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A SIMPLE APPROACH TO FUNCTIONAL INEQUALITIES FOR NON-LOCAL DIRICHLET FORMS

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Abstract. With direct and simple proofs, we establish Poincaré type inequalities (including Poincaré inequalities, weak Poincaré inequalities and super Poincaré inequalities), entropy inequalities and Beckner-type inequalities for non-local Dirichlet forms. The proofs are efficient for non-local Dirichlet forms with general jump kernel, and also work for $L^p (p > 1)$ settings. Our results yield a new sufficient condition for fractional Poincaré inequalities, which were recently studied in [P.T. Gressman, *J. Funct. Anal.* **265** (2013) 867–889. C. Mouhot, E. Russ and Y. Sire, *J. Math. Pures Appl.* **95** (2011) 72–84.] To our knowledge this is the first result providing entropy inequalities and Beckner-type inequalities for measures more general than Lévy measures.

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1. Introduction and main results

The question of obtaining Poincaré-type inequalities (or more generally entropy inequalities) for pure jump Lévy processes was studied in the last decades, *e.g.* see [1, 2, 10]. In particular, it was proved by Corollary 4.2 of [10] and Theorem 23 of [2] that

$$\mathrm{Ent}_\mu^\Phi(f) \leqslant \iint D_\Phi(f(x), f(x+z))\, \nu_\mu(\mathrm{d}z)\, \mu(\mathrm{d}x), \quad f \in C_b^\infty(\mathbb{R}^d), f > 0 \tag{1}$$

with

$$\mathrm{Ent}_\mu^\Phi(f) = \int \Phi(f)\, \mathrm{d}\mu - \Phi\left(\int f \mathrm{d}\mu\right)$$

and D_Φ is the so-called Bergman distance associated with Φ:

$$D_\Phi(a, b) = \Phi(a) - \Phi(b) - \Phi'(b)(a - b),$$

where μ is a rather general probability measure and ν_μ is the (singular) Lévy measure associated to μ. By setting $\Phi(x) = x^2$ and $\Phi(x) = x \log x$, $\mathrm{Ent}_\mu^\Phi(f)$ becomes the classical variance $\mathrm{Var}_\mu(f)$ and entropy $\mathrm{Ent}_\mu(f)$ respectively, and so (1) yields the Poincaré inequality and the entropy inequality for the choice of measure (μ, ν_μ). Note that either one of the measures μ and ν_μ in (1) uniquely specifies the other, and so this is a strong

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constraint to study functional inequalities for general non-local Dirichlet forms. The first breakthrough in this direction was established in [7] by using the methods from harmonic analysis, and then was extended in [6] to L^p weighted Poincaré inequalities and generalized logarithmic Sobolev inequalities in an abstract situation.

Let V be a locally bounded measurable function on $\mathbb{R}^d$ such that $\int \mathrm{e}^{-V(x)}\,\mathrm{d}x = 1$; that is, $\mu_V(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. The main result in [7] (see [7], Thm. 1.2) states that, if $V \in C^2(\mathbb{R}^d)$ such that for some constant $\varepsilon > 0$,

$$\frac{(1-\varepsilon)|\nabla V(x)|^2}{2} - \Delta V(x) \to \infty, \qquad |x| \to \infty,$$

then there exist two positive constants δ and C_0 such that for all $f \in C_b^\infty(\mathbb{R}^d)$,

$$\int (f - \mu_V(f))^2 \big(1 + |\nabla V|^\alpha\big)\,\mu_V(\mathrm{d}x) \leqslant C_0 D_{\alpha,V,\delta}(f,f),$$

where

$$D_{\alpha,V,\delta}(f,f) = \iint \frac{(f(y)-f(x))^2}{|y-x|^{d+\alpha}} \mathrm{e}^{-\delta|y-x|}\,\mathrm{d}y\,\mu_V(\mathrm{d}x). \tag{2}$$

According to the paragraph below ([7], Rem. 1.3) (2) is natural in the sense that: we should regard the measure $|y-x|^{-(d+\alpha)}\mathrm{e}^{-\delta|y-x|}\,\mathrm{d}y$ as the Lévy measure, and $\mu_V(\mathrm{d}x)$ as the ambient measure. Namely, $D_{\alpha,V,\delta}$ does get rid of the constraint of (μ,ν_μ) in (1), and it should be a typical example in study functional inequalities for non-local Dirichlet forms. This leds us to consider the Dirichlet form $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$ as follows. Let ρ be a strictly positive measurable function on $(0,\infty)$ such that $\int_{(0,\infty)} \rho(r)(1\wedge r^2) r^{d-1}\,dr < \infty$. Let $L^2(\mu_V)$ be the space of Borel measurable functions f on $\mathbb{R}^d$ such that $\mu_V(f^2) := \int f^2(x)\,\mu_V(\mathrm{d}x) < \infty$. Set

$$D_{\rho,V}(f,f) := \iint_{x\neq y} \big(f(x)-f(y)\big)^2 \rho(|x-y|)\,\mathrm{d}y\,\mu_V(\mathrm{d}x)$$

$$\mathscr{D}(D_{\rho,V}) := \Big\{ f \in L^2(\mu_V) : D_{\rho,V}(f,f) < \infty \Big\}.$$

