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GENERAL DRAWDOWN-BASED DE FINETTI OPTIMIZATION FOR SPECTRALLY NEGATIVE LÉVY RISK PROCESSES

WENYUAN WANG,* *Xiamen University*
XIAOWEN ZHOU,** *Concordia University*

Abstract

For spectrally negative Lévy risk processes we consider a general version of de Finetti's optimal dividend problem in which the ruin time is replaced with a general drawdown time from the running maximum in its value function. We identify a condition under which a barrier dividend strategy is optimal among all admissible strategies if the underlying process does not belong to a small class of compound Poisson processes with drift, for which the take-the-money-and-run dividend strategy is optimal. It generalizes the previous results on dividend optimization from ruin time based to drawdown time based. The associated drawdown functions are discussed in detail for examples of spectrally negative Lévy processes.

Keywords: Spectrally negative Lévy process; drawdown time; de Finetti's optimal dividend problem; stochastic control; HJB inequality

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1. Introduction

De Finetti [15] first introduced a dividend model to risk theory. In the discrete-time model, an insurance company has the option to pay out dividends from its surplus to its beneficiaries up to the time of ruin. It was argued that this should be done in an optimal way so that the expected total amount of discounted dividends paid until ruin is maximized. De Finetti [15] proved that the optimal way of paying out dividends is by adopting a barrier strategy, i.e. there exists a nonnegative constant such that at each time the excess of the net surplus is above the constant level it is paid out as dividend to shareholders.

In recent years, there has been much progress in the study of dividend optimization when the underlying surplus processes are spectrally one-sided Lévy processes, which we summarize in the following.

For spectrally negative Lévy risk processes with the present value of dividends paid until ruin as its value function, applying the fluctuation theory, Avram *et al.* [5] found an explicit analytical description of the optimal barrier in the set of barrier strategies. The authors also identified conditions under which the barrier dividend policy is optimal among all admissible strategies. It was later pointed out by Loeffen [26] that the condition is satisfied if the Lévy measure allows a completely monotone density. Loeffen [27] further considered a modified

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* Postal address: School of Mathematical Sciences, Xiamen University, Fujian, 361005, China.
Email address: wwywang@xmu.edu.cn
** Postal address: Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. W., Montreal, H3G 1M8, Canada. Email address: xzhou@mathstat.concordia.ca

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version of the de Finetti optimal dividend problem by adding to the value function an extra term which takes account of the ruin time. The author showed that, with the exception of a small class of compound Poisson processes with drift, a barrier strategy is still optimal if the Lévy measure has a completely monotone density. In addition, Loeffen [28] considered an optimal dividend problem with transaction costs for the spectrally negative Lévy reserve processes, where an optimal strategy is to pay a dividend in order to reduce the reserve to a certain level c_1 whenever it is above another level $c_2 > c_1$. Loeffen and Renaud [29] further showed that either the barrier strategy or the take-the-money-and-run strategy is optimal among all admissible strategies if the dividend payments are subject to an affine penalty function of the ruin time. An expression for the nth moment of the present value of dividends barrier strategy can be found in [22] and [31].

Kyprianou *et al.* [23] also considered de Finetti's control problem, but with a restriction that the control strategies are absolutely continuous with respect to the Lebesgue measure. The authors investigated the robustness of the condition that the Lévy measure has a completely monotone density and established an explicit optimal strategy for this case that covers some existing results. The explicit optimal strategy in question is the so-called refraction strategy. Avram *et al.* [6] considered maximizing the sum of expected cumulative discounted dividend payments received until the moment of ruin and a penalty payment that is a function of the deficit at ruin, with or without the cost on dividend payments. The optimality of single-dividend-band strategies and multi-dividend-band strategies were discussed.

For models based on spectrally positive Lévy processes, which are also referred to as the dual models, Bayraktar *et al.* [10] demonstrated the optimality of a barrier dividend strategy for such a dual model. Moreover, they characterized the optimal barrier using the functional inverse of a scale function. In [2], [3], [11], [23], and [37]–[39], the optimal dividend problems were also studied for a spectrally positive Lévy risk model.

In addition, business activities of the insurer have also been taken into consideration. To avoid ruin, Bayraktar *et al.* [10] and Zhae *et al.* [39] considered rescuing the company by means of capital injection in the dual model. Proportional and fixed transaction costs for dividend and capital injection have been introduced to the optimal dividend problem; see, for example, [2], [6], [11], [36], and [39].

Dividends optimization problems have also been studied for the Brownian motion setting and the (dual) Cramér–Lundberg setting; see, for example, [8], [13], [16], [33], and [35].

Li *et al.* [25] recently proved results on exit problems involving a general drawdown time from the running maximum for a spectrally negative Lévy process. Using an approximating method, the Laplace transforms involving the drawdown time, the hitting time, and the creeping time over a maximum related drawdown level, as well as an associated potential measure, were all expressed in terms of the scale functions. The general drawdown times found interesting applications in [9] in which they were used to solve the Skorokhod embedding problem. We refer the reader to [4], [14], [32], and [34] for applications of drawdown times in finance. For more results on a general drawdown time, see [7], [24], and [30].

Motivated by [5] and [25]–[27], in this paper we extend the previous de Finetti's dividend problem for the spectrally negative Lévy risk processes from a ruin time based to a drawdown time based dividend optimization. In the classical de Finetti's dividend problem, the optimal dividend strategy is often the barrier strategy under which ruin occurs with probability 1. However, for the drawdown time based dividend optimization, one can easily adjust the drawdown function so that the ruin occurs with probability less than 1 at the drawdown time. It thus provides an interesting alternative to de Finetti's dividend problem in that it achieves a

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balance between dividend optimization and solvency.

The approach in this paper follows the standard treatment of the Hamilton–Jacobi–Bellman (HJB) inequality in control theory that had been previously adopted in [5], [26], and [27]. However, our drawdown time related arguments are more involved. For example, the derivation of the value function for the barrier dividend strategy is more demanding. We also adopt a new martingale argument in verifying the HJB inequalities. When the drawdown time is reduced to the ruin time, we are able to recover the main results of [26] and [27].

This paper is arranged as follows. Some preliminary results concerning the spectrally negative Lévy processes and the mathematical setup of the dividend optimization problem are presented in Section 2. In Section 3 we express the value function of a barrier dividend strategy in terms of the scale functions for the spectrally negative Lévy processes. In Section 4 we then prove that a solution to the HJB inequities coincides with the optimal dividend value function. We further identify a condition under which the solution to the HJB inequities can be constructed and the optimal strategy is also found, which is either a barrier strategy or a take-the-money-and-run strategy. In Section 5, for some well-known spectrally negative Lévy processes, we identify those drawdown functions for which the corresponding barrier strategies are optimal.

2. The optimization problem

To formulate the optimization problem, write $X = \{X(t);\ t \geq 0\}$, defined on a probability space with probability laws $\{\mathbb{P}_x;\ x \in [-l, \infty)\}$ and natural filtration $\{\mathcal{F}_t;\ t \geq 0\}$, for a spectrally negative Lévy process that is not a purely increasing linear drift or the negative of a subordinator. Denote its running supremum as $\bar{X}(t) := \sup_{0\leq s\leq t} X(s)$ for $t \geq 0$. We assume that in the case of no control (no dividend is deducted from the surplus process), the risk process evolves as the process X.

A dividend or control strategy, denoted by $D = \{D(t);\ t \geq 0\}$, is a one-dimensional nondecreasing, left-continuous, and $\{\mathcal{F}_t;\ t \geq 0\}$-adapted process started at 0, i.e. $D(0) = 0$. The random variable $D(t)$ represents the cumulative dividends that the company has paid out until time t. Then the controlled aggregate surplus process for dividend strategy D is defined as

$$U^D(t) := X(t) - D(t), \qquad t \geq 0.$$

The general drawdown time of the risk process $\{U^D(t);\ t \geq 0\}$ with respect to the drawdown function ξ is defined as

$$\tau_\xi^D := \inf\{t > 0 \mid U^D(t) < \xi(\bar{U}^D(t)) - l\}. \tag{2.1}$$

Here, $\bar{U}^D(t) := \sup_{0\leq s\leq t} U^D(s)$ is the running supremum of the process $\{U^D(t);\ t \geq 0\}$, $l \in [0, \infty)$ is a predetermined constant, and ξ is a drawdown function satisfying

$$\xi(y) = 0 \quad \text{for all } y \in (-\infty, -l] \quad \text{and} \quad -l \leq \xi(y) - l < y \quad \text{for } y \in (-l, \infty).$$

The parameter l is introduced to take into account the situation that the insurer could continue with the business when the reserve is below 0 with a deficit not exceeding l. In particular, the classical ruin time is recovered if we choose $\xi \equiv 0$ and $l = 0$ in (2.1).

A dividend strategy D is called admissible if the drawdown time cannot be reached due to a lump sum dividend payment, i.e.

$$D(t^+) - D(t) \leq [U^D(t) - (\xi(\bar{U}^D(t)) - l)] \vee 0 \quad \text{for all } t < \tau_\xi^D.$$

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Let $\mathcal{D}$ be the set of all admissible dividend strategies. For a given admissible dividend strategy $D \in \mathcal{D}$, we define the corresponding value function V_D by

$$V_D(x) := \mathbb{E}_x\left(\int_0^{\tau_\xi^D} \mathrm{e}^{-qt}\,\mathrm{d}D(t) + S\mathrm{e}^{-q\tau_\xi^D}\right), \qquad x \in [-l, \infty),$$

where $q > 0$ is a discount factor and $S \in \mathbb{R}$ is the terminal value. For $S > 0$, S can be thought of as the amount of salvage that is distributed to the beneficiaries or shareholders of the company when bankruptcy takes place; while for $S < 0$, S is regarded as the amount of penalty imposed on the shareholders. More specifically, this extra negative term added to the value function penalizes early ruin and so this model is applicable when, besides the value of the dividend payments, we are also concerned with the length of the lifetime of the risk process. We refer the reader to [27] for a more detailed explanation of the extra term S.

Our goal is to find the optimal value function defined by

$$V(x) := \sup_{D\in\mathcal{D}} V_D(x) = \sup_{D\in\mathcal{D}} \mathbb{E}_x\left(\int_0^{\tau_\xi^D} \mathrm{e}^{-qt}\,\mathrm{d}D(t) + S\mathrm{e}^{-q\tau_\xi^D}\right), \qquad x \in [-l, \infty), \tag{2.2}$$

and to find an optimal dividend strategy D^* that satisfies $V(x) = V_{D^*}(x)$ for all $x \in [-l, \infty)$.

The Laplace exponent of X is given by

$$\psi(\theta) := \ln(\mathbb{E}_x(\mathrm{e}^{\theta(X_1-x)})) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} (1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{(0,1)}(x))\nu(\mathrm{d}x),$$

where ν is the Lévy measure satisfying $\int_{(0,\infty)}(1\wedge x^2)\nu(\mathrm{d}x) < \infty$ and $\mathbf{1}$ is the indicator function. For $\theta \in [0, \infty)$, the function $\psi(\theta)$ is finite, strictly convex, and infinitely differentiable. The process X can be represented as

$$X(t) = \gamma t + \sigma B(t) - \int_0^t \int_{(0,1)} x\bar{N}(\mathrm{d}s,\, \mathrm{d}x) - \int_0^t \int_{[1,\infty)} xN(\mathrm{d}s,\, \mathrm{d}x), \qquad t \geq 0,$$

where $B(t)$ is the standard Brownian motion, $N(\mathrm{d}s,\, \mathrm{d}x)$ is an independent Poisson random measure on $[0, \infty) \times (0, \infty)$ with intensity measure $\mathrm{d}s\nu(\mathrm{d}x)$, and $\bar{N}(\mathrm{d}s,\, \mathrm{d}x) = N(\mathrm{d}s,\, \mathrm{d}x) - \mathrm{d}s\nu(\mathrm{d}x)$ denotes the compensated random measure.

The scale functions $\{W^{(q)};\, q \geq 0\}$ of X are defined as follows. For each $q \geq 0$,

$$W^{(q)} : [0, \infty) \to [0, \infty)$$

is the unique strictly increasing and continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\theta) - q} \quad \text{for } \theta > \Psi(q),$$

where $\Psi(q)$ is the largest solution of $\psi(\theta) = q$; see [12]. Further, define $W^{(q)}(x) = 0$ for $x < 0$, and write W for $W^{(0)}$. For $q \geq 0$ and $x \in \mathbb{R}$, set

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y.$$

We refer to reader to [20] for discussions on the differentiability of the scale function $W^{(q)}$. For example, $W^{(q)} \in C^2(0, \infty)$ if $\sigma^2 > 0$, i.e. the Lévy process has a nontrivial Brownian component.

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For any $x \in \mathbb{R}$ and $\vartheta \geq 0$, we can introduce a new probability measure $\mathbb{P}_x^\vartheta$ by

$$\left.\frac{\mathbb{P}_x^\vartheta}{\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathrm{e}^{\vartheta(X(t)-x)-\psi(\vartheta)t}, \qquad t \geq 0.$$

Under $\mathbb{P}_x^\vartheta$, process X remains a spectrally negative Lévy process and we denote by $W_\vartheta^{(q)}$ the scale function for X under $\mathbb{P}_x^\vartheta$.

