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OPTIMAL CONTROL PROBLEMS GOVERNED BY MARGUERRE–VON KÁRMÁN EQUATIONS

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Abstract

We consider the Marguerre–von Kármán equations that model the deformation of a thin, nonlinearly elastic, shallow shell, subjected to a specific class of boundary conditions of von Kármán's type. Next, we reduce these equations to a single equation with a cubic operator following Berger's classical method, whose second member depends on the function defining the middle surface of the shallow shell and the resultant of the vertical forces acting on the shallow shell. We also prove the existence and uniqueness of a weak solution to the reduced equation. Then, we prove the existence theorem for the optimal control problem governed by Marguerre–von Kármán equations, with a control variable on the resultant of the vertical forces. Using the Fréchet differentiability of the state function with respect to the control variable, we prove the uniqueness of the optimal control and derive the necessary optimality condition. As a result, this work addresses the more general geometry of Marguerre–von Kármán shallow shells to study the quadratic cost optimal control problems governed by these equations.

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1. Introduction

The Marguerre–von Kármán equations are a fourth-order system of nonlinear partial differential equations, constituting a mathematical model for a thin Marguerre–von Kármán shallow shell. These equations have their origin in mechanics, dating back to the first half of the 20th century with the works of Marguerre [31] and von Kármán and Tsien [38]. The Marguerre–von Kármán equations for a nonlinearly elastic shallow shell represented an extension of von Kármán equations for a nonlinearly elastic plate

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due to von Kármán [37]. The von Kármán and Marguerre–von Kármán theories play an important role in applied mathematics, and have attracted a great deal of interest in engineering and materials science. Detailed and mathematical modelling related to these equations can be found in the works of Ciarlet [14], Ciarlet and Gratie [16], and the historical references therein. For both models, the stationary and evolutionary von Kármán equations, we refer to the book of Chueshov and Lasiecka [12] and the references quoted therein for more details. In addition, we refer to the works of the second author and the others, for example, [1, 19] for stationary Marguerre–von Kármán equations, [10, 20] for dynamical Marguerre–von Kármán equations, and [33] for linear shallow shells.

The control theory is a basic mathematical strategy that is crucial in applied mathematics and solving problems in different disciplines. The importance of this theory appeared after the Second World War in response to practical guidance needs, particularly in aeronautics and flight mechanics. The control problem of multi-structures is of great practical interest to modern engineering, particularly in aerospace engineering, where the stabilisation of large multi-structures is a crucial problem. Many results have been obtained in the context of partial differential equations (PDEs; see, for example, the books of Lasiecka and Triggiani [27, 28], Lions [29], and Tröltzsch [36]). Especially, we refer to the book of Lasiecka [26] for the control theory of coupled PDEs.

Historically, there are some results for von Kármán equations in the context of control theory. In particular, we refer to the works of Bock et al. [6–8] for analytical studies focusing on existence, uniqueness and optimality conditions. More specifically, the authors addressed different boundary conditions, ranging from homogeneous Dirichlet to mixed types in the works [6, 7]. In [8], they discussed the case of arbitrarily large perpendicular loads. We also refer to Chowdhury et al. [11], Hou and Turner [23], and Mallik et al. [30] for numerical work. For evolutionary systems, we refer to Bock [5], Hwang [24, 25] and Park et al. [34]. Recently, the control problems governed by Marguerre–von Kármán evolution equations with long-range memory term were studied by Mechaouf et al. [32].

Thin shells are extensively used across various industries due to their advantageous structural properties, including adaptability to complex geometries. These structures are common in aerospace engineering (for example, aircraft fuselages, rocket interstage sections and piezoelectric sensor patches), civil engineering (for example, dome roofs and curved bridge components) and microsystems (for example, micro-electromechanical systems actuators).

The mathematical modelling frameworks for thin shells provide powerful tools for analysing their behaviour. These models are particularly suitable for studying thin-structure applications in advanced technological areas. There is significant potential in the actuation and control of thin structures to perform network cycles. By applying optimal control theory, particularly in the context of models such as the von Kármán equations for nonlinear thin shells, one can design control strategies that precisely manipulate these structures through cyclic deformations. This enables

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the structures to perform net mechanical work. The mathematical framework provides the tools to optimise these cycles for efficiency and stability, which is valuable in applications like energy harvesting, adaptive morphing surfaces and soft robotics.

