t,R,g)^2 \leq Rg(1-\Theta_i(\boldsymbol{\alpha})),$$

where

$$\Theta_i(\boldsymbol{\alpha}) := \left(1-\frac{1}{g}\right)\left\{1-\left(1-\frac{2}{g^2(g-1)}\sum_{1\leq h<g}\left|\sum_{0\leq n,n+h<g} e(\alpha_i(n+h)-\alpha_i(n))\right|^2\right)^{\frac{1}{2}}\right\}.$$

Also, for $\Theta_i(\boldsymbol{\alpha})$, we have

$$\Theta_g := \inf_{\substack{\boldsymbol{\alpha}\in\mathscr{A}_g \\ i\in\mathbb{Z}_{\geq 0}}} \Theta_i(\boldsymbol{\alpha}) \geq \left(1-\frac{1}{g}\right)\left\{1-\left(1-\frac{2}{g^2(g-1)}\right)^{\frac{1}{2}}\right\} > \frac{1}{g^3} > 0.$$

Proof. By putting $S = g$ in the definition (4.7), we have

$$\Psi_i(t,R,g) = \frac{1}{g^2}\sum_{0\leq r<R}\varphi_{i+1}\left(\frac{t+r}{R}\right)\sum_{0\leq s<g}\varphi_i\left(\frac{t+r}{gR}+\frac{s}{g}\right).$$

By using the Cauchy–Schwarz inequality and Lemma 4.6, we obtain

$$\Psi_i(t,R,g)^2 \leq \frac{1}{g^2}\sum_{0\leq r<R}\left(\sum_{0\leq s<g}\varphi_i\left(\frac{t+r}{gR}+\frac{s}{g}\right)\right)^2. \tag{4.11}$$

Since the function

$$\mathbb{Z}\to\mathbb{R};\ s\mapsto \varphi_i\left(\frac{t+r}{gR}+\frac{s}{g}\right)$$

has a period g, the sum over s can be thought of as a sum over residues $s \pmod g$. Thus, by (4.11),

$$\begin{aligned}
\Psi_i(t,R,g)^2 &\leq \frac{1}{g^2}\sum_{0\leq r<R}\sum_{0\leq k<g}\sum_{0\leq \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)\varphi_i\left(\frac{t+r}{gR}+\frac{\ell}{g}\right)\\
&= \frac{1}{g^2}\sum_{0\leq r<R}\sum_{0\leq k<g}\sum_{0\leq \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)\varphi_i\left(\frac{t+r}{gR}+\frac{k+\ell}{g}\right)\\
&= \frac{1}{g^2}\sum_{0\leq r<R}\sum_{0\leq k<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^2\\
&\quad+\frac{1}{g^2}\sum_{0\leq r<R}\sum_{0\leq k<g}\sum_{1\leq \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)\varphi_i\left(\frac{t+r}{gR}+\frac{k+\ell}{g}\right).
\end{aligned}$$

We can estimate the first sum by Lemma 4.6 to get

$$\Psi_i(t,R,g)^2 \leq R+\frac{1}{g^2}\sum_{0\leq r<R}\sum_{0\leq k<g}\sum_{1\leq \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)\varphi_i\left(\frac{t+r}{gR}+\frac{k+\ell}{g}\right).$$

By applying the Cauchy–Schwarz inequality, this gives

$$\Psi_i(t,R,g)^2 \leq R+\frac{1}{g^2}\left(Rg(g-1)\Upsilon_i(t,R,g)\right)^{\frac{1}{2}}, \tag{4.12}$$

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where

$$\Upsilon_i(t,R,g) := \sum_{0\le r<R}\sum_{0\le k<g}\sum_{1\le \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^2\varphi_i\left(\frac{t+r}{gR}+\frac{k+\ell}{g}\right)^2.$$

We can rewrite $\Upsilon_i(t,R,g)$ as

$$\begin{aligned}\Upsilon_i(t,R,g) = &\sum_{0\le r<R}\sum_{0\le k<g}\sum_{0\le \ell<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^2\varphi_i\left(\frac{t+r}{gR}+\frac{k+\ell}{g}\right)^2\\ &-\sum_{0\le r<R}\sum_{0\le k<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^4.\end{aligned}\tag{4.13}$$

For the first sum in (4.13), we use Lemma 4.6 to get

$$\Upsilon_i(t,R,g) \le Rg^4 - \sum_{0\le r<R}\sum_{0\le k<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^4.\tag{4.14}$$

The last sum can be rewritten as

$$\sum_{0\le r<R}\sum_{0\le k<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^4 = \sum_{0\le r<R}\sum_{0\le k<g}\varphi_i\left(\frac{t+r+Rk}{gR}\right)^4 = \sum_{0\le \ell<gR}\varphi_i\left(\frac{t+\ell}{gR}\right)^4.$$

By the assumption $R\ge 2$, we have $gR\ge 2g-1$ and so Lemma 4.7 implies

$$\begin{aligned}\sum_{0\le r<R}\sum_{0\le k<g}\varphi_i\left(\frac{t+r}{gR}+\frac{k}{g}\right)^4 &= Rg\sum_{|h|<g}\left|\sum_{0\le n,n+h<g}e(\alpha_i(n+h)-\alpha_i(n))\right|^2\\ &= Rg^3+2Rg\sum_{1\le h<g}\left|\sum_{0\le n,n+h<g}e(\alpha_i(n+h)-\alpha_i(n))\right|^2.\end{aligned}$$

On inserting this formula into (4.14), we get

$$\Upsilon_i(t,R,g)\le Rg^3(g-1)-2Rg\sum_{1\le h<g}\left|\sum_{0\le n,n+h<g}e(\alpha_i(n+h)-\alpha_i(n))\right|^2.$$

On inserting this estimate into (4.12), we have

$$\Psi_i(t,R,g)^2\le R+R(g-1)\left(1-\frac{2}{g^2(g-1)}\sum_{1\le h<g}\left|\sum_{0\le n,n+h<g}e(\alpha_i(n+h)-\alpha_i(n))\right|^2\right)^{\frac12}.$$

By rewriting the right-hand side, we obtain the first result.

For the estimate of $\Theta_i(\alpha)$, it suffices to pick up the contribution of the term with $h=g-1$, which implies $n=0$. The subsequent bound follows by $(1-x)^{\frac12}<1-\frac12 x$ valid for $|x|<1$. □

We then combine the above two bounds for $\Psi_i(t,R,S)$. Let us introduce

$$\eta_g := \max\left(\frac12-\frac{\log\frac32}{4\log g-2\log 2},\frac12+\frac{\log(1-\Theta_g)}{4\log g}\right).$$

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Since

$$\frac{1}{2}+\frac{\log(1-\Theta_g)}{4\log g}\le\frac{1}{2}-\frac{\Theta_g}{4\log g}<\frac{1}{2}-\frac{1}{4g^3\log g}$$

and

$$0.2075187<\frac{1}{2}-\frac{\log\frac{3}{2}}{2\log 2}\le\frac{1}{2}-\frac{\log\frac{3}{2}}{4\log g-2\log 2}<\frac{1}{2}-\frac{1}{16\log g}<\frac{1}{2}-\frac{1}{4g^3\log g},$$

we have

$$0.2075187<\eta_g<\frac{1}{2}-\frac{1}{4g^3\log g}. \tag{4.15}$$

Lemma 4.10. *For $g\in\mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha}\in\mathscr{A}_g$, $i\in\mathbb{Z}_{\ge 0}$, $t\in\mathbb{R}$, and $R,S\in\mathbb{N}$ with $R\mid g$, $S\mid g$, $(R,g/S)=1$, and $R\ge 2$, we have*

$$\Psi_i(t,R,S)\le(RS)^{\eta_g}.$$

Proof. Let us first consider the case $S\ne g$. Then, since $R\ge 2$, Lemma 4.8 gives

$$\Psi_i(t,R,S)\le(RS)^{\frac{1}{2}-\frac{\log\frac{3}{2}}{2\log RS}}.$$

Since $S\mid g$ and $S\ne g$ implies $S\le g/2$, we have

$$\log RS\le\log(g^2/2)=2\log g-\log 2$$

and so

$$\Psi_i(t,R,S)\le(RS)^{\frac{1}{2}-\frac{\log\frac{3}{2}}{4\log g-2\log 2}}\le(RS)^{\eta_g}.$$

Thus, the assertion holds if $S\ne g$.

When $S=g$, since $R\ge 2$, Lemma 4.9 gives

$$\Psi_i(t,R,S)\le(Rg)^{\frac{1}{2}+\frac{\log(1-\Theta_g)}{2\log Rg}}\le(Rg)^{\frac{1}{2}+\frac{\log(1-\Theta_g)}{4\log g}}\le(RS)^{\eta_g}.$$

Thus, the assertion also holds when $S=g$. □

We can now prove the following bound for an L^1-moment with a congruence condition.

Lemma 4.11. *For $g\in\mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha}\in\mathscr{A}_g$, $\lambda,\delta,j\in\mathbb{Z}_{\ge 0}$, $k\in\mathbb{N}$, $a\in\mathbb{Z}$, and $\beta\in\mathbb{R}$ satisfying*

$$0\le\delta\le\lambda,\quad kg^{\delta}\mid g^{\lambda},\quad g\nmid k,$$

we have

$$\sum_{\substack{0\le h<g^{\lambda}\\ h\equiv a\pmod{kg^{\delta}}}}\left|F_{\lambda}^{[j]}\left(\frac{h+\beta}{g^{\lambda}}\right)\right|\le g\left(\frac{g^{\lambda}}{kg^{\delta}}\right)^{\eta_g}\left|F_{\delta}^{[j+\lambda-\delta]}\left(\frac{a+\beta}{g^{\delta}}\right)\right|.$$

In particular, for $k=1$, $\delta=0$, we have

$$\sum_{0\le h<g^{\lambda}}\left|F_{\lambda}^{[j]}\left(\frac{h+\beta}{g^{\lambda}}\right)\right|\le g^{\eta_g\lambda+1}. \tag{4.16}$$

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Proof. When $\lambda = \delta$, then the condition $kg^{\delta} \mid g^{\lambda}$ implies $k = 1$. Then, using the periodicity of $F_{\lambda}^{[j]}(\frac{h+\beta}{g^{\lambda}})$, the left-hand side of the assertion is $|F_{\delta}^{[j]}(\frac{a+\beta}{g^{\delta}})|$, whereas the right-hand side of the assertion is $g|F_{\delta}^{[j]}(\frac{a+\beta}{g^{\delta}})|$, so that the inequality holds trivially. We may thus assume $\lambda > \delta$.