We also briefly recall concepts in excursion theory for the reflected process

$$\{\bar{X}(t) - X(t);\ t \geq 0\},$$

and we refer the reader to [12] for more details. For each $x \geq -l$, the process

$$\{L(t) = \bar{X}(t) - x,\ t \geq 0\}$$

serves as a local time at 0 for the Markov process $\{\bar{X}(t) - X(t);\ t \geq 0\}$ under $\mathbb{P}_x$. The Poisson point process of excursions indexed by this local time is denoted by $\{(t, \varepsilon_t);\ t \geq 0\}$, where, for t with $L^{-1}(t) - L^{-1}(t^-) > 0$,

$$\varepsilon_t(s) := X(L^{-1}(t)) - X(L^{-1}(t^-) + s), \qquad s \in (0, L^{-1}(t) - L^{-1}(t^-)],$$

where

$$L^{-1}(t) := \inf\{s \geq 0 \mid L(s) > t\} = \sup\{s \geq 0 \mid L(s) \leq t\}$$

and

$$L^{-1}(t^-) = \lim_{s \uparrow t} L^{-1}(s) = \inf\{s \geq 0 \mid L(s) \geq t\} = \sup\{s \geq 0 \mid L(s) < t\}$$

satisfy

$$s < L^{-1}(t^-) \quad \Longleftrightarrow \quad L(s) < t. \tag{2.3}$$

Accordingly, we denote a generic excursion as $\varepsilon(\cdot)$ (or, ε for short) belonging to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the process $\{(t, \varepsilon_t);\ t \geq 0\}$ is given by $\mathrm{d}t \times \mathrm{d}n$, where n is a measure on the space of excursions. In particular, $\bar{\varepsilon} = \sup_{s \geq 0} \varepsilon(s)$ is an example of an n-measurable functional of the canonical excursion. It is known that

$$n(\bar{\varepsilon} > x) = \frac{W'(x)}{W(x)} \tag{2.4}$$

(see [21]), providing that x is not a point of discontinuity for the derivative of W.

We do not impose the safety loading condition $\psi'(0^+) \geq 0$. Instead, $\psi'(0^+) > -\infty$ is assumed throughout the paper.

3. The barrier type dividend strategy and its value function

In the following we first express the value function of the barrier dividend strategy in terms of the scale functions. Recall that a barrier dividend strategy with barrier level $a \in [-l, \infty)$ is defined as $D_a(t) := (\bar{X}(t) - a) \vee 0$. Denote by $V_a(x)(x \in [-l, \infty))$ the value function of the barrier dividend strategy with barrier level $a \in [-l, \infty)$, i.e. $V_a(x) = V_{D_a}(x), x \in [-l, \infty)$. For notational convenience, write $\bar{\xi}(x) := x - \xi(x) + l$, $U^a := U^{D_a}$, and

$$\tau_\xi^a := \tau_\xi^{D_a} = \inf\{t > 0 \mid U^a(t) < \xi(\bar{U}^a(t)) - l.\}.$$

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Further, define the general drawdown time for the drawdown function ξ and the first up-crossing time of the given process X, respectively, as

$$T_\xi := \inf\{t > 0 \mid X(t) < \xi(\bar{X}(t)) - l\} \quad \text{and} \quad T_b^+ := \inf\{t > 0 \mid X(t) > b\}$$

with the convention that $\inf \varnothing = \infty$.

Lemma 3.1. *When $S = 0$, we have, for $a \in [-l, \infty)$,*

$$V_a(a) = \mathbb{E}_a \int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\, \mathrm{d}(\bar{X}(t) - a) = \frac{W^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}. \tag{3.1}$$

Proof. By (2.3), we have, for $b > a$,

$$\begin{aligned}
\mathbb{E}_a \int_0^{T_b^+ \wedge \tau_\xi^a} \mathrm{e}^{-qt}\, \mathrm{d}(\bar{X}(t) - a) &= \mathbb{E}_a \int_0^\infty \mathbf{1}_{\{t < T_b^+ \wedge \tau_\xi^a\}} \mathrm{e}^{-qt}\, \mathrm{d}(\bar{X}(t) - a) \\
&= \mathbb{E}_a \int_0^\infty \mathbf{1}_{\{L(t) < L(T_b^+ \wedge \tau_\xi^a)\}} \mathrm{e}^{-qt}\, \mathrm{d}L(t) \\
&= \mathbb{E}_a \int_0^\infty \mathbf{1}_{\{y < L(T_b^+ \wedge \tau_\xi^a)\}} \mathrm{e}^{-qL^{-1}(y)}\, \mathrm{d}y \\
&= \int_0^\infty \mathbb{E}_a(\mathrm{e}^{-qL^{-1}(y)} \mathbf{1}_{\{y < L(T_b^+) = b-a,\, y < L(\tau_\xi^a)\}})\, \mathrm{d}y \\
&= \int_0^{b-a} \mathbb{E}_a(\mathrm{e}^{-qL^{-1}(y)} \mathbf{1}_{\{y < L(\tau_\xi^a)\}})\, \mathrm{d}y.
\end{aligned} \tag{3.2}$$

In addition, $\{y < L(\tau_\xi^a)\} = \{\bar{\varepsilon}_t \le a - \xi(a) + l \text{ for all } t \in [0, y]\}$, which, together with (3.2), implies that

$$\begin{aligned}
&\mathbb{E}_a \int_0^{T_b^+ \wedge \tau_\xi^a} \mathrm{e}^{-qt}\, \mathrm{d}(\bar{X}(t) - a) \\
&\quad = \int_0^{b-a} \mathbb{E}_a(\mathrm{e}^{-qL^{-1}(y)} \mathbf{1}_{\{\bar{\varepsilon}_t \le a - \xi(a) + l \text{ for all } t \in [0,y]\}})\, \mathrm{d}y \\
&\quad = \int_0^{b-a} \mathrm{e}^{-\Psi(q)(X(L^{-1}(y)) - a)} \\
&\qquad\qquad \times \mathbb{E}_a(\mathrm{e}^{\Psi(q)(X(L^{-1}(y)) - a) - qL^{-1}(y)} \mathbf{1}_{\{\bar{\varepsilon}_t \le a - \xi(a) + l \text{ for all } t \in [0,y]\}})\, \mathrm{d}y \\
&\quad = \int_0^{b-a} \mathrm{e}^{-\Psi(q)y} \mathbb{E}_a^{\Psi(q)}(\mathbf{1}_{\{\bar{\varepsilon}_t \le a - \xi(a) + l \text{ for all } t \in [0,y]\}})\, \mathrm{d}y \\
&\quad = \int_0^{b-a} \mathrm{e}^{-\Psi(q)y} \exp\left\{ - \int_0^y n_{\Psi(q)}(\bar{\varepsilon} > a - \xi(a) + l)\, \mathrm{d}t \right\} \mathrm{d}y \\
&\quad = \int_0^{b-a} \mathrm{e}^{-\Psi(q)y} \exp\left\{ - \int_0^y \frac{W'_{\Psi(q)}(a - \xi(a) + l)}{W_{\Psi(q)}(a - \xi(a) + l)}\, \mathrm{d}t \right\} \mathrm{d}y \\
&\quad = \int_0^{b-a} \exp\left\{ - \int_0^y \frac{W^{(q)'}(a - \xi(a) + l)}{W^{(q)}(a - \xi(a) + l)}\, \mathrm{d}t \right\} \mathrm{d}y,
\end{aligned}$$

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where $n_{\Psi(q)}$ denotes the excursion measure under the new measure $\mathbb{P}_x^{\Psi(q)}$, and (2.4) and $W_{\Psi(q)}(x) = \mathrm{e}^{-\Psi(q)x}W^{(q)}(x)$ are used in the second to last equality. Letting b go to $+\infty$ in the above equation, we have

$$\begin{aligned}\mathbb{E}_a\int_0^{\tau_\xi^a}\mathrm{e}^{-qt}\,\mathrm{d}(\bar{X}(t)-a) &= \int_a^\infty \exp\left\{-\int_a^y \frac{W^{(q)'}(a-\xi(a)+l)}{W^{(q)}(a-\xi(a)+l)}\,\mathrm{d}s\right\}\mathrm{d}y\\ &= \frac{W^{(q)}(a-\xi(a)+l)}{W^{(q)'}(a-\xi(a)+l)},\end{aligned}$$

which is (3.1). □

In the following lemma we solve the drawdown-based two-sided exit problem.

Lemma 3.2. (Two-sided exit problem.) *For $a \in [-l, \infty)$ and $x \in [-l, a)$, we have*

$$\mathbb{E}_x(\mathrm{e}^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}}) = \exp\left\{-\int_x^a \frac{W^{(q)'}(\bar{\xi}(y))}{W^{(q)}(\bar{\xi}(y))}\,\mathrm{d}y\right\}. \tag{3.3}$$

Proof. On the event $\{T_a^+ < T_\xi\}$, the process X up-crosses level a before the general drawdown time T_ξ, i.e. $X(t) \geq \xi(\bar{X}(t)) - l$ for all $t \in [0, T_a^+]$. Therefore,

$$\{T_a^+ < T_\xi\} = \{\bar{\varepsilon}_t \leq x + t - \xi(x+t) + l \text{ for all } t \in [0, a-x]\}, \tag{3.4}$$

where we use the equivalence of the fact that, for all $s \in [0, L_t^{-1} - L_{t-}^{-1}]$,

$$X(L_{t-}^{-1}+s) \geq \xi(\bar{X}(L_{t-}^{-1}+s)) - l = \xi(X(L_t^{-1})) - l$$

and the fact that, for all $s \in [0, L_t^{-1} - L_{t-}^{-1}]$,

$$\varepsilon_t(s) = X(L_t^{-1}) - X(L_{t-}^{-1}+s) \leq X(L_t^{-1}) - \xi(X(L_t^{-1})) + l = x + t - \xi(x+t) + l.$$

By (3.4), we have

$$\begin{aligned}\mathbb{P}_x(T_a^+ < T_\xi) &= \mathbb{P}_x(\bar{\varepsilon}_t \leq x + t - \xi(x+t) + l \text{ for all } t \in [0, a-x])\\ &= \exp\left\{-\int_0^{a-x} n(\bar{\varepsilon} > x + t - \xi(x+t) + l)\,\mathrm{d}t\right\}\\ &= \exp\left\{-\int_0^{a-x} \frac{W'(x+t-\xi(x+t)+l)}{W(x+t-\xi(x+t)+l)}\,\mathrm{d}t\right\}\\ &= \exp\left\{-\int_x^a \frac{W'(\bar{\xi}(s))}{W(\bar{\xi}(s))}\,\mathrm{d}s\right\}.\end{aligned}$$

On the one hand,

$$\mathbb{P}_x^{\Psi(q)}(T_a^+ < T_\xi) = \mathbb{E}_x(\mathrm{e}^{\Psi(q)(X(T_a^+)-x)-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}}) = \mathrm{e}^{\Psi(q)(a-x)}\mathbb{E}_x(\mathrm{e}^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}}). \tag{3.5}$$

On the other hand,

$$\begin{aligned}\mathbb{P}_x^{\Psi(q)}(T_a^+ < T_\xi) &= \exp\left\{-\int_x^a \frac{W'_{\Psi(q)}(y-\xi(y)+l)}{W_{\Psi(q)}(y-\xi(y)+l)}\,\mathrm{d}y\right\}\\ &= \exp\left\{-\int_x^a \frac{W^{(q)'}(y-\xi(y)+l)}{W^{(q)}(y-\xi(y)+l)}\,\mathrm{d}y + \Psi(q)(a-x)\right\},\end{aligned} \tag{3.6}$$

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where the identity $W_{\Psi(q)}(x) = \mathrm{e}^{-\Psi(q)x} W^{(q)}(x)$ is used in the above display. Combining (3.5) and (3.6), we arrive at

$$\mathbb{E}_x(\mathrm{e}^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}}) = \mathrm{e}^{-\Psi(q)(a-x)}\mathbb{P}_x^{\Psi(q)}(T_a^+ < T_\xi) = \exp\left\{-\int_x^a \frac{W^{(q)'}(y-\xi(y)+l)}{W^{(q)}(y-\xi(y)+l)}\,\mathrm{d}y\right\},$$

which is (3.3). □

Proposition 3.1. *Suppose that $W^{(q)}$ is continuously differentiable on $(0,\infty)$. Then*

$$V_a(x) = \begin{cases} \exp\left(-\int_x^a \frac{W^{(q)'}(\bar\xi(t))}{W^{(q)}(\bar\xi(t))}\,\mathrm{d}t\right) \\ \quad \times \left(SZ^{(q)}(\bar\xi(a)) + W^{(q)}(\bar\xi(a))\left(\frac{1-qSW^{(q)}(\bar\xi(a))}{W^{(q)'}(\bar\xi(a))}\right)\right) \\ \quad + S\int_x^a \exp\left(-\int_x^t \frac{W^{(q)'}(\bar\xi(s))}{W^{(q)}(\bar\xi(s))}\,\mathrm{d}s\right) \\ \quad \times \left(\frac{W^{(q)'}(\bar\xi(t))}{W^{(q)}(\bar\xi(t))}Z^{(q)}(\bar\xi(t)) - qW^{(q)}(\bar\xi(t))\right)\mathrm{d}t, & x\in[-l,a], \\ x-a+SZ^{(q)}(\bar\xi(a)) + W^{(q)}(\bar\xi(a))\left(\frac{1-qSW^{(q)}(\bar\xi(a))}{W^{(q)'}(\bar\xi(a))}\right), & x\in[a,\infty). \end{cases} \tag{3.7}$$

Proof. Recall that $\tau_\xi^a = \tau_\xi^{D_a}$ and let $U^a(t)$ denote the risk process $\{U^D(t);\ t\ge 0\}$ with $D(t) \equiv D_a(t) := (\bar X(t) - a)\vee 0$, i.e. with the barrier dividend strategy with barrier level $a\in[-l,\infty)$. Define the first up-crossing time of $U^a(t)$ as $\tau_b^+ := \inf\{t>0 \mid U^a(t) > b\}$ with the convention that $\inf\varnothing = \infty$.

