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Codimension-one attracting sets in $\mathbb{P}^k(\mathbb{C})$

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Abstract. We are interested in attracting sets of $\mathbb{P}^k(\mathbb{C})$ which are of small topological degree and of codimension one. We first show that there exists a large family of examples. Then we study their ergodic and pluripotential theoretic properties.

1. *Introduction*

The aim of this article is to study a special class of attracting sets for holomorphic endomorphisms of the complex projective space $\mathbb{P}^k$. Let f be such an endomorphism, of algebraic degree $d \geq 2$. Recall that an *attracting set* of f is a compact set $\mathscr{A} \subset \mathbb{P}^k$ admitting an open neighborhood U which is a trapping region, i.e. $f(U) \Subset U$ and $\mathscr{A} := \bigcap_{n\geq 0} f^n(U)$. These sets play an important role in dynamical systems and they have been studied by many authors, see e.g. [**FS98**, **FW99**, **JW00**, **Din07**, **R12**] for the complex setting.

In [**Dau14**], the first author introduced a notion of *dimension* for attracting sets and the notion of attracting sets of *small topological degree*, see Definitions 2.1 and 2.2 below. In this framework several interesting dynamical and geometric properties have been pointed out. Among other things, it was shown in this paper that such attracting sets are non-algebraic and also that such examples are quite abundant in $\mathbb{P}^2$. Our purpose here is to answer questions that were left open.

Our first result is a generalization of the construction [**Dau14**] to arbitrary dimensions $k \geq 2$. To be more specific, let $k \geq 2$ be an integer and for $d \geq 2$, let $\mathscr{F}_d$ denote the set of pairs $f = (f_\infty, R)$ where R is an homogeneous polynomial of degree d in $\mathbb{C}^k$ and $f_\infty = (F_0, \ldots, F_{k-1})$ is a k-tuple of homogeneous polynomials which defines a holomorphic endomorphism of $\mathbb{P}^{k-1}$ of degree d. The set $\mathscr{F}_d$ is a quasi-projective variety. If $f = (f_\infty, R) \in \mathscr{F}_d$ and $\varepsilon \in \mathbb{C}$, then we denote by f_ε the endomorphism of $\mathbb{P}^k$ defined by

$$[z_0 : \cdots : z_{k-1} : z_k] = [z' : z_k] \mapsto [f_\infty(z') : z_k^d + \varepsilon R(z')].$$

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When $\varepsilon = 0$, these maps admit the hyperplane $\{z_k = 0\}$ as an attracting set, which is algebraic and thus cannot be of small topological degree. However, a generic perturbation f_ε has an attracting set with this property.

THEOREM 1.1. *There exists a non-empty Zariski open set $\Omega \subset \mathscr{F}_d$, such that if $f \in \Omega$, then for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \neq 0$ small enough, the maps f_ε is of small topological degree on an attracting set $\mathscr{A}_\varepsilon$ close to the hyperplane $\{z_k = 0\}$.*

The idea of the construction of the Zariski open set Ω is the same as in [**Dau14**]. In particular, the proof uses the fact that the maps f_ε preserve the pencil of lines passing through $[0 : \cdots : 0 : 1]$. The new delicate part here is to provide examples in any dimension and any degree to ensure that Ω is not empty.

Note that the property to possess an attracting set of small topological degree is stable under small perturbations. It is also the case for all our other assumptions below. Hence, Theorem 1.1 provides a lot of attracting sets of small topological degree of codimension one, i.e. they carry non-trivial positive closed currents of bidegree (1, 1) and they are not equal to the whole space $\mathbb{P}^k$. It would be interesting to understand the nature of the open set in the whole parameter space where the small topological degree condition holds.

Our other results are related to the ergodic and pluripotential theoretic properties of such attracting sets. Let $\mathscr{A}$ be an attracting set of codimension one with a trapping region U. Let $\mathscr{C}_1(U)$ be the set of all positive closed (1, 1)-currents S supported in U and of mass 1. Under a mild assumption on U, which will be referred to as (HD), Dinh constructed in [**Din07**] an *attracting current* $\tau \in \mathscr{C}_1(U)$ such that any continuous form $S \in \mathscr{C}_1(U)$ satisfies

$$\lim_{n \to +\infty} \Lambda^n S = \tau,$$

where $\Lambda := d^{1-k} f_*$ is the normalized push-forward operator acting on $\mathscr{C}_1(U)$. In particular, τ is invariant under Λ. He also obtained that if T denotes the Green current of f, then the measure $\nu := \tau \wedge T^{k-1}$ is mixing and of entropy $(k-1)\log d$.

The first author showed that when f is of small topological degree on $\mathscr{A}$, then τ has bounded potential and asked whether this potential was continuous. The answer is positive, provided that the geometric assumption (HD) holds.

THEOREM 1.2. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$. If f is of small topological degree on an attracting set $\mathscr{A}$ of codimension one satisfying (HD), then the attracting current τ on $\mathscr{A}$ has continuous potential.*

More precisely, we are going to prove that there exist constants $A > 0$ and $\alpha > 0$ such that, for all $x, y \in \mathbb{P}^k$, $|u(x) - u(y)| \leq A|\log \operatorname{dist}(x, y)|^{-\alpha}$ where u is such that $\tau - \omega = dd^c u$.

This result gives information on the size of the attracting set in the sense that the more regular the potential of τ is, the bigger its support is. However, we were not able to obtain Hölder continuity which would give a lower bound on the Hausdorff dimension of $\mathscr{A}$.

Another interesting question about currents in $\mathscr{C}_1(U)$ is whether τ is the unique current in $\mathscr{C}_1(U)$ invariant by Λ. Actually, under the same assumptions, this is indeed the case

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and, moreover, we obtain the following equidistribution result towards τ with uniform exponential speed.

THEOREM 1.3. *Let f and $\mathscr{A}$ be as in Theorem 1.2. There exist constants $0 < \delta < 1$ and $c \geq 0$ such that*

$$|\langle \Lambda^n S - \tau, \phi \rangle| \leq c\delta^n \|\phi\|_{\mathcal{C}^2},$$

for all $S \in \mathscr{C}_1(U)$ and test forms ϕ of class $\mathcal{C}^2$. In particular, τ is the unique current in $\mathscr{C}_1(U)$ invariant by Λ.

Both theorems can be reformulated as speed-of-convergence results for potentials, in the sup norm for the first and in the L^1 norm for the second. In addition to the assumption of small topological degree, the proofs are based on the 'geometric' properties of the space $\mathscr{C}_1(U)$ which come from (HD). Briefly, the assumptions (HD) imply that the set $\mathscr{C}_1(U)$ is arcwise connected by 'structural discs' in a uniform way. We refer to [**Din07**, §3, p. 233] for more details.

Our last result is about the hyperbolicity of certain invariant measures. In [**deT08**], de Thélin obtained a powerful result which relates the entropy of an invariant measure, its Lyapunov exponents and the dynamical degrees of f. For an endomorphism of $\mathbb{P}^k$, the dynamical degree of order i is just d^i. The last one, d^k, is the topological degree of f on $\mathbb{P}^k$. We will substitute it by a local version to prove the following result.

THEOREM 1.4. *Let ν be an ergodic measure and of entropy $h(\nu) = (k-1)\log d$. If there exists an open neighborhood V of* $\operatorname{supp}(\nu)$ *such that*

$$d_{k,\mathrm{loc}} := \limsup_{n \to +\infty} \left(\int_V (f^n)^*(\omega^k) \right)^{1/n} < d^{k-1},$$

then ν is hyperbolic and its Lyapunov exponents satisfy

$$\chi_1 \geq \cdots \geq \chi_{k-1} \geq \frac{1}{2}\log d > 0 > \frac{1}{2}\log\left(\frac{d_t}{d^{k-1}}\right) \geq \chi_k \geq -\infty.$$

The assumption $d_{k,\mathrm{loc}} < d^{k-1}$ holds in all examples known by the authors. Moreover, to be of small topological degree on $\mathscr{A}$ is much stronger than $d_{k,\mathrm{loc}} < d^{k-1}$. The first is a uniform pointwise property while the latter deals with an average on V.