For integers θ with $\delta \le \theta \le \lambda$, we write

$$d_{\theta} := (g^{\theta}, kg^{\delta}) \quad \text{and} \quad u_{\theta} := \frac{g^{\theta}}{d_{\theta}}.$$

For $\delta < \theta \le \lambda$, since $d_{\theta-1} = (g^{\theta-1}, kg^{\delta}) \mid (g^{\theta}, kg^{\delta}) = d_{\theta}$, we also write

$$\rho_{\theta} := \frac{d_{\theta}}{d_{\theta-1}} \in \mathbb{N}.$$

Note that we then have

$$d_{\theta-1}(\rho_{\theta}, g) = (d_{\theta}, gd_{\theta-1}) = (g^{\theta}, kg^{\delta}, g^{\theta}, kg^{\delta+1}) = (g^{\theta}, kg^{\delta}) = d_{\theta} = \rho_{\theta} d_{\theta-1}$$

so that

$$\rho_{\theta} \mid g. \tag{4.17}$$

Also, we have

$$\rho_{\theta} = g \implies g^{\delta+1} \mid g \cdot g^{\delta}(g^{\theta-\delta-1}, k) = gd_{\theta-1} = \rho_{\theta} d_{\theta-1} = d_{\theta} = g^{\delta}(g^{\theta-\delta}, k)$$

and so $g \mid (g^{\theta-\delta}, k) \mid k$, which contradicts the assumption $g \nmid k$. Thus, we have

$$\rho_{\theta} < g. \tag{4.18}$$

Also, we have

$$d_{\theta-1}(\rho_{\theta}, u_{\theta-1}) = (d_{\theta}, g^{\theta-1}) = (g^{\theta}, kg^{\delta}, g^{\theta-1}) = (g^{\theta-1}, kg^{\delta}) = d_{\theta-1}$$

and so

$$(\rho_{\theta}, u_{\theta-1}) = 1. \tag{4.19}$$

We first prove a reduction formula for

$$G_{\theta}^{[j]}(a, d_{\theta}) := \sum_{\substack{0 \le h < g^{\theta} \\ h \equiv a \pmod{d_{\theta}}}} \left| F_{\theta}^{[j]}\left(\frac{h+\beta}{g^{\theta}}\right) \right|.$$

Assume $\delta < \theta \le \lambda$. By the periodicity, we can write

$$G_{\theta}^{[j]}(a, d_{\theta}) = \sum_{0 \le u < u_{\theta}} \left| F_{\theta}^{[j]}\left(\frac{a + ud_{\theta} + \beta}{g^{\theta}}\right) \right|.$$

Note that

$$\frac{u_{\theta}}{u_{\theta-1}} = \frac{g^{\theta}}{g^{\theta-1}} \frac{d_{\theta-1}}{d_{\theta}} = \frac{g}{\rho_{\theta}} \in \mathbb{N} \tag{4.20}$$

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since $\rho_\theta \mid g$ as in (4.17). We can thus change variables via $u = su_{\theta-1} + v$ to obtain

$$G_\theta^{[j]}(a, d_\theta) = \sum_{0 \le v < u_{\theta-1}} \sum_{0 \le s < g/\rho_\theta} \left| F_\theta^{[j]}\left(\frac{a + vd_\theta + su_{\theta-1}d_\theta + \beta}{g^\theta} \right) \right|.$$

Since $u_{\theta-1}d_\theta = g^{\theta-1}\rho_\theta$, we get

$$G_\theta^{[j]}(a, d_\theta) = \sum_{0 \le v < u_{\theta-1}} \sum_{0 \le s < g/\rho_\theta} \left| F_\theta^{[j]}\left(\frac{a + vd_\theta + sg^{\theta-1}\rho_\theta + \beta}{g^\theta} \right) \right|. \tag{4.21}$$

By using Lemma 4.2, periodicity, and the trivial bound $\varphi_j(x) \le g$ for $x \in \mathbb{R}$, we have

$$\begin{aligned} &\left| F_\theta^{[j]}\left(\frac{a + vd_\theta + sg^{\theta-1}\rho_\theta + \beta}{g^\theta} \right) \right| \\ &= \left| F_{\theta-1}^{[j+1]}\left(\frac{a + vd_\theta + sg^{\theta-1}\rho_\theta + \beta}{g^{\theta-1}} \right) \right| \cdot \frac{1}{g} \varphi_j\left(\frac{a + vd_\theta + sg^{\theta-1}\rho_\theta + \beta}{g^\theta} \right) \\ &\le \left| F_{\theta-1}^{[j+1]}\left(\frac{a + vd_\theta + \beta}{g^{\theta-1}} \right) \right|. \end{aligned}$$

In particular, we have the bound

$$G_\theta^{[j]}(a, d_\theta) \le \frac{g}{\rho_\theta} \sum_{0 \le v < u_{\theta-1}} \left| F_{\theta-1}^{[j+1]}\left(\frac{a + vd_\theta + \beta}{g^{\theta-1}} \right) \right| = \frac{g}{\rho_\theta} \sum_{0 \le v < u_{\theta-1}} \left| F_{\theta-1}^{[j+1]}\left(\frac{a + v\rho_\theta d_{\theta-1} + \beta}{g^{\theta-1}} \right) \right|.$$

Since $(\rho_\theta, u_{\theta-1}) = 1$ and the function

$$\mathbb{Z} \to \mathbb{R};\ v \mapsto \left| F_{\theta-1}^{[j+1]}\left(\frac{a + vd_{\theta-1} + \beta}{g^{\theta-1}} \right) \right|$$

has a period $u_{\theta-1}$, we can change variables to obtain

$$G_\theta^{[j]}(a, d_\theta) \le \frac{g}{\rho_\theta} \sum_{0 \le v < u_{\theta-1}} \left| F_{\theta-1}^{[j+1]}\left(\frac{a + vd_{\theta-1} + \beta}{g^{\theta-1}} \right) \right| = \frac{g}{\rho_\theta} G_{\theta-1}^{[j+1]}(a, d_{\theta-1}). \tag{4.22}$$

This is a trivial one-step reduction.

We next establish a two-step reduction assuming $\delta + 1 < \theta \le \lambda$. We change variables via

$$v = ru_{\theta-2} + u$$

in (4.21). This gives

$$G_\theta^{[j]}(a, d_\theta) = \sum_{0 \le u < u_{\theta-2}} \sum_{0 \le r < g/\rho_{\theta-1}} \sum_{0 \le s < g/\rho_\theta} \left| F_\theta^{[j]}\left(\frac{a + ud_\theta + ru_{\theta-2}d_\theta + sg^{\theta-1}\rho_\theta + \beta}{g^\theta} \right) \right|.$$

Since $u_{\theta-2}d_\theta = g^{\theta-2}\rho_{\theta-1}\rho_\theta$, by writing

$$R_\theta := \frac{g}{\rho_\theta},$$

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we get

$$G_\theta^{[j]}(a,d_\theta)=\sum_{0\le u<u_{\theta-2}}\sum_{0\le r<g/\rho_{\theta-1}}\sum_{0\le s<g/\rho_\theta}\left|F_\theta^{[j]}\left(\frac{a+ud_\theta+rg^{\theta-2}\rho_\theta\rho_{\theta-1}+sg^{\theta-1}\rho_\theta+\beta}{g^\theta}\right)\right|$$
$$=\sum_{0\le u<u_{\theta-2}}\sum_{0\le r<R_{\theta-1}}\sum_{0\le s<R_\theta}\left|F_\theta^{[j]}\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\right|.$$

By applying Lemma 4.2 twice and using periodicity together with $R_\theta \mid g$ and $R_{\theta-1}\mid g$, we have

$$\begin{aligned}
&\left|F_\theta^{[j]}\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\right|\\
&=\left|F_{\theta-1}^{[j+1]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-1}}+\frac{gr}{R_{\theta-1}R_\theta}+\frac{gs}{R_\theta}\right)\right|\cdot\frac{1}{g}\varphi_j\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\\
&=\left|F_{\theta-1}^{[j+1]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-1}}+\frac{gr}{R_{\theta-1}R_\theta}\right)\right|\cdot\frac{1}{g}\varphi_j\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\\
&=\left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-2}}+\frac{g^2r}{R_{\theta-1}R_\theta}\right)\right|\\
&\qquad\cdot\frac{1}{g^2}\varphi_{j+1}\left(\frac{a+ud_\theta+\beta}{g^{\theta-1}}+\frac{gr}{R_{\theta-1}R_\theta}\right)\varphi_j\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\\
&=\left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-2}}\right)\right|\\
&\qquad\cdot\frac{1}{g^2}\varphi_{j+1}\left(\frac{a+ud_\theta+\beta}{g^{\theta-1}}+\frac{gr}{R_{\theta-1}R_\theta}\right)\varphi_j\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right).
\end{aligned}$$

We thus have

$$\begin{aligned}
&G_\theta^{[j]}(a,d_\theta)\\
&\le\sum_{0\le u<u_{\theta-2}}\left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-2}}\right)\right|\\
&\qquad\cdot\frac{1}{g^2}\sum_{0\le r<R_{\theta-1}}\sum_{0\le s<R_\theta}\varphi_{j+1}\left(\frac{a+ud_\theta+\beta}{g^{\theta-1}}+\frac{gr}{R_{\theta-1}R_\theta}\right)\\
&\qquad\qquad\cdot\varphi_j\left(\frac{a+ud_\theta+\beta}{g^\theta}+\frac{r}{R_{\theta-1}R_\theta}+\frac{s}{R_\theta}\right)\\
&=\sum_{0\le u<u_{\theta-2}}\left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-2}}\right)\right|\Psi_j(t_\theta,R_{\theta-1},R_\theta)
\end{aligned}\tag{4.23}$$

with

$$t_\theta:=\frac{(a+ud_\theta+\beta)R_{\theta-1}R_\theta}{g^\theta}.$$

We then use Lemma 4.10 with $i:=j$, $t:=t_\theta$, $R:=R_{\theta-1}$, and $S:=R_\theta$. We then have $R\mid g$ and $S\mid g$. By (4.18) and $\delta+1<\theta\le\lambda$, we know that $R\ge 2$. Also, since

$$(gd_{\theta-2},d_\theta)=(g^{\theta-1},kg^{\delta+1},g^\theta,kg^\delta)=(g^{\theta-1},kg^\delta)=d_{\theta-1},$$

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we have

$$(R, g/S) = (R_{\theta-1}, g/R_\theta) = (g/\rho_{\theta-1}, \rho_\theta) = (gd_{\theta-2}, d_\theta)/d_{\theta-1} = 1.$$

Thus, by using Lemma 4.10 in (4.23), we have

$$\begin{aligned} G_\theta^{[j]}(a, d_\theta) &\le (R_{\theta-1}R_\theta)^{\eta_g} \sum_{0\le u<u_{\theta-2}} \left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_\theta+\beta}{g^{\theta-2}}\right)\right| \\ &= \left(\frac{g^2}{\rho_{\theta-1}\rho_\theta}\right)^{\eta_g} \sum_{0\le u<u_{\theta-2}} \left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_{\theta-2}\rho_{\theta-1}\rho_\theta+\beta}{g^{\theta-2}}\right)\right|. \end{aligned}$$

Since (4.19) and (4.20) imply $(\rho_\theta\rho_{\theta-1}, u_{\theta-2}) = 1$ and the function

$$\mathbb{Z} \to \mathbb{R};\ u \mapsto \left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_{\theta-2}+\beta}{g^{\theta-2}}\right)\right|$$

has a period $u_{\theta-2}$, we can change variables to get

$$\begin{aligned} G_\theta^{[j]}(a, d_\theta) &\le \left(\frac{g^2}{\rho_{\theta-1}\rho_\theta}\right)^{\eta_g} \sum_{0\le u<u_{\theta-2}} \left|F_{\theta-2}^{[j+2]}\left(\frac{a+ud_{\theta-2}+\beta}{g^{\theta-2}}\right)\right| \\ &= \left(\frac{g^2}{\rho_{\theta-1}\rho_\theta}\right)^{\eta_g} G_{\theta-2}^{[j+2]}(a, d_{\theta-2}). \end{aligned} \tag{4.24}$$

This is a two-step reduction.