The value function for the barrier strategy at level $a\in[-l,\infty)$ is expressed as

$$\begin{aligned} V_a(x) &= \mathbb{E}_x\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\,\mathrm{d}D_a(t) + S\mathrm{e}^{-q\tau_\xi^a}\right) \\ &= \mathbb{E}_x\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\,\mathrm{d}([\bar X(t)-a]\vee 0) + S\mathrm{e}^{-q\tau_\xi^a}\right) \\ &= \mathbb{E}_x\left(\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\int_0^{\tau_\xi^a-\tau_a^+} \mathrm{e}^{-qt}\,\mathrm{d}(\bar X(t+\tau_a^+)-a)\mathbf{1}_{\{\tau_a^+<\tau_\xi^a\}}\,\middle|\,\mathcal{F}_{\tau_a^+}\right)\right) + S\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}) \\ &= \mathbb{E}_x(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_\xi^a\}})\mathbb{E}_a\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\,\mathrm{d}(\bar X(t)-a)\right) + S\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}) \\ &= \mathbb{E}_x(\mathrm{e}^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}})\mathbb{E}_a\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\,\mathrm{d}(\bar X(t)-a)\right) + S\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}), \end{aligned} \tag{3.8}$$

where the first and second expectations on the right-hand side of the last equality are given by (3.3) and (3.1), respectively.

We now consider $\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a})$. By the strong Markov property, we have

$$\begin{aligned} \mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}) &= \mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}\mathbf{1}_{\{\tau_\xi^a<\tau_a^+\}}) + \mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^a}\mathbf{1}_{\{\tau_\xi^a\ge\tau_a^+\}}) \\ &= \mathbb{E}_x(\mathrm{e}^{-qT_\xi}\mathbf{1}_{\{T_\xi<T_a^+\}}) + \mathbb{E}_x(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_\xi^a>\tau_a^+\}})\mathbb{E}_a(\mathrm{e}^{-q\tau_\xi^a}) \\ &= \mathbb{E}_x(\mathrm{e}^{-qT_\xi}\mathbf{1}_{\{T_\xi<T_a^+\}}) + \mathbb{E}_x(\mathrm{e}^{-qT_a^+}\mathbf{1}_{\{T_a^+<T_\xi\}})\mathbb{E}_a(\mathrm{e}^{-q\tau_\xi^a}), \end{aligned} \tag{3.9}$$

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where we have used the fact that the process U^a evolves in the same way as the process X without dividend, before τ_a^+.

From Proposition 3.2 of [25], the first expectation on the right-hand side of the last line of (3.9) can be expressed as

$$\mathbb{E}_x(\mathrm{e}^{-qT_{\bar\xi}}\mathbf{1}_{\{T_{\bar\xi}<T_a^+\}}) = \int_x^a \exp\bigg(-\int_x^t \frac{W^{(q)'}(\bar\xi(s))}{W^{(q)}(\bar\xi(s))}\,\mathrm{d}s\bigg) \times\bigg(\frac{W^{(q)'}(\bar\xi(t))}{W^{(q)}(\bar\xi(t))}Z^{(q)}(\bar\xi(t)) - qW^{(q)}(\bar\xi(t))\bigg)\,\mathrm{d}t. \tag{3.10}$$

The expression in the second expectation on the right-hand side of the last line of (3.9) is given by (3.3). In order to deal with the third expectation on the right-hand side of the last line of (3.9), we first note that

$$\mathbb{E}_a(\exp(-q\tau_\xi^a)) = \mathbb{E}_{a-\xi(a)+l}(\exp(-q\sigma_0^{a-\xi(a)+l})), \tag{3.11}$$

where

$$\sigma_0^{a-\xi(a)+l} := \inf\{t>0 \mid U^{a-\xi(a)+l}(t) = X(t) - D_{a-\xi(a)+l}(t) < 0\}.$$

From the proof of Proposition 2 of [27],

$$\mathbb{E}_{a-\xi(a)+l}(\exp -q\sigma_0^{a-\xi(a)+l}) = Z^{(q)}(\bar\xi(a)) - qW^{(q)}(\bar\xi(a))\frac{W^{(q)}(\bar\xi(a))}{W^{(q)'}(\bar\xi(a))}. \tag{3.12}$$

Combining (3.1), (3.3), and (3.8)–(3.12) finally leads to (3.7). □

Remark 3.1. Given $W^{(q)}$ continuously differentiable, for $\xi(\cdot)\equiv 0$ and $l=0$, we see that (3.7) reduces to that of Proposition 2 of [27]. Indeed, we have, for all $x\in[-l,a]=[0,a]$,

$$\begin{aligned}
V_a(x) &= \exp\bigg(-\int_x^a \frac{W^{(q)'}(t)}{W^{(q)}(t)}\,\mathrm{d}t\bigg)\frac{W^{(q)}(a)}{W^{(q)'}(a)} \\
&\quad + S\int_x^a \exp\bigg(-\int_x^t \frac{W^{(q)'}(s)}{W^{(q)}(s)}\,\mathrm{d}s\bigg)\bigg(\frac{W^{(q)'}(t)}{W^{(q)}(t)}Z^{(q)}(t) - qW^{(q)}(t)\bigg)\,\mathrm{d}t \\
&\quad + S\exp\bigg(-\int_x^a \frac{W^{(q)'}(t)}{W^{(q)}(t)}\,\mathrm{d}t\bigg)\bigg(Z^{(q)}(a) - qW^{(q)}(a)\frac{W^{(q)}(a)}{W^{(q)'}(a)}\bigg) \\
&= \frac{W^{(q)}(x)}{W^{(q)'}(a)} - S\int_x^a \bigg(Z^{(q)}(t) - qW^{(q)}(t)\frac{W^{(q)}(t)}{W^{(q)'}(t)}\bigg)\,\mathrm{d}\bigg(\exp\bigg(-\int_x^t \frac{W^{(q)'}(s)}{W^{(q)}(s)}\,\mathrm{d}s\bigg)\bigg) \\
&\quad + S\exp\bigg(-\int_x^a \frac{W^{(q)'}(t)}{W^{(q)}(t)}\,\mathrm{d}t\bigg)\bigg(Z^{(q)}(a) - qW^{(q)}(a)\frac{W^{(q)}(a)}{W^{(q)'}(a)}\bigg) \\
&= \frac{W^{(q)}(x)}{W^{(q)'}(a)} + S\bigg(Z^{(q)}(x) - qW^{(q)}(x)\frac{W^{(q)}(a)}{W^{(q)'}(a)}\bigg),
\end{aligned}$$

which coincides with that of Proposition 2 of [27]. In addition, for $\xi(\cdot)\equiv 0$ and $l=0$, $V_a(x)$ in (3.7) also has the same expression as the function $V_a(x)$ in Proposition 2 of [27] for all $x\in(0,\infty)$.

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Proposition 3.2. *Suppose that the drawdown function $\xi(x)$ is continuously differentiable, and $W^{(q)}(x)$ is twice differentiable on $(0,\infty)$. Given any $q > 0$ and $b \in [-l,\infty)$, we have $V_a'(x) \geq 1$ for all $x, a \in [-l,b]$ with $x \leq a$ if and only if*

$$\Gamma(z) := -1 + \left(1 - qSW^{(q)}(\bar{\xi}(z))\right)\left[1 - \frac{W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{[W^{(q)'}(\bar{\xi}(z))]^2}\right](1-\xi'(z)) \geq 0 \quad (3.13)$$

for all $z \in [-l,b]$. In particular, if (3.13) holds for all $z \in [-l,a]$, then $V_a'(x) \geq 1$ for all $x \in [-l,a]$.

Proof. From the expression of the function $V_a(x)$, we have $V_a'(x) = 1$ for $x \geq a$. For $x \leq a$,

$$\begin{aligned}
V_a'(x) &= \frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))} \\
&\quad \times \left(\exp\left(-\int_x^a \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\frac{W^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right. \\
&\qquad + S\exp\left(-\int_x^a \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\left(Z^{(q)}(\bar{\xi}(a)) - qW^{(q)}(\bar{\xi}(a))\frac{W^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right) \\
&\qquad + S\int_x^a \exp\left(-\int_x^t \frac{W^{(q)'}(\bar{\xi}(s))}{W^{(q)}(\bar{\xi}(s))}\,\mathrm{d}s\right) \\
&\qquad\quad \left.\times\left(\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}Z^{(q)}(\bar{\xi}(t)) - qW^{(q)}(\bar{\xi}(t))\right)\mathrm{d}t\right) \\
&\quad - S\left(\frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\right) \\
&= \frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}V_a(x) - S\left(\frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\right). \qquad (3.14)
\end{aligned}$$

We then have

$$V_a'(x) \geq 1 \quad \text{for all } x, a \in [-l,b],\ x \leq a,$$

if and only if

$$\begin{aligned}
V_a(x) &\geq \frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))} \\
&\quad + S\left(Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))}\right) \quad \text{for all } x, a \in [-l,b],\ x \leq a,
\end{aligned}$$

if and only if

$$\begin{aligned}
&\exp\left(-\int_x^a \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\left(SZ^{(q)}(\bar{\xi}(a)) + W^{(q)}(\bar{\xi}(a))\left(\frac{1-qSW^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right)\right) \\
&\quad + S\int_x^a \exp\left(-\int_x^t \frac{W^{(q)'}(\bar{\xi}(s))}{W^{(q)}(\bar{\xi}(s))}\,\mathrm{d}s\right)\left(\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}Z^{(q)}(\bar{\xi}(t)) - qW^{(q)}(\bar{\xi}(t))\right)\mathrm{d}t \\
&\qquad \geq \frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))} \\
&\qquad\quad + S\left(Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))}\right) \quad \text{for all } x, a \in [-l,b],\ x \leq a,
\end{aligned}$$

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if and only if

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\left(\exp\left(-\int_x^z \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\left(SZ^{(q)}(\bar{\xi}(z))+W^{(q)}(\bar{\xi}(z))\left(\frac{1-qSW^{(q)}(\bar{\xi}(z))}{W^{(q)'}(\bar{\xi}(z))}\right)\right)\right.\\&\quad\left.+S\int_x^z \exp\left(-\int_x^t \frac{W^{(q)'}(\bar{\xi}(s))}{W^{(q)}(\bar{\xi}(s))}\,\mathrm{d}s\right)\left(\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}Z^{(q)}(\bar{\xi}(t))-qW^{(q)}(\bar{\xi}(t))\right)\mathrm{d}t\right)\\&\quad\geq 0\quad\text{for all } x,z\in[-l,b],\end{aligned}$$

if and only if

$$\begin{aligned}&\exp\left(-\int_x^z \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\\&\quad\times\left(-1+(1-qSW^{(q)}(\bar{\xi}(z)))\left[1-\frac{W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{[W^{(q)'}(\bar{\xi}(z))]^2}\right](1-\xi'(z))\right)\\&\quad\geq 0\quad\text{for all } x,z\in[-l,b],\end{aligned}$$

if and only if

$$\exp\left(-\int_x^z \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\Gamma(z)\geq 0\quad\text{for all } x,z\in[-l,b],$$

if and only if (3.13) holds for all $z\in[-l,b]$. □

Remark 3.2. Given $W^{(q)}(x)$ continuously differentiable and $a\in[-l,\infty)$, it follows from (3.14) and Proposition 3.1 that $\lim_{x\uparrow a}V_a'(x)=1$ for $a>-l$, and $\lim_{x\downarrow a}V_a'(x)=1$ is obvious from (3.7). The function $V_a(x)$ is thus continuously differentiable in x and $V_a'(a)=1$, and the differentiability of the drawdown function $\xi(x)$ is not required. In fact, we also have $V_{-l}'(-l)=1$ if $V_{-l}'(-l)$ is understood as the right derivative of $V_{-l}(x)$ at $x=-l$.

However, $V_a(x)$ is not twice continuously differentiable in x in general even if $\xi(x)$ is continuously differentiable and $W^{(q)}(x)$ is twice continuously differentiable on $(0,\infty)$. Indeed, $\lim_{x\downarrow a}V_a''(x)=0$ is obvious from (3.7). But taking derivatives and then taking limits on both sides of (3.14) yields, for $a>-l$,

$$\begin{aligned}\lim_{x\uparrow a}V_a''(x)&=\frac{W^{(q)'}(\bar{\xi}(a))}{W^{(q)}(\bar{\xi}(a))}V_a'(a)+(V_a(a)-SZ^{(q)}(\bar{\xi}(a)))\left[\frac{W^{(q)'}(\bar{\xi}(a))}{W^{(q)}(\bar{\xi}(a))}\right]'\\&=\frac{W^{(q)'}(\bar{\xi}(a))}{W^{(q)}(\bar{\xi}(a))}\\&\quad+\left(\frac{1-qSW^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right)\frac{W^{(q)''}(\bar{\xi}(a))W^{(q)}(\bar{\xi}(a))-[W^{(q)'}(\bar{\xi}(a))]^2}{W^{(q)}(\bar{\xi}(a))}(1-\xi'(a))\\&=-\frac{W^{(q)'}(\bar{\xi}(a))}{W^{(q)}(\bar{\xi}(a))}\\&\quad\times\left[-1+(1-qSW^{(q)}(\bar{\xi}(a)))\left[1-\frac{W^{(q)}(\bar{\xi}(a))W^{(q)''}(\bar{\xi}(a))}{[W^{(q)'}(\bar{\xi}(a))]^2}\right](1-\xi'(a))\right]\\&=-\frac{W^{(q)'}(\bar{\xi}(a))}{W^{(q)}(\bar{\xi}(a))}\Gamma(a),\end{aligned}\tag{3.15}$$

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which is not equal to 0 unless either $W^{(q)'}(\bar{\xi}(a)) = 0$ or $\Gamma(a) = 0$. In addition, $V_a(x)$ is indeed twice continuously differentiable in x when $a = -l$.