Hence, Theorem 1.4 applies to the measure ν constructed by Dinh, as long as $d_{k,\mathrm{loc}} < d^{k-1}$, so in particular for all of the examples obtained in Theorem 1.1.

To conclude, let us underline here that results similar to Theorems 1.3 and 1.4 have been obtained by the second author in [**Taf13**] in a different setting well adapted to perturbations of super-attractive attracting sets of any codimension, which are all of the known examples. However, this set of examples is quite small and we believe these theorems, or at least their proofs, could be applied to a very large class of codimension-one attracting sets. Note furthermore that, in Theorem 1.4, the support of the measure ν is not necessary contained in an attracting set.

The paper is organized as follows. In §2, we recall Dinh's setting (HD) and the notions of dimension and of small topological degree attached to attracting sets. Section 3 is devoted to the proof of Theorem 1.1 on the existence of examples in any dimension. Then, in §4 we prove Theorems 1.2 and 1.3 about the attracting current τ. Finally, we discuss in §5 the hyperbolicity of measures of entropy $(k-1)\log d$.

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2. *Attracting sets in* $\mathbb{P}^k$

In this section, we summarize several notions and results about attracting sets.

For $0 \leq p \leq k$ and $E \subset \mathbb{P}^k$, let denote by $\mathscr{C}_p(E)$ the set of all positive closed (p, p)-currents of mass 1 and supported on E. Here, the mass of a (p, p)-current S is $\|S\| := \langle S, \omega^{k-p} \rangle$ where ω is the standard Fubini-study form on $\mathbb{P}^k$.

Following [**Dau14**], we define the dimension of an attracting set.

Definition 2.1. Let $\mathscr{A}$ be an attracting set. We say that $\mathscr{A}$ has dimension $k - p$, or equivalently codimension p, if $\mathscr{C}_p(\mathscr{A}) \neq \varnothing$ and $\mathscr{C}_{p-1}(\mathscr{A}) = \varnothing$.

The whole space $\mathbb{P}^k$ is always an attracting set of maximal dimension k and it is the only one. In the same way, the attracting periodic points correspond exactly to attracting sets of dimension zero. Between these two extremes, the possibilities for attracting sets of dimension p with $0 < p < k$ are much more varied. In what follows we will focus on the codimension-one case.

The key notion in this paper is the following.

Definition 2.2. Let f be an endomorphism of $\mathbb{P}^k$ of degree d admitting an attracting set $\mathscr{A}$ of dimension s. The endomorphism f is said to be of small topological degree on $\mathscr{A}$ if there exists a trapping region U such that

$$\limsup_{n \to +\infty} (\sharp f^{-n}(x) \cap U)^{1/n} < d^s,$$

for any $x \in \mathbb{P}^k$.

Note that if the inequality above holds for a trapping region U, it also holds for any other trapping region. Sometimes, we will abbreviate this into '$\mathscr{A}$ is an attracting set of small topological degree'.

This notion is stable under small deformations and it ensures that $\mathscr{A}$ is not algebraic (cf. [**Dau14**]).

To conclude this section, let us introduce a set of the geometric assumption on U used by Dinh [**Din07**], which imply good path connectedness properties for $\mathscr{C}_p(U)$. We say that an attracting set $\mathscr{A}$ with a trapping region U satisfies the assumptions (HD) if the following holds:

There exist an integer $1 \leq p \leq k$ *and two projective subspaces* I *and* L *of dimension* $p - 1$ *and* $k - p$ *respectively such that* $I \cap U = \varnothing$ *and* $L \subset U$. *Since* $I \cap L = \varnothing$, *for each* $x \in L$ *there exists a unique projective subspace* $I(x)$ *of dimension* p *which contains* I *and such that* $L \cap I(x) = \{x\}$. *We ask that, for each* $x \in L$, *the set* $U \cap I(x)$ *is star-shaped with respect to* x *as a subset of* $I(x) \backslash I \simeq \mathbb{C}^p$.

It is easy to check that in this situation $\mathscr{A}$ has codimension p.

Under these assumptions, Dinh showed in [**Din07**] that continuous forms in $\mathscr{C}_p(U)$ equidistribute under push-forward. To be more precise, let Λ be the operator on $\mathscr{C}_p(U)$ defined by

$$\langle \Lambda S, \phi \rangle := \langle d^{p-k} f_* S, \phi \rangle := \langle S, d^{p-k} f^* \phi \rangle.$$

Then, the following result holds.

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THEOREM 2.3. (Dinh [**Din07**]) *Let f, $\mathscr{A}$ and U be as above. There exists a current $\tau \in \mathscr{C}_p(U)$ such that if S is a continuous form in $\mathscr{C}_p(U)$, then*

$$\lim_{n\to+\infty} \Lambda^n S = \tau.$$

Moreover, τ is woven, invariant under Λ and extremal relative to this last property.

3. *Existence of attracting sets of small topological degree in* $\mathbb{P}^k$

This section is devoted to the proof of Theorem 1.1. We assume that the dimension k is at least two. From now on and in the rest of the paper, we will always denote by s the dimension of our attracting sets, i.e. $s = k - 1 \geq 1$. Recall from the introduction that if $d \geq 2$, then $\mathscr{F}_d$ denotes the set of pairs $f = (f_\infty, R)$ where R is an homogeneous polynomial of degree d in $\mathbb{C}^k$ and $f_\infty = (F_0, \ldots, F_s)$ is a k-tuple of homogeneous polynomials which defines a holomorphic endomorphism of $\mathbb{P}^s$ of degree d. By an abuse of notation, we will also denote this endomorphism by f_∞. Moreover, if $f \in \mathscr{F}_d$ and $\varepsilon \in \mathbb{C}$, f_ε stands for the endomorphism of $\mathbb{P}^k$ defined by

$$[z_0 : \cdots : z_s : z_k] = [z' : z_k] \mapsto [f_\infty(z') : z_k^d + \varepsilon R(z')].$$

When $|\varepsilon|$ is small enough, f_ε admits an attracting set near the hyperplane $\{z_k = 0\}$. Indeed, for $|\varepsilon| \neq 0$ small, we will use the trapping region

$$U_\varepsilon = \{[z_0 : \cdots : z_s : z_k] \in \mathbb{P}^k;\ |z_k| \leq c|\varepsilon| \max(|z_0|, \ldots, |z_s|)\},$$

where $c > 0$ is a small number depending on $f \in \mathscr{F}_d$, see [**Dau14**, Theorem 4.1, Step 1] for more details. It satisfies the assumptions of Dinh (HD), and we denote by $\mathscr{A}_\varepsilon := \bigcap_{n\in\mathbb{N}} f_\varepsilon^n(U_\varepsilon)$ the associated attracting set.

The proof of Theorem 1.1 will be given in the next two subsections. In the first, we will define the Zariski open set $\Omega \subset \mathscr{F}_d$ and show that if $f \in \Omega$ then f_ε is of small topological degree on $\mathscr{A}_\varepsilon$ for $|\varepsilon| \neq 0$ small enough. This part is just an adaptation of [**Dau14**] to higher dimension.

The second subsection is devoted to showing that Ω is not empty and, therefore, it is a dense Zariski open set of $\mathscr{F}_d$. Even if it relies on elementary arguments, the construction is quite delicate when $k \geq 3$.