To the best of the author's knowledge, no existing results in the literature address optimal control problems governed by the stationary Marguerre–von Kármán equations. This work therefore aims to study such problems, generalising the approach of Bock et al. [6] for von Kármán plates to the more general geometry of Marguerre–von Kármán shallow shells.

After this introduction, in Section 2, problem formulation and contributions are given. Section 3 describes some preliminary results. Section 4 is devoted to the reduced Marguerre–von Kármán equations, including the existence and uniqueness of a weak solution. Section 5 presents an existence result for the optimal control problem. Section 6 is concerned with the characterization of the optimal controls. Conclusions and commentary are found in Section 7.

2. Problem formulation and contributions

As shown by Gratie [21] and Rao [35], the classical Marguerre–von Kármán equations for a thin nonlinearly elastic shallow shell with boundary conditions of von Kármán's type (known as the Marguerre–von Kármán equations) is a nonlinear system of fourth-order partial differential equations and written as

$$\begin{cases} \Delta^2\zeta = [\phi, \zeta + \theta] + v & \text{in } \omega, \\ \Delta^2\phi = -[\zeta, \zeta + 2\theta] & \text{in } \omega, \\ \zeta = \partial_\nu\zeta = 0 & \text{on } \gamma, \\ \phi = \phi_0 \text{ and } \partial_\nu\phi = \phi_1 & \text{on } \gamma, \end{cases} \tag{2.1}$$

where ω is a bounded and simply connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ. The unit outer normal vector along γ is denoted by ν and the outer normal derivative operator $\sum_{\alpha=1}^{\alpha=2} \nu_\alpha \partial_\alpha$ along γ is denoted by ∂_ν. The nonlinear symmetric von Kármán bracket $[.,.]$ (also called the Monge–Ampère form) is defined by

$$[\eta, \varphi] = \partial_{11}^2\eta\partial_{22}^2\varphi + \partial_{22}^2\eta\partial_{11}^2\varphi - 2\partial_{12}^2\eta\partial_{12}^2\varphi.$$

The known function θ plays a key role in Marguerre's model, as it measures the deviation of the middle surface of the reference shell configuration from a plane. The known function v is the resultant of the vertical forces acting on the shell. The traces ϕ_0 and ϕ_1 are known functions to be calculated through the von Kármán forces acting on the lateral face of the shell. The unknown $\zeta : \bar{\omega} \to \mathbb{R}$ is the vertical component of the displacement field of the middle surface of the shell and the unknown $\phi : \bar{\omega} \to \mathbb{R}$ is the Airy-stress function.

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First, we consider here the problem of buckling of a thin nonlinearly elastic shallow shell subjected to von Kármán forces of magnitude λ acting on the lateral face, λ is also called the bifurcation parameter. Accordingly (see, for example, [3, page 177] and [2, 4] for further details), the Airy function in the (possibly deflected) shallow shell is given by

$$\phi = \Phi + \lambda\theta_0, \tag{2.2}$$

where the stress function θ_0 in the undeflected shallow shell (under the action of ϕ_0 and ϕ_1) is a solution to the boundary value problem:

$$\begin{cases} \Delta^2\eta = 0 & \text{in } \omega, \\ \eta = \phi_0 \text{ and } \partial_\nu\eta = \phi_1 & \text{on } \gamma. \end{cases} \tag{2.3}$$

To write (2.1) in the canonical form, we denote

$$\zeta = \xi - \theta. \tag{2.4}$$

Inserting the change of unknowns (2.2) and (2.4) into (2.1) gives the canonical Marguerre–von Kármán equations:

$$\begin{cases} \Delta^2\xi = [\Phi, \xi] + \lambda[\theta_0, \xi] + v + \Delta^2\theta & \text{in } \omega, \\ \Delta^2\Phi = [\theta, \theta] - [\xi, \xi] & \text{in } \omega, \\ \xi = \partial_\nu\xi = \Phi = \partial_\nu\Phi = 0 & \text{on } \gamma. \end{cases} \tag{2.5}$$

When restricting our consideration to the important case that the shallow shell becomes a plate, that is,

$$\theta = 0 \quad \text{in } \bar{\omega}, \tag{2.6}$$

we recover the canonical von Kármán equations as a special case.