Finally, by using (4.24) $[(\lambda-\delta)/2]$ times and by using (4.22) if $\lambda-\delta$ is odd, we obtain

$$\sum_{\substack{0\le h<g^\lambda \\ h\equiv a \pmod{kg^\delta}}} \left|F_\lambda^{[j]}\left(\frac{h+\beta}{g^\lambda}\right)\right| = G_\lambda^{[j]}(a, d_\lambda) \le \cdots \le (\rho_\lambda\cdots\rho_{\delta+1})^{-\eta_g} g^{2[(\lambda-\delta)/2]\eta_g+1} G_\delta^{[j+\lambda-\delta]}(a, d_\delta).$$

Here, we used $0 < \eta_g < 1$, which follows from (4.15). Since $\rho_\lambda\cdots\rho_{\delta+1} = \frac{d_\lambda}{d_\delta} = \frac{kg^\delta}{g^\delta} = k$ and

$$G_\delta^{[j+\lambda-\delta]}(a, d_\delta) = G_\delta^{[j+\lambda-\delta]}(a, g^\delta) = \sum_{\substack{0\le h<g^\delta \\ h\equiv a \pmod{g^\delta}}} \left|F_\delta^{[j+\lambda-\delta]}\left(\frac{h+\beta}{g^\delta}\right)\right| = \left|F_\delta^{[j+\lambda-\delta]}\left(\frac{a+\beta}{g^\delta}\right)\right|,$$

we obtain the assertion. □

4.4. A hybrid bound

We finally prove a hybrid bound by combining the L^∞ and L^1 bounds. We first deduce continuous analogs of the discrete bounds obtained above.

Lemma 4.12. *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, and $\lambda, j \in \mathbb{Z}_{\ge 0}$, we have*

$$\int_0^1 |F_\lambda^{[j]}(\beta)|d\beta \ll g^{-(1-\eta_g)\lambda},$$

where the implicit constant depends only on g.

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Proof. By decomposing the integral, we have

$$\int_0^1 |F_\lambda^{[j]}(\beta)|d\beta = \sum_{0\le h<g^\lambda} \int_{\frac{h}{g^\lambda}}^{\frac{h+1}{g^\lambda}} |F_\lambda^{[j]}(\beta)|d\beta$$
$$= \sum_{0\le h<g^\lambda} \int_0^{\frac{1}{g^\lambda}} \left|F_\lambda^{[j]}\left(\frac{h+\beta g^\lambda}{g^\lambda}\right)\right| d\beta = \int_0^{\frac{1}{g^\lambda}} \left(\sum_{0\le h<g^\lambda} \left|F_\lambda^{[j]}\left(\frac{h+\beta g^\lambda}{g^\lambda}\right)\right|\right) d\beta.$$

Then, Lemma 4.11 gives the result. □

Lemma 4.13. *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, and $\lambda, j \in \mathbb{Z}_{\ge 0}$, we have*

$$\int_0^1 \left|\frac{\partial}{\partial\beta} F_\lambda^{[j]}(\beta)\right| d\beta \ll g^{\eta_g \lambda},$$

where the implicit constant depends only on g.

Proof. By (4.1), we have

$$\frac{\partial}{\partial\beta} F_\lambda^{[j]}(\beta) = -\frac{2\pi i}{g^\lambda} \sum_{0\le n<g^\lambda} n \cdot e(f_\lambda^{[j]}(n) - \beta n).$$

By partial summation and Lemma 3.2, we get

$$\left|\frac{\partial}{\partial\beta} F_\lambda^{[j]}(\beta)\right| \ll \sup_{0\le x\le g^\lambda} \left|\sum_{0\le n<x} e(f_\lambda^{[j]}(n) - \beta n)\right| \ll \sum_{0\le \ell\le \lambda} g^\ell |F_\ell^{[j]}(\beta)|.$$

Thus, by Lemma 4.12, we get

$$\int_0^1 \left|\frac{\partial}{\partial\beta} F_\lambda^{[j]}(\beta)\right| d\beta \ll \sum_{0\le \ell\le \lambda} g^\ell \int_0^1 |F_\ell^{[j]}(\beta)|d\beta \ll \sum_{0\le \ell\le \lambda} g^{\eta_g \ell} \ll g^{\eta_g \lambda},$$

where, in the last step, we used $\eta_g > 0$, which follows from (4.15). □

Lemma 4.14 (Gallagher–Sobolev inequality). *Let $f\colon [0,1] \to \mathbb{C}$ be a function of class C^1 with a period 1, let $\delta \in (0,1]$, and let $(\alpha_i)_{i=1}^R$ be a sequence of real numbers which is δ-spaced* (mod 1)*, that is,*

$$\|\alpha_i - \alpha_j\| \ge \delta \quad \text{for any } i, j \in \{1, \dots, R\} \text{ with } i \ne j.$$

We then have

$$\sum_{i=1}^R |f(\alpha_i)| \le \frac{1}{\delta}\int_0^1 |f(\alpha)|d\alpha + \frac{1}{2}\int_0^1 |f'(\alpha)|d\alpha.$$

Proof. See Lemma 1.2 of [11, p. 2]. □

The next lemma gives a hybrid bound combining the L^∞ and L^1 bounds. For the L^1 part, we use the Gallagher–Sobolev inequality as in Gallagher's proof [5, Section 2] of the large sieve inequality.

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Lemma 4.15. *Let $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $\lambda, j \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{R}_{\geq 1}$, and $\mu := [\frac{\log M}{\log g}]$. Then we have*

$$\sum_{M \leq m \leq 2M} \sum_{\substack{0 \leq k < m \\ (k,m)=1}} \left| F_\lambda^{[j]}\left(\frac{k}{m}\right) \right| \ll \begin{cases} M g^{-(\frac{1}{2}-\eta_g)2\mu - \sigma_{\lambda-2\mu}^{[j+2\mu]}(\boldsymbol{\alpha})} & \text{if } 2\mu \leq \lambda, \\ M^2 g^{-(1-\eta_g)\lambda} & \text{if } 2\mu > \lambda, \end{cases}$$

where the implicit constant depends only on g.

Proof. Take a parameter $\lambda_1 \in [0, \lambda] \cap \mathbb{Z}$ to be chosen later. By Lemma 4.1, we have

$$\begin{aligned} \left| F_\lambda^{[j]}\left(\frac{k}{m}\right) \right| &= \frac{1}{g^{\lambda_1}} \prod_{0 \leq i < \lambda_1} \varphi_i^{[j]}\left(\frac{k}{m} g^i\right) \cdot \frac{1}{g^{\lambda-\lambda_1}} \prod_{\lambda_1 \leq i < \lambda} \varphi_i^{[j]}\left(\frac{k}{m} g^i\right) \\ &= \frac{1}{g^{\lambda_1}} \prod_{0 \leq i < \lambda_1} \varphi_i^{[j]}\left(\frac{k}{m} g^i\right) \cdot \frac{1}{g^{\lambda-\lambda_1}} \prod_{0 \leq i < \lambda-\lambda_1} \varphi_i^{[j+\lambda_1]}\left(\frac{k}{m} g^{\lambda_1} \cdot g^i\right) \\ &= \left| F_{\lambda_1}^{[j]}\left(\frac{k}{m}\right) \right| \left| F_{\lambda-\lambda_1}^{[j+\lambda_1]}\left(\frac{k}{m} g^{\lambda_1}\right) \right|. \end{aligned}$$

By using Lemma 4.5, we thus have

$$S := \sum_{M \leq m \leq 2M} \sum_{\substack{0 \leq k < m \\ (k,m)=1}} \left| F_\lambda^{[j]}\left(\frac{k}{m}\right) \right| \ll g^{-\sigma_{\lambda-\lambda_1}^{[j+\lambda_1]}(\boldsymbol{\alpha})} \sum_{M \leq m \leq 2M} \sum_{\substack{0 \leq k < m \\ (k,m)=1}} \left| F_{\lambda_1}^{[j]}\left(\frac{k}{m}\right) \right|.$$

Since the reduced fractions

$$\frac{k}{m} \quad \text{for } k, m \in \mathbb{Z} \text{ with } M \leq m \leq 2M, 0 \leq k < m, (k,m) = 1$$

are $(2M)^{-2}$-spaced, the Gallagher–Sobolev inequality (Lemma 4.14) gives

$$S \ll g^{-\sigma_{\lambda-\lambda_1}^{[j+\lambda_1]}(\boldsymbol{\alpha})} \left(M^2 \int_0^1 |F_{\lambda_1}^{[j]}(\beta)| d\beta + \int_0^1 \left| \frac{\partial}{\partial \beta} F_{\lambda_1}^{[j]}(\beta) \right| d\beta \right).$$

By using Lemmas 4.12 and 4.13, we further have

$$S \ll g^{\eta_g \lambda_1 - \sigma_{\lambda-\lambda_1}^{[j+\lambda_1]}(\boldsymbol{\alpha})} (M^2 g^{-\lambda_1} + 1). \tag{4.25}$$

When $2\mu \leq \lambda$, by taking $\lambda_1 := 2\mu \in [0, \lambda] \cap \mathbb{Z}$ in (4.25), so that

$$g^{\lambda_1} \asymp M^2 \quad \text{and} \quad g^{\eta_g \lambda_1} = g^{\frac{1}{2}\lambda_1} \cdot g^{-(\frac{1}{2}-\eta_g)\lambda_1} \asymp M g^{-(\frac{1}{2}-\eta_g)2\mu},$$

we obtain

$$S \ll M g^{-(\frac{1}{2}-\eta_g)2\mu - \sigma_{\lambda-2\mu}^{[j+2\mu]}(\boldsymbol{\alpha})}. \tag{4.26}$$

When $\lambda < 2\mu$, we have $M^2 g^{-\lambda} \gg 1$ and so, by taking $\lambda_1 := \lambda$ in (4.25), we obtain

$$S \ll M^2 g^{-(1-\eta_g)\lambda}. \tag{4.27}$$

By combining (4.26) and (4.27), we obtain the result. □

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5. The Type I sum estimate

As usual, we need to estimate bilinear sums called the Type I and Type II sums. In this section, we prepare the Type I sum estimate for weakly digital functions.

We start with a small lemma on $\sigma_\lambda^{[j]}(\boldsymbol{\alpha})$.

Lemma 5.1. *Let $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$.*

(i) *For $\lambda \in \mathbb{N}$, $j \in \mathbb{Z}_{\geq 0}$, we have*

$$\sigma_\lambda^{[j]}(\boldsymbol{\alpha}) - \frac{\log 2}{4g^3(\log g)^2} \leq \sigma_{\lambda-1}^{[j+1]}(\boldsymbol{\alpha}) \leq \sigma_\lambda^{[j]}(\boldsymbol{\alpha}).$$

(ii) *For $\lambda, j \in \mathbb{Z}_{\geq 0}$ and $A \geq \frac{\log 2}{\log g}$, the function*

$$f\colon [0,\lambda] \cap \mathbb{Z} \to \mathbb{R};\ \mu \mapsto A\left(\frac{1}{2} - \eta_g\right)\mu + \sigma_{\lambda-\mu}^{[j+\mu]}(\boldsymbol{\alpha})$$

is increasing.