3.1. *An algebraic sufficient condition.*

All the maps f_ε preserve the pencil of lines passing through $[0 : \cdots : 0 : 1]$, which is naturally parametrized by $\mathbb{P}^s$. Moreover, via this parametrization, the dynamics induced by f_ε on this pencil is exactly f_∞. The four following sets of lines play an important role in order to control the number of preimages which are in an attractive region (cf. [**Dau14**]). Denote

$$\begin{aligned}
X_{-1}(f) &:= \{z \in \mathbb{P}^s;\ \sharp(f_\infty^+)^{-1}(f_\infty^+(z)) \geq d^{s-1} + 1 = d^{k-2} + 1\},\\
X(f) &:= f_\infty(X_{-1}(f)),\\
Y_{-2}(f) &:= \{z \in \mathbb{P}^s;\ R(z) = 0\}\\
&\quad \cup \{z \in \mathbb{P}^s;\ \exists z' \in \mathbb{P}^s,\ f_\infty^+(z) \neq f_\infty^+(z'),\ f_\infty(z) = f_\infty(z'),\\
&\qquad f \circ f_\infty^+(z) = f \circ f_\infty^+(z')\},\\
Y(f) &:= f_\infty^2(Y_{-2}(f)).
\end{aligned}$$

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Here f_∞^+ is the map from $\mathbb{P}^s$ to $\mathbb{P}^k$ defined by

$$f_\infty^+(z) = [f_\infty(z) : R(z)].$$

Observe that these sets only depend on $f \in \mathscr{F}_d$, not on ε. When no confusion is possible, we shall just write X, Y, etc. Note that in the definition of $X_{-1}(f)$ the cardinality is counted with multiplicity and

$$(f_\infty^+)^{-1}(f_\infty^+(z)) = \{z' \in (f_\infty)^{-1}(f_\infty(z)) \mid R(z') = R(z)\}.$$

The following result gives a sufficient condition on X and Y to ensure the existence of an attracting set with small topological degree. It is analogous to [**Dau14**, Proposition 4.3].

PROPOSITION 3.1. *Let $Z(f) := X(f) \cup Y(f)$. If $f \in \mathscr{F}_d$ is such that:*

- *$X(f)$ and $Y(f)$ are disjoint;*
- *there exists $n \in \mathbb{N}$ such that $\bigcap_{i=0}^n f_\infty^{-i}(Z(f)) = \varnothing$;*

then for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \neq 0$ small enough, f_ε is of small topological degree on $\mathscr{A}_\varepsilon$.

Denote by Ω the subset of pairs $(f_\infty, R) \in \mathscr{F}_d$ satisfying the assumptions of this proposition. Before proving the result, let us point out the important fact that the set Ω is a Zariski open set. Indeed, the sets X_{-1} and Y_{-2} are analytic subsets of $\mathbb{P}^s$, and so the same holds for X, Y, Z and $f_\infty^{-i}(Z)$, for all $i \geq 1$. When f varies in $\mathscr{F}_d$, it is easy to check that they form analytic subsets of $\mathbb{P}^s \times \mathscr{F}_d$. The set Ω is exactly the complement of the projection on $\mathscr{F}_d$ of an analytic subset obtained by intersecting and taking the finite union of such sets, see Step 3 of the proof of [**Dau14**, Theorem 4.1] for more details.

Proof. The proof is essentially the same as that of [**Dau14**, Proposition 4.3]. The fact that the pencil of lines is parametrized by $\mathbb{P}^s$ instead of $\mathbb{P}^1$ does not affect the arguments. However, for the convenience of the reader we recall the main ideas.

Throughout the proof, by a line we mean a line passing through $[0 : \cdots : 0 : 1]$. We will identify such a line l with the associated point in $\mathbb{P}^s$, i.e. the unique point in the intersection of l with the hyperplane at infinity. The image of l by f_ε is $f_\infty(l)$, for all $\varepsilon \in \mathbb{C}$. With this identification, the sets X and Y can be seen as sets of points in $\mathbb{P}^s$ or sets of lines in $\mathbb{P}^k$.

As in [**Dau14**, Step 2, Theorem 4.1] we reduce the estimation of the number of preimages staying in $f_\varepsilon(U_\varepsilon)$ to a combinatorial problem by analyzing the geometry of $f_\varepsilon(U_\varepsilon)$, as well as its self-crossings, and the behavior of the preimages.

A generic line has d^s preimages and a generic point p included in a line ℓ has $d^{s+1} = d^k$ preimages (in $\mathbb{P}^k$), d in each line $\ell_{-1} \in f^{-1}(\ell)$. So we may:

- control the number of lines ℓ_{-1} that contain preimages staying in U_ε;
- for each line ℓ_{-1}, bound the number of preimages that belong to $f_\varepsilon(U_\varepsilon)$.

As in [**Dau14**, Step 2, Theorem 4.1], this is done by defining sets A, B of 'good' lines.

The set A is the set of lines l such that the preimages of a point $x \in f_\varepsilon(U_\varepsilon) \cap l$, which lie in U_ε, are contained in at most d^{s-1} lines. Here and in what follows, everything is counted with multiplicity.

We say that l is in B if for all $x \in l \cap f_\varepsilon^2(U_\varepsilon)$ and each $l_{-1} \in f_\infty^{-1}(l)$ the point x has at most one preimage in $l_{-1} \cap f_\varepsilon(U_\varepsilon)$.

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We summarize as follows.

- If $x \in l \in A$, then there are at most d^{s-1} lines intersecting $f_\varepsilon^{-1}(x) \cap U_\varepsilon$, in at most d points for each of them.
- If $x \in l \in B$, then there are at most d^s lines intersecting $f_\varepsilon^{-1}(x) \cap f_\varepsilon(U_\varepsilon)$, in at most 1 point for each of them.
- If $x \in l \in A \cap B$, then there are at most d^{s-1} lines intersecting $f_\varepsilon^{-1}(x) \cap f_\varepsilon(U_\varepsilon)$, in at most 1 point for each of them.

In the two first cases, $f_\varepsilon^{-1}(x) \cap f_\varepsilon(U_\varepsilon)$ contains at most d^s points. In the last one, it contains at most d^{s-1} points.

By our assumptions, there exists $r > 0$ such that

$$X^r \cap Y^r = \varnothing \quad \text{and} \quad \bigcap_{i=0}^{n} f_\infty^{-i}(Z^r) = \varnothing,$$

where X^r (respectively Y^r, Z^r) denotes the r-neighborhood of X (respectively Y, Z) in $\mathbb{P}^s$.

Note that the sets A and B depend on ε. Indeed, a key point is that the lines in X (respectively Y) are exactly those which are not in A (respectively B) for any $\varepsilon \in \mathbb{C}$. This fact is illustrated by the following result which can be proved in the same way as [**Dau14**, Lemma 4.4].

LEMMA 3.2. *Fix r as above. Then if $|\varepsilon| \neq 0$ is small enough, we have that*

$$A^c \subset X^r, \quad B^c \subset Y^r \quad \text{and} \quad X \cup Y = Z \subset (A \cap B)^c \subset Z^r. \tag{3.1}$$

Now we have reduced the problem to a combinatorial problem. We have to make the same disjunction of cases as in the end of the proof of [**Dau14**, Proposition 4.3], except that there are now n steps instead of three. By the previous lemma, we have the following.

- If $x \in l \notin X^r$, then x has at most d preimages in at most d^{s-1} lines.
- If $x \in l \notin Y^r$, then x has at most one preimage in at most d^s lines.

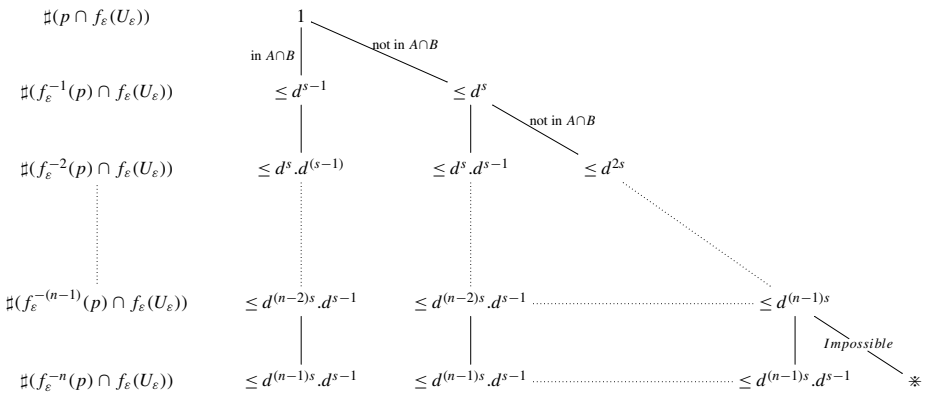
The condition $X^r \cap Y^r = \varnothing$ implies that for all $x \in f^2(U_\varepsilon)$ either:

- there are at most d^{s-1} lines intersecting $f_\varepsilon^{-1}(x) \cap U_\varepsilon$, in at most d points for each of them; or
- there are up to d^s lines l_{-1} intersecting $f_\varepsilon^{-1}(x) \cap f_\varepsilon(U_\varepsilon)$, but in each such line there is at most one preimage of x.