According to ([3], Example 2.2) we know that $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$ is a symmetric Dirichlet form such that $C_b^\infty(\mathbb{R}^d) \subset \mathscr{D}(D_{\rho,V})$, where $C_b^\infty(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ with bounded derivatives for all orders.

The purpose of this note is to present sufficient conditions for Poincaré type inequalities (*i.e.* Poincaré inequalities, weak Poincaré inequalities and super Poincaré inequalities), entropy inequalities and Beckner-type inequalities for $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$. We first state the main result for Poincaré type inequalities of $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$.

Theorem 1.1.

(1) *If there exists a constant $c > 0$ such that for any x, $y \in \mathbb{R}^d$ with $x \neq y$,*

$$\big(\mathrm{e}^{V(x)} + \mathrm{e}^{V(y)}\big)\rho(|x-y|) \geqslant c, \tag{3}$$

then the following Poincaré inequality

$$\mu_V(f-\mu_V(f))^2 \leqslant c^{-1} D_{\rho,V}(f,f), \quad f \in \mathscr{D}(D_{\rho,V}) \tag{4}$$

holds.

(2) *For any probability measure μ_V, the following weak Poincaré inequality*

$$\mu_V(f-\mu_V(f))^2 \leqslant \alpha(r) D_{\rho,V}(f,f) + r\|f\|_\infty^2, \quad r > 0, f \in \mathscr{D}(D_{\rho,V}) \tag{5}$$

holds with

$$\alpha(r) = \inf\left\{ \frac{1}{\inf\limits_{0<|x-y|\leqslant s} \big[(\mathrm{e}^{V(x)} + \mathrm{e}^{V(y)})\rho(|x-y|)\big]} : \iint_{|x-y|>s} \mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \leqslant \frac{r}{2} \right\}.$$

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(3) *Suppose that there exists a nonnegative locally bounded measurable function w on $\mathbb{R}^d$ such that*

$$\lim_{|x|\to\infty} w(x) = \infty,$$

and for any x, $y \in \mathbb{R}^d$ with $x \neq y$,

$$\mathrm{e}^{V(x)} + \mathrm{e}^{V(y)} \geqslant \frac{w(x)+w(y)}{\rho(|x-y|)}. \tag{6}$$

Then the following super Poincaré inequality

$$\mu_V(f^2) \leqslant rD_{\rho,V}(f,f) + \beta(r)\mu_V(|f|)^2, \quad r>0, f \in \mathscr{D}(D_{\rho,V}) \tag{7}$$

holds with

$$\beta(r) = \inf\left\{ \frac{2\mu_V(\omega)}{\inf_{|x|\geqslant t}\omega(x)} + \beta_t(t\wedge s) : \frac{2}{\inf_{|x|\geqslant t} w(x)} + s \leqslant r \text{ and } t,s>0 \right\},$$

where for any $t>0$,

$$\beta_t(s) = \inf\left\{ \frac{c_0\Big(\sup_{|z|\leqslant 2t}\mathrm{e}^{V(z)}\Big)^2}{u^d\Big(\inf_{|z|\leqslant t}\mathrm{e}^{V(z)}\Big)} : \frac{c_0\Big(\sup_{0<\varepsilon\leqslant u}\rho(\varepsilon)^{-1}\Big)\Big(\sup_{|z|\leqslant 2t}\mathrm{e}^{V(z)}\Big)}{u^d\Big(\inf_{|z|\leqslant t}\mathrm{e}^{V(z)}\Big)} \leqslant s \text{ and } u>0 \right\}.$$

To illustrate the power of Theorem 1.1, we will consider the following examples.

Example 1.2. Let $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x := C_{d,\varepsilon}(1+|x|)^{-(d+\varepsilon)}\,\mathrm{d}x$ with $\varepsilon>0$, and $\rho(r) = r^{-(d+\alpha)}$ with $\alpha \in (0,2)$.

(1) If $\varepsilon \geqslant \alpha$, then the Poincaré inequality (4) holds with $c = \frac{2^{1-(d+\alpha)}}{C_{d,\varepsilon}}$.
(2) If $0<\varepsilon<\alpha$, then the weak Poincaré inequality (5) holds with

$$\alpha(r) = c_1\left(1 + r^{-(\alpha-\varepsilon)/\varepsilon}\right)$$

for some constant $c_1>0$.
(3) If $\varepsilon>\alpha$, then the super Poincaré inequality (7) holds with

$$\beta(r) = c_2\left(1 + r^{-\frac{d}{\alpha} - \frac{(d+\varepsilon)(d+2\alpha)}{\alpha(\varepsilon-\alpha)}}\right)$$

for some constant $c_2>0$.