Remark 3.3. Suppose that the drawdown function ξ is differentiable and $W^{(q)}$ is twice differentiable on $(0, \infty)$. Given $a \geq -l$, since for all $x \in [-l, a]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}V_a(x) &= \exp\left(-\int_x^a \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\left(-1 + (1 - qSW^{(q)}(\bar{\xi}(a)))\left[\frac{W^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right]'\right)\\ &= \exp\left(-\int_x^a \frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\Gamma(a) \qquad (3.16)\end{aligned}$$

and, for all $x \in (a, \infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}a}V_a(x) = \Gamma(a), \qquad (3.17)$$

we know that $(\mathrm{d}/\mathrm{d}a)V_a(x)$ has the same sign as the function $\Gamma(a)$ for all $x \in [-l, \infty)$ and $a \in [-l, \infty)$.

Proposition 3.2 is useful in the following definition.

Definition 3.1. Suppose that the drawdown function ξ is differentiable and $W^{(q)}$ is twice differentiable. Let

$$a^* := \sup\{a \in [-l, \infty) \mid \Gamma(z) \geq 0 \text{ for all } z \in [-l, a]\} \qquad (3.18)$$

with the convention that $\sup \varnothing = -l$.

Remark 3.4. Suppose that $\xi(x)$ is continuously differentiable and $W^{(q)} \in C^2(0, \infty)$, then $\Gamma(x) \in C[-l, \infty)$. If $a^* \in (-l, \infty)$, by the definition of a^*, we have $\Gamma(a^*) = 0$. It follows from (3.7) that $\lim_{x \downarrow a^*} V''_{a^*}(x) = 0$. In addition, by (3.15), we have

$$\lim_{x \uparrow a^*} V''_{a^*}(x) = -\frac{W^{(q)'}(\bar{\xi}(a^*))}{W^{(q)}(\bar{\xi}(a^*))}\Gamma(a^*) = 0.$$

That is to say, if $a^* \in (-l, \infty)$ then $V_{a^*} \in C^2(-l, \infty)$ and $V''_{a^*}(a^*) = 0$. If $a^* = -l$, we have $V_{a^*}(x) = x + l + V_{-l}(-l)$ for all $x \in [-l, \infty)$. Thus, we also have $V_{a^*} \in C^2(-l, \infty)$ and $V''_{a^*}(a^*) = 0$ for $a^* = -l$. Here, we take $V''_{-l}(-l)$ as the right derivative of $V'_{-l}(x)$ at $x = -l$ after additionally defining $V'_{-l}(-l)$ as the right derivative of $V_{-l}(x)$ at $x = -l$.

In the following proposition, we provide a sufficient condition for $a^* < \infty$.

Proposition 3.3. *Given the function ξ continuously differentiable with $\sup_{x \in [-l, \infty)} |\xi'(x)| < \infty$, we have $a^* < \infty$ if the Lévy measure ν of the process X has a completely monotone density.*

Proof. Under the assumption of this lemma, the q-scale function can be written as

$$W^{(q)}(x) = \frac{\mathrm{e}^{\Psi(q)x}}{\psi'(\Psi(q))} - \int_{0+}^{\infty} \mathrm{e}^{-xt}\mu(\mathrm{d}t + \Psi(q)),$$

where μ is a measure on $(0, \infty)$ satisfying $\int_{0+}^{\infty}(t \wedge 1)\mu(\mathrm{d}t) < \infty$ and Ψ is the inverse function of ψ; see Theorem 1 of [27].

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Since the functions t^2e^{-xt}, te^{-xt}, and e^{-xt} are all monotone with respect to x satisfying

$$\lim_{x\to+\infty} t^2e^{-xt} = \lim_{x\to+\infty} te^{-xt} = \lim_{x\to+\infty} e^{-xt} = 0,$$

then $\mu(dt+\Psi(q))$ is a finite measure on $(0,\infty)$. Hence, by the monotone convergence theorem, we have

$$\begin{aligned}\lim_{x\to+\infty}\int_{0+}^{\infty} t^2e^{-xt}\mu(dt+\Psi(q)) &= \lim_{x\to+\infty}\int_{0+}^{\infty} te^{-xt}\mu(dt+\Psi(q))\\ &= \lim_{x\to+\infty}\int_{0+}^{\infty} e^{-xt}\mu(dt+\Psi(q))\\ &= 0.\end{aligned}$$

It follows that

$$\begin{aligned}&1-\frac{W^{(q)}(x)W^{(q)''}(x)}{[W^{(q)'}(x)]^2}\\ &= 1-\frac{(e^{\Psi(q)x}/\psi'(\Psi(q))-\int_{0+}^{\infty}e^{-xt}\mu(dt+\Psi(q)))(((\Psi(q))^2/\psi'(\Psi(q)))e^{\Psi(q)x}-\int_{0+}^{\infty}t^2e^{-xt}\mu(dt+\Psi(q)))}{[(\Psi(q)/\psi'(\Psi(q)))e^{\Psi(q)x}+\int_{0+}^{\infty}te^{-xt}\mu(dt+\Psi(q))]^2}\\ &= o(e^{-\Psi(q)x}) \quad \text{as } x\to+\infty. \end{aligned} \tag{3.19}$$

From (3.19) and the facts that

$$(1-qSW^{(q)}(x)) = O(e^{\Psi(q)x}), \qquad |1-\xi'(x)| \le 1+\sup_{x\in[-l,\infty)}|\xi'(x)| < +\infty,$$

and from the assumption, we arrive at

$$\lim_{a\to\infty}\left|(1-qSW^{(q)}(\bar\xi(a)))\left[1-\frac{W^{(q)}(\bar\xi(a))W^{(q)''}(\bar\xi(a))}{[W^{(q)'}(\bar\xi(a))]^2}\right](1-\xi'(a))\right| = 0.$$

Therefore, we can claim that there exists a large enough $x_0\in(0,+\infty)$ such that $\Gamma(z)<0$ for all $z\ge x_0$. The proof is complete. □

4. Verification arguments and characterization of the optimal value functions and the optimal dividend strategies

To find the optimal dividend strategy and the optimal value function, the following results concerning properties of the optimal value function are useful.

Proposition 4.1. *The function $V(x)=\sup_{D\in\mathcal{D}}V_D(x)$ is nondecreasing and continuous in $[-l,\infty)$. In particular, $V(y)-V(x)\ge y-x$ for $y\ge x\ge -l$.*

Proof. The proof is left to the interested reader. □

Proposition 4.2. *Suppose that the function ξ is continuously differentiable with*

$$\sup_{x\in[-l,\infty)}\xi'(x)\le 1,$$

and $W^{(q)}$ is twice differentiable. Then $V_{a^*}(-l)<S$ if and only if X is a compound Poisson process with drift and

$$0<\frac{\gamma+\int_{(0,1)}x\nu(dx)}{q}<S. \tag{4.1}$$

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Proof. Define an auxiliary function

$$\begin{aligned}\phi(x) &:= V_x(-l)\\ &= \exp\left(-\int_{-l}^{x}\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\mathrm{d}t\right)\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))}\\ &\quad + S\int_{-l}^{x}\exp\left(-\int_{-l}^{t}\frac{W^{(q)'}(\bar{\xi}(s))}{W^{(q)}(\bar{\xi}(s))}\mathrm{d}s\right)\left(\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}Z^{(q)}(\bar{\xi}(t)) - qW^{(q)}(\bar{\xi}(t))\right)\mathrm{d}t\\ &\quad + S\exp\left(-\int_{-l}^{x}\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\left(Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))}\right). \quad (4.2)\end{aligned}$$

Differentiating both sides of (4.2) with respect to x leads to

$$\phi'(x) = \exp\left(-\int_{-l}^{x}\frac{W^{(q)'}(\bar{\xi}(t))}{W^{(q)}(\bar{\xi}(t))}\,\mathrm{d}t\right)\Gamma(x).$$

Recalling the definition of a^* (see (3.18)) and Assumption 4.1, we have

$$\begin{aligned}S + W^{(q)}(0)\left(\frac{1-qSW^{(q)}(0)}{W^{(q)'}(0)}\right) &= SZ^{(q)}(0) + W^{(q)}(0)\left(\frac{1-qSW^{(q)}(0)}{W^{(q)'}(0)}\right)\\ &= SZ^{(q)}(\bar{\xi}(-l)) + W^{(q)}(\bar{\xi}(-l))\left(\frac{1-qSW^{(q)}(\bar{\xi}(-l))}{W^{(q)'}(\bar{\xi}(-l))}\right)\\ &= V_{-l}(-l)\\ &= \phi(-l)\\ &\le \phi(a^*)\\ &= V_{a^*}(-l)\\ &< S,\end{aligned}$$

which happens only if $S > 0$, $W^{(q)}(0)/W^{(q)'(0)} \in (0,\infty)$, and $1 - qSW^{(q)}(0) < 0$. Then $W^{(q)}(0) > 0$ implies that X has sample paths of bounded variation, which together with $W^{(q)'}(0) \in (0,\infty)$ further implies that $\nu(0,\infty) < \infty$; see, for example, [5] and [27]. Consequently, X is a compound Poisson process with drift and

$$0 > 1 - qSW^{(q)}(0) = 1 - \frac{qS}{\gamma + \int_{(0,1)} x\nu(\mathrm{d}x)} \quad\Longleftrightarrow\quad 0 < \frac{\gamma + \int_{(0,1)} x\nu(\mathrm{d}x)}{q} < S. \quad (4.3)$$

By the definition of the function $\Gamma(x)$ (see (3.13)), we have, for any $x \in [-l,\infty)$,

$$\Gamma(x) = -1 + (1 - qSW^{(q)}(\bar{\xi}(x)))\left[\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)'}(\bar{\xi}(x))}\right]' < 0, \quad (4.4)$$

which requires $a^* = -l$; see (3.18). Here we have used the facts that

$$1 - qSW^{(q)}(\bar{\xi}(-l)) = 1 - qSW^{(q)}(0) < 0 \quad\text{and}\quad \left[\frac{W^{(q)}(\bar{\xi}(a))}{W^{(q)'}(\bar{\xi}(a))}\right]' \ge 0$$

since $[W^{(q)}(x)/W^{(q)'(x)}]' \ge 0$, $\bar{\xi}'(x) = 1 - \xi'(x) \ge 0$ by the assumption, and $W^{(q)}(\bar{\xi}(x))$ is increasing for $x \in [-l,\infty)$.

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Conversely, if X is a compound Poisson process with drift and (4.1) holds (hence, the first inequality in (4.3) holds), then by (3.7), we have

$$\begin{aligned} V_{-l}(-l) &= \frac{W^{(q)}(\bar{\xi}(-l))}{W^{(q)'}(\bar{\xi}(-l))} + S\left(Z^{(q)}(\bar{\xi}(-l)) - qW^{(q)}(\bar{\xi}(-l))\frac{W^{(q)}(\bar{\xi}(-l))}{W^{(q)'}(\bar{\xi}(-l))}\right) \\ &= \frac{W^{(q)}(0)}{W^{(q)'}(0)} + S\left(Z^{(q)}(0) - qW^{(q)}(0)\frac{W^{(q)}(0)}{W^{(q)'}(0)}\right) \\ &< S. \end{aligned} \tag{4.5}$$

Here, $W^{(q)}(0)/W^{(q)'(0)} \in (0, \infty)$ since $X(t)$ is a compound Poisson process with positive drift. Also, $(1 - qSW^{(q)}(0)) < 0$ due to the equivalence in (4.3). Hence, (4.4) also holds for any $x \in [-l, \infty)$, which implies that $a^* = -l$. Therefore, $V_{a^*}(-l) < S$ by (4.5). □

Proposition 4.3. *For all $x \in [-l, \infty)$, we have $V(x) = \sup_{D\in\mathcal{D}} V_D(x) \geq S$.*

Proof. We first suppose that X has paths of bounded variation or, equivalently,

$$\int_{(0,1)} y\nu(\mathrm{d}y) < \infty \quad \text{and} \quad \sigma = 0.$$

Then

$$X(t) = x + \left(\gamma + \int_{(0,1)} y\nu(\mathrm{d}y)\right)t - \int_0^t \int_{(0,\infty)} yN(\mathrm{d}s,\, \mathrm{d}y)$$

with $\gamma + \int_{(0,1)} y\nu(\mathrm{d}y) \in (0, \infty)$. Define an admissible dividend strategy D_1 as

$$D_1(0) = 0, \qquad D_1(t) = x - \xi(x) + l + ct \quad \text{for all } t > 0$$

with $c > \gamma + \int_{(0,1)} y\nu(\mathrm{d}y)$ being a constant. It is not difficult to see that $\tau_\xi^{D_1} = 0$ and

$$\sup_{D\in\mathcal{D}} V_D(x) \geq V_{D_1}(x) = \mathbb{E}_x\left(\int_0^{\tau_\xi^{D_1}} \mathrm{e}^{-qt}\,\mathrm{d}D_1(t) + S\mathrm{e}^{-q\tau_\xi^{D_1}}\right) = x - \xi(x) + l + S \geq S.$$