So the number of preimages of a point staying in $f_\varepsilon(U_\varepsilon)$ is less than d^s. Since $\bigcap_{i=0,\dots,n} f_\infty^{-i}(Z^r) = \varnothing$, there exists one step, between 0 and n, such that the preimages of a line are in $A \cap B$ and so they have less than d^{s-1} preimages.

This is summarized in the following diagram:

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Thus, in each case $\sharp(f_\varepsilon^{-n}(p) \cap f_\varepsilon(U_\varepsilon)) \leq d^{(n-1)s} \cdot d^{s-1} < d^{ns}$ and we conclude that f_ε^n is of small topological degree in $f_\varepsilon(U_\varepsilon)$, and hence in $\mathscr{A}_\varepsilon$ too. This finishes the proof. □

3.2. *Construction of examples.* In order to exhibit an element $f \in \Omega \subset \mathscr{F}_d$, the first step is to find an example for which $X(f)$ is finite. To this aim, until the end of this section, we only consider pairs $f = (f_\infty, R) \in \mathscr{F}_d$ such that only monomials of the form z_i^d appear in the expression of f_∞.

Let $\xi \in \mathbb{C}$ be a primitive dth root of unity. Let $G := \langle \xi \rangle \simeq \mathbb{Z}/d\mathbb{Z}$. The group G^s acts on $\mathbb{P}^s$ by

$$(\xi_1, \ldots, \xi_s).[z_0 : z_1 : \cdots : z_s] = [z_0 : \xi_1 z_1 : \cdots : \xi_s z_s].$$

The special form of f_∞ ensures that if z is a preimage of $w \in \mathbb{P}^s$, then $f_\infty^{-1}(w) = G^s.\{z\}$. Therefore, in this case, the set X_{-1} consists of points $z \in \mathbb{P}^s$ such that there exists $A \subset G^s$ satisfying $\sharp A = d^{s-1} + 1$, $\mathrm{Id} \in A$ and $R(g \cdot z) = R(h \cdot z)$ for all $g, h \in A$. It is easy to see that the $s + 1$ points in $\mathbb{P}^s$ which have only one non-zero coordinate are fixed by all of the elements of G^s. In particular, they are all in X_{-1}. We call these $s + 1$ points the *extremal points* of $\mathbb{P}^s$.

The following result exhibits an example where X_{-1}, and therefore X, is finite.

PROPOSITION 3.3. *If $R = (\sum_{i=0}^s z_i)^d$, then X_{-1} is finite.*

Proof. We have to prove that if $A \subset G^s$ satisfies $\sharp A = d^{s-1} + 1$, then the set X_A of points $z \in \mathbb{P}^s$ such that $R(g \cdot z) = R(h \cdot z)$, for all $g, h \in A$, is finite.

The choice of R provides two advantages. The first is that if $R(z) = R(w)$, then there exists $\eta \in \mathbb{C}$ such that $\sum_{i=0}^s z_i = \eta \sum_{i=0}^s w_i$, which is a linear equation. Hence, the problem turns out to be reduced to basic linear algebra. The second is that the restriction of R to $\{z_i = 0\}$ gives the counterpart of R in dimension $s - 1$ on $\{z_i = 0\} \simeq \mathbb{P}^{s-1}$. It allows us to proceed by induction on s.

For $s = 0$, there is nothing to prove since $\mathbb{P}^0$ is reduced to a single point. Now, let $s \geq 1$ and assume that the result holds for $s - 1$. To be more specific, assume that for any subset $A' \subset G^{s-1}$ with $\sharp A' = d^{s-2} + 1$, there is only a finite set $X'_{A'}$ of points $z' \in \mathbb{P}^{s-1}$ such that $R'(g \cdot z') = R'(h \cdot z')$ for all $g, h \in A'$, where $R'(z') := (\sum_{i=0}^{s-1} z_i')^d$.

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Let $A \subset G^s$ with $\sharp A = d^{s-1} + 1$. The first step is to show that $X_A \cap \{z_i = 0\}$ is finite.

Let $1 \leq i \leq s$ and $\alpha \in G$. If $A_{i,\alpha}$ denotes the set of points $(\xi_1, \ldots, \xi_s) \in A$ such that $\xi_i = \alpha$, then $\sum_{\alpha \in G} \sharp A_{i,\alpha} = d^{s-1} + 1$. Hence, there exists at least one $\tilde{\alpha} \in G$ such that $\sharp A_{i,\tilde{\alpha}} \geq d^{s-2} + 1$. Since the projection

$$\pi : (\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_s) \mapsto (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_s)$$

is injective restricted to $A_{i,\tilde{\alpha}}$, the induction assumption implies, via the natural isomorphisms $\{z_i = 0\} \simeq \mathbb{P}^{s-1}$, $X_A \cap \{z_i = 0\} \simeq X'_{A'}$ with $A' := \pi(A_{i,\tilde{\alpha}})$, that $X_A \cap \{z_i = 0\}$ is finite. For $X_A \cap \{z_0 = 0\}$ the same argument holds with $A' := \pi(A_{i,\tilde{\alpha}})$ for an arbitrary $1 \leq i \leq s$.

It remains to prove the result outside $\bigcup_{i=0}^{s} \{z_i = 0\}$. Let $z = [z_0 : \cdots : z_s] \in X_A$ such that none of its coordinates vanish. In particular, $z_0 \neq 0$ so we can normalize by $z_0 = 1$. Up to exchange of z by $h \cdot z$ and A by $h^{-1}A$, where h is an arbitrary element of A, we can assume that $\mathrm{Id} \in A$. By the definition of X_A, z has to satisfy $R(z) = R(g \cdot z)$. Hence, there exists a function $\eta : A \to G$ such that

$$1 + \sum_{i=1}^{s} z_i = \eta(g)\left(1 + \sum_{i=1}^{s} \xi_i z_i\right), \tag{3.2}$$

for each $g = (\xi_1, \ldots, \xi_s)$ in A.

With the same argument as above, there exists $\alpha \in G$ such that the set $g = (\xi_1, \ldots, \xi_s) \in A$ such that $\eta(g)\xi_s = \alpha$ has more than $d^{s-2} + 1$ elements. Up to a change of variables, we can assume that $\alpha = 1$. By induction, we obtain a filtration of A,

$$A_1 \subset \cdots \subset A_{s-1} \subset A_s := A,$$

such that $\sharp A_i = d^{i-1} + 1$ and $\eta(g)\xi_j = 1$ if $j > i$ and $g = (\xi_1, \ldots, \xi_s) \in A_i$.

We will now show that there exists $g_i = (\xi_{i,1}, \ldots, \xi_{i,s}) \in A_i$, $1 \leq i \leq s$, such that $(z_1, \ldots, z_s)$ is the unique solution of the system of s linear equations obtained from (3.2), whose matrix form is

$$\left(\begin{array}{ccccc|c} \alpha_{1,1} & 0 & 0 & \cdots & 0 & \beta_1 \\ \alpha_{2,1} & \alpha_{2,2} & 0 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \alpha_{s-1,1} & \alpha_{s-1,2} & \cdots & \alpha_{s-1,s-1} & 0 & \beta_{s-1} \\ \alpha_{s,1} & \alpha_{s,2} & \cdots & \cdots & \alpha_{s,s} & \beta_s \end{array}\right),$$

where $\alpha_{i,j} = \eta(g_i)\xi_{i,j} - 1$ and $\beta_i = 1 - \eta(g_i)$. Indeed, we only have to show that we can find $g_i \in A_i$ such that $\alpha_{i,i} \neq 0$ for $1 \leq i \leq s$.

For $i = 1$, since our system has at least one solution $(z_1, \ldots, z_s)$, $\alpha_{1,1} = 0$ implies that $\beta_1 = 0$, i.e. $\eta(g_1) = 1$. Hence, $\xi_{1,1} = 1$ and g_1 has to be equal to Id. But A_1 has two distinct elements, so we can choose $g_1 \in A_1$ such that $\alpha_{1,1} \neq 0$.