(iii) *For $j \in \mathbb{Z}_{\geq 0}$ and $A \geq \frac{\log 2}{4g^3(\log g)^2}$, the function*

$$f\colon \mathbb{Z}_{\geq 0} \to \mathbb{R};\ \lambda \mapsto A\lambda - \sigma_\lambda^{[j]}(\boldsymbol{\alpha})$$

is nondecreasing.

Proof.

(i). By the definition (4.5) of $\sigma_\lambda^{[j]}(\boldsymbol{\alpha})$, we have

$$\sigma_{\lambda-1}^{[j+1]}(\boldsymbol{\alpha}) = \sum_{0\leq i<\lambda-1} \gamma_i^{[j+1]}(\boldsymbol{\alpha}) = \sum_{0\leq i<\lambda-1} \gamma_{i+1}^{[j]}(\boldsymbol{\alpha}) = \sum_{1\leq i<\lambda} \gamma_i^{[j]}(\boldsymbol{\alpha}) = \sigma_\lambda^{[j]}(\boldsymbol{\alpha}) - \gamma_0^{[j]}(\boldsymbol{\alpha}).$$

By using (4.4), we obtain (i).

(ii). By (i), proven above, for $\mu \in [0, \lambda-1] \cap \mathbb{Z}$, we have

$$\begin{aligned} f(\mu+1) - f(\mu) &= A\left(\frac{1}{2} - \eta_g\right) + \sigma_{\lambda-(\mu+1)}^{[j+\mu+1]}(\boldsymbol{\alpha}) - \sigma_{\lambda-\mu}^{[j+\mu]}(\boldsymbol{\alpha}) \\ &\geq \frac{\log 2}{\log g}\left(\frac{1}{2} - \eta_g\right) - \frac{\log 2}{4g^3(\log g)^2} > 0 \end{aligned}$$

by (4.15). This proves (ii).

(iii). By (4.4) and (4.5), we have

$$f(\lambda+1) - f(\lambda) = A - \gamma_\lambda^{[j]}(\boldsymbol{\alpha}) \geq 0,$$

and so (iii) follows. □

Lemma 5.2. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $L \in \mathbb{N}$, and $x, M \in \mathbb{R}_{\geq 1}$ with $2 \leq x \leq g^L$, we have*

$$S_{\mathrm{I}} := \sum_{1\leq m\leq M} \sup_{1\leq t\leq x/m} \left| \sum_{1\leq n\leq t} e(f_L(mn)) \right| \ll x g^{-\kappa_{\mathrm{I}}} (\log x)^2$$

provided $M \leq x^{\frac{1}{2}}$. Here,

$$\kappa_{\mathrm{I}} := \sigma_{\xi_{\mathrm{I}}}(\boldsymbol{\alpha}) \quad \textit{with} \quad \xi_{\mathrm{I}} := \left[\frac{\log x}{\log g}\right]$$

and the implicit constant depends only on g.

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Proof. By rewriting mn as n, mt as t, and taking care of the terms with $n = 0$ and $n = t$, we have

$$S_{\mathrm{I}} \ll \sum_{1\le m\le M} \sup_{m\le t\le x} \Bigg| \sum_{\substack{0\le n<t \\ n\equiv 0 \pmod m}} e(f_L(n)) \Bigg| + M.$$

By the orthogonality, we have

$$\begin{aligned} S_{\mathrm{I}} &\ll \sum_{1\le m\le M} \sup_{m\le t\le x} \left| \frac{1}{m} \sum_{0\le k<m} \sum_{0\le n<t} e\left(f_L(n) - \frac{kn}{m}\right) \right| + M \\ &\ll \sum_{1\le m\le M} \frac{1}{m} \sum_{0\le k<m} \sup_{m\le t\le x} \left| \sum_{0\le n<t} e\left(f_L(n) - \frac{kn}{m}\right) \right| + M. \end{aligned} \tag{5.1}$$

We may replace x by $[x]$ in (5.1), and then, by applying Lemma 3.2, we can bound (5.1) as

$$\begin{aligned} S_{\mathrm{I}} &\ll \sum_{0\le\lambda\le \frac{\log x}{\log g}} \sum_{1\le m\le M} \frac{1}{m} \sum_{0\le k<m} \left| \sum_{0\le n<g^\lambda} e\left(f_\lambda(n) - \frac{kn}{m}\right) \right| + M \\ &= \sum_{0\le\lambda\le \frac{\log x}{\log g}} \sum_{1\le m\le M} \frac{g^\lambda}{m} \sum_{0\le k<m} \left| F_\lambda\left(\frac{k}{m}\right) \right| + M. \end{aligned}$$

By classifying the terms according to the values of $(k, m) = d$, we have

$$\begin{aligned} S_{\mathrm{I}} &\ll \sum_{0\le\lambda\le \frac{\log x}{\log g}} \sum_{1\le d\le M} \sum_{\substack{1\le m\le M \\ d\mid m}} \frac{g^\lambda}{m} \sum_{\substack{0\le k<m \\ (k,m)=d}} \left| F_\lambda\left(\frac{k}{m}\right) \right| + M \\ &= \sum_{0\le\lambda\le \frac{\log x}{\log g}} \sum_{1\le d\le M} \frac{1}{d} \sum_{1\le m\le M/d} \frac{g^\lambda}{m} \sum_{\substack{0\le k<m \\ (k,m)=1}} \left| F_\lambda\left(\frac{k}{m}\right) \right| + M, \end{aligned} \tag{5.2}$$

where, in the last equality, we made the change of variables $k \rightsquigarrow dk$ and $m \rightsquigarrow dm$.

In order to bound the inner sum of (5.2), we bound the sum

$$T := \sum_{U\le m\le 2U} \frac{g^\lambda}{m} \sum_{\substack{0\le k<m \\ (k,m)=1}} \left| F_\lambda\left(\frac{k}{m}\right) \right|$$

for $1 \le U \le M/d$ with $1 \le d \le M$ by using Lemma 4.15. Write

$$\mu_U := \left[\frac{\log U}{\log g} \right].$$

When $2\mu_U \le \lambda$, Lemma 4.15 gives

$$T \ll g^\lambda U^{-1} \sum_{U\le m\le 2U} \sum_{\substack{0\le k<m \\ (k,m)=1}} \left| F_\lambda\left(\frac{k}{m}\right) \right| \ll g^{\lambda-(\frac{1}{2}-\eta_g)2\mu_U - \sigma^{[2\mu_U]}_{\lambda-2\mu_U}(\alpha)}.$$

By (ii) and (iii) of Lemma 5.1, we have

$$T \ll g^{\frac{1}{2}\lambda} g^{\frac{1}{2}\lambda - \sigma_\lambda(\alpha)} \le g^{\frac{1}{2}\lambda} g^{\frac{1}{2}\xi_1 - \sigma_{\xi_1}(\alpha)} \ll g^{\frac{1}{2}\lambda} x^{\frac{1}{2}} g^{-\kappa_1}. \tag{5.3}$$

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When $0 \le \lambda < 2\mu_U$, Lemma 4.15 gives

$$T \ll g^{\lambda} U^{-1} \sum_{U \le m \le 2U} \sum_{\substack{0 \le k < m \\ (k,m)=1}} \left| F_\lambda\left(\frac{k}{m}\right) \right| \ll U g^{\eta_g \lambda} \ll \frac{M}{d} g^{\eta_g \lambda}. \tag{5.4}$$

By (5.3) and (5.4), in any case, we have

$$T \ll g^{\frac{1}{2}\lambda} x^{\frac{1}{2}} g^{-\kappa_{\mathrm{I}}} + \frac{M}{d} g^{\eta_g \lambda}. \tag{5.5}$$

Inserting (5.5) into (5.2) and using $\eta_g > 0$ as seen in (4.15), we get

$$S_{\mathrm{I}} \ll (\log x)^2 \sum_{0 \le \lambda \le \frac{\log x}{\log g}} (g^{\frac{1}{2}\lambda} x^{\frac{1}{2}} g^{-\kappa_{\mathrm{I}}} + M g^{\eta_g \lambda}) + M \ll (x g^{-\kappa_{\mathrm{I}}} + M x^{\eta_g})(\log x)^2. \tag{5.6}$$

By using (4.6) and (4.15), we have

$$\kappa_{\mathrm{I}} \le \frac{\xi_{\mathrm{I}}}{4g^3 \log g} \le \frac{\log x}{\log g}\left(\frac{1}{2} - \eta_g\right).$$

Thus, the assumption $M \le x^{\frac{1}{2}}$ gives

$$M x^{\eta_g} \le x^{\frac{1}{2}+\eta_g} = x^{1-(\frac{1}{2}-\eta_g)} \le x g^{-\kappa_{\mathrm{I}}}.$$

On inserting this estimate into (5.6), we obtain the result. □

6. The Type II sum estimate

For the Type II sum estimate, we first recall some known lemmas.

Lemma 6.1 (Van der Corput's inequality). *For a sequence of complex numbers $(z_n)_{n=1}^N$ and $R \in \mathbb{N}$,*

$$\left| \sum_{n=1}^{N} z_n \right|^2 \le \frac{N+R-1}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \sum_{1 \le n, n+r \le N} z_{n+r} \overline{z_n}.$$

Proof. See, for example, Montgomery–Vaughan [12, Lemma 16.8, p. 10]. □

Lemma 6.2. *Let $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $L \in \mathbb{N}$, $r \in \mathbb{Z}_{\ge 0}$, and $M, N, R \in \mathbb{R}_{\ge 1}$ with $0 \le r \le R$. Let λ be the unique positive integer satisfying*

$$g^{\lambda-1} \le MR^2 < g^{\lambda}. \tag{6.1}$$

Assume that

$$R \le N^{\frac{1}{2}} \quad \text{and} \quad \lambda \le L. \tag{6.2}$$

Consider the set

$$\mathscr{E} := \left\{ (m,n) \in \mathbb{N}^2 \;\middle|\; \begin{array}{c} M < m \le 2M,\ N < n \le 2N \\ f_L(m(n+r)) - f_L(mn) \ne f_\lambda(m(n+r)) - f_\lambda(mn) \end{array} \right\}.$$

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Then we have the bound

$$\#\mathscr{E} \ll \frac{MN}{R},$$

where the implicit constant depends only on g.