In addition, suppose that the process X has paths of unbounded variation, or

$$\int_{(0,1)} y\nu(\mathrm{d}y) = \infty.$$

Define an admissible dividend strategy as $D_2(0) = 0$ and $D_2(t) = x - \xi(x) + l$ for $t > 0$. Note that for a Lévy process X with paths of unbounded variation, the point $\xi(x) - l$ is regular for the open set $(-\infty, \xi(x) - l)$. Then $\tau_\xi^{D_2} = 0$ and

$$\sup_{D\in\mathcal{D}} V_D(x) \geq V_{D_2}(x) = \mathbb{E}_x\left(\int_0^{\tau_\xi^{D_2}} \mathrm{e}^{-qt}\,\mathrm{d}D_1(t) + S\mathrm{e}^{-q\tau_\xi^{D_2}}\right) = x - \xi(x) + l + S \geq S,$$

completing the proof. □

Let $\mathcal{A}$ be an operator acting on the functions $f \in C^2(\mathbb{R})$ and defined as

$$\mathcal{A}f(x) := \gamma f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{(0,\infty)} (f(x - y) - f(x) + f'(x)y\mathbf{1}_{(0,1)}(y))\nu(\mathrm{d}y).$$

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Lemma 4.1. (Verification lemma.) *Suppose that D^* is an admissible dividend strategy such that $V_{D^*}(x)$ is twice continuously differentiable, i.e. $V_{D^*}(x) \in C^2(-l, \infty)$. Further, suppose that, for all $x \in [-l, \infty)$,*

$$\max\{\mathcal{A}V_{D^*}(x) - qV_{D^*}(x), 1 - V'_{D^*}(x)\} \le 0 \quad \textit{and} \quad V_{D^*}(x) \ge S. \tag{4.6}$$

Then D^* is the optimal dividend strategy, i.e. $V_{D^*}(x) = \sup_{D\in\mathcal{D}} V_D(x)$ for all $x \in [-l, \infty)$.

Proof. Recall that τ_b^+ is the first up-crossing time of level b by the process U^D. For a given dividend strategy $D \in \mathcal{D}$, define $\Pi^D(t) := \{s \in [0, t) \mid D(s+) - D(s) \neq 0\}$ as the set of time s (equal to or less than t) when a jump occurs for the dividend process. Let the continuous parts of the processes $\{U^D(t);\ t \ge 0\}$ and $\{D(t);\ t \ge 0\}$ be denoted by $\{U_c^D(t);\ t \ge 0\}$ and $\{D_c(t);\ t \ge 0\}$, respectively. Suppose that $f(x)$ is any sufficiently smooth function satisfying condition (4.6) with V_{D^*} replaced by f. From Theorem 4.57 (Itô's formula) of [19, p. 57], we have

$$\begin{aligned}
&\mathrm{e}^{-q(t\wedge\tau_\xi^D)} f(U^D(t\wedge\tau_\xi^D)) \\
&\quad = f(x) - \int_{0-}^{t\wedge\tau_\xi^D} q\mathrm{e}^{-qs} f(U^D(s))\,\mathrm{d}s + \int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}U^D(s) \\
&\qquad + \frac{1}{2}\int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs} f''(U^D(s))\,\mathrm{d}\langle U_c^D(\cdot), U_c^D(\cdot)\rangle_s \\
&\qquad + \sum_{s\le t\wedge\tau_\xi^D} \mathrm{e}^{-qs}[f(U^D(s-) + \Delta U^D(s)) - f(U^D(s-)) - f'(U^D(s-))\Delta U^D(s)] \\
&\qquad + \sum_{s\in\Pi^D(t\wedge\tau_\xi^D)} \mathrm{e}^{-qs}(f(U^D(s+)) - f(U^D(s)) + f'(U^D(s))(D(s+) - D(s))) \\
&\quad = f(x) + \int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs}(\mathcal{A} - q) f(U^D(s))\,\mathrm{d}s + \int_{0-}^{t\wedge\tau_\xi^D} \sigma\mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}B_s \\
&\qquad - \int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}D_c(s) - \sum_{s\le t\wedge\tau_\xi^D} \mathrm{e}^{-qs}(D(s+) - D(s)) \\
&\qquad + \int_{0-}^{t\wedge\tau_\xi^D}\int_0^\infty \mathrm{e}^{-qs}[f(U^D(s-) - y) - f(U^D(s-))](N(\mathrm{d}s,\,\mathrm{d}y) - \nu(\mathrm{d}y)\,\mathrm{d}s) \\
&\qquad + \sum_{s\in\Pi^D(t\wedge\tau_\xi^D)} \mathrm{e}^{-qs}[f(U^D(s+)) - f(U^D(s+) + (D(s+) - D(s))) \\
&\qquad\qquad + (D(s+) - D(s))],
\end{aligned} \tag{4.7}$$

where $\Delta U^D(s) = U^D(s) - U^D(s-)$ and $x \in (-l, \infty)$. Note that the second equation of (4.7) is indeed an alternative form of Itô's formula; see Theorem 4.4.7 of [1].

Since $f'(x) \ge 1$, we have

$$f(U^D(s+)) - f(U^D(s+) + (D(s+) - D(s))) + (D(s+) - D(s)) \le 0. \tag{4.8}$$

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Therefore, from (4.7) and (4.8), we have

$$\begin{aligned}
&\mathrm{e}^{-q(t\wedge\tau_\xi^D)} f(U^D_{t\wedge\tau_\xi^D}) \\
&\quad \le f(x) + \int_0^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs}(\mathcal{A}-q) f(U^D(s))\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_\xi^D}\int_0^\infty \mathrm{e}^{-qs}[f(U^D(s-)-y) - f(U^D(s-))](N(\mathrm{d}s,\,\mathrm{d}y) - \nu(\mathrm{d}y)\,\mathrm{d}s) \\
&\qquad + \int_{0-}^{t\wedge\tau_\xi^D} \sigma \mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}B_s - \int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}D_c(s) \\
&\qquad - \sum_{s\le t\wedge\tau_\xi^D} \mathrm{e}^{-qs}(D(s+) - D(s)) \\
&\quad \le f(x) + \int_0^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs}(\mathcal{A}-q) f(U^D(s))\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_\xi^D}\int_0^\infty \mathrm{e}^{-qs}[f(U^D(s-)-y) - f(U^D(s-))](N(\mathrm{d}s,\,\mathrm{d}y) - \nu(\mathrm{d}y)\,\mathrm{d}s) \\
&\qquad + \int_{0-}^{t\wedge\tau_\xi^D} \sigma \mathrm{e}^{-qs} f'(U^D(s))\,\mathrm{d}B_s - \int_{0-}^{t\wedge\tau_\xi^D} \mathrm{e}^{-qs}\,\mathrm{d}D(s), \qquad x \in (-l,\infty). \qquad (4.9)
\end{aligned}$$

In addition, it follows from [18, p. 62] that the compensated sum

$$t \mapsto \int_0^t \int_0^\infty \mathrm{e}^{-qs}[f(U^D(s-)-y) - f(U^D(s-))](N(\mathrm{d}s,\,\mathrm{d}y) - \nu(\mathrm{d}y)\,\mathrm{d}s)$$

is an $(\mathcal{F}_t)$-local martingale.

Define a sequence of localization stopping times $\{T_{m,n};\ m \ge 1, n \ge 1\}$ such that

$$T_{m,n} = \inf\left\{t \ge 0 \mid \text{either } U^D(t) > n \text{ or } U^D(t) < \xi(\bar{U}^D(t)) - l + \frac{1}{m}\right\}.$$

Then, for all $t < T_{m,n}$, we have

$$-l + \frac{1}{m} \le \xi(\bar{U}^D(t)) - l + \frac{1}{m} \le U^D(t) \le n,$$

which implies that the associated $f(U^D(t))$, $f(U^D(t^-))$, and $f'(U^D(t))$ are all bounded since f and f' are both continuous functions and $U^D(t^-)$ and $U^D(t)$ are both restricted to the bounded closed interval $[-l + 1/m, n]$. Taking expectations on both sides of (4.9) after localizing, and recalling that $(\mathcal{A}-q) f(U^D(s)) \le 0$, we have, for $x \in (-l,\infty)$,

$$f(x) \ge \mathbb{E}_x\big(\mathrm{e}^{-q(t\wedge T_{m,n}\wedge\tau_\xi^D)} f(U^D_{t\wedge T_{m,n}\wedge\tau_\xi^D})\big) + \mathbb{E}_x\left(\int_0^{t\wedge T_{m,n}\wedge\tau_\xi^D} \mathrm{e}^{-qs}\,\mathrm{d}D(s)\right).$$

Finally, letting t, m, and n go to ∞ in the above inequality and noting that $f(x) \ge S$ for all $x \in [-l,\infty)$, we have

$$f(x) \ge \mathbb{E}_x\left(\int_0^{\tau_\xi^D} \mathrm{e}^{-qs}\,\mathrm{d}D(s)\right) + S\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi^D}) = V_D(x), \qquad x \in (-l,\infty).$$

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Since the strategy D is arbitrary, it follows that

$$f(x) \geq \sup_{D\in\mathcal{D}} V_D(x), \qquad x \in (-l, \infty),$$

which, together with the continuity of $f(x)$ and $\sup_{D\in\mathcal{D}} V_D(x)$ (see Proposition 4.1), implies that

$$f(x) \geq \sup_{D\in\mathcal{D}} V_D(x), \qquad x \in [-l, \infty),$$

which, in turn, implies that

$$V_{D^*}(x) \geq \sup_{D\in\mathcal{D}} V_D(x), \qquad x \in [-l, \infty), \tag{4.10}$$

from the assumption. The reverse inequality of (4.10) is trivial. □

Before presenting Theorem 4.1, the main result of this paper, a candidate of sufficient condition under which the barrier type dividend strategy turns out to be the optimal dividend strategy is given in the following assumption.

Assumption 4.1. *The drawdown function ξ is continuously differentiable and $W^{(q)}$ is twice continuously differentiable. Given the a^* in (3.18) and the function Γ in (3.13), for all $x \in [a^*, \infty)$, we have*

$$\Gamma(x) \leq 0. \tag{4.11}$$

Remark 4.1. For $\xi(x) \equiv 0$ and $l = 0$, Assumption 4.1 degenerates to

$$-1 + (1 - qSW^{(q)}(x))\left[1 - \frac{W^{(q)}(x)W^{(q)''}(x)}{[W^{(q)'}(x)]^2}\right] \leq 0 \quad \text{for all } x \in [a^*, \infty) \subseteq [0, \infty).$$

After some calculation, we arrive at

$$\begin{aligned} &-1 + (1 - qSW^{(q)}(x))\left[1 - \frac{W^{(q)}(x)W^{(q)''}(x)}{[W^{(q)'}(x)]^2}\right] \\ &\quad = W^{(q)}(x)\left(-qS - (1 - qSW^{(q)}(x))\frac{W^{(q)''}(x)}{[W^{(q)'}(x)]^2}\right) \\ &\quad = W^{(q)}(x)\left(\frac{1 - qSW^{(q)}(x)}{W^{(q)'}(x)}\right)', \end{aligned}$$

from which we know that, for $\xi(x) \equiv 0$ and $l = 0$, Assumption 4.1 is equivalent to

$$\frac{1 - qSW^{(q)}(x)}{W^{(q)'}(x)} \leq \frac{1 - qSW^{(q)}(y)}{W^{(q)'}(y)} \quad \text{for all } x \text{ and } y \text{ such that } 0 \leq a^* \leq y \leq x < +\infty,$$

which coincides with Equation (9) of [27] and holds if the Lévy measure has a completely monotone density.

Lemma 4.2. *Under Assumption 4.1, we have $V_b'(x) - V_{a^*}'(x) \leq 0$ for all $b \geq a^*$ and $x \geq -l$. In addition, under Assumption 4.1, we have $\lim_{x\uparrow a} V_a''(x) \geq 0$ for all $a > a^*$.*

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Proof. We consider the following cases separately.

Case 1. For $-l \le x \le a^* \le b$. From (3.14), we have

$$V_b'(x) - V_{a^*}'(x) = \frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}(V_b(x) - V_{a^*}(x)) \le 0 \tag{4.12}$$

since, by Remark 3.3, the value of the function $(\mathrm{d}/\mathrm{d}a)V_a(x)(x \in [-l, \infty))$ has the same sign as $\Gamma(a)$, and the latter is nonpositive for all $a \in [a^*, \infty)$ by Assumption 4.1.

Case 2. For $a^* \le x \le b$. We have

$$\begin{aligned} V_b'(x) - V_{a^*}'(x) &= \frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}V_b(x) - S\left(\frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}Z^{(q)}(\bar{\xi}(x)) - qW^{(q)}(\bar{\xi}(x))\right) - 1 \\ &= \frac{W^{(q)'}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(x))}(V_b(x) - V_x(x)) \\ &\le 0 \end{aligned} \tag{4.13}$$

since, from (3.16), (3.17), and Assumption 4.1, $V_b(x)$ is a nonincreasing function with respect to the variable $b \in [a^*, \infty)$.