According to Corollary 1.2 of [9], we know that all the conclusions above are optimal.

Example 1.3. Let $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x := C_{d,\alpha,\varepsilon}(1+|x|)^{-(d+\alpha)}\log^{\varepsilon}(e+|x|)\,\mathrm{d}x$ with $\varepsilon \in \mathbb{R}$, and $\rho(r) = r^{-(d+\alpha)}$ with $\alpha \in (0,2)$.

(1) If $\varepsilon \leqslant 0$, then the Poincaré inequality (4) holds with $c = \frac{2^{1-(d+\alpha)}}{C_{d,\varepsilon}}$.
(2) If $\varepsilon>0$, then the weak Poincaré inequality (5) holds with

$$\alpha(r) = c_3\left(1 + \log^{\varepsilon}\left(1 + r^{-1}\right)\right)$$

for some constant $c_3>0$.

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(3) If $\varepsilon < 0$, then the super Poincaré inequality (7) holds with

$$\beta(r) = \exp\left(c_4\left(1 + r^{1/\varepsilon}\right)\right)$$

for some constant $c_4 > 0$.

By Corollary 1.3 of [9], all the conclusions above are also sharp.

Example 1.4. Let $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x := C_\lambda \mathrm{e}^{-\lambda|x|}\,\mathrm{d}x$ with $\lambda > 0$, and $\rho(r) = \mathrm{e}^{-\delta r} r^{-(d+\alpha)}$ with $\delta \geqslant 0$ and $\alpha \in (0,2)$. Therefore, if $\lambda > 2\delta$, then the super Poincaré inequality (7) holds with $\beta(r) = c_5\left(1 + r^{-\frac{d}{\alpha} - \frac{2\lambda(d+2\varepsilon)}{\alpha(\lambda-2\delta)}}\right)$ for some constant $c_5 > 0$. In particular, the Poincaré inequality (4) holds. Note that, this conclusion can not be deduced from Theorem 1.1 of [7], see also the statement before (2).

Next, we turn to study entropy inequalities and Beckner-type inequalities for $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$. Recall that for any $f \in \mathscr{D}(D_{\rho,V})$ with $f > 0$,

$$\mathrm{Ent}_{\mu_V}(f) := \mu_V(f\log f) - \mu_V(f)\log\mu_V(f).$$

Theorem 1.5. *Suppose that (3) is satisfied. Then the following entropy inequality*

$$Ent_{\mu_V}(f) \leqslant c^{-1} D_{\rho,V}(f, \log f) \tag{8}$$

holds for all $f \in \mathscr{D}(D_{\rho,V})$ with $f > 0$; and moreover, the following Beckner-type inequality also holds: for any $p \in (1,2]$ and $f \in \mathscr{D}(D_{\rho,V})$ with $f \geqslant 0$,

$$\mu_V(f^p) - \mu_V(f)^p \leqslant c^{-1} D_{\rho,V}\left(f, f^{p-1}\right). \tag{9}$$

The entropy inequality (8) and Beckner-type inequality (9) are stronger than the Poincaré inequality (4) (to see this, one can apply these inequalities to the function $1 + \varepsilon f$ and then take the limit as $\varepsilon \to 0$). Clearly, the Beckner-type inequality (9) reduces to the Poincaré inequality (4) if $p = 2$, whereas dividing both sides by $p - 1$ and taking the limit as $p \to 1$ we obtain the entropy inequality (8). As mentioned in the remarks below (2), comparing Theorem 1.5 with (1) the improvement is due to that we do not impose any link between the measure $\mu_V(\mathrm{d}x)$ on x and the singular measure $\rho(|z|)\,\mathrm{d}z$ on $z = y - z$. This is to our knowledge the first result that gets rid of the strong constraint for entropy inequalities and Beckner-type inequalities of non-local Dirichlet forms.

Example 1.6 (Continuation of Example 1.2).
Let

$$\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x := C_{d,\varepsilon}(1 + |x|)^{-(d+\varepsilon)}\,\mathrm{d}x$$

with $\varepsilon > 0$, and $\rho(r) = r^{-(d+\alpha)}$ with $\alpha \in (0,2)$.

(1) If $\varepsilon \geqslant \alpha$, then, according to Theorem 1.5, the entropy inequality (8) and Beckner-type inequality (9) hold with $c = \frac{2^{1-(d+\alpha)}}{C_{d,\varepsilon}}$.
(2) If $0 < \varepsilon < \alpha$, then, according to Corollary 1.2 of [9], the Poincaré inequality (4) does not hold. Hence, by the remark below Theorem 1.5, both the entropy inequality (8) and Beckner-type inequality (9) do not hold.