Now assume that $\alpha_{j,j} \neq 0$ for $1 \leq j \leq i-1$. After simplifications, we obtain the subsystem

$$\left(\begin{array}{ccccc|c} \alpha_{1,1} & 0 & 0 & \cdots & 0 & \tilde{\beta}_1 \\ 0 & \alpha_{2,2} & 0 & \cdots & 0 & \tilde{\beta}_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{i-1,i-1} & 0 & \tilde{\beta}_{i-1} \\ \alpha_{i,1} & \alpha_{i,2} & \cdots & \cdots & \alpha_{i,i} & \beta_i \end{array}\right),$$

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where $\tilde{\beta}_j \neq 0$ since $z_j \neq 0$. Since $\alpha_{i,i} = \eta(g_i)\xi_{i,i} - 1$, it vanishes if and only if $\eta(g_i) = \xi_{i,i}^{-1}$. As above, if this is the case, then

$$\beta_i - \sum_{j=1}^{i-1} \alpha_{i,j}\tilde{\beta}_j = 0,$$

which can be rewritten, using $\eta(g_i) = \xi_{i,i}^{-1}$, as

$$\xi_{i,i} \sum_{j=0}^{i-1} \tilde{\beta}_j - \sum_{j=0}^{i-1} \xi_{i,j}\tilde{\beta}_j = 0,$$

where $\tilde{\beta}_0 := 1$ and $\xi_{i,0} := 1$. If $\sum_{j=0}^{i-1} \tilde{\beta}_j \neq 0$, then

$$\xi_{i,i} = \frac{\sum_{j=0}^{i-1} \xi_{i,j}\tilde{\beta}_j}{\sum_{j=0}^{i-1} \tilde{\beta}_j},$$

and, therefore, for each choice of $\xi_{i,j}$, $1 \leq j \leq i-1$, there is a unique possible value for $\xi_{i,i}$, and so there are at most d^{i-1} choices for which $\alpha_{i,i} = 0$. But A_i has $d^{i-1} + 1$ distinct elements, and hence there exists $g_i \in A_i$ such that $\alpha_{i,i} \neq 0$. Finally, if $\sum_{j=0}^{i-1} \tilde{\beta}_j = 0$, the same argument holds with the equation

$$\xi_{i,i-1} = \frac{\sum_{j=0}^{i-2} \xi_i,\, j\tilde{\beta}_j}{\tilde{\beta}_{i-1}}.$$

Therefore, by induction, the system is invertible and admits a unique solution $(z_1, \ldots, z_s)$.

At the end, we have proved that for each function $\eta : A \to G$ there exists at most one point in $X_A \setminus \bigcup_{i=0}^{s} \{z_i = 0\}$. In particular, this set is finite. □

Remark 3.4. When f_∞ is of the same form as above, then X_{-1} has at least $s(d-1)+1$ points counted with multiplicity. We do not know whether X_{-1} can be empty in the general case.

We will need the following result.

LEMMA 3.5. *Let Z be an analytic subset of $\mathbb{P}^s$. Let $Z_0 := Z$ and $Z_{n+1} := f_\infty^{-1}(Z_n) \cap Z_n$. If the analytic subsets Z_n are all non-empty, then Z contains a periodic point.*

Proof. Observe that $Z_n = \bigcap_{i=0}^{n} f_\infty^{-i}(Z)$. Therefore, it is a decreasing sequence of analytic subsets in $\mathbb{P}^s$ and so must be stationary, i.e. there exists $n_0 \in \mathbb{N}$ such that $Z_{n_0} \subset f_\infty^{-1}(Z_{n_0})$, or equivalently $f_\infty(Z_{n_0}) \subset Z_{n_0}$. Again, the sequence $f_\infty^n(Z_{n_0})$ is decreasing and so there exists $n_1 \in \mathbb{N}$ such that $f_\infty^{n_1+1}(Z_{n_0}) = f_\infty^{n_1}(Z_{n_0})$. If $Z_{n_0} \neq \varnothing$, then $f_\infty^{n_1}(Z_{n_0})$ is a non-empty analytic subset, invariant by f_∞, so it must contain a periodic point (cf. e.g. [**Fak03**, Theorem 5.1]). □

Using this lemma with $Z = X \cup Y$, in order to have an example satisfying the assumptions of Proposition 3.1, it remains to find a map such that X and Y are disjoint and do not contain periodic points. Let $\mathrm{Per}_n(f_\infty)$ denote the set of periodic point of f_∞ of period n.

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PROPOSITION 3.6. *If* $k \geq 2$, *there exists* $f = (f_\infty, R) \in \mathscr{F}_d$ *such that:*

- $X \cap Y = \varnothing$;
- $X \cap \mathrm{Per}_n(f_\infty) = \varnothing$ *for all* $n \in \mathbb{N}$;
- $Y \cap \mathrm{Per}_n(f_\infty) = \varnothing$ *for all* $n \in \mathbb{N}$.

Proof. We first exhibit an example where $X \cap \mathrm{Per}_n(f_\infty) = \varnothing$ for all $n \in \mathbb{N}$. Let f_∞ be an endomorphism of $\mathbb{P}^s$ where only monomials of the form z_i^d appear. Moreover, assume that none of the $s+1$ extremal points are preperiodic. It is easy to check that

$$f_\infty[z_0 : \cdots : z_s] = [z_1^d : z_2^d : \cdots : z_s^d : z_s^d + z_0^d]$$

is such a map when $s \geq 1$. By Proposition 3.3, if $R = (\sum_{i=0}^s z_i)^d$, then $X_{-1}(f_\infty, R)$ is finite. Moreover, if $\alpha = (\alpha_0, \ldots, \alpha_s) \in (\mathbb{C}^*)^{s+1}$, then $X_{-1}(f_\infty, R_\alpha)$ associated to $R_\alpha = (\sum_{i=0}^s \alpha_i^{-1} z_i)^d$ is the image of $X_{-1}(f_\infty, R)$ by α, via the standard action of $(\mathbb{C}^*)^{s+1}$ on $\mathbb{P}^s$. But, the orbits of this action are all uncountable, except for the extremal points. Hence, by a cardinality argument, we can find $\alpha \in (\mathbb{C}^*)^{s+1}$ such that no point in $X_{-1}(f_\infty, R_\alpha)$ is preperiodic. In particular, $X(f_\infty, R_\alpha) \cap \mathrm{Per}_n(f_\infty) = \varnothing$ for all $n \in \mathbb{N}$.

Now, observe that adding $\sum_{i=0}^s c_i z_i^d$ to R does not change X or, of course, $\mathrm{Per}_n(f_\infty)$. And this operation allows us to choose R such that Y, or equivalently Y_{-2}, avoids a prescribed countable set. Indeed, let $E \subset \mathbb{P}^s$ be a countable set. We can assume that $f_\infty^{-1}(f_\infty(E)) = E$. For Y_{-2}, it is clear that, for a very generic choice of $\sum_{i=0}^s c_i z_i^d$, the resulting R will satisfy $\{R(z) = 0\} \cap E = \varnothing$. For the second part, it is enough to choose $\sum_{i=0}^s c_i z_i^d$ in such a way that $f \circ f_\infty^+(z) \neq f \circ f_\infty^+(z')$ for all $z, z' \in E$ with $z \neq z'$. Again, it is satisfied for a very generic choice.

Starting from $f = (f_\infty, R_\alpha)$ as above, these arguments with

$$E = \bigcup_{n \geq 1} \mathrm{Per}_n(f_\infty) \cup X$$

give the desired result. □

4. *Pluripotential theoretic properties and equidistribution*

We refer to [**Dem12**] for basics on currents and plurisubharmonic functions and to [**DS10**] for a survey on their use in complex dynamics and, in particular, for the construction of the Green current T and the equilibrium measure $\mu := T^k$.