Case 3. For $a^* \le b \le x$. We have

$$V_b'(x) - V_{a^*}'(x) = 1 - 1 = 0. \tag{4.14}$$

Case 4. For all $a > a^*$. Combining (3.15) and Assumption 4.1, we have $\lim_{x \uparrow a} V_a''(x) \ge 0$.

The proof is complete. □

Remark 4.2. For the $V_{a^*}(-l) < S$ case, however, from Proposition 4.3 we know that the barrier dividend strategy with barrier level a^* is not the optimal dividend strategy.

In the following theorem we characterize the optimal dividend strategy and the optimal value function, where a similar proof argument as in [26] is adopted. We show that Assumption 4.1 is a sufficient condition under which the barrier type dividend strategy is the optimal dividend strategy whenever the underlying process does not belong to a small class of compound Poisson processes with drift, for which the take-the-money-and-run dividend strategy is optimal.

Theorem 4.1. *Suppose that the drawdown function ξ is continuously differentiable and the scale function $W^{(q)}$ is twice differentiable. Given the a^* from (3.18), the optimal dividend strategies of the optimization problem (2.2) can be characterized in the following two cases.*

(i) *If Assumption 4.1 holds and $V_{a^*}(-l) \ge S$, then the value function $V_{a^*}(x)$ satisfies the HJB inequalities*

$$\begin{aligned} &\max\{\mathcal{A}V_{a^*}(x) - qV_{a^*}(x), 1 - V_{a^*}'(x)\} \le 0, \\ &V_{a^*}(x) \ge S \quad \text{for all } x \in [-l, +\infty). \end{aligned} \tag{4.15}$$

Consequently, the barrier dividend strategy with barrier level a^* is the optimal dividend strategy among all admissible strategies.

(ii) *If $\sup_{x \in [-l, \infty)} \xi'(x) \le 1$ and $V_{a^*}(-l) < S$, then the function $V_{\mathrm{run}}(x) := x + l + S$, $x \in [-l, +\infty)$ satisfies the HJB inequalities (4.15) and the take-the-money-and-run strategy is the optimal dividend strategy.*

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Proof. (i) We need to prove $\max\{\mathcal{A}V_{a^*}(x) - qV_{a^*}(x), 1 - V'_{a^*}(x)\} \le 0$ and $V_{a^*}(x) \ge S$ for all $x \in [-l, \infty)$. By definition, $V'_{a^*}(x) = 1$ for all $x \ge a^*$. In addition, from the definition of a^*, the continuity of function $\Gamma(x)$, and Proposition 3.2, we obtain $V'_{a^*}(x) \ge 1$ for $x \in [-l, a^*]$. We then have $1 - V'_{a^*}(x) \le 0$ for $x \in [-l, \infty)$.

By the strong Markov property of the process X, we have, for all $x \in [-l, a)$,

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qt}\,\mathrm{d}([\bar{X}(t) - a] \vee 0) + S\mathrm{e}^{-q\tau_\xi^D} \;\middle|\; \mathcal{F}_{r\wedge\tau_a^+\wedge\tau_\xi^a}\right)\\
&\quad= \mathbb{E}_x\left(\int_0^{\tau_\xi^a - r\wedge\tau_a^+\wedge\tau_\xi^a} \mathrm{e}^{-q(s+r\wedge\tau_a^+\wedge\tau_\xi^a)}\right.\\
&\qquad\qquad\left.\times\,\mathrm{d}\Big(\Big[\sup_{r\wedge\tau_a^+\wedge\tau_\xi^a \le u \le s+r\wedge\tau_a^+\wedge\tau_\xi^a} X(u) - a\Big] \vee 0\Big) \;\middle|\; \mathcal{F}_{r\wedge\tau_a^+\wedge\tau_\xi^a}\right)\\
&\qquad+ \mathrm{e}^{-q(r\wedge\tau_a^+\wedge\tau_\xi^a)}\mathbb{E}_{X(r\wedge\tau_a^+\wedge\tau_\xi^a)}(S\mathrm{e}^{-q\tau_\xi^a})\\
&\quad= \mathrm{e}^{-q(r\wedge\tau_a^+\wedge\tau_\xi^a)}\mathbb{E}_{X(r\wedge\tau_a^+\wedge\tau_\xi^a)}\left(\int_0^{\tau_\xi^a} \mathrm{e}^{-qs}\,\mathrm{d}([\bar{X}(s) - a] \vee 0) + S\mathrm{e}^{-q\tau_\xi^a}\right)\\
&\quad= \mathrm{e}^{-q(r\wedge\tau_a^+\wedge\tau_\xi^a)}V_a(U^a(r \wedge \tau_a^+ \wedge \tau_\xi^a)),
\end{aligned}$$

which implies that the right-hand side of the above display is a martingale. Here, we have used the fact that no dividends are paid out during the time interval $[0, \tau_a^+] \supseteq [0, r \wedge \tau_a^+ \wedge \tau_\xi^a]$; i.e. $U^a(r \wedge \tau_a^+ \wedge \tau_\xi^a) = X(r \wedge \tau_a^+ \wedge \tau_\xi^a)$ for all $r \ge 0$. The martingale property of the process

$$(\mathrm{e}^{-q(r\wedge\tau_a^+\wedge\tau_\xi^a)}V_a(U^a(r \wedge \tau_a^+ \wedge \tau_\xi^a)))_{r\ge 0}$$

implies that

$$\mathcal{A}V_a(x) - qV_a(x) = 0 \quad \text{for } x \in [-l, a). \tag{4.16}$$

For all $x \in (-l, a)$, an application of Itô's formula yields

$$\begin{aligned}
&\mathrm{e}^{-q(r\wedge\tau_a^+\wedge\tau_\xi^a)}V_a(U^a(r \wedge \tau_a^+ \wedge \tau_\xi^a)) - V_a(x)\\
&\quad= \int_{0-}^{r\wedge\tau_a^+\wedge\tau_\xi^a} \mathrm{e}^{-qs}(\mathcal{A} - q)V_a(U^a(s))\,\mathrm{d}s + \int_{0-}^{r\wedge\tau_a^+\wedge\tau_\xi^a} \sigma\mathrm{e}^{-qs}V'_a(U^a(s))\,\mathrm{d}B_s\\
&\qquad+ \int_{0-}^{r\wedge\tau_a^+\wedge\tau_\xi^a}\int_0^\infty \mathrm{e}^{-qs}[V_a(U^a(s-) - y) - V_a(U^a(s-))](N(\mathrm{d}s,\,\mathrm{d}y) - \nu(\mathrm{d}y)\,\mathrm{d}s).
\end{aligned}$$

Taking the expectation on both sides of the above display after localization, we have

$$0 = \mathbb{E}_x\int_{0-}^{r\wedge\tau_a^+\wedge\tau_\xi^a\wedge T_{m,n}} \mathrm{e}^{-qs}(\mathcal{A} - q)V_a(U^a(s))\,\mathrm{d}s, \qquad r > 0, \tag{4.17}$$

with $\{T_{m,n};\ m, n \ge 1\}$ being the sequence of localizing stopping times defined in Lemma 4.1. Dividing both sides of (4.17) by $\mathbb{E}_x(r \wedge \tau_a^+ \wedge \tau_\xi^a \wedge T_{m,n})$ and then letting $r \downarrow 0$, we obtain (4.16) for $x \in (-l, a)$. For a more detailed proof of (4.16), we can also turn to Proposition 2.1 of [17]. Letting $x \downarrow -l$, we know that (4.16) holds also for $x = -l$ since $(\mathcal{A} - q)V_a(x)$ is continuous over $[-l, a)$ by the definition of $V_a(x)$.

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Equation (4.16) implies that if $a^* > -l$ then $\mathcal{A}V_{a^*}(x) - qV_{a^*}(x) = 0$ for $x \in [-l, a^*)$. Hence, we only need to prove that

$$\mathcal{A}V_{a^*}(x) - qV_{a^*}(x) \le 0 \quad \text{for } x \in [a^*, \infty). \tag{4.18}$$

From (4.16), we have

$$\lim_{y \uparrow x}(\mathcal{A}V_x(y) - qV_x(y)) = 0 \quad \text{for } x > a^*. \tag{4.19}$$

In addition, from the continuity of the function $\mathcal{A}V_{a^*}(y) - qV_{a^*}(y)$ due to $V_{a^*}(y) \in C^2(-l, \infty)$ (see Remark 3.4), we have

$$\lim_{y \uparrow x}(\mathcal{A}V_{a^*}(y) - qV_{a^*}(y)) = \mathcal{A}V_{a^*}(x) - qV_{a^*}(x) \quad \text{for } x > a^*. \tag{4.20}$$

Combining (4.19) and (4.20), in order to prove (4.18) we need to prove

$$\lim_{y \uparrow x}(\mathcal{A}[V_{a^*}(y) - V_x(y)] - q[V_{a^*}(y) - V_x(y)]) \le 0 \quad \text{for } x > a^*.$$

We can use the dominated convergence theorem and (4.14) to show that, for $x > a^*$,

$$\begin{aligned}
&\lim_{y \uparrow x}(\mathcal{A}[V_{a^*}(y) - V_x(y)] - q[V_{a^*}(y) - V_x(y)]) \\
&\quad = \gamma([V'_{a^*}(x) - V'_x(x)]) + \frac{\sigma^2}{2}\Big[V''_{a^*}(x) - \lim_{y \uparrow x} V''_x(y)\Big] - q[V_{a^*}(x) - V_x(x)] \\
&\qquad + \int_{(0,\infty)} ([V_{a^*}(x-y) - V_x(x-y)] - [V_{a^*}(x) - V_x(x)] \\
&\qquad\qquad + [V'_{a^*}(x) - V'_x(x)]y\mathbf{1}_{(0,1)}(y))\nu(\mathrm{d}y) \\
&\quad = -\frac{\sigma^2}{2}\lim_{y \uparrow x} V''_x(y) - q[V_{a^*}(x) - V_x(x)] \\
&\qquad + \int_{(0,\infty)} ([V_{a^*}(x-y) - V_x(x-y)] - [V_{a^*}(x) - V_x(x)])\nu(\mathrm{d}y),
\end{aligned} \tag{4.21}$$

where the last equality holds since $V'_{a^*}(x) = V'_x(x) = 1$ and $V''_{a^*}(x) = 0$ for all $x > a^*$ from (3.7), (3.14), and Remark 3.2.

From Lemma 4.2, we know that $\lim_{y \uparrow x} V''_x(y) \ge 0$ for $x > a^*$. From (3.16) and (3.17), we have $V_{a^*}(x) - V_x(x) \ge 0$ for $x > a^*$ since, for $x \in [-l, \infty)$, $(\mathrm{d}/\mathrm{d}a)V_a(x)$ has the same sign as $\Gamma(x)$ which is nonpositive for all $a \in [a^*, \infty)$ from Assumption 4.1.

For $x > a^*$, define $g(z) := V_{a^*}(z) - V_x(z)$ for $z \in [-l, \infty)$. From (4.12), we have, for $z \le a^* < x$,

$$g'(z) = [V_{a^*}(z) - V_x(z)]' = V'_{a^*}(z) - V'_x(z) \ge 0.$$

From (4.13), we also have $g'(z) \ge 0$ for $z \in (a^*, x]$. It then follows that, for $y \in [0, \infty)$,

$$[V_{a^*}(x-y) - V_x(x-y)] - [V_{a^*}(x) - V_x(x)] = g(x-y) - g(x) \le 0.$$

Combining these results, we conclude that the right-hand side of (4.21) is nonpositive, which proves (4.18) for $x \in (a^*, +\infty)$.

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Equation (4.18) for $x = a^*$ can be proved by letting $x \downarrow a^*$ and noting that $V_{a^*}(x) \in C^2(-l, \infty)$; see Remark 3.4.

In order to complete the proof that $V_{a^*}(x)$ satisfies the HJB inequalities (4.15), we need to prove that $V_{a^*}(x) \geq S$ for all $x \in [-l, \infty)$. Indeed, since $V_{a^*}(x)$ is nondecreasing in x, we have $V_{a^*}(x) \geq V_{a^*}(-l) \geq S$.

From Lemma 4.1, we conclude that the barrier strategy with barrier level a^* is the optimal dividend strategy for (i).

(ii) The following arguments are devoted to verifying that the function $V_{\text{run}}(x)$ satisfies the HJB inequalities

$$\max\{\mathcal{A}V_{\text{run}}(x) - qV_{\text{run}}(x), 1 - V'_{\text{run}}(x)\} \leq 0 \quad \text{and} \quad V_{\text{run}}(x) \geq S.$$

Actually, $1 - V'_{\text{run}}(x) \leq 0$ and $V_{\text{run}}(x) \geq S$ are obvious from the definition of $V_{\text{run}}(x)$. It remains to prove that $\mathcal{A}V_{\text{run}}(x) - qV_{\text{run}}(x) \leq 0$ for $x \in (-l, \infty)$. After some calculation, for $x \in (-l, \infty)$,

$$\begin{aligned}\mathcal{A}V_{\text{run}}(x) - qV_{\text{run}}(x) &= \gamma + \int_{(0,\infty)} (-y + y\mathbf{1}_{(0,1)}(y))\nu(\mathrm{d}y) - q(x + l + S) \\ &\leq \gamma + \int_{(0,1)} y\nu(\mathrm{d}y) - qS \\ &< 0,\end{aligned}$$

where the last inequality follows from Proposition 4.2. Therefore, the take-the-money-and-run strategy is the optimal dividend strategy. □

Remark 4.3. In the proof of Theorem 4.1(ii), we see that the corresponding result can be generalized as follows. If (4.1) holds, i.e. $(\gamma + \int_{(0,1)} x\nu(\mathrm{d}x))/q < S$ then the function $V_{\text{run}}(x) = x + l + S$, $x \in [-l, +\infty)$, satisfies the HJB inequalities (4.15) and the take-the-money-and-run strategy is the optimal dividend strategy where the differentiability of ξ or $W^{(q)}$ is not needed.