According to Examples 1.2, 1.6 and Corollary 1.2 of [9], we know that for the probability measure $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x := C_{d,\alpha}(1 + |x|)^{-(d+\alpha)}\,\mathrm{d}x$, it fulfills the entropy inequality (8) and the Beckner-type inequality (9), but not the super Poincaré inequality (7).

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2. Proofs of Theorems and Example 1.2

Proof of Theorem 1.1.
(1) For any $f \in \mathscr{D}(D_{\rho,V})$,

$$\begin{aligned}\frac{1}{2}\iint \big(f(x)-f(y)\big)^2\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) &= \frac{1}{2}\iint \big(f^2(x)+f^2(y)-2f(x)f(y)\big)\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &= \mu_V(f^2)-\mu_V(f)^2 = \mu_V(f-\mu_V(f))^2. \end{aligned} \tag{10}$$

On the other hand, by (3), we find that

$$\begin{aligned}\frac{1}{2}\iint \big(f(x)-f(y)\big)^2\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) &= \frac{1}{2}\iint_{x\neq y} \big(f(x)-f(y)\big)^2\rho(|x-y|)\rho(|x-y|)^{-1}\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &\leqslant c^{-1}\iint_{x\neq y} \big(f(x)-f(y)\big)^2\rho(|x-y|)\,\frac{\mathrm{e}^{-V(x)}+\mathrm{e}^{-V(y)}}{2}\,\mathrm{d}y\,\mathrm{d}x\\ &= c^{-1}D_{\rho,V}(f,f),\end{aligned}$$

which, along with (10), yields the required assertion.
(2) According to (10), for any $s>0$ and $f\in\mathscr{D}(D_{\rho,V})$,

$$\begin{aligned}\mu_V(f-\mu_V(f))^2 &= \frac{1}{2}\iint (f(x)-f(y))^2\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &= \frac{1}{2}\iint_{0<|x-y|\leqslant s} (f(x)-f(y))^2\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &\quad+\frac{1}{2}\iint_{|x-y|>s} (f(x)-f(y))^2\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &\leqslant \iint_{0<|x-y|\leqslant s} (f(x)-f(y))^2\rho(|x-y|)\\ &\quad\times\left[\frac{\mathrm{e}^{-V(x)-V(y)}}{(\mathrm{e}^{-V(x)}+\mathrm{e}^{-V(y)})\rho(|x-y|)}\right]\frac{\mathrm{e}^{-V(x)}+\mathrm{e}^{-V(y)}}{2}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad+2\|f\|_\infty^2\iint_{|x-y|>s}\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\\ &\leqslant\left(\sup_{0<|x-y|\leqslant s}\frac{1}{(\mathrm{e}^{V(x)}+\mathrm{e}^{V(y)})\rho(|x-y|)}\right)D_{\rho,V}(f,f)\\ &\quad+\left(2\iint_{|x-y|>s}\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x)\right)\|f\|_\infty^2.\end{aligned}$$

The desired assertion follows from the definition of α.
(3) For any $f\in\mathscr{D}(D_{\rho,V})$, by Jensen's inequality,

$$\begin{aligned}\mu_V((f-\mu_V(f))^2w) &= \int\left(f(x)-\int f(y)\,\mu_V(\mathrm{d}y)\right)^2 w(x)\,\mu_V(\mathrm{d}x)\\ &= \int\left(\int (f(x)-f(y))\,\mu_V(\mathrm{d}y)\right)^2 w(x)\,\mu_V(\mathrm{d}x)\\ &\leqslant \iint (f(x)-f(y))^2 w(x)\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x).\end{aligned}$$

This implies that

$$\mu_V((f-\mu_V(f))^2w)\leqslant\iint (f(x)-f(y))^2\,\frac{w(x)+w(y)}{2}\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x).$$

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Thus, by (6), we arrive at

$$\begin{aligned}\mu_V((f-\mu_V(f))^2 w) &\leqslant \iint (f(x)-f(y))^2 \\ &\quad \times \rho(|x-y|)\frac{\mathrm{e}^{V(x)}+\mathrm{e}^{V(y)}}{2}\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \\ &\leqslant D_{\rho,V}(f,f).\end{aligned} \tag{11}$$