Proof. For $(m,n) \in \mathbb{N}^2$, if there is no $k \in \mathbb{N}$ with

$$mn < kg^{\lambda} \le m(n+r), \tag{6.3}$$

that is, if $[\frac{mn}{g^{\lambda}}] = [\frac{m(n+r)}{g^{\lambda}}]$, then we have

$$\varepsilon_i(m(n+r)) = \varepsilon_i(mn) \quad \text{for } i \ge \lambda$$

since

$$\sum_{i\ge\lambda} \varepsilon_i(m(n+r))g^{i-\lambda} = \left[\frac{m(n+r)}{g^{\lambda}}\right] = \left[\frac{mn}{g^{\lambda}}\right] = \sum_{i\ge\lambda} \varepsilon_i(mn)g^{i-\lambda}.$$

Therefore, if there is no $k \in \mathbb{N}$ with (6.3), then since $\lambda \le L$, we have

$$\begin{aligned} f_L(m(n+r)) - f_L(mn) &= \sum_{0\le i<L} \big(\alpha_i(\varepsilon_i(m(n+r))) - \alpha_i(\varepsilon_i(mn))\big) \\ &= \sum_{0\le i<\lambda} \big(\alpha_i(\varepsilon_i(m(n+r))) - \alpha_i(\varepsilon_i(mn))\big) = f_{\lambda}(m(n+r)) - f_{\lambda}(mn). \end{aligned}$$

Thus, if $(m,n) \in \mathscr{E}$, there exists $k \in \mathbb{N}$ satisfying (6.3). Since such k must satisfy

$$mN < kg^{\lambda} \le m(2N+r),$$

we have

$$\begin{aligned} \#\mathscr{E} &\le \sum_{M<m\le 2M} \sum_{\frac{mN}{g^{\lambda}}<k\le\frac{m(2N+r)}{g^{\lambda}}} \sum_{\substack{N<n\le 2N \\ mn<kg^{\lambda}\le m(n+r)}} 1 \\ &\le \sum_{M<m\le 2M} \sum_{\frac{mN}{g^{\lambda}}<k\le\frac{m(2N+r)}{g^{\lambda}}} \sum_{\frac{kg^{\lambda}}{m}-r\le n<\frac{kg^{\lambda}}{m}} 1 \\ &\ll r \sum_{M<m\le 2M} \left(\frac{m(N+r)}{g^{\lambda}} + 1\right) \ll MR\left(\frac{M(N+R)}{g^{\lambda}} + 1\right). \end{aligned} \tag{6.4}$$

By (6.1) and (6.2), we have $g^{\lambda} \asymp MR^2 \ll MN$. Thus, we can bound (6.4) further as

$$\#\mathscr{E} \ll \frac{M^2NR}{g^{\lambda}} \ll \frac{MN}{R}$$

as desired. This completes the proof. □

Lemma 6.3. *Let $a \in \mathbb{Z}$, $m \in \mathbb{N}$, $b \in \mathbb{R}$, and $d := (a,m)$. For $M \in \mathbb{R}_{>0}$, we have*

$$\sum_{0\le n<m} \min\left(M, \frac{1}{|\sin(\pi\frac{an+b}{m})|}\right) \le d\min\left(M, \frac{1}{\sin(\pi\frac{d}{m}\|\frac{b}{d}\|)}\right) + \frac{d}{\sin(\frac{\pi d}{2m})} + \frac{2m}{\pi}\log\frac{2m}{\pi d}.$$

Proof. See Lemma 6 of [7, p. 1609]. □

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We can now prove the following Type II sum estimate.

Lemma 6.4. *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $L \in \mathbb{N}$, and $x, M, N, \theta \in \mathbb{R}$ with $M, N \geq 1$, $\theta > 0$, $2 \leq x \leq g^L$ and complex coefficients $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ with $|a_m|, |b_n| \leq 1$ for any m and n, we have*

$$S_{\mathrm{II}} := \sum_{\substack{M<m\leq 2M\\ N<n\leq 2N\\ mn\leq x}} a_m b_n e(f_L(mn)) \ll x g^{-\kappa_{\mathrm{II}}} \log x,$$

provided

$$M, N \geq x^{\theta}, \tag{6.5}$$

where

$$\kappa_{\mathrm{II}} := \frac{1}{10}\sigma_{\xi_{\mathrm{II}}}(\boldsymbol{\alpha}) \quad \textit{with} \quad \xi_{\mathrm{II}} := \left[\theta \cdot \frac{\log x}{\log g}\right]$$

and the implicit constant depends only on g.

Proof. We may assume $MN < x$ since otherwise S_{II} is an empty sum and the assertion is trivial. By symmetry, we may assume $M \leq N$. Since $MN < x$, we then have $M \leq x^{\frac{1}{2}}$. Take $\mu \in \mathbb{N}$ with

$$g^{\mu-1} \leq M < g^{\mu}. \tag{6.6}$$

By (6.5) and $M \leq x^{\frac{1}{2}}$, we have

$$\theta \cdot \frac{\log x}{\log g} < \mu \leq \frac{1}{2}\frac{\log x}{\log g} + 1. \tag{6.7}$$

By the Cauchy–Schwarz inequality, we have

$$|S_{\mathrm{II}}|^2 \leq M \sum_{M<m\leq 2M} \left| \sum_{\substack{N<n\leq 2N\\ mn\leq x}} b_n e(f_L(mn)) \right|^2. \tag{6.8}$$

Take a real parameter

$$R \in [1, N^{\frac{1}{2}}]. \tag{6.9}$$

By applying Lemma 6.1 to (6.8) with R replaced by $[R]+1$, we have

$$\begin{aligned} |S_{\mathrm{II}}|^2 &\ll \frac{M(N+[R])}{[R]+1} \sum_{M<m\leq 2M} \sum_{|r|\leq R} \left(1 - \frac{|r|}{[R]+1}\right) \\ &\qquad \cdot \sum_{\substack{N<n,n+r\leq 2N\\ mn,m(n+r)\leq x}} b_{n+r}\overline{b_n} e\big(f_L(m(n+r)) - f_L(mn)\big) \\ &\ll \frac{MN}{R} \sum_{|r|\leq R} \sum_{N<n,n+r\leq 2N} \left| \sum_{\substack{M<m\leq 2M\\ mn,m(n+r)\leq x}} e\big(f_L(m(n+r)) - f_L(mn)\big) \right|. \end{aligned}$$

Note that the contribution from the terms with $r < 0$ is the same as that from the terms with $r > 0$. This can be seen by changing variables via $n \rightsquigarrow n - r$ and taking the complex conjugate inside the absolute

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value sign. Thus, separating off the contribution of $r = 0$ and enlarging the range of n by dropping the condition $n + r \le 2N$, we have

$$|S_{\mathrm{II}}|^2 \ll \frac{MN}{R} \sum_{1 \le r \le R} \sum_{N < n \le 2N} \Bigg| \sum_{\substack{M < m \le 2M \\ m(n+r) \le x}} e\big(f_L(m(n+r)) - f_L(mn)\big) \Bigg| + \frac{M^2N^2}{R}.$$

We now use Lemma 6.2 to truncate f_L. By (6.9), the condition $R \le N^{\frac{1}{2}}$ is satisfied. Take $\lambda \in \mathbb{N}$ by

$$g^{\lambda-1} \le MR^2 < g^{\lambda} \tag{6.10}$$

as in Lemma 6.2. Since we are assuming $MN < x$, by using $x \le g^L$ and $R \le N^{\frac{1}{2}}$, we have

$$g^{\lambda-1} \le MR^2 \le MN < x \le g^L$$

and so the condition $\lambda \le L$ is also satisfied. Thus, we can use Lemma 6.2 to truncate f_L as

$$\begin{aligned} |S_{\mathrm{II}}|^2 &\ll \frac{MN}{R} \sum_{1 \le r \le R} \sum_{N < n \le 2N} \Bigg| \sum_{\substack{M < m \le 2M \\ m(n+r) \le x}} e\big(f_\lambda(m(n+r)) - f_\lambda(mn)\big) \Bigg| + \frac{M^2N^2}{R} \\ &\ll MN \sup_{1 \le r \le R} S_{\mathrm{II}}(r) + \frac{M^2N^2}{R}, \end{aligned} \tag{6.11}$$

where the sum $S_{\mathrm{II}}(r)$ is defined by

$$S_{\mathrm{II}}(r) := \sum_{N < n \le 2N} |T(r,n)| \quad \text{with} \quad T(r,n) := \sum_{\substack{M < m \le 2M \\ m(n+r) \le x}} e\big(f_\lambda(m(n+r)) - f_\lambda(mn)\big). \tag{6.12}$$

Since f_λ has a period g^λ, by using the exponential sum (4.1), we can write

$$\begin{aligned} T(r,n) &= \sum_{0 \le u < g^\lambda} e(f_\lambda(u)) \sum_{0 \le v < g^\lambda} e(-f_\lambda(v)) \sum_{\substack{M < m \le 2M \\ m(n+r) \le x \\ m(n+r) \equiv u \pmod{g^\lambda} \\ mn \equiv v \pmod{g^\lambda}}} 1 \\ &= \sum_{0 \le u < g^\lambda} e(f_\lambda(u)) \sum_{0 \le v < g^\lambda} e(-f_\lambda(v)) \\ &\qquad \cdot \frac{1}{g^{2\lambda}} \sum_{0 \le h < g^\lambda} \sum_{0 \le k < g^\lambda} \sum_{\substack{M < m \le 2M \\ m(n+r) \le x}} e\left(\frac{h(m(n+r)-u)}{g^\lambda} + \frac{k(mn-v)}{g^\lambda} \right) \\ &= \sum_{0 \le h < g^\lambda} \sum_{0 \le k < g^\lambda} F_\lambda\left(\frac{h}{g^\lambda}\right) \overline{F_\lambda\left(-\frac{k}{g^\lambda}\right)} \sum_{\substack{M < m \le 2M \\ m(n+r) \le x}} e\left(\frac{(h+k)n + hr}{g^\lambda} m \right). \end{aligned}$$

Since the inner sum over m is a sum of geometric sequence, we have

$$|T(r,n)| \ll \sum_{0 \le h < g^\lambda} \sum_{0 \le k < g^\lambda} \left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right| \left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right| \min\left(M, \frac{1}{\left|\sin\left(\pi \frac{(h+k)n+hr}{g^\lambda}\right)\right|}\right).$$

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By taking the sum over n and recalling (6.12), we have

$$S_{\mathrm{II}}(r) \ll \sum_{0\le h<g^\lambda}\sum_{0\le k<g^\lambda}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right|\sum_{N<n\le 2N}\min\left(M,\frac{1}{|\sin(\pi\frac{(h+k)n+hr}{g^\lambda})|}\right). \tag{6.13}$$

Since the term

$$\min\left(M,\frac{1}{|\sin(\pi\frac{(h+k)n+hr}{g^\lambda})|}\right)$$

depends only on $n \pmod{g^\lambda}$, we can bound (6.13) as

$$\begin{aligned} S_{\mathrm{II}}(r) \ll \left(\frac{N}{g^\lambda}+1\right)\sum_{0\le h<g^\lambda}\sum_{0\le k<g^\lambda}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right| \\ \cdot\sum_{0\le n<g^\lambda}\min\left(M,\frac{1}{|\sin(\pi\frac{(h+k)n+hr}{g^\lambda})|}\right) \end{aligned} \tag{6.14}$$

by adding extra terms if necessary. By using Lemma 6.3 in (6.14), we have

$$\begin{aligned} S_{\mathrm{II}}(r) &\ll \left(\frac{N}{g^\lambda}+1\right)\sum_{d\mid g^\lambda} d\sum_{\substack{0\le h,k<g^\lambda\\(h+k,g^\lambda)=d}}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right|\min\left(M,\frac{1}{\sin(\pi\frac{d}{g^\lambda}\|\frac{hr}{d}\|)}\right) \\ &\quad+\left(\frac{N}{g^\lambda}+1\right)g^\lambda(\log x)\sum_{0\le h<g^\lambda}\sum_{0\le k<g^\lambda}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right| \\ &=: \left(\frac{N}{g^\lambda}+1\right)(S_{\mathrm{II},1}(r)+S_{\mathrm{II},2}(r)). \end{aligned} \tag{6.15}$$