Let R and S be two currents in $\mathscr{C}_1(\mathbb{P}^k)$. There exists a unique d.s.h. function $u_{R,S}$, i.e. defined outside a pluripolar set of $\mathbb{P}^k$ and which is locally the difference of two plurisubharmonic functions, such that

$$dd^c u_{R,S} = R - S \quad \text{and} \quad \langle \mu, u_{R,S} \rangle = 0.$$

The operator Λ also acts on these potentials by

$$\Lambda u_{R,S}(x) = \frac{1}{d^s} \sum_{y \in f^{-1}(x)} u_{R,S}(y),$$

where $s = k - 1$ and the points in $f^{-1}(x)$ are counted with multiplicity. It sends bounded (respectively continuous) functions to bounded (respectively continuous) functions and it satisfies $\Lambda dd^c u_{R,S} = dd^c \Lambda u_{R,S}$.

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A current R has bounded (respectively continuous) potential if there exists a smooth form $S \in \mathscr{C}_1(\mathbb{P}^k)$ such that $u_{R,S}$ is bounded (respectively continuous). It implies that $u_{R,S}$ is bounded (respectively continuous) for any current $S \in \mathscr{C}_1(\mathbb{P}^k)$ with bounded (respectively continuous) potential. The operator Λ preserves these two classes of currents. Moreover, $u_{\Lambda R,\Lambda S} = \Lambda u_{R,S}$ since dd^c commutes with Λ and

$$\langle \mu, \Lambda u_{R,S}\rangle = \langle d^{-s} f^* \mu, u_{R,S}\rangle = d\langle \mu, u_{R,S}\rangle = 0.$$

From now on and for the rest of this section, f is a holomorphic endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$ which has an attracting set $\mathscr{A}$ of codimension one and of small topological degree. Moreover, we assume that $\mathscr{A}$ has two trapping regions U and U' such that

$$f(U) \Subset f(U') \Subset U \Subset U'$$

and such that U satisfies (HD).

The main result in this section is that, under these assumptions, the attracting current constructed by Dinh has continuous potential.

THEOREM 4.1. *The attracting current τ of $\mathscr{A}$ has continuous potential. Moreover, there exist constants $A > 0$ and $\alpha > 0$ such that, for all $x, y \in \mathbb{P}^k$,*

$$|u_{\tau,\omega}(x) - u_{\tau,\omega}(y)| \leq A|\log(\operatorname{dist}(x, y))|^{-\alpha}.$$

The proof is based on two facts. The first one is that τ has bounded potential [**Dau14**, Theorem 3.7]. The second one is a lemma on the speed of convergence for pluriharmonic forms. To be more specific, let $\mathcal{H}$ be the set of all continuous real (s, s)-forms ϕ on U such that $dd^c\phi = 0$ and $|\langle R - \tau, \phi\rangle| \leq 1$ for all $R \in \mathscr{C}_1(U)$. We have the following result [**Taf13**, Lemma 4.1], based on the geometry of $\mathscr{C}_1(U)$.

LEMMA 4.2. *There exists a constant $0 < \lambda < 1$ such that for any R in $\mathscr{C}_1(U)$, ϕ in $\mathcal{H}$ and n in $\mathbb{N}$, we have*

$$|\langle \Lambda^n R - \tau, \phi\rangle| \leq \lambda^n.$$

It has the following corollaries.

COROLLARY 4.3. *There exist constants $C > 0$ and $0 < \lambda < 1$ such that if $R \in \mathscr{C}_1(U)$ is a current with bounded potential and x is a point outside U' then*

$$|u_{\Lambda^n R,\tau}(x)| \leq C\lambda^n.$$

Proof. If $x \notin U'$ then there exists a real (s, s)-current ϕ_x which is smooth on U' and such that $dd^c\phi_x = \delta_x - \mu$. Moreover, there exists a constant $C > 0$, independent of x, such that $|\langle R - \tau, \phi_x\rangle| \leq C$ for all x outside U'. Indeed,

$$\langle R - \tau, \phi_x\rangle = \langle dd^c u_{R,\tau}, \phi_x\rangle = \langle u_{R,\tau}, dd^c\phi_x\rangle = u_{R,\tau}(x) \tag{4.1}$$

and $\{u_{R,\tau}\}_{R\in\mathscr{C}_1(\overline{U})}$ is a compact family of pluriharmonic functions on $\mathbb{P}^k\backslash\overline{U}$ and then, which is therefore uniformly bounded on $\mathbb{P}^k\backslash U'$. Therefore, the restriction of ϕ_x/C to U is a form in $\mathcal{H}$ and Lemma 4.2 implies that

$$|u_{\Lambda^n R,\tau}(x)| = |\langle \Lambda^n R - \tau, \phi_x\rangle| \leq C\lambda^n. \qquad \square$$

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Remark 4.4. The second equality in (4.1) is *a priori* just a formal equality since both sides in the brackets involve currents. Anyway, in our case the singular locus of these currents are disjoint so it is easy, using regularization, to give a rigorous meaning to this equality. In the general case, it is related to the theory of super-potential of Dinh and Sibony.

Proof of Theorem 4.1. If S is a current in $\mathscr{C}_1(U)$ with continuous potential then it is also the case for $\Lambda^n S$, for all $n \in \mathbb{N}$. Hence $(u_{S,\Lambda^n S})_{n\in\mathbb{N}}$ is a sequence of continuous functions. We will show that it converges uniformly to $u_{S,\tau}$ which will therefore be continuous. Indeed, as $u_{S,\tau} = u_{S,\Lambda^n S} + u_{\Lambda^n S,\tau}$, for all $n \in \mathbb{N}$, we have to prove that $u_n := u_{\Lambda^n S,\tau}$ converge uniformly to 0. Observe that $u_n = \Lambda^n u_0$.

By the definition of Λ acting on functions, we have

$$|u_{n+1}(x)| = |\Lambda u_n(x)| = \frac{1}{d^s}\left| \sum_{y\in f^{-1}(x)\cap U'} u_n(y) + \sum_{y\in f^{-1}(x)\cap \mathbb{P}^k\setminus U'} u_n(y)\right|.$$

Since f is of small topological degree on $\mathscr{A}$, we can assume, possibly by exchanging f by an iterate f^N, that $f^{-1}(x) \cap U'$ has strictly less that d^s points, for all $x \in \mathbb{P}^k$. This and Corollary 4.3 give

$$|u_{n+1}(x)| \leq \frac{d^s - 1}{d^s}\|u_n\|_\infty + \frac{d^k}{d^s} C\lambda^n.$$

Without loss of generality, we can assume that $(d^s - 1)/d^s \leq \lambda < 1$ and then obtain by induction that

$$\|u_{n+1}\|_\infty \leq \lambda^n(\|u_0\|_\infty + (n+1)dC).$$

As τ has bounded potential [**Dau14**], $\|u_0\|_\infty$ is finite and therefore $(\|u_n\|_\infty)_{n\in\mathbb{N}}$ converges exponentially to zero. This finishes the proof of the continuity of the potential of τ.

To estimate the modulus of continuity, we assume now that $S \in \mathscr{C}_1(U)$ is a current which has Lipschitz continuous potential and such that the same holds for ΛS. By replacing U by $f^{-1}(f(U))$, we can choose $S = d^{-1} f^* R$ where $R \in \mathscr{C}_1(f(U))$ is smooth. If $v_n := u_{\Lambda^n S,S}$ then it easy to check that $v_{n+1} = \Lambda v_n + v_1$, $v_n - u_1 = u_n$ and that v_1 is a Lipschitz continuous function. Our goal is to estimate the modulus of continuity of $u_1 = u_{\tau,S}$.