Remark 4.4. If $\xi(x) \equiv 0$ and $l = 0$, our dividend optimization model reduces to that of [27]. It was shown in [27] that $\Gamma(x)\big|_{\xi\equiv 0, l=0} \geq 0$ for all $x \in [-l, a^*(S)]$ and $\Gamma(x)\big|_{\xi\equiv 0, l=0} \leq 0$ for all $x \in [a^*(S), \infty)$ with $a^*(S)$ defined right above Proposition 3 of [27]. Our a^* is indeed equal to $a^*(S)$ of [27], and Assumption 4.1 holds automatically for $\xi(x) \equiv 0$ and $l = 0$. Consequently, the conclusions of Theorem 4.1 (combined with Proposition 4.2) coincide with those of Theorem 1 of [27]. The results in this paper thus generalize those of [27].

Remark 4.5. The function $W^{(q)}(x)/W^{(q)'}(x)$ is nondecreasing with respect to x since

$$\frac{W^{(q)}(x)}{W^{(q)'}(x)} = \frac{\mathrm{e}^{\Psi(q)x} W_{\Psi(q)}(x)}{[\mathrm{e}^{\Psi(q)x} W_{\Psi(q)}(x)]'} = \frac{1}{\Psi(q) + n_{\Psi(q)}(\bar{\varepsilon} \geq x)},$$

where $n_{\Psi(q)}$ is the excursion measure corresponding to the probability measure $\mathbb{P}_x^{\Psi(q)}$. Then

$$1 - \frac{W^{(q)}(x)W^{(q)''}(x)}{[W^{(q)'}(x)]^2} = \left[\frac{W^{(q)}(x)}{W^{(q)'}(x)}\right]' \geq 0.$$

In addition, since $1 - \xi'(x) \geq 0$ by the assumption, it follows from (3.13) that

$$1 - qSW^{(q)}(x) \leq 0 \quad \text{for all } x \in [0, \infty) \quad \Longleftrightarrow \quad S > 0, \qquad W^{(q)}(0) \geq \frac{1}{qS} > 0,$$

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which happens only if X has sample paths of bounded variation. If we assume that

$$S > 0, \qquad W^{(q)}(0) \geq \frac{1}{qS} > 0, \tag{4.22}$$

then (4.11) holds for all $x \in [0, \infty)$. Therefore, (4.22) serves as a sufficient condition which guarantees that Assumption 4.1 for Theorem 4.1 holds.

Remark 4.6. Given that $S > 0$, the continuity and strictly increasing property of the functions $\bar{\xi}$ and $W^{(q)}$ imply that the function $1 - qSW^{(q)}(\bar{\xi}(z))$ is continuous and strictly decreasing with respect to z. An alternative sufficient condition for Assumption 4.1 is

$$1 - qSW^{(q)}(\bar{\xi}(a^*)) \leq 0. \tag{4.23}$$

Indeed, under (4.23), we must have $1 - qSW^{(q)}(\bar{\xi}(z)) \leq 0$ for all $z \in [a^*, \infty)$, and then, for $z \in [a^*, \infty)$,

$$\Gamma(z) = -1 + (1 - qSW^{(q)}(\bar{\xi}(z)))\left[1 - \frac{W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{[W^{(q)'}(\bar{\xi}(z))]^2}\right](1 - \xi'(z)) \leq -1 \leq 0.$$

In particular, when $a^* = \infty$, Assumption 4.1 needed in Theorem 4.1 is satisfied *a priori*, i.e. the conclusions of Theorem 4.1 hold *a priori*.

Remark 4.7. Given that $S \leq 0$, Assumption 4.1 holds if and only if

$$1 - (1 - qSW^{(q)}(\bar{\xi}(z)))^{-1}\left[1 - \frac{W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{[W^{(q)'}(\bar{\xi}(z))]^2}\right]^{-1} \leq \xi'(z) \quad \text{for } z \in [a^*, \infty)$$

or, equivalently,

$$1 - (1 - qSW^{(q)}(x))^{-1}\left[1 - \frac{W^{(q)}(x)W^{(q)''}(x)}{[W^{(q)'}(x)]^2}\right]^{-1} \leq \xi'((\bar{\xi})^{-1}(x)) \quad \text{for } x \in [\bar{\xi}(a^*), \infty)$$

with the convention that $\frac{1}{0} = +\infty$.

5. Examples

It is often not immediately evident that for general spectrally negative Lévy process X and for general drawdown function ξ, the barrier dividend strategy is typically the optimal strategy. In this section we consider several examples of processes X and show that Assumption 4.1 holds for certain choices of drawdown function ξ, and find the corresponding optimal dividend level a^* defined in (3.18).

For simplicity, we assume that $l = 0$ and $S = 0$ in the following examples. Then

$$\Gamma(z) = -1 + \left(\frac{W^{(q)}}{W^{(q)'}}\right)'(\bar{\xi}(z))(1 - \xi'(z)).$$

Example 5.1. In this example, $X_t = \mu t + B_t$ for constant μ and Brownian motion B.

We first consider the function $\xi(x) = kx$ with $k \in [0, 1)$. The q-scale function of X is given by

$$W^{(q)}(x) = \frac{1}{\sqrt{\mu^2 + 2q}}\left(\exp\left(\left(-\mu + \sqrt{\mu^2 + 2q}\right)x\right)\right.$$
$$\left. - \exp\left(-\left(\mu + \sqrt{\mu^2 + 2q}\right)x\right)\right), \qquad x \geq 0, \tag{5.1}$$

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which satisfies

$$W^{(q)''}(x) = 2(qW^{(q)}(x) - \mu W^{(q)'}(x)), \qquad x \geq 0. \tag{5.2}$$

For simplicity of notation, write

$$W_\xi(z) := \frac{W^{(q)}(\bar{\xi}(z))}{W^{(q)'}(\bar{\xi}(z))} \quad \text{and} \quad W_0(z) = \frac{W^{(q)}(z)}{W^{(q)'}(z)}.$$

From (5.2), we can verify that

$$\begin{aligned}
\Gamma(z) \leq 0 \quad &\Longleftrightarrow \quad W^{(q)'}(\bar{\xi}(z))^2 - W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z)) \leq \frac{W^{(q)'}(\bar{\xi}(z))^2}{1-k} \\
&\Longleftrightarrow \quad \frac{k}{1-k} - 2\mu W_\xi(z) + 2q W_\xi(z)^2 \geq 0 \\
&\Longleftrightarrow \quad \left(\sqrt{2q}W_\xi(z) - \frac{\mu}{\sqrt{2q}}\right)^2 + \frac{k}{1-k} - \frac{\mu^2}{2q} \geq 0
\end{aligned} \tag{5.3}$$

with $W_\xi(z) = 1/[\Psi(q) + n_{\Psi(q)}(\bar{\varepsilon} \geq \bar{\xi}(z))]$ being an increasing nonnegative function of $z \in [0, \infty)$.

There are four cases to be considered.

Case 1. For $\mu \leq 0$. From the second equivalence of (5.3), we have $\Gamma(z) < 0$ for any $z > 0$. Hence, $a^* = 0$ and the barrier strategy with barrier level 0 is the optimal dividend strategy.

Case 2. For $\mu > 0$ and $k = 0$. From the first equivalence of (5.2) and (5.3), we have $\Gamma(z) < 0$ if and only if

$$z \in \{y > 0 \mid W^{(q)''}(\bar{\xi}(y)) > 0\} = \left\{y > 0 \mid W_\xi(y) > \frac{\mu}{q}\right\} = \left(W_\xi^{-1}\frac{\mu}{q}, \infty\right),$$

where W_ξ^{-1} denotes the inverse function of W_ξ. Here, $W_\xi^{-1}(\mu/q) > 0$ since $W_\xi(z)$ is increasing in z. In addition, from (5.1), we have

$$\lim_{z\downarrow 0} W_\xi(z) = 0 < \frac{\mu}{q}, \qquad \lim_{z\uparrow\infty} W_\xi(z) = \frac{1}{-\mu + \sqrt{\mu^2 + 2q}} = \frac{\mu + \sqrt{\mu^2 + 2q}}{2q} > \frac{\mu}{q}.$$

Then the barrier strategy with barrier level $a^* = W_\xi^{-1}(\mu/(q) > 0$ is the optimal dividend strategy, which coincides with Example 6.1 of [5].

Case 3. For $\mu > 0$ and $k/(1-k) - \mu^2/2q \geq 0$, i.e. $k \in [\mu^2/(\mu^2 + 2q), 1)$. From (5.3), we have $\Gamma(z) \leq 0$ for all $z \geq 0$. Hence, $a^* = 0$ and the barrier strategy with barrier level 0 is the optimal dividend strategy.

Case 4. For $\mu > 0$ and $k \in (0, \mu^2/(\mu^2 + 2q)$, i.e. $k/(1-k) - \mu^2/2q < 0$. We claim that there is no $a^* \geq 0$ such that $\Gamma(z) \leq 0$ for all $z \geq a^*$ and $\Gamma(z) \geq 0$ for all $z \in [0, a^*]$. Indeed, for $x \geq 0$, define a convex function

$$P(x) := \left(\sqrt{2q}x - \frac{\mu}{\sqrt{2q}}\right)^2 + \frac{k}{1-k} - \frac{\mu^2}{2q},$$

and the left-hand side of (5.3) becomes $P(W_\xi(z))$. Some algebra yields the two roots of $P(x) = 0$ as

$$x_1(k) = \frac{\mu - \sqrt{\mu^2 - 2qk/(1-k)}}{2q}, \qquad x_2(k) = \frac{\mu + \sqrt{\mu^2 - 2qk/(1-k)}}{2q}.$$

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Since

$$\lim_{z\downarrow 0} W_\xi(z) = 0 < x_1(k) < x_2(k) < \frac{\mu}{q} < \lim_{z\uparrow\infty} W_\xi(z),$$

we have, for $\mu > 0$, $P(W_\xi(z)) > 0$ for z satisfying $W_\xi(z) \in (0, x_1(k)) \cup (x_2(k), \infty)$ and $P(W_\xi(z)) < 0$ for z satisfying $W_\xi(z) \in (x_1(k), x_2(k))$, which together with (5.3) imply our claim. Therefore, Assumption 4.1 fails to hold.

In the following, instead of considering the particular function $\xi(z) = kz$, we proceed to identify a class of functions ξ that satisfy Assumption 4.1 for the above mentioned process X with $\mu > 0$, where the exact monotonicity of the scale function and its derivative play a key role.

Note that, for $\theta_1 := -\mu + \sqrt{\mu^2 + 2q}$ and $\theta_2 := -\mu - \sqrt{\mu^2 + 2q}$,

$$W_0'(z) = \frac{(\theta_1 - \theta_2)^2 \mathrm{e}^{(\theta_1+\theta_2)z}}{[\theta_1 \mathrm{e}^{\theta_1 z} - \theta_2 \mathrm{e}^{\theta_2 z}]^2}, \qquad W_0''(z) = (\theta_2 - \theta_1)\mathrm{e}^{(\theta_1+\theta_2)z} \frac{\theta_1 \mathrm{e}^{\theta_1 z} + \theta_2 \mathrm{e}^{\theta_2 z}}{[\theta_1 \mathrm{e}^{\theta_1 z} - \theta_2 \mathrm{e}^{\theta_2 z}]^3/(\theta_1 - \theta_2)^2}.$$

It follows that

$$W_0''(z) > 0 \quad \Longleftrightarrow \quad z < \frac{1}{\theta_1 - \theta_2} \ln\left(-\frac{\theta_2}{\theta_1}\right) := a,$$

which implies that the function $W_0'(z)$ is strictly increasing over $[0, a]$, strictly decreasing over $[a, \infty)$, and reaches its global maximum at a. In addition, $W_0'(0^+) = 1$, $W_0'(z) > 1$ for all $z \in (0, a]$, and $W_0'(z) \to 0$ as $z \to \infty$. There is a unique $b \in (a, \infty)$ satisfying $W_0'(b) = 1$.

We also need the observation that

$$\begin{aligned}
\Gamma(z) < 0 \quad &\Longleftrightarrow \quad \frac{W^{(q)'}(\bar\xi(z))^2 - W^{(q)}(\bar\xi(z)) W^{(q)''}(\bar\xi(z))}{W^{(q)'}(\bar\xi(z))^2}(1 - \xi'(z)) < 1 \\
&\Longleftrightarrow \quad W_0'(\bar\xi(z))(1 - \xi'(z)) < 1 \\
&\Longleftrightarrow \quad \xi'(z) > 1 - \frac{1}{W_0'(\bar\xi(z))}.
\end{aligned}$$

Choose a nonnegative function ξ such that $\xi' < 1$, and $\bar\xi$ is thus a strictly increasing function with inverse $\bar\xi^{-1}$. Further, choose ξ to satisfy

(i) for $z \in [0, \bar\xi^{-1}(a)]$, there exists $0 \le \varepsilon_1 < 1$ such that $\bar\xi(z) \ge \varepsilon_1 z$ or, equivalently, $\xi(z) \le (1 - \varepsilon_1)z$ and $\xi'(z) \le 1 - 1/W_0'(\varepsilon_1 z)$;

(ii) for $z \in (\bar\xi^{-1}(a), \bar\xi^{-1}(b)]$, $\xi'(z)$ is increasing;

(iii) for $z \in (\bar\xi^{-1}(b), \infty)$, there exists $0 \le \varepsilon_2 < 1$ such that $\bar\xi(z) \ge b + \varepsilon_2(z - \bar\xi^{-1}(b))$ and $\xi'(z) \ge 1 - 1/W_0'(b + \varepsilon_2(z - \bar\xi^{-1}(b)))$.