We first estimate the sum $S_{\mathrm{II},1}(r)$. Since $d \mid g^\lambda$ in the sum $S_{\mathrm{II},1}(r)$, we can bound $S_{\mathrm{II},1}(r)$ as

$$S_{\mathrm{II},1}(r) \le \sum_{d\mid g^\lambda} d\sum_{\substack{0\le h,k<g^\lambda\\ h\equiv -k \pmod d}}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(-\frac{k}{g^\lambda}\right)\right|\min\left(M,\frac{1}{\sin(\pi\frac{d}{g^\lambda}\|\frac{hr}{d}\|)}\right)$$

by weakening the condition $(h+k, g^\lambda) = d$ to $h \equiv -k \pmod d$. Since the function

$$\mathbb{Z}\to\mathbb{R};\ k\mapsto\left|F_\lambda\left(\frac{k}{g^\lambda}\right)\right|$$

has a period g^λ, we can change variables via $k \rightsquigarrow -k$ to get

$$S_{\mathrm{II},1}(r) \le \sum_{d\mid g^\lambda} d\sum_{\substack{0\le h,k<g^\lambda\\ h\equiv k \pmod d}}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\left|F_\lambda\left(\frac{k}{g^\lambda}\right)\right|\min\left(M,\frac{1}{\sin(\pi\frac{d}{g^\lambda}\|\frac{hr}{d}\|)}\right).$$

By classifying the values of $h \equiv k \pmod d$, we obtain

$$S_{\mathrm{II},1}(r) \le \sum_{d\mid g^\lambda} d\sum_{0\le a<d}\min\left(M,\frac{1}{\sin(\pi\frac{d}{g^\lambda}\|\frac{ar}{d}\|)}\right)\left(\sum_{\substack{0\le h<g^\lambda\\ h\equiv a \pmod d}}\left|F_\lambda\left(\frac{h}{g^\lambda}\right)\right|\right)^2.$$

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Let $v_g(d)$ be the largest integer satisfying $g^{v_g(d)} \mid d$. By Lemma 4.11, we then have

$$S_{\mathrm{II},1}(r) \ll \sum_{d \mid g^\lambda} d^{1-2\eta_g} g^{2\eta_g \lambda} \sum_{0 \le a < d} \min\left(M, \frac{1}{\sin(\pi \frac{d}{g^\lambda} \| \frac{ar}{d} \|)}\right) \left| F_{v_g(d)}^{[\lambda - v_g(d)]}\left(\frac{a}{g^{v_g(d)}}\right)\right|^2.$$

By using Lemma 4.5, we have

$$S_{\mathrm{II},1}(r) \ll \sum_{d \mid g^\lambda} d^{1-2\eta_g} g^{2\eta_g \lambda - \sigma_{v_g(d)}^{[\lambda - v_g(d)]}(\alpha)} \sum_{0 \le a < d} \min\left(M, \frac{1}{\sin(\pi \frac{d}{g^\lambda} \| \frac{ar}{d} \|)}\right). \tag{6.16}$$

Since $x \mapsto \sin \pi x$ is concave for $x \in [0, \frac{1}{2}]$, we have

$$\sin\left(\pi \frac{d}{g^\lambda} \left\| \frac{ar}{d} \right\|\right) \ge \frac{d}{g^\lambda} \sin\left(\pi \left\| \frac{ar}{d} \right\|\right) = \frac{d}{g^\lambda} \left| \sin\left(\pi \frac{ar}{d}\right)\right|,$$

and so we can apply Lemma 6.3 to the inner sum of (6.16) to obtain

$$\begin{aligned} \sum_{0 \le a < d} \min\left(M, \frac{1}{\sin(\pi \frac{d}{g^\lambda} \| \frac{ar}{d} \|)}\right) &\le \frac{g^\lambda}{d} \sum_{0 \le a < d} \min\left(\frac{dM}{g^\lambda}, \frac{1}{|\sin(\pi \frac{ar}{d})|}\right) \\ &\ll \frac{g^\lambda}{d}\left((r,d)\frac{dM}{g^\lambda} + d \log d\right) \ll MR + g^\lambda \log d \ll g^\lambda \log x \end{aligned}$$

since $g^\lambda \asymp MR^2$ by (6.10). On inserting this estimate into (6.16), we get

$$S_{\mathrm{II},1}(r) \ll (\log x) \sum_{d \mid g^\lambda} d^{1-2\eta_g} g^{(1+2\eta_g)\lambda - \sigma_{v_g(d)}^{[\lambda - v_g(d)]}(\alpha)}.$$

We now write $d = k g^\delta$ with $0 \le \delta \le \lambda$ and $g \nmid k$ so that $v_g(d) = \delta$. This gives

$$S_{\mathrm{II},1}(r) \ll (\log x) \sum_{0 \le \delta \le \lambda} g^{(1+2\eta_g)\lambda + \delta(1-2\eta_g) - \sigma_\delta^{[\lambda-\delta]}(\alpha)} \sum_{\substack{k \mid g^{\lambda-\delta} \\ g \nmid k}} k^{1-2\eta_g}. \tag{6.17}$$

For $k \mid g^{\lambda-\delta}$ with $g \nmid k$, we have $(k, g) \le g/2$, and so

$$k = (k, g^{\lambda-\delta}) \le (k, g)^{\lambda-\delta} \le (g/2)^{\lambda-\delta}.$$

This gives

$$\sum_{\substack{k \mid g^{\lambda-\delta} \\ g \nmid k}} k^{1-2\eta_g} \le (g/2)^{(1-2\eta_g)(\lambda-\delta)} \tau(g^{\lambda-\delta}) \ll g^{(1-2\eta_g)(\lambda-\delta) - \omega_g(\lambda-\delta)}$$

with

$$\omega_g := \frac{\log 2}{\log g}\left(\frac{1}{2} - \eta_g\right)$$

by using the bound $\tau(n) \ll_\varepsilon n^\varepsilon$. On inserting this estimate into (6.17), we get

$$S_{\mathrm{II},1}(r) \ll g^{2\lambda} (\log x)^2 \max_{0 \le \delta \le \lambda} g^{-\omega_g(\lambda-\delta) - \sigma_\delta^{[\lambda-\delta]}(\alpha)} = g^{2\lambda} (\log x)^2 \max_{0 \le \delta \le \lambda} g^{-\omega_g \delta - \sigma_{\lambda-\delta}^{[\delta]}(\alpha)}, \tag{6.18}$$

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where we changed variables via $\delta \rightsquigarrow \lambda - \delta$ in the last equality. By (6.6) and (6.10), we have

$$g^{\mu-1} \le M \le MR^2 < g^{\lambda}, \quad \text{and so} \quad \mu \in [0, \lambda] \cap \mathbb{Z}. \tag{6.19}$$

Thus, by (ii) of Lemma 5.1 and (6.7), we have

$$\omega_g \delta + \sigma_{\lambda-\delta}^{[\delta]}(\boldsymbol{\alpha}) \ge \sigma_{\lambda}(\boldsymbol{\alpha}) \ge \sigma_{\mu}(\boldsymbol{\alpha}) \ge 10\kappa_{\mathrm{II}}$$

in (6.18) and so

$$S_{\mathrm{II},1}(r) \ll g^{2\lambda - 10\kappa_{\mathrm{II}}} (\log x)^2. \tag{6.20}$$

For the sum $S_{\mathrm{II},2}(r)$, we can simply use (4.16) of Lemma 4.11 to get

$$S_{\mathrm{II},2}(r) = g^{\lambda}(\log x) \left(\sum_{0 \le h < g^{\lambda}} \left| F_{\lambda}\left(\frac{h}{g^{\lambda}}\right) \right| \right)^2 \ll g^{(1+2\eta_g)\lambda} \log x = g^{2\lambda - (1-2\eta_g)\lambda} \log x.$$

By (4.6), (4.15), (6.7), and (6.19), we have

$$(1 - 2\eta_g)\lambda \ge \left(\frac{1}{2} - \eta_g\right) \xi_{\mathrm{II}} \ge \frac{\xi_{\mathrm{II}}}{4g^3 \log g} \ge 10\kappa_{\mathrm{II}},$$

we have

$$S_{\mathrm{II},2}(r) \ll g^{2\lambda - 10\kappa_{\mathrm{II}}} \log x. \tag{6.21}$$

On inserting (6.20) and (6.21) into (6.15) and recalling (6.10), we obtain

$$S_{\mathrm{II}}(r) \ll (Ng^{\lambda} + g^{2\lambda}) g^{-10\kappa_{\mathrm{II}}} (\log x)^2 \ll (MNR^2 + M^2R^4) g^{-10\kappa_{\mathrm{II}}} (\log x)^2.$$

Since we are assuming $M \le N$, this can be bounded as

$$S_{\mathrm{II}}(r) \ll MNR^4 g^{-10\kappa_{\mathrm{II}}} (\log x)^2.$$

On inserting this estimate into (6.11), we obtain

$$S_{\mathrm{II}} \ll MNR^2 g^{-5\kappa_{\mathrm{II}}} \log x + MNR^{-\frac{1}{2}}. \tag{6.22}$$

We now choose $R = g^{2\kappa_{\mathrm{II}}}$. By (4.6), we have

$$2\kappa_{\mathrm{II}} \le \frac{\xi_{\mathrm{II}}}{2g^3 \log g} \le \frac{\theta}{2} \frac{\log x}{\log g},$$

and so, by (6.5), we have

$$R = g^{2\kappa_{\mathrm{II}}} \le x^{\frac{\theta}{2}} \le N^{\frac{1}{2}}$$

as required in (6.9). On inserting $R = g^{2\kappa_{\mathrm{II}}}$ and $MN < x$ into (6.22), we obtain the assertion. □

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7. Exponential sum over primes with weakly digital functions

We now prove an estimate for exponential sums over primes involving a weakly digital function.

Lemma 7.1 (Vaughan's identity). *For $z > 0$ and $n \in \mathbb{N}$, we have*

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n)$$

with

$$a_1(n) := \sum_{\substack{dm=n\\ d\le z}} \mu(d)\log m, \qquad a_2(n) := \sum_{\substack{dm=n\\ d,m>z}} \mu(d)c_2(m),$$
$$a_3(n) := -\sum_{\substack{dm=n\\ d\le z^2}} c_3(d), \qquad a_4(n) := \Lambda(n)\mathbf{1}_{n\le z},$$
$$c_2(n) := \sum_{\substack{dm=n\\ d>z}} \Lambda(d), \qquad c_3(n) := \sum_{\substack{dm=n\\ d,m\le z}} \mu(d)\Lambda(m).$$

Proof. See formula (17.5) and the argument on pp. 55–56 of [12]. □

Theorem 7.1. *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}_g$, $L \in \mathbb{N}$, and $2 \le x \le g^L$, we have*

$$S := \sum_{n\le x} \Lambda(n)e(f_L(n)) \ll xg^{-\kappa}(\log x)^4,$$

where

$$\kappa := \frac{1}{10}\sigma_\xi(\boldsymbol{\alpha}) \quad \textit{with} \quad \xi := \left[\frac{1}{4}\frac{\log x}{\log g}\right]$$

and the implicit constant depends only on g.