For justification of classical and generalized models of von Kármán and Marguerre–von Kármán equations, we refer to the literature [13, 15, 17, 22]. In particular, see the work of Chueshov and Lasiecka [12] for stationary von Kármán equations with a variety of boundary conditions.

One of the important physical cases corresponding to a uniform lateral loading along the shallow shell boundary (which is most relevant in buckling analysis) is obtained by the choice of von Kármán forces (h_α) as

$$h_\alpha = -\lambda\nu_\alpha \quad \text{on } \gamma, \ \alpha = 1, 2.$$

Accordingly (see, for example, [4] and [14, Section 5.9] for more details), we obtain

$$[\theta_0, \xi] = -\Delta\xi. \tag{2.7}$$

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If we restrict our analysis to cases (2.6) and (2.7), then (2.5) reduces to the simplest form of canonical von Kármán equations:

$$\begin{cases} \Delta^2 \xi = [\Phi, \xi] - \lambda \Delta \xi + v & \text{in } \omega, \\ \Delta^2 \Phi = -[\xi, \xi] & \text{in } \omega, \\ \xi = \partial_\nu \xi = \Phi = \partial_\nu \Phi = 0 & \text{on } \gamma. \end{cases} \tag{2.8}$$

Consequently, (2.8) are not exactly the classical Föpl–von Kármán equations, but rather constitute a modified formulation incorporating an additional linear term $(-\lambda \Delta \xi)$. Such extensions are standard in shell and plate theories when considering pre-stressed or loaded cases. The optimal control problems governed by (2.8) were studied by Bock et al. [6].

Our first contribution is to study the existence and uniqueness of a weak solution of (2.5). To do this, we reduce the Marguerre–von Kármán equations to a single equation with a cubic operator; its second member depends on the functions θ and v. This approach was first introduced by Berger [2] and has been widely used in the literature for both models: for von Kármán equations (see, for example, [3, 6–8, 14, 16, 18, 21, 35, 39]) and for Marguerre–von Kármán equations (see, [1, 19]). We prove a general existence theorem for the reduced equation by using the theory of pseudomonotone operators due to Zeidler [40]. Then, we extend the uniqueness result for weak solutions of the von Kármán equations, as given by Ciarlet and Rabier [18], to the more general case of Marguerre–von Kármán equations.

Second, we consider the control problem governed by Marguerre–von Kármán equations, with the resultant of the vertical forces acting on the shell v as a control variable defined by

$$\begin{cases} \Delta^2 \xi(v) = [\Phi(v), \xi(v)] + \lambda[\theta_0, \xi(v)] + v + \Delta^2 \theta & \text{in } \omega, \\ \Delta^2 \Phi(v) = [\theta, \theta] - [\xi(v), \xi(v)] & \text{in } \omega, \\ \xi(v) = \partial_\nu \xi(v) = \Phi(v) = \partial_\nu \Phi(v) = 0 & \text{on } \gamma, \end{cases} \tag{2.9}$$

where the known functions θ_0 and θ do not depend on v.

The second and main contribution of this work lies in studying the quadratic cost optimal control problems for (2.9). To this end, by adapting the ideas presented by Bock et al. [6], we prove the existence of the optimal control and the Fréchet differentiability of the state function with respect to the control variable. Next, we prove the uniqueness of the optimal control and derive the necessary condition of the optimality.

3. Preliminary results

In this section, we present notation, definitions and some properties which are used throughout this work.

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The standard notation for Lebesgue and Sobolev spaces are employed. To simplify notation, the following abbreviations are used: $L^2 = L^2(\omega)$, $H^m = H^m(\omega)$ $(m = 2, 4)$, $H_0^2 = H_0^2(\omega)$, $H^s = H^s(\gamma)$ $(s = 1/2, 3/2)$.