Observe that $\xi'(0^+) = 0$ is necessary for (i) to hold. In (iii) we used the fact that $W_0'(z)$ is decreasing in $[b, \infty)$. Also note that $W_0'(b+\varepsilon_2(z-\bar\xi^{-1}(b))) < W_0'(b) = 1$ for $z \in (\bar\xi^{-1}(b), \infty)$ by the strict monotonicity of W_0' on $[b, \infty)$.

Combining these results, we have the following:

- $\Gamma(z) \ge 0$ for $z \in (0, \bar\xi^{-1}(a)]$;
- $W_0'(\bar\xi(z))(1 - \xi'(z))$ is strictly decreasing for $z \in (\bar\xi^{-1}(a), \bar\xi^{-1}(b)]$;
- $W_0'(\bar\xi(z))(1 - \xi'(z)) < 1 - \xi'(z) \le 1$ for $z \in (\bar\xi^{-1}(b), \infty)$.

Then there exists a unique $z^* \in (\bar\xi^{-1}(a), \bar\xi^{-1}(b)]$ such that $\Gamma(z^*) = 0$, $\Gamma(z) \ge 0$ for $z \in [0, z^*)$, and $\Gamma(z) < 0$ for $z \in (z^*, \infty)$. Assumption 4.1 thus holds.

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In particular, we can simply choose a nonnegative, nondecreasing, and differentiable function ξ satisfying

(i) $\xi(x) \equiv 0$ for all $x \in [0, a]$;

(ii) $\xi'(z) < 1$ and $\xi'(z)$ is increasing over $(a, \bar{\xi}^{-1}(b))$;

(iii) $\xi'(z)$ remains constant over $(\bar{\xi}^{-1}(b), \infty)$, i.e.

$$\xi'(z) \equiv \xi'(\bar{\xi}^{-1}(b)) \in (0, 1) \quad \text{on } (\bar{\xi}^{-1}(b), \infty).$$

Then the barrier strategy at level $z^* \in (\bar{\xi}^{-1}(a), \bar{\xi}^{-1}(b)] = (a, \bar{\xi}^{-1}(b)]$ is optimal.

Example 5.2. Let $X_t = x + pt - \sum_{i=1}^{N_t} e_i$, where $p > 0$, $\{e_i;\ i \geq 1\}$ are independent and identically distributed exponential random variables with mean $1/\mu$, and $\{N_t,\ t \geq 0\}$ is an independent Poisson process with intensity λ. We assume that $\xi(x) = kx$ for $k \in [0, 1)$. The q-scale function of X is

$$W^{(q)}(x) = p^{-1}(A_+ \mathrm{e}^{\theta_+ x} - A_- \mathrm{e}^{\theta_- x}), \qquad x \geq 0,$$

where

$$A_\pm = \frac{\mu + \theta_\pm}{\theta_+ - \theta_-} > 0 \quad \text{and} \quad \theta_\pm = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}$$

with $\theta_+ > 0, \theta_- < 0$. We can verify that $\Gamma(z) \leq 0$ if and only if

$$W^{(q)'}(\bar{\xi}(z))^2 - W^{(q)}(\bar{\xi}(z)) W^{(q)''}(\bar{\xi}(z)) \leq \frac{W^{(q)'}(\bar{\xi}(z))^2}{1 - k}, \tag{5.4a}$$

if and only if

$$A_+ A_- (\theta_+ - \theta_-)^2 \mathrm{e}^{(\theta_+ + \theta_-)\bar{\xi}(z)} \leq \frac{1}{1 - k}(A_+ \theta_+ \mathrm{e}^{\theta_+ \bar{\xi}(z)} - A_- \theta_- \mathrm{e}^{\theta_- \bar{\xi}(z)})^2, \tag{5.4b}$$

if and only if

$$\sqrt{(1 - k) A_+ A_-}(\theta_+ - \theta_-) \exp\left(\frac{\theta_+ + \theta_-}{2} \bar{\xi}(z)\right) \leq A_+ \theta_+ \mathrm{e}^{\theta_+ \bar{\xi}(z)} - A_- \theta_- \mathrm{e}^{\theta_- \bar{\xi}(z)}, \tag{5.4c}$$

if and only if

$$A_+ \theta_+ \mathrm{e}^{(\theta_+ - \theta_-)\bar{\xi}(z)} - \sqrt{(1 - k) A_+ A_-}(\theta_+ - \theta_-) \exp\left(\frac{\theta_+ - \theta_-}{2} \bar{\xi}(z)\right) - A_- \theta_- \geq 0, \tag{5.4d}$$

if and only if

$$\left(\sqrt{A_+ \theta_+} \exp\left(\frac{\theta_+ - \theta_-}{2} \bar{\xi}(z)\right) - \sqrt{(1 - k) A_-} \frac{\theta_+ - \theta_-}{2\sqrt{\theta_+}}\right)^2 - A_- \theta_- - \frac{(1 - k) A_- (\theta_+ - \theta_-)^2}{4\theta_+} \geq 0. \tag{5.4e}$$

There are four cases to be considered.

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Case 1. If $c_1 := -A_-\theta_- - (1-k)A_-(\theta_+ - \theta_-)^2/4\theta_+ \geq 0$ then $\Gamma(z) \leq 0$ for all $z \geq 0$, and the optimal dividend level is $a^* = 0$.

Case 2. If $c_1 < 0$ and

$$\begin{aligned} c_2 &:= \frac{\sqrt{(1-k)A_-}(\theta_+ - \theta_-)}{2\sqrt{A_+}\theta_+} - \frac{\sqrt{A_-\theta_- + (1-k)A_-(\theta_+ - \theta_-)^2/4\theta_+}}{\sqrt{A_+\theta_+}} \\ &\leq 1 \\ &< \frac{\sqrt{(1-k)A_-}(\theta_+ - \theta_-)}{2\sqrt{A_+}\theta_+} + \frac{\sqrt{A_-\theta_- + (1-k)A_-(\theta_+ - \theta_-)^2/4\theta_+}}{\sqrt{A_+\theta_+}} \\ &=: c_3. \end{aligned}$$

By the definition of a^* (see (3.18)),

$$a^* = 2(1-k)^{-1}(\theta_+ - \theta_-)^{-1}\ln(c_3) > 0.$$

From (5.4a), we know that $\Gamma(z) \geq 0$ for all $z \in [0, a^*]$ and $\Gamma(z) < 0$ for all $z \in (a^*, \infty)$. Hence, the barrier dividend strategy with barrier level $a^* > 0$ is the optimal dividend strategy.

Case 3. If $c_1 < 0$ and $c_3 \leq 1$ then $a^* = 0$ and $\Gamma(z) < 0$ for all $z \in (a^*, \infty)$ ($\Gamma(0) \leq 0$), and the barrier dividend strategy with barrier level 0 is the optimal dividend strategy.

Case 4. If $c_1 < 0$ and $c_2 > 1$ then Assumption 4.1 fails.

If $k = 0$ then $c_1 < 0$ and $c_3 \leq 1$ if and only if $p\lambda\mu \leq (q+\lambda)^2$. Therefore, the above results coincide with those of Example 6.3 of [5]. In particular, c_3 is reduced to

$$\begin{aligned} c_3 &= \frac{\sqrt{A_-}(\theta_+ - \theta_-)}{2\sqrt{A_+}\theta_+} + \frac{\sqrt{A_-\theta_- + A_-(\theta_+ - \theta_-)^2/4\theta_+}}{\sqrt{A_+\theta_+}} \\ &= \sqrt{\frac{A_-}{A_+}}\left(\frac{\theta_+ - \theta_-}{2\theta_+} + \frac{\sqrt{(\theta_+ + \theta_-)^2}}{2\theta_+}\right) \\ &= \sqrt{\frac{A_-}{A_+}}\frac{-\theta_-}{\theta_+}, \end{aligned}$$

and a^* is reduced to

$$a^* = 2(\theta_+ - \theta_-)^{-1}\ln\left(\sqrt{\frac{A_-}{A_+}}\frac{-\theta_-}{\theta_+}\right) = \frac{1}{\theta_+ - \theta_-}\ln\left(\frac{A_-}{A_+}\frac{\theta_-^2}{\theta_+^2}\right) = \frac{1}{\theta_+ - \theta_-}\ln\left(\frac{\theta_-^2}{\theta_+^2}\frac{\mu + \theta_-}{\mu + \theta_+}\right),$$

which recovers the c^* in Example 6.3 of [5] for the $p\lambda\mu > (q+\lambda)^2$ case.

Example 5.3. In this example we consider a spectrally negative Lévy process X whose Lévy measure has a completely monotone density. From [26], $W^{(q)'''}(z) \geq 0$ for $z \geq 0$ and consequently, $W(q)''(z)$ increases in z. Let $a_2 := \inf\{z \geq 0 \colon W(q)''(z) > 0\}$. We want to choose a drawdown function ξ that initially satisfies $\xi'(z) < 1$ for $z \geq 0$ and $\xi'(z) \geq 0$ for $z \in (\bar{\xi}^{-1}(a_2), \infty)$. As a result, $\bar{\xi}(z)$ is increasing in z and we always have $\Gamma(z) < 0$ for $z \in (\bar{\xi}^{-1}(a_2), \infty)$.

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Note that, for $z \in [0, \bar{\xi}^{-1}(a_2)]$ or, equivalently, $\bar{\xi}(z) \in [0, a_2]$, we have

$$\begin{aligned}\Gamma(z) \geq 0 \quad &\Longleftrightarrow \quad 1 - \frac{W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{W^{(q)'}(\bar{\xi}(z))^2} \geq \frac{1}{1-\xi'(z)} \\ &\Longleftrightarrow \quad \xi'(z) \leq \frac{-W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}{W^{(q)'}(\bar{\xi}(z))^2 - W^{(q)}(\bar{\xi}(z))W^{(q)''}(\bar{\xi}(z))}.\end{aligned}$$

Define $a_1 := \inf\{z \geq 0 \colon W^{(q)''}(z) \geq 0\}$. We consider the following three cases and identify further conditions on $\xi(z)$ for $z \in [0, \bar{\xi}^{-1}(a_2)]$, respectively.

Case 1. For $a_1 = 0$, choose ξ satisfying $\xi(z) = 0$ for $0 = \bar{\xi}^{-1}(a_1) \leq z \leq \bar{\xi}^{-1}(a_2) = a_2$. Then the drawdown function defined in this way satisfies $\Gamma(z) = 0$ for $z \in [0, a_2]$ and $a^* = a_2$ is the optimal dividend barrier level.

Case 2. For $a_1 > 0$ and X of bounded variation and for $\varepsilon \in (0, a_1)$, choose the function ξ such that

$$\xi'(z) \leq k_0 := \inf_{z \in [0, a_1 - \varepsilon]} \frac{-W^{(q)}(z)W^{(q)''}(z)}{W^{(q)'}(z)^2 - W^{(q)}(z)W^{(q)''}(z)} > 0 \quad \text{for } z \in [0, \bar{\xi}^{-1}(a_1 - \varepsilon)],$$

$\bar{\xi}^{-1}(a_1 - \varepsilon) < \infty$, and $\xi'(z) \leq 0$ for $z \in (\bar{\xi}^{-1}(a_1 - \varepsilon), \bar{\xi}^{-1}(a_2)]$. Then $\Gamma(z) \geq 0$ for $z \in [0, \bar{\xi}^{-1}(a_2)]$ and $a^* = \bar{\xi}^{-1}(a_2) > 0$ is the optimal dividend barrier level.

Case 3. For $a_1 > 0$ and X of unbounded variation, given any $\varepsilon \in (0, \frac{1}{2}a_1)$ we choose ξ satisfying $\xi(z) = 0$ for $z \in [0, \bar{\xi}^{-1}(\varepsilon))$, $\bar{\xi}^{-1}(a_1 - \varepsilon) < \infty$,

$$\xi'(z) \leq k_1 := \inf_{z \in [\varepsilon, a_1 - \varepsilon]} \frac{-W^{(q)}(z)W^{(q)''}(z)}{W^{(q)'}(z)^2 - W^{(q)}(z)W^{(q)''}(z)} > 0 \quad \text{for } z \in [\bar{\xi}^{-1}(\varepsilon), \bar{\xi}^{-1}(a_1 - \varepsilon)],$$

and $\xi'(z) \leq 0$ for $z \in (\bar{\xi}^{-1}(a_1 - \varepsilon), \bar{\xi}^{-1}(a_2)]$.

Then, for such a function ξ, $\Gamma(z) \geq 0$ for $z \in [0, \bar{\xi}^{-1}(a_2)]$ and $a^* = \bar{\xi}^{-1}(a_2) > 0$ is the optimal dividend barrier level.

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