Proof. Let $z = x^{\frac{1}{4}}$. By Lemma 7.1, we can decompose the sum as

$$S = S_1 + S_2 + S_3 + S_4, \tag{7.1}$$

where

$$S_1 := \sum_{\substack{mn\le x\\ m\le z}} \mu(m)(\log n)e(f_L(mn)), \qquad S_2 := \sum_{\substack{mn\le x\\ m,n>z}} \mu(m)c_2(n)e(f_L(mn)),$$
$$S_3 := -\sum_{\substack{mn\le x\\ m\le z^2}} c_3(m)e(f_L(mn)), \qquad S_4 := \sum_{n\le z} \Lambda(n)e(f_L(n)).$$

We estimate these sums separately.

For the sum S_1, since $z \le x^{\frac{1}{2}}$, we can use partial summation and Lemma 5.2 to get

$$S_1 \ll (\log x)\sum_{m\le z} \sup_{1\le t\le x/m} \left|\sum_{n\le t} e(f_L(mn))\right| \ll xg^{-\kappa}(\log x)^3, \tag{7.2}$$

where we used $\kappa_{\mathrm{I}} \ge \kappa$.

For the sum S_2, we can use Lemma 6.4. Note that

$$|c_2(n)|, |c_3(n)| \le \sum_{d|n} \Lambda(d) = \log n. \tag{7.3}$$

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We can take $\theta = \frac{1}{4}$ in Lemma 6.4 so $\kappa_{\mathrm{II}} = \kappa$. Thus, by decomposing the sum dyadically, we get

$$S_2 \ll (\log x)^3 \sup_{\substack{M,N \geq x^{\frac{1}{4}} \\ |a(m)|,|b(n)| \leq 1}} \left| \sum_{\substack{mn \leq x \\ M < m \leq 2M \\ N < n \leq 2N}} a(m)b(n)e(f_L(mn)) \right| \ll xg^{-\kappa}(\log x)^4. \tag{7.4}$$

For the sum S_3, since $z^2 \leq x^{\frac{1}{2}}$, Lemma 5.2 together with (7.3) gives

$$S_3 \ll (\log x) \sum_{m \leq z^2} \left| \sum_{n \leq x/m} e(f_L(mn)) \right| \ll xg^{-\kappa}(\log x)^3.$$

Finally, the sum S_4 is bounded trivially as

$$S_4 \ll z \log x \ll x^{1-\frac{3}{4}} \log x \ll xg^{-\kappa} \log x, \tag{7.5}$$

where we used the bound

$$\kappa \leq \frac{\xi}{20} \leq \frac{1}{80} \frac{\log x}{\log g} < \frac{3}{4} \frac{\log x}{\log g},$$

which follows by (4.6).

On inserting (7.2) and (7.4)–(7.5) into (7.1), we obtain the assertion. □

8. Application to the digital reverse

We finally apply Theorem 7.1 to digital reverse to obtain Theorems 1.1 and 1.2. In order to make the digital reverse fit into the setting of Theorem 7.1, we introduce a slightly modified digital reverse. For $L, n \in \mathbb{Z}_{\geq 0}$, we define *the relative digital reverse* $\mathrm{rev}_L(n)$ of order L by

$$\mathrm{rev}_L(n) := \sum_{0 \leq i < L} \varepsilon_i(n) g^{L-i-1}.$$

Note that then the "absolute" digital reverse defined by (1.1) can be expressed as

$$\overleftarrow{n} := \mathrm{rev}_{\mathrm{len}(n)}(n)$$

for $n \in \mathbb{Z}_{\geq 0}$. Note that the relative and absolute digital reverses coincide on $[g^{L-1}, g^L) \cap \mathbb{Z}$, that is,

$$\overleftarrow{n} = \mathrm{rev}_L(n) \quad \text{for} \quad n \in [g^{L-1}, g^L) \cap \mathbb{Z}.$$

To fit into the picture of weakly digital function, consider $\boldsymbol{\alpha}_L = (\alpha_{L,i})_{i=0}^{\infty} \in \mathscr{A}_g$ given by

$$\alpha_{L,i}(n) := \alpha n g^{L-i-1}$$

for $\alpha \in \mathbb{R}$. This seed $\boldsymbol{\alpha}_L$ gives

$$f_{L,\boldsymbol{\alpha}_L}(n) = \alpha \, \mathrm{rev}_L(n).$$

Since $\boldsymbol{\alpha}_L$ depends on a parameter L, the uniformity over $\boldsymbol{\alpha} \in \mathscr{A}_g$ in Theorem 7.1 is necessary.

Lemma 8.1. *For $g \in \mathbb{Z}_{\geq 2}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, we have*

$$\|g^{i_0}\alpha\| \geq \frac{1}{g+1} \quad \textit{with} \quad i_0 := \left\lfloor \frac{\log \frac{g}{(g+1)\|\alpha\|}}{\log g} \right\rfloor.$$

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Proof. This is Lemma 2.7 of [3]. For completeness, we give a proof. We have

$$g^{i_0}\|\alpha\| \geq g^{\frac{\log \frac{g}{(g+1)\|\alpha\|}}{\log g}-1}\|\alpha\| = \frac{1}{g}\frac{g}{(g+1)\|\alpha\|}\cdot\|\alpha\| = \frac{1}{g+1}$$

and

$$g^{i_0}\|\alpha\| \leq g^{\frac{\log \frac{g}{(g+1)\|\alpha\|}}{\log g}}\|\alpha\| \leq \frac{g}{(g+1)\|\alpha\|}\cdot\|\alpha\| = 1-\frac{1}{g+1}.$$

The result now follows since $\|g^{i_0}\alpha\| = \|g^{i_0}\|\alpha\|\|$. □

Lemma 8.2. *Let $g \in \mathbb{Z}_{\geq 2}$, $L \in \mathbb{Z}_{\geq 0}$, and $\alpha \in \mathbb{R}$ satisfy*

$$\sigma := \min_{0\leq i\leq L}\|g^i(g^2-1)\alpha\| > 0. \tag{8.1}$$

Define $\boldsymbol{\alpha}_L = (\alpha_{L,i}) \in \mathscr{A}_g$ by $\alpha_{L,i}(n) := \alpha n g^{L-i-1}$. For $\lambda \in \mathbb{Z}_{\geq 0}$ with $0 \leq \lambda \leq L$, we have

$$\sigma_\lambda(\boldsymbol{\alpha}_L) \gg \frac{\lambda}{\log\frac{1}{\sigma}} + O(1),$$

where the implicit constants depend only on g.

Proof. By considering the contribution of $(m,n) = (0,1)$ to (4.3), for $0 \leq i < \lambda$, we have

$$\begin{aligned}\gamma_i(\boldsymbol{\alpha}_L) &\gg \|(g\alpha_{L,i}(0)-\alpha_{L,i+1}(0))-(g\alpha_{L,i}(1)-\alpha_{L,i+1}(1))\|^2\\ &= \|(g^{L-i}-g^{L-(i+1)-1})\alpha\|^2 = \|g^{L-i-2}(g^2-1)\alpha\|^2 \gg \|g^{L-i-1}(g^2-1)\alpha\|^2,\end{aligned}$$

where we used the inequality $g\|x/g\| \geq \|x\|$ in the last step. We thus have

$$\sigma_\lambda(\boldsymbol{\alpha}_L) = \sum_{0\leq i<\lambda}\gamma_i(\boldsymbol{\alpha}_L) \gg \sum_{0\leq i<\lambda}\|g^{L-i-1}(g^2-1)\alpha\|^2 = \sum_{L-\lambda\leq i<L}\|g^i(g^2-1)\alpha\|^2. \tag{8.2}$$

Let

$$J := 1+\left\lfloor\frac{\log\frac{g}{(g+1)\sigma}}{\log g}\right\rfloor \geq 1, \quad K := \left\lfloor\frac{\lambda}{J}\right\rfloor \quad \text{and} \quad \alpha_k := g^{L-\lambda+(k-1)J}(g^2-1)\alpha.$$

We then have

$$\begin{aligned}\sum_{L-\lambda\leq i<L}\|(g^2-1)g^i\alpha\|^2 &\geq \sum_{1\leq k\leq K}\sum_{L-\lambda+(k-1)J\leq i<L-\lambda+kJ}\|g^i(g^2-1)\alpha\|^2\\ &\geq \sum_{1\leq k\leq K}\sum_{0\leq i<J}\|g^i\alpha_k\|^2.\end{aligned} \tag{8.3}$$

By the assumption (8.1) and $0 \leq \lambda \leq L$, we have $\alpha_k \notin \mathbb{Z}$ for all $k \in \{1,\dots,K\}$ and so Lemma 8.1 is applicable to the inner sum with $\alpha := \alpha_k$. Also, (8.1) implies $\|\alpha_k\| \geq \sigma$. For $1 \leq k < K$, let

$$i_{0,k} := \left\lfloor\frac{\log\frac{g}{(g+1)\|\alpha_k\|}}{\log g}\right\rfloor.$$

We then have

$$0 \leq i_{0,k} \leq \left\lfloor\frac{\log\frac{g}{(g+1)\sigma}}{\log g}\right\rfloor < J.$$

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Therefore, we can pick up the contribution of $i = i_{0,k}$ and use Lemma 8.1 in (8.3) to get

$$\sum_{L-\lambda\le i<L} \|(g^2-1)g^i\alpha\|^2 \ge \sum_{1\le k\le K} \|g^{i_{0,k}}\alpha_k\|^2 \ge \frac{K}{(g+1)^2} \gg K. \tag{8.4}$$

By using the estimate

$$K = \frac{\lambda}{J} + O(1) \ge \frac{\lambda}{1+\frac{\log\frac{1}{\sigma}}{\log 2}} + O(1) \gg \frac{\lambda}{\log\frac{1}{\sigma}} + O(1)$$

in (8.4), we get

$$\sum_{L-\lambda\le i<L} \|(g^2-1)g^i\alpha\|^2 \gg \frac{\lambda}{\log\frac{1}{\sigma}} + O(1).$$

On inserting this estimate into (8.2), we obtain the result. □

For $g \in \mathbb{Z}_{\ge 2}$, $L \in \mathbb{Z}_{\ge 0}$, and $a, q \in \mathbb{Z}$ with $q \ge 1$ and $x \in \mathbb{R}_{\ge 1}$, let us write

$$\overleftarrow{\psi}_L(x,a,q) := \sum_{\substack{n\le x\\ \mathrm{rev}_L(n)\equiv a \pmod q}} \Lambda(n), \qquad \overleftarrow{\psi}_{\sharp,L}(x,a,q) := \sum_{\substack{n\le x\\ \mathrm{rev}_L(n)\equiv a \pmod{(q,g^L(g^2-1))}}} \Lambda(n),$$

$$\overleftarrow{\vartheta}_L(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}_L(p)\equiv a \pmod q}} \log p, \qquad \overleftarrow{\vartheta}_{\sharp,L}(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}_L(p)\equiv a \pmod{(q,g^L(g^2-1))}}} \log p,$$

$$\overleftarrow{\pi}_L(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}_L(p)\equiv a \pmod q}} 1, \qquad \overleftarrow{\pi}_{\sharp,L}(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}_L(p)\equiv a \pmod{(q,g^L(g^2-1))}}} 1.$$

Theorem 8.1. *For $g \in \mathbb{Z}_{\ge 2}$, $L, a, q \in \mathbb{Z}$, and $x \in \mathbb{R}$ with $L, q \ge 1$ and $4 \le x \le g^L$, we have*

$$\overleftarrow{\psi}_L(x,a,q) = \frac{(q,g^L(g^2-1))}{q}\overleftarrow{\psi}_{\sharp,L}(x,a,q) + O\left(x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right)\right), \tag{8.5}$$

$$\overleftarrow{\vartheta}_L(x,a,q) = \frac{(q,g^L(g^2-1))}{q}\overleftarrow{\vartheta}_{\sharp,L}(x,a,q) + O\left(x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right)\right), \tag{8.6}$$

$$\overleftarrow{\pi}_L(x,a,q) = \frac{(q,g^L(g^2-1))}{q}\overleftarrow{\pi}_{\sharp,L}(x,a,q) + O\left(x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right)\right) \tag{8.7}$$

provided

$$q \le \exp\left(c\cdot\frac{\log x}{\log\log x}\right) \tag{8.8}$$

with some constant $c > 0$, where c and the implicit constant depend only on g.