Next, we will follow the proof of Proposition 1.6 from [3] to obtain the super Poincaré inequality from (11). We first claim that $\mu_V(\omega)<\infty$. In fact, let $C_c^\infty(\mathbb{R}^d)$ be the set of smooth functions on $\mathbb{R}^d$ with compact support. Choose a function $g\in C_c^\infty(\mathbb{R}^d)$ such that $g(x)=0$ for every $|x|\geqslant 1$ and $\mu_V(g)=1$. Then, applying this test function g into (11) and noting the fact that $C_c^\infty(\mathbb{R}^d)\subset C_b^\infty(\mathbb{R}^d)\subset\mathscr{D}(D_{\rho,V})$, we have

$$\int_{\{|x|\geqslant 1\}}\omega(x)\,\mu_V(\mathrm{d}x)\leqslant\int\big(g(x)-\mu_V(g)\big)^2\omega(x)\,\mu_V(\mathrm{d}x)\leqslant D_{\rho,V}(g,g)<\infty.$$

Since the function ω is bounded on $\{x\in\mathbb{R}^d:|x|\leqslant 1\}$, $\int_{\{|x|\leqslant 1\}}\omega(x)\,\mu_V(\mathrm{d}x)<\infty$. Combining both estimates above, we prove the desired claim.

For any $t>1$ large enough and $f\in\mathscr{D}(D_{\rho,V})$, by (11), we have

$$\begin{aligned}\int_{\{|x|\geqslant t\}} f^2(x)\,\mu_V(\mathrm{d}x) &\leqslant \frac{1}{\inf\limits_{|x|\geqslant t}\omega(x)}\int f^2(x)\omega(x)\,\mu_V(\mathrm{d}x) \\ &\leqslant \frac{2}{\inf\limits_{|x|\geqslant t}\omega(x)}\int\big(f(x)-\mu_V(f)\big)^2\omega(x)\,\mu_V(\mathrm{d}x) \\ &\quad+\frac{2}{\inf\limits_{|x|\geqslant t}\omega(x)}\int\mu_V(f)^2\omega(x)\,\mu_V(\mathrm{d}x) \\ &\leqslant \frac{2}{\inf\limits_{|x|\geqslant t}\omega(x)}\Big(D_{\rho,V}(f,f)+\mu_V(\omega)\,\mu_V(|f|)^2\Big),\end{aligned}$$

where the second inequality follows from the inequality that for any $a,b\in\mathbb{R}$, $a^2\leqslant 2(a-b)^2+2b^2$.

On the other hand, Lemma 2.1 below shows that the local super Poincaré inequality

$$\int_{\{|x|\leqslant t\}} f^2(x)\,\mu_V(\mathrm{d}x)\leqslant sD_{\rho,V}(f,f)+\beta_t(t\wedge s)\mu_V(|f|)^2,\quad s>0 \tag{12}$$

holds for any $t>1$ and $f\in\mathscr{D}(D_{\rho,V})$.

Combining both estimates above, we get that for $t>1$ large enough and any $f\in\mathscr{D}(D_{\rho,V})$,

$$\mu_V(f^2)\leqslant\left(\frac{2}{\inf\limits_{|x|\geqslant t}\omega(x)}+s\right)D_{\rho,V}(f,f)+\left(\frac{2\mu_V(\omega)}{\inf\limits_{|x|\geqslant t}\omega(x)}+\beta_t(t\wedge s)\right)\mu_V(|f|)^2,\quad s>0.$$

This, along with $\lim\limits_{|x|\to\infty}\omega(x)=\infty$ and the definition of β, yields the required super Poincaré inequality. □

For the local super Poincaré inequality (12) in part (3) of the proof above, we can see from the following

Lemma 2.1. *For any $f\in\mathscr{D}(D_{\rho,V})$ and $r>0$, we have*

$$\int_{B(0,r)} f^2(x)\,\mu_V(\mathrm{d}x)\leqslant sD_{\rho,V}(f,f)+\beta_r(r\wedge s)\mu_V(|f|)^2\quad s>0,$$

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where

$$\beta_r(s)=\inf\left\{\frac{2\left(\sup\limits_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\right)^2}{|B(0,t)|\left(\inf\limits_{|z|\leqslant r}\mathrm{e}^{V(z)}\right)}:\frac{2\left(\sup\limits_{0<\varepsilon\leqslant t}\rho(\varepsilon)^{-1}\right)\left(\sup\limits_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\right)}{|B(0,t)|\left(\inf\limits_{|z|\leqslant r}\mathrm{e}^{V(z)}\right)}\leqslant s \text{ and } t>0\right\},$$

and $|B(0,t)|$ *denotes the volume of the ball with radius* t.