The scalar products on L^2 and H_0^2 are denoted respectively by

$$(\eta, \varphi)_0 = \int_\omega \eta\varphi \, dx \quad \text{for all } \eta, \varphi \in L^2,$$
$$(\eta, \varphi)_\Delta = (\Delta\eta, \Delta\varphi)_0 \quad \text{for all } \eta, \varphi \in H_0^2,$$

with respective norms

$$\|\eta\|_0 = [(\eta, \eta)_0]^{1/2} \quad \text{for all } \eta \in L^2,$$
$$\|\eta\|_\Delta = \|\Delta\eta\|_0 \quad \text{for all } \eta \in H_0^2.$$

There exists a constant $\underline{\mu} > 0$ such that

$$\|\eta\|_0 \leq \underline{\mu}\|\eta\|_\Delta \quad \text{for all } \eta \in H_0^2. \tag{3.1}$$

The dual spaces of L^2 and H_0^2 are denoted by $(L^2)^\star$ and H^{-2}, respectively. The norm of H^{-2} and the duality pairing between H^{-2} and H_0^2 are denoted by $\|.\|_{-2}$ and $\langle ., . \rangle_{-2,2}$, respectively. Also, the duality pairing between $(L^2)^\star$ and L^2 is denoted by $\langle ., . \rangle_0$.

Let $G = (\Delta^2)^{-1}$ denote the Green operator of the Dirichlet problem for the biharmonic operator Δ^2 in ω. We note that the operator

$$\Delta^2 : H^s \cap H_0^2 \longrightarrow H^{s-4} \quad \text{for } s \geq 2, \tag{3.2}$$

is an isomorphism. Thus (see, for example, [12, Remark 1.3.7]),

$$G \in \mathcal{L}(H^s; H^{s+4} \cap H_0^2) \quad \text{for } s \geq -2. \tag{3.3}$$

In the following, we denote strong convergence by $\rightarrow$ and weak convergence by $\rightharpoonup$.

We assume that

$$(\phi_0, \phi_1) \in H^{3/2} \times H^{1/2}. \tag{3.4}$$

Then, by the classical elliptic theory [14, page 395], there exists a unique function $\theta_0 \in H^2$ which satisfies (2.3).

We also assume that

$$\theta \in H_0^2 \tag{3.5}$$

and

$$v \in L^2. \tag{3.6}$$

The regularity condition $\theta \in H_0^2$ imposed in Sections 3 and 4 is appropriate for the well-posedness of the problem. In Sections 5 and 6, additional regularity is required to derive further properties.

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We define the bilinear operator $B : H^2 \times H^2 \longrightarrow H_0^2$ as follows: for each pair $(\eta, \varphi) \in H^2 \times H^2$, the function $B(\eta, \varphi) \in H_0^2$ is the unique solution to the problem:

$$\Delta^2 B(\eta, \varphi) = [\eta, \varphi] \quad \text{in } \omega.$$

Next, we define the cubic nonlinear operator $C : H_0^2 \longrightarrow H_0^2$ by

$$C(\eta) = B(B(\eta, \eta), \eta).$$

Then, we define the linear operators $L_1, L_2 \in \mathcal{L}(H_0^2; H_0^2)$ by

$$L_1(\eta) = B(B(\theta, \theta), \eta) \quad \text{and} \quad L_2(\eta) = B(\theta_0, \eta).$$

In the following lemmas, we gather the properties of the operators G, B, C, L_1 and L_2.

LEMMA 3.1. *The Green operator G satisfies the following estimate:*

$$\|G\eta\|_\Delta \leq \underline{\mu} \|\eta\|_0 \quad \textit{for all } \eta \in L^2, \tag{3.7}$$

where $\underline{\mu}$ is defined in (3.1).