By a result of Dinh and Sibony, see [**DS08**, Lemma 4.3], there exist constants $r > 0$ and $C_0 > 1$ such that, for all $x, y \in \mathbb{P}^k$, if $\operatorname{dist}(x, y) < r$ then it is possible to order the sets $f^{-1}(x) = \{x_i\}_{1\leq i\leq d^k}$ and $f^{-1}(y) = \{y_i\}_{1\leq i\leq d^k}$ in such a way that

$$\operatorname{dist}(x_i, y_i) \leq C_0 \operatorname{dist}(x, y)^{d^{-k}}.$$

This implies that there exists a constant $C_1 > 0$ such that, for all $x, y \in \mathbb{P}^k$ and all $n \in \mathbb{N}$,

$$|v_n(x) - v_n(y)| \leq nd^n C_0^{\sum_{i=0}^{n-1} d^{-ik}} \operatorname{dist}(x, y)^{d^{-kn}} \leq nd^n C_1 \operatorname{dist}(x, y)^{d^{-kn}}.$$

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Indeed, this is clear for $n = 1$ as v_1 is Lipschitz with a constant which can be assumed smaller than C_0 and C_1. And if the statement holds for v_n, then if $\mathrm{dist}(x, y) < r$ we have

$$\begin{aligned}
|v_{n+1}(x) - v_{n+1}(y)| &\leq |\Lambda v_n(x) - \Lambda v_n(y)| + |v_1(x) - v_1(y)| \\
&\leq d^{-s} \sum_{j=1}^{d^k} |v_n(x_j) - v_n(y_j)| + C_1 \,\mathrm{dist}(x, y) \\
&\leq d^{-s} \sum_{j=1}^{d^k} n d^n C_0^{\sum_{i=0}^{n-1} d^{-ik}} \,\mathrm{dist}(x_j, y_j)^{d^{-kn}} + C_1 \,\mathrm{dist}(x, y) \\
&\leq n d^{n+1} C_0^{\sum_{i=0}^{n} d^{-ik}} \,\mathrm{dist}(x, y)^{d^{-k(n+1)}} + C_1 \,\mathrm{dist}(x, y) \\
&\leq (n+1) d^{n+1} C_0^{\sum_{i=0}^{n} d^{-ik}} \,\mathrm{dist}(x, y)^{d^{-k(n+1)}}.
\end{aligned}$$

The case where $\mathrm{dist}(x, y) \geq r$ comes from the fact that the functions $v_n = u_1 + u_n$ are uniformly bounded since u_1 is continuous and $\|u_n\|_\infty \leq C\lambda^n$.

We deduce that, for all $x, y \in \mathbb{P}^k$,

$$\begin{aligned}
|u_1(x) - u_1(y)| &\leq \inf_{n \in \mathbb{N}} (2\|u_1 - v_n\|_\infty + |v_n(x) - v_n(y)|) \\
&\leq \inf_{n \in \mathbb{N}} (2C\lambda^n + n d^n C_1 \,\mathrm{dist}(x, y)^{d^{-kn}}).
\end{aligned}$$

Let x, y be in $\mathbb{P}^k$ such that $\mathrm{dist}(x, y) \ll 1$. Then, if

$$n = \left\lfloor \frac{-\log(\sqrt{-\log(\mathrm{dist}(x, y))})}{\log(1/d^k)} \right\rfloor = \left\lfloor \frac{\log(-\log(\mathrm{dist}(x, y)))}{2k \log(d)} \right\rfloor,$$

we have

$$\begin{aligned}
d^{kn} &= e^{nk \log(d)} \leq e^{\log(-\log(\mathrm{dist}(x,y)))/2k \log(d) \times k \log(d)} \\
&= e^{(1/2) \log(-\log(\mathrm{dist}(x,y)))} = (-\log(\mathrm{dist}(x, y)))^{1/2}
\end{aligned}$$

so

$$n d^n \,\mathrm{dist}(x, y)^{1/d^{kn}} \leq d^{kn} e^{\log(\mathrm{dist}(x,y))/d^{kn}} \leq \sqrt{-\log(\mathrm{dist}(x, y))} e^{-\sqrt{-\log(\mathrm{dist}(x,y))}}$$

and

$$\lambda^n \leq e^{\log(\lambda)((\log(-\log(\mathrm{dist}(x,y)))/2k \log(d))-1)} \leq \frac{(-\log \mathrm{dist}(x, y))^{\log(\lambda)/2k \log(d)}}{\lambda}.$$

If $\mathrm{dist}(x, y)$ is small enough then

$$\sqrt{-\log(\mathrm{dist}(x, y))} e^{-\sqrt{-\log(\mathrm{dist}(x,y))}} \leq (-\log \mathrm{dist}(x, y))^{\log(\lambda)/2k \log(d)}.$$

As u_1 is bounded [**Dau14**], there exists $A > 0$ such that, for all $x, y \in \mathbb{P}^k$,

$$|u_1(x) - u_1(y)| \leq A |\log \mathrm{dist}(x, y)|^{-\alpha}$$

where $\alpha = -\log(\lambda)/2k \log(d)$. This finishes the proof of Theorem 4.1. □

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Remark 4.5. Without (HD), similar arguments, using

$$|u_{\Lambda^n R,\tau}(x)| \leq C \quad \text{for all } x \notin U'$$

instead of Corollary 4.3, give an alternative proof to [**Dau14**, Theorem 3.7], i.e. if $S \in \mathscr{C}_1(U)$ has bounded potential then any limit value of $\Lambda^n S$ also has bounded potential.

We can deduce from the proof of Theorem 4.1 that under the same assumptions τ is the unique current in $\mathscr{C}_1(U)$ invariant by Λ and with bounded potential. In fact, the boundedness assumption can be dropped and, moreover, Theorem 1.3 gives a uniform speed of equidistribution of any current in $\mathscr{C}_1(U)$ towards τ.

Proof of Theorem 1.3. Let S be a current in $\mathscr{C}_1(U)$. The functions $u_n := u_{\Lambda^n S,\tau}$ satisfy $\Lambda u_n = u_{n+1}$. Observe that if ϕ is a $\mathcal{C}^2$ test form then

$$|\langle \Lambda^n S - \tau, \phi\rangle| = |\langle u_n, dd^c\phi\rangle| \leq \|u_n\|_{L^1(\mathbb{P}^k)}\|\phi\|_{\mathcal{C}^2}.$$

So, in order to prove the theorem, it is sufficient to obtain an estimate on $\|u_n\|_{L^1(\mathbb{P}^k)}$.

We now introduce some notation. First, notice that, as the constants below will not depend on S, considering an iterate f^N instead of f does not affect the arguments. As in the proof of Theorem 4.1 we can assume, possibly by exchanging f by f^N, that $f^{-1}(x) \cap U$ has strictly less that d^s points, for all $x \in \mathbb{P}^k$. Let C and λ be the constants of Corollary 4.3 where, as above, we can assume that $(d^s - 1)/d^s < \lambda < 1$. Finally, let $\alpha > 0$ and $\delta > 0$ be such that $\lambda < \delta < 1$ and $(\alpha - 1)\delta^n > Cdn\lambda^{n-1}$, for all $n \geq 0$.

Since

$$|u_{n+1}(x)| = |\Lambda u_n(x)| = \frac{1}{d^s}\left|\sum_{y\in f^{-1}(x)\cap U'} u_n(y) + \sum_{y\in f^{-1}(x)\cap \mathbb{P}^k\setminus U'} u_n(y)\right|,$$

in the same way as in the proof of Theorem 1.2, we obtain that

$$|u_{n+1}(x)| \leq \lambda \max_{y\in f^{-1}(x)} |u_n(y)| + Cd\lambda^n,$$

and by induction

$$|u_{n+1}(x)| \leq \lambda^{n+1} \max_{y\in f^{-(n+1)}(x)} |u_0(y)| + Cd(n+1)\lambda^n.$$

Therefore,

$$\{|u_n| \geq \alpha\delta^n\} \subset f^n\left(\left\{|u_0| \geq \frac{\alpha\delta^n - Cdn\lambda^{n-1}}{\lambda^n}\right\}\right) \subset f^n\left(\left\{|u_0| \geq \frac{\delta^n}{\lambda^n}\right\}\right),$$

where the last inclusion comes from the choices of α and δ.