Proof. (1) For any $0<s\leqslant r$ and $f\in\mathscr{D}(D_{\rho,V})$, define

$$f_s(x):=\frac{1}{|B(0,s)|}\int_{B(x,s)}f(z)\,\mathrm{d}z,\quad x\in B(0,r).$$

We have

$$\sup_{x\in B(0,r)}|f_s(x)|\leqslant\frac{1}{|B(0,s)|}\int_{B(0,2r)}|f(z)|\,\mathrm{d}z,$$

and

$$\begin{aligned}\int_{B(0,r)}|f_s(x)|\,\mathrm{d}x&\leqslant\int_{B(0,r)}\frac{1}{|B(0,s)|}\int_{B(x,s)}|f(z)|\,\mathrm{d}z\,\mathrm{d}x\\&\leqslant\int_{B(0,2r)}\left(\frac{1}{|B(0,s)|}\int_{B(z,s)}\mathrm{d}x\right)|f(z)|\,\mathrm{d}z\leqslant\int_{B(0,2r)}|f(z)|\,\mathrm{d}z.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{B(0,r)}f_s^2(x)\,\mathrm{d}x&\leqslant\left(\sup_{x\in B(0,r)}|f_s(x)|\right)\int_{B(0,r)}|f_s(x)|\,\mathrm{d}x\\&\leqslant\frac{1}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,\mathrm{d}z\right)^2.\end{aligned}$$

Therefore, for any $f\in\mathscr{D}(D_{\rho,V})$ and $0<s\leqslant r$, by Jensen's inequality,

$$\begin{aligned}&\int_{B(0,r)}f^2(x)\,\mathrm{d}x\\&\quad\leqslant 2\int_{B(0,r)}\big(f(x)-f_s(x)\big)^2\,\mathrm{d}x+2\int_{B(0,r)}f_s^2(x)\,\mathrm{d}x\\&\quad\leqslant 2\int_{B(0,r)}\frac{1}{|B(0,s)|}\int_{B(x,s)}(f(x)-f(y))^2\,\mathrm{d}y\,\mathrm{d}x+\frac{2}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,\mathrm{d}z\right)^2\\&\quad\leqslant\left(\frac{2\sup_{0<\varepsilon\leqslant s}\rho(\varepsilon)^{-1}}{|B(0,s)|}\right)\int_{B(0,r)}\int_{B(x,s)}(f(x)-f(y))^2\rho(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\&\qquad+\frac{2}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,\mathrm{d}z\right)^2\\&\quad\leqslant\left(\frac{2\sup_{0<\varepsilon\leqslant s}\rho(\varepsilon)^{-1}}{|B(0,s)|}\right)\int_{B(0,2r)}\int_{B(0,2r)}(f(x)-f(y))^2\rho(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\&\qquad+\frac{2}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,\mathrm{d}z\right)^2.\end{aligned}$$

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(2) According to the inequality above, for any $f \in \mathscr{D}(D_{\rho,V})$ and $0 < s \leqslant r$,

$$\begin{aligned}
&\int_{B(0,r)} f^2(x)\,\mu_V(\mathrm{d}x)\\
&\quad\leqslant \frac{1}{\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}}\int_{B(0,r)} f^2(x)\,\mathrm{d}x\\
&\quad\leqslant \left(\frac{2\big(\sup_{0<\varepsilon\leqslant s}\rho(\varepsilon)^{-1}\big)}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\right)\int_{B(0,2r)}\int_{B(0,2r)}(f(x)-f(y))^2\rho(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\frac{2}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\left(\int_{B(0,2r)}|f(z)|\,\mathrm{d}z\right)^2\\
&\quad\leqslant \left(\frac{2\big(\sup_{0<\varepsilon\leqslant s}\rho(\varepsilon)^{-1}\big)\left(\sup_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\right)}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\right)\\
&\qquad\times\int_{B(0,2r)}\int_{B(0,2r)}(f(x)-f(y))^2\rho(|x-y|)\,\mathrm{d}y\,\mu_V(\mathrm{d}x)\\
&\qquad+\frac{2\left(\sup_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\right)^2}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\left(\int_{B(0,2r)}|f(x)|\,\mu_V(\mathrm{d}x)\right)^2\\
&\quad\leqslant \left(\frac{2\left(\sup_{0<\varepsilon\leqslant s}\rho(\varepsilon)^{-1}\right)\left(\sup_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\right)}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\right)D_{\rho,V}(f,f)\\
&\qquad+\frac{2\big(\sup_{|z|\leqslant 2r}\mathrm{e}^{V(z)}\big)^2}{|B(0,s)|\big(\inf_{|z|\leqslant r}\mathrm{e}^{V(z)}\big)}\mu_V(|f|)^2.
\end{aligned}$$

The desired assertion for the case $0 < s \leqslant r$ follows from the conclusion above and the definition of β_r.

(3) When $s > r$, by (2),

$$\int_{B(0,r)} f^2(x)\,\mu_V(\mathrm{d}x) \leqslant rD_{\rho,V}(f,f)+\beta_r(r)\mu_V(|f|)^2 \leqslant sD_{\rho,V}(f,f)+\beta_r(r\wedge s)\mu_V(|f|)^2.$$

The proof is completed. □

We present the following two remarks for the proof of Theorem 3.