PROOF. For any $\eta \in L^2$ and $\varphi \in H_0^2$,

$$\begin{aligned} (G\eta, \varphi)_\Delta &= (\Delta G\eta, \Delta\varphi)_0 \\ &= (\eta, \varphi)_0. \end{aligned} \tag{3.8}$$

Then, by (3.1),

$$\begin{aligned} \|G\eta\|_\Delta^2 &= (G\eta, G\eta)_\Delta \\ &= (\eta, G\eta)_0 \\ &\leq \|\eta\|_0 \|G\eta\|_0 \\ &\leq \underline{\mu} \|\eta\|_0 \|G\eta\|_\Delta. \end{aligned}$$

This proves (3.7). □

LEMMA 3.2. *If one of the arguments η, φ, ψ belongs to H_0^2, then*

$$(B(\eta, \varphi), \psi)_\Delta = (B(\psi, \varphi), \eta)_\Delta.$$

PROOF. See the paper by Bendob and Ghezal [1, Section 3.2]. □

Here, this property requires weaker assumptions imposed on three arguments than those imposed by Bock et al. [6, Proof of Lemma 4.1] (that all arguments are in H_0^2). In particular, this is needed to show (4.4), due to the presence of the function $\theta_0 \in H^2$, which solves (2.3).

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LEMMA 3.3.

(i) *Let* $\eta \in H_0^2$, *then*

$$(C(\eta), \eta)_\Delta = (B(\eta, \eta), B(\eta, \eta))_\Delta = \|B(\eta, \eta)\|_\Delta^2 \geq 0$$

and

$$(C(\eta), \eta)_\Delta = 0 \quad \textit{if and only if } \eta = 0.$$

(ii) *The operators B and C are sequentially compact (also called strongly continuous by Zeidler* [40]*) and hence, a fortiori, continuous.*

PROOF. See the paper by Ciarlet [14, Theorem 5.8-2]. □

LEMMA 3.4. *Let* $(\eta, \varphi) \in H_0^2 \times H_0^2$, *then*

$$(C(\eta) - C(\varphi), \varphi - \eta)_\Delta \leq \|B\|^2 \max\{\|\eta\|_\Delta^2, \|\varphi\|_\Delta^2\}\|\eta - \varphi\|_\Delta^2,$$

where

$$\|B\| = \sup_{\eta, \varphi \in H_0^2, \|\eta\| = \|\varphi\| = 1} \|B(\eta, \varphi)\|.$$

PROOF. See Ciarlet and Rabier [18, Lemma 2.2-5]. □

LEMMA 3.5. *The linear operators* L_1 *and* L_2 *are continuous, compact and symmetric with respect to the inner product* $(.,.)_\Delta$.

PROOF. The continuity and compactness of operators L_1 and L_2 are immediately deduced from the properties of the operator B. The fact that L_2 is symmetric with respect to the inner product $(.,.)_\Delta$ was proved by Ciarlet [14, Theorem 5.8-2(d)].

Let $\eta, \varphi \in H_0^2$. According to Lemma 3.2, we deduce that

$$\begin{aligned}(L_1(\eta), \varphi)_\Delta &= (B(B(\theta, \theta), \eta), \varphi)_\Delta \\ &= (B(B(\theta, \theta), \varphi), \eta)_\Delta \\ &= (L_1(\varphi), \eta)_\Delta.\end{aligned}$$

Thus, L_1 is symmetric with respect to the inner product $(.,.)_\Delta$. □

4. Reduced Marguerre–von Kármán equation

4.1. Cubic operator equation First, on the one hand, the definition of the bilinear operator B implies that the equations

$$\begin{cases}\Delta^2 \Phi = [\theta, \theta] - [\xi, \xi] & \text{in } \omega, \\ \Phi = \partial_\nu \Phi = 0 & \text{on } \gamma,\end{cases}$$