On the other hand, the family $\mathscr{P}$ of functions u such that $\langle \mu, u\rangle = 0$, $dd^c u = S - \tau$ with $S \in \mathscr{C}_1(\mathbb{P}^k)$ is a compact family of d.s.h. functions. Therefore, by an inequality of Hörmander (it can be deduced from [**Hör90**, Theorem 4.4.5]), there exists a constant $a > 0$ such that

$$\int_{\mathbb{P}^k} e^{a|u|}\omega^k \leq a^{-1},$$

for any $u \in \mathscr{P}$. As a consequence of this inequality, we have

$$\mathrm{Vol}\left(\left\{|u_0| \geq \frac{\delta^n}{\lambda^n}\right\}\right) \leq a^{-1}e^{-a\delta^n\lambda^{-n}},$$

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since $u_0 \in \mathscr{P}$. And, by the change of variables formula, there exists a constant $M > 0$, which is essentially equal to $\|\mathrm{Jac}(f)\|_\infty$, such that

$$\mathrm{Vol}\left(f^n\left(\left\{ |u_0| \geq \frac{\delta^n}{\lambda^n} \right\} \right) \right) \leq a^{-1} M^n e^{-a\delta^n \lambda^{-n}},$$

which implies

$$\mathrm{Vol}(\{|u_n| \geq \alpha\delta^n\}) \leq a^{-1} M^n e^{-a\delta^n \lambda^{-n}},$$

as $\{|u_n| \geq \alpha\delta^n\} \subset f^n(\{|u_0| \geq \delta^n/\lambda^n\})$.

Hence, by the Cauchy–Schwarz inequality,

$$\|u_n\|_{L^1(\mathbb{P}^k)} \leq \mathrm{Vol}(\{|u_n| \geq \alpha\delta^n\})^{1/2} \|u_n\|_{L^2(\mathbb{P}^k)} + \alpha\delta^n,$$

and then

$$\|u_n\|_{L^1(\mathbb{P}^k)} \leq a^{-1/2} M^{n/2} e^{-a\delta^n \lambda^{-n}/2} \|u_n\|_{L^2(\mathbb{P}^k)} + \alpha\delta^n \leq c\delta^n,$$

for a constant $c > 0$ large enough since $\|u\|_{L^2(\mathbb{P}^k)}$ is uniformly bounded for $u \in \mathscr{P}$ and $\lambda < \delta$. □

Remark 4.6. Using interpolation between Banach spaces (see e.g. [**Tri95**]), we can obtain that

$$|\langle \Lambda^n S - \tau, \phi \rangle| \leq c\delta^{\alpha n/2} \|\phi\|_{\mathcal{C}^\alpha},$$

for any test form ϕ of class $\mathcal{C}^\alpha$, with $0 < \alpha \leq 2$. Here, c depends on α.

5. *Negative Lyapunov exponent*

In this section, we consider an endomorphism f of $\mathbb{P}^k$ of degree $d \geq 2$ with an attracting set $\mathscr{A}$ of codimension one. We will study hyperbolicity properties of measures of entropy $s \log d$ on $\mathscr{A}$.

Let ν be such a measure and assume that ν is ergodic. We will use the ideas of de Thélin [**deT08**] to obtain a relation between the Lyapunov exponents of ν and a certain 'topological degree'. Indeed, as the dynamics near an attracting set stay confined in a trapping region, we consider the following local version of topological degree. Let V be a small neighborhood of $\mathrm{supp}(\nu)$ and define

$$d_{k,\mathrm{loc}} := \limsup_{n \to +\infty} \left(\int_V (f^n)^*(\omega^k) \right)^{1/n}.$$

We will now prove Theorem 1.4 which says that if $d_{k,\mathrm{loc}} < d^s$, then ν is a hyperbolic measure.

Remark 5.1. As we said in the introduction, to be of small topological degree on $\mathscr{A}$ is much stronger than $d_{k,\mathrm{loc}} < d^s$. Moreover, the assumption $d_{k,\mathrm{loc}} < d^s$ holds in all examples known by the authors. We do not know whether this assumption or the conclusion on the hyperbolicity of ν always holds.

Proof of Theorem 1.4. If $\log(\mathrm{Jac}\, f) \notin L^1(\nu)$, then $\chi_k = -\infty$ and we have the upper bound on χ_k. By the Margulis–Ruelle inequality, we have $\chi_1 > 0$. Thus, $\chi_1 \neq \chi_k$ and the lower bound $\chi_{k-1} \geq 2^{-1} \log d$ is a direct consequence of [**Dup**, Theorem D].

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So we assume that $\log(\mathrm{Jac}\, f) \in L^1(\nu)$. By [**deT08**], we have $\chi_{k-1} > 0$ and the lower bound $\chi_{k-1} \geq 2^{-1} \log d$ as long as we know that $\chi_k \leq 0$.

The proof of the upper bound on χ_k follows closely the one of [**deT08**]. We will recall the technical intermediary results de Thélin obtains and will only give all details when the assumption about the dynamical degree $d_{k,\mathrm{loc}}$ appears. Actually, everything in what follows, as in the proofs of [**deT08**], will take place in a fixed compact set $K \subset V$.

Let $\gamma > 0$ be an arbitrary small constant. If γ is small enough, there exists a compact $K \subset V$ containing a γ-neighborhood of $\mathrm{supp}(\nu)$. Hereafter, $\delta > 0$ will be a small constant depending on γ. Recall that two points x and y in $\mathbb{P}^k$ are (n, δ)-separated if $\mathrm{dist}_n(x, y) \geq \delta$ where $\mathrm{dist}_n(x, y) := \max_{0 \leq i \leq n} \mathrm{dist}(f^i(x), f^i(y))$. Using a theorem of Brin and Katok [**BK83**], de Thélin constructs, for n large enough, a set $E = \{x_i\}_{1 \leq i \leq N} \subset \mathrm{supp}(\nu)$ such that:

- $N \geq \frac{1}{2} e^{h(\nu)n - \gamma n}$;
- the points x_i and x_j are (n, δ)-separated if $i \neq j$.

Then for each $x_i \in E$, using Pesin theory and graph transform, he constructs a 'thick unstable manifold' $W(x_i) \subset K$, which in our case is an open subset, such that $f^n(x_i) \in W(x_i)$ and $\mathrm{Vol}(W(x_i)) \geq c^{-1} e^{2\chi_k^- n - c\gamma n}$ for some constant $c \geq 1$ independent of n. With the terms of [**BLS93**], $W(x_i)$ is the nth cut-off iterate of a Lyapunov chart.

Moreover, we can deduce from his construction that if $x_i \in E$, then there exists an open set $W_{-n}(x_i) \subset K$ which contains x_i and such that:

- $f^n(W_{-n}(x_i)) = W(x_i)$;
- the diameter of $f^l(W_{-n}(x_i))$ is less than $\delta/4$ for $l = 0, \ldots, n$.

These two properties and the fact that the points x_i are (n, δ)-separated imply that the sets $W_{-n}(x_i)$ are pairwise disjoint. In particular, if, for $a \in K$, $n(a)$ denotes the number of points $x_i \in E$ such that $a \in W(x_i)$, then there exist at least $n(a)$ points in $f^{-n}(a) \cap V$, i.e. $n(a) \leq (f^n)_* \mathbf{1}_V(a)$.

We can now use our assumption that $d_{k,\mathrm{loc}} < d^s$. The last observation implies that

$$\int_K n(a)\omega^k \leq \int (f^n)_*(\mathbf{1}_V)\omega^k = \int_V (f^n)^*(\omega^k). \tag{5.1}$$

Moreover, by the definition of $d_{k,\mathrm{loc}}$, for all $\varepsilon > 0$, we have, for n large enough,

$$\int_V (f^n)^*(\omega^k) \leq (d_{k,\mathrm{loc}} + \varepsilon)^n. \tag{5.2}$$

On the other hand, the estimates on N and on $\mathrm{Vol}(W(x_i))$ imply that

$$\int_K n(a)\omega^k \geq \frac{1}{2c} e^{h(\nu)n + 2\chi_k^- - (c+1)\gamma n}. \tag{5.3}$$

Hence, we deduce from equations (5.1)–(5.3) that

$$(d_{k,\mathrm{loc}} + \varepsilon)^n \geq \frac{1}{2c} e^{h(\nu)n + 2\chi_k^- - (c+1)\gamma n},$$

for n large enough. As the constant c is independent of n, we obtain, when n goes to infinity, that $h(\nu) \leq \log(d_{k,\mathrm{loc}} + \varepsilon) + (c+1)\gamma - 2\chi_k^-$. But $\varepsilon > 0$ and $\gamma > 0$ are arbitrary small, and hence $h(\nu) \leq \log(d_{k,\mathrm{loc}}) - 2\chi_k^-$, which was the desired result. □

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