(1) The proof above is efficient for the following more general non-local Dirichlet form

$$\begin{aligned}
\widetilde{D}_{j,V}(f,f) &:= \iint_{x\neq y}\big(f(x)-f(y)\big)^2 j(x,y)\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x),\\
\mathscr{D}(\widetilde{D}_{j,V}) &:= \left\{f\in L^2(\mu_V):\widetilde{D}_{j,V}(f,f)<\infty\right\},
\end{aligned}$$

where j is a Borel measurable function on $\mathbb{R}^{2d}\setminus\{(x,y)\in\mathbb{R}^{2d}: x=y\}$ such that $j(x,y)>0$ and $j(x,y)=j(y,x)$. See ([3], Sect. 2) for details.

(2) The argument above also works for L^p $(p>1)$ setting. For instance, it can yield the statement as follows. If (3) holds, then the following L^p-Poincaré inequality

$$\mu_V(|f-\mu_V(f)|^p) \leqslant 2c^{-1}\iint_{x\neq y}\frac{\big|f(x)-f(y)\big|^p}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mu_V(\mathrm{d}x) =: 2c^{-1}D_{\rho,V,p}(f,f)$$

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holds, where

$$f \in \mathscr{D}(D_{\rho,V,p}) := \left\{ f \in L^p(\mu_V) : D_{\rho,V,p}(f,f) < \infty \right\},$$

and $L^p(\mu_V)$ denotes the set of Borel measurable functions f on $\mathbb{R}^d$ such that $\int |f|^p(x)\,\mu_V(\mathrm{d}x) < \infty$. The proof is based on the argument of Theorem 1.1 (1) and the fact that for any $f \in L^p(\mu_V)$,

$$\mu_V(|f - \mu_V(f)|^p) \leqslant \iint |f(x) - f(y)|^p\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x),$$

due to the Hölder inequality. The readers can refer to [5] for related discussion about L^p-Poincaré inequalities of local Dirichlet forms.

Now, we are in a position to give the

Proof of Theorem 1.5. (a) For any $f \in \mathscr{D}(D_{\rho,V})$ with $f > 0$, by the Jensen inequality,

$$\begin{aligned}
\mathrm{Ent}_{\mu_V}(f) &= \mu_V(f\log f) - \mu_V(f)\log\mu_V(f) \\
&\leqslant \mu_V(f\log f) - \mu_V(f)\mu_V(\log f) \\
&= \frac{1}{2}\iint \Big[f(x)\log f(x) + f(y)\log f(y) \\
&\quad - f(x)\log f(y) - f(y)\log f(x) \Big]\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \\
&= \frac{1}{2}\iint (f(x) - f(y))(\log f(x) - \log f(y))\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x). \qquad (13)
\end{aligned}$$

Next, following the argument of Theorem 1.1 (1), we can obtain that under (3), for any $f \in \mathscr{D}(D_{\rho,V})$ with $f > 0$,

$$\frac{1}{2}\iint (f(x) - f(y))(\log f(x) - \log f(y))\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \leqslant c^{-1} D_{\rho,V}(f, \log f),$$

which, along with (13), completes the proof of the inequality (8).

(b) For any $p \in (1,2]$, $f \in \mathscr{D}(D_{\rho,V})$ with $f \geqslant 0$, by the Hölder inequality,

$$\begin{aligned}
&\mu_V(f^p) - \mu_V(f)^p \\
&\quad \leqslant \mu_V(f^p) - \mu_V(f)\mu_V(f^{p-1}) \\
&\quad = \frac{1}{2}\iint \Big[f^p(x) + f^p(y) - f(x)f^{p-1}(y) - f(y)f^{p-1}(x) \Big]\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \\
&\quad = \frac{1}{2}\iint (f(x) - f(y))(f^{p-1}(x) - f^{p-1}(y))\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x). \qquad (14)
\end{aligned}$$

Therefore, the desired Beckner-type inequality (9) follows from (14) and the following fact

$$\frac{1}{2}\iint (f(x) - f(y))(f^{p-1}(x) - f^{p-1}(y))\,\mu_V(\mathrm{d}y)\,\mu_V(\mathrm{d}x) \leqslant c^{-1} D_{\rho,V}(f, f^{p-1}),$$

where we have used (3) again. □

To close this section, we present

Sketch of the proof of Example 1.2. In this setting, $\mathrm{e}^{-V(x)} = C_{d,\varepsilon}(1+|x|)^{-(d+\varepsilon)}$ and $\rho(r) = r^{-d-\alpha}$. By the C_r-inequality, for any $x, y \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$|x-y|^{d+\varepsilon} \leqslant 2^{d+\varepsilon-1}(|x|^{d+\varepsilon} + |y|^{d+\varepsilon}) \leqslant 2^{d+\varepsilon-1}\big((1+|x|)^{d+\varepsilon} + (1+|y|)^{d+\varepsilon}\big).$$