Proof. We may assume that x is large and $q \ge 2$ since otherwise the assertions are trivial.

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We first prove (8.5). By the orthogonality of additive characters, we have

$$\begin{aligned}\overleftarrow{\psi}_L(x,a,q) &= \frac{1}{q}\sum_{0\le h<q} e\Big(-\frac{ha}{q}\Big)\sum_{n\le x}\Lambda(n)e\Big(\frac{h\mathrm{rev}_L(n)}{q}\Big)\\ &= \sum_{\substack{0\le h<q\\ q\mid g^L(g^2-1)h}} + \sum_{\substack{0\le h<q\\ q\nmid g^L(g^2-1)h}} =: S_{\mathfrak{M}} + S_{\mathfrak{m}}.\end{aligned} \tag{8.9}$$

For the sum $S_{\mathfrak{M}}$, by using

$$q \mid g^L(g^2-1)h \iff \frac{q}{(q, g^L(g^2-1))} \mid h,$$

by rewriting h to $\frac{q}{(q,g^L(g^2-1))}h$ and using the orthogonality, we have

$$\begin{aligned}S_{\mathfrak{M}} &= \frac{1}{q}\sum_{0\le h<(q,g^L(g^2-1))} e\Big(-\frac{ha}{(q,g^L(g^2-1))}\Big)\sum_{n\le x}\Lambda(n)e\Big(\frac{h\mathrm{rev}_L(n)}{(q,g^L(g^2-1))}\Big)\\ &= \frac{(q,g^L(g^2-1))}{q}\overleftarrow{\psi}_{\sharp,L}(x,a,q).\end{aligned} \tag{8.10}$$

For the sum $S_{\mathfrak{m}}$, we have

$$|S_{\mathfrak{m}}| \le \max_{\substack{0\le h<q\\ q\nmid g^L(g^2-1)h}}\Big|\sum_{n\le x}\Lambda(n)e\Big(\frac{h\mathrm{rev}_L(n)}{q}\Big)\Big|. \tag{8.11}$$

We then use Theorem 7.1 with $\boldsymbol{\alpha}_L = (\alpha_{L,i}) \in \mathscr{A}_g$ given by

$$\alpha_{L,i}(n) := \alpha n g^{L-i-1} \quad \text{with} \quad \alpha := \frac{h}{q}.$$

This $\boldsymbol{\alpha}_L$ gives

$$f_{L,\boldsymbol{\alpha}_L}(n) = \frac{h}{q}\sum_{0\le i<L}\varepsilon_i(n)g^{L-i-1} = \frac{h\mathrm{rev}_L(n)}{q}.$$

Also, since $q \nmid g^L(g^2-1)h$ in $S_{\mathfrak{m}}$, we have

$$\min_{0\le i\le L}\Big\|g^i(g^2-1)\cdot\frac{h}{q}\Big\| \ge \frac{1}{q}$$

for the quantity σ defined by (8.1). Thus, by Lemma 8.2, κ in Theorem 7.1 satisfies

$$\kappa = \frac{1}{10}\sigma_\xi(\boldsymbol{\alpha}_L) \gg \frac{\xi}{\log\frac{1}{\sigma}} + O(1) \gg \frac{\log x}{\log q} + O(1), \quad \text{where} \quad \xi := \Big[\frac{1}{4}\frac{\log x}{\log g}\Big].$$

Applying Theorem 7.1 to (8.11), we obtain

$$S_{\mathfrak{m}} \ll x(\log x)^4\exp\Big(-5c\cdot\frac{\log x}{\log q}\Big). \tag{8.12}$$

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Under the condition (8.8), we have

$$(\log x)^4 = \exp(4\log\log x) \le \exp\left(4c\cdot\frac{\log x}{\log q}\right). \tag{8.13}$$

On inserting this estimate into (8.12), we obtain

$$S_{\mathfrak{m}} \ll x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right). \tag{8.14}$$

On inserting (8.10) and (8.14) into (8.9), we obtain (8.5).

We next deduce (8.6) from (8.5). In (8.5), we may assume $c \in (0, \frac{\log 2}{2})$ to assure

$$x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right) \ge x^{\frac{1}{2}}.$$

We can then use

$$\sum_{\substack{p^\nu \le x\\ \nu\ge 2}} \log p \ll x^{\frac{1}{2}}$$

in (8.5) to get (8.6).

We finally deduce (8.7) from (8.6). We first change the value of c to replace it in (8.6) and the associated (8.8) by $4c$. (We however use the same c for (8.8) associated with (8.7).) We may also assume that $c \in (0, \frac{\log 2}{8})$. Then, for $x^{\frac{1}{2}} \le u \le x$, we have $1 \le u \le g^L$ and

$$q \le \exp\left(c\cdot\frac{\log x}{\log\log x}\right) \le \exp\left(4c\cdot\frac{\log u}{\log\log u}\right).$$

Thus, we can apply (8.6) with x replaced by u to get

$$\overleftarrow{\vartheta}_L(u,a,q) = \frac{(q, g^L(g^2-1))}{q}\overleftarrow{\vartheta}_{\sharp,L}(u,a,q) + O\left(x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right)\right)$$

for $1 \le u \le x$ since this is trivial if $u \le x^{\frac{1}{2}}$. We can then use partial summation to deduce (8.7). □

Define

$$\overleftarrow{\pi}(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}(p)\equiv a \pmod q}} 1 \quad\text{and}\quad \overleftarrow{\pi}_\sharp(x,a,q) := \sum_{\substack{p\le x\\ \mathrm{rev}(p)\equiv a \pmod{(q, g^L(g^2-1))}}} 1,$$

where $L := [\frac{\log x}{\log g}] + 1$.

Theorem 8.2. *For $g, a, q \in \mathbb{Z}$ and $x \in \mathbb{R}_{\ge 4}$ with $g \ge 2$ and $q \ge 1$, there is a constant $c \in (0,1)$ such that*

$$\overleftarrow{\pi}(x,a,q) = \frac{(q,(g^2-1)g^L)}{q}\overleftarrow{\pi}_\sharp(x,a,q) + O\left(x\exp\left(-c\cdot\frac{\log x}{\log(q+1)}\right)\right)$$

provided

$$q \le \exp\left(c\cdot\frac{\log x}{\log\log x}\right), \tag{8.15}$$

where $L := [\frac{\log x}{\log g}] + 1$ and c and the implicit constant depend only on g.

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Proof. We may assume that x is large in terms of g and $q \geq 2$ since otherwise the assertion is trivial. Let $\widetilde{c}$ be the constant c of Theorem 8.1 and let $c := \frac{\widetilde{c}}{5}$. Since

$$g^{\ell-1} \leq n < g^{\ell} \implies \mathrm{rev}_{\ell}(n) = \mathrm{rev}(n) \tag{8.16}$$

for any $\ell \in \mathbb{N}$, we have

$$\overleftarrow{\pi}(u,a,q) - \overleftarrow{\pi}(g^{\ell-1}-1,a,q) = \overleftarrow{\pi}_{\ell}(u,a,q) - \overleftarrow{\pi}_{\ell}(g^{\ell-1}-1,a,q) \tag{8.17}$$

for $g^{\ell-1} \leq u < g^{\ell}$. For $x^{\frac{1}{2}} \leq g^{\ell-1} \leq u < g^{\ell}$, we have

$$q \leq \exp\left(c \cdot \frac{\log x}{\log\log x}\right) \leq \exp\left(\widetilde{c} \cdot \frac{\log u}{\log\log u}\right).$$

Thus, we can apply (8.7) of Theorem 8.1 to (8.17) and use (8.16) again to get

$$\begin{aligned}&\overleftarrow{\pi}(u,a,q) - \overleftarrow{\pi}(g^{\ell-1}-1,a,q)\\ &\quad = \frac{(q,g^{\ell}(g^2-1))}{q} \sum_{\substack{g^{\ell-1} \leq p \leq u\\ \mathrm{rev}(p) \equiv a \pmod{(q,g^{\ell}(g^2-1))}}} 1 + O\left(x \exp\left(-\widetilde{c} \cdot \frac{\log x}{\log q}\right)\right)\end{aligned} \tag{8.18}$$

for $g^{\ell-1} \leq u < g^{\ell}$ with $x^{\frac{1}{2}} \leq g^{\ell-1}$ and $g^{\ell} \leq gx$. By (8.15) and $\ell \gg \log x$, we indeed have

$$(q, g^{\ell}(g^2-1)) = (q, g^{L}(g^2-1))$$

for all ℓ with $x^{\frac{1}{2}} \leq g^{\ell-1}$ and $g^{\ell} \leq gx$. Thus, we can rewrite (8.18) by replacing ℓ by L to get

$$\begin{aligned}&\overleftarrow{\pi}(u,a,q) - \overleftarrow{\pi}(g^{\ell-1}-1,a,q)\\ &\quad = \frac{(q,g^{L}(g^2-1))}{q} \sum_{\substack{g^{\ell-1} \leq p \leq u\\ \mathrm{rev}(p) \equiv a \pmod{(q,g^{L}(g^2-1))}}} 1 + O\left(x \exp\left(-\widetilde{c} \cdot \frac{\log x}{\log q}\right)\right)\end{aligned} \tag{8.19}$$

for $g^{\ell-1} \leq u < \min(g^{\ell}, x)$ for any $\ell \in \mathbb{N}$ since this is trivial if $u \leq x^{\frac{1}{2}}$. Thus, by summing up (8.19) over ℓ and using the bound similar to (8.13), we obtain the theorem. □

Finally, Theorems 1.1 and 1.2 follow from Theorem 8.2 by the argument of Section 11 of [1].

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