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(a) For any $\varepsilon \geqslant \alpha$,

$$(\mathrm{e}^{V(x)} + \mathrm{e}^{V(y)})\rho(|x-y|) \geqslant \frac{C_{d,\varepsilon}^{-1}\big((1+|x|)^{d+\varepsilon} + (1+|y|)^{d+\varepsilon}\big)}{2^{d+\alpha-1}\big((1+|x|)^{d+\alpha} + (1+|y|)^{d+\alpha}\big)} \geqslant \frac{2^{1-(d+\alpha)}}{C_{d,\varepsilon}}.$$

Combining it with Theorem 1.1 (1), we get the first assertion.

(b) For any $\varepsilon < \alpha$,

$$\inf_{0<|x-y|\leqslant s}\big[(\mathrm{e}^{V(x)} + \mathrm{e}^{V(y)})\rho(|x-y|)\big] \geqslant C_{d,\varepsilon}^{-1}2^{1-(d+\varepsilon)}s^{\varepsilon-\alpha}.$$

Then, choosing $s = cr^{-1/\varepsilon}$ in the definition of α, we arrive at the second assertion.

(c) For any $\varepsilon > \alpha$, (6) holds with $\omega(x) = c_1(1+|x|)^{\varepsilon-\alpha}$, and

$$\beta_t(s) \leqslant c_2(1 + s^{-d/\alpha}t^{(d+\varepsilon)(2+d/\alpha)}).$$

Then, the third assertion follows from the definition of β by taking $s = c_3 r$ and $t = c_4 r^{-1/(\varepsilon-\alpha)}$. □

3. Applications: Porous media equations

Functional inequalities for non-local Dirichlet forms appear throughout the probability literature, and also are interesting in analysis, *e.g.* see references in [6,7]. This section is mainly motivated by [4,8] for the description of the convergence rate of porous media equations by using L^p functional inequalities. Let $(L_{\rho,V}, \mathscr{D}(L_{\rho,V}))$ be the generator corresponding to Dirichlet form $(D_{\rho,V}, \mathscr{D}(D_{\rho,V}))$. Consider the following equation

$$\partial_t u(t,\cdot) = L_{\rho,V}\{u(t,\cdot)^m\}, \quad u(0,\cdot) = f, \tag{15}$$

where $m > 1$, f is a bounded measurable function on $\mathbb{R}^d$ and $u^m := \mathrm{sgn}(u)|u|^m$. We call $T_t f := u(t,\cdot)$ a solution to the equation (15), if $u(t,\cdot)^m \in \mathscr{D}(L_{\rho,V})$ for all $t > 0$ and $u^m \in L^1_{\mathrm{loc}}([0,\infty) \to \mathscr{D}(D_{\rho,V}); \mathrm{d}t)$ such that, for any $g \in \mathscr{D}(D_{\rho,V})$,

$$\mu_V(u(t,\cdot)g) = \mu_V(fg) - \int_0^t D_{\rho,V}(u(s,\cdot)^m, g)\,ds, \quad t > 0.$$

Theorem 3.1. *Assume that for any bounded measurable function $f \in \mathscr{D}(L_{\rho,V})$ the equation (15) has a unique solution $T_t f$. If (3) holds, then*

$$\mu_V((T_t f)^2) \leqslant \Big[\mu_V(f^2)^{-(m-1)/2} + c^{-1}(m-1)t\Big]^{-2/(m-1)}, \quad t \geqslant 0,\ \mu_V(f) = 0.$$

Proof. The argument of Theorem 1.5 gives us that, under (3) for any $m > 1$ and $f \in \mathscr{D}(L_{\rho,V})$ with $\mu_V(f) = 0$,

$$\mu_V(f^{m+1}) \leqslant c^{-1}D_{\rho,V}(f, f^m). \tag{16}$$

Now, let f be a function such that $\mu_V(f) = 0$. Then, by the definition of the solution to the equation (15), $\mu_V(T_t f) = 0$ for all $t \geqslant 0$. According to (16), we obtain that

$$\begin{aligned}\frac{\mathrm{d}\mu_V(T_t f)^2}{\mathrm{d}t} &= 2\mu_V\Big(T_t f \partial_t T_t f\Big) = 2\mu_V(T_t f L\{(T_t f)^m\}) \\ &= -2D_{\rho,V}(T_t f, (T_t f)^m) \leqslant -2c^{-1}\mu_V\left((T_t f)^{m+1}\right) \\ &\leqslant -2c^{-1}\Big[\mu_V((T_t f)^2)\Big]^{\frac{m+1}{2}},\end{aligned}$$

where in the inequality we have used the Hölder inequality. The required assertion easily follows from the inequality above. □

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