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which are part of the canonical Marguerre–von Kármán equations (2.5), are equivalent to the operator equation in H_0^2:

$$\Phi = B(\theta, \theta) - B(\xi, \xi). \tag{4.1}$$

On the other hand, the definition of the bilinear operator B and that of the operator G imply that the remaining equations in the canonical Marguerre–von Kármán equations (2.5), namely

$$\begin{cases} \Delta^2 \xi = [\Phi, \xi] + \lambda[\theta_0, \xi] + v + \Delta^2 \theta & \text{in } \omega, \\ \xi = \partial_\nu \xi = 0 & \text{on } \gamma, \end{cases}$$

are equivalent to the operator equation in H_0^2:

$$\xi - Gv - \theta = B(\Phi, \xi) + \lambda B(\theta_0, \xi). \tag{4.2}$$

Next, substituting (4.1) into (4.2) and using the bilinearity of the operator B, we derive the following single operator equation in H_0^2:

$$\xi - Gv - \theta = B(B(\theta, \theta), \xi) - B(B(\xi, \xi), \xi) + \lambda B(\theta_0, \xi).$$

From the definition of the operators C, L_1 and L_2, we infer that the pair (ξ, Φ) satisfies the canonical Marguerre–von Kármán equations (2.5) if and only if the unknown $\xi \in H_0^2$ satisfies the reduced Marguerre–von Kármán equation:

$$\xi - \lambda L_2(\xi) + C(\xi) - L_1(\xi) = Gv + \theta, \tag{4.3}$$

and the Airy function $\Phi \in H_0^2$ satisfies (4.1).

Therefore, we conclude that finding the solution ξ to the operator equation (4.3) is equivalent to solving the following variational problem (see [1]).

$$\begin{cases} \text{Find } \xi \in H_0^2 \text{ such that} \\ (\xi - \lambda L_2(\xi) + C(\xi) - L_1(\xi) - Gv - \theta, \eta)_\Delta = 0 \quad \text{for all } \eta \in H_0^2. \end{cases}$$

4.2. General existence result We consider the operator $A : H_0^2 \to H_0^2$ defined by

$$A = id_{H_0^2} - \lambda L_2 + C - L_1.$$

LEMMA 4.1. *The operator A is coercive.*

PROOF. From the definition of the operators L_1, L_2, C,

$$(A(\xi), \xi)_\Delta = \|\xi\|_\Delta^2 - \lambda(B(\theta_0, \xi), \xi)_\Delta + (B(B(\xi, \xi), \xi), \xi)_\Delta - (B(B(\theta, \theta), \xi), \xi)_\Delta.$$

According to Lemma 3.2,

$$(A(\xi), \xi)_\Delta = \|\xi\|_\Delta^2 - \lambda(B(\xi, \xi), \theta_0)_\Delta + \|B(\xi, \xi)\|_\Delta^2 - (B(\xi, \xi), B(\theta, \theta))_\Delta. \tag{4.4}$$

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By Cauchy–Schwarz inequality and Young's inequality, we obtain

$$\lambda(B(\xi,\xi),\theta_0)_\Delta = (B(\xi,\xi),\lambda\theta_0)_\Delta \leq \frac{1}{2}\|B(\xi,\xi)\|_\Delta^2 + \frac{\lambda^2}{2}\|\theta_0\|_\Delta^2$$

and

$$(B(\xi,\xi),B(\theta,\theta))_\Delta \leq \tfrac{1}{2}\|B(\xi,\xi)\|_\Delta^2 + \tfrac{1}{2}\|B(\theta,\theta)\|_\Delta^2.$$

Consequently,

$$\begin{aligned}(A(\xi),\xi)_\Delta &\geq \|\xi\|_\Delta^2 - \frac{1}{2}\|B(\xi,\xi)\|_\Delta^2 - \frac{\lambda^2}{2}\|\theta_0\|_\Delta^2 + \|B(\xi,\xi)\|_\Delta^2 - \frac{1}{2}\|B(\xi,\xi)\|_\Delta^2 - \frac{1}{2}\|B(\theta,\theta)\|_\Delta^2 \\ &\geq \|\xi\|_\Delta^2 - \frac{\lambda^2}{2}\|\theta_0\|_\Delta^2 - \frac{1}{2}\|B(\theta,\theta)\|_\Delta^2.\end{aligned}$$

Hence,

$$\frac{(A(\xi),\xi)_\Delta}{\|\xi\|_\Delta} \geq \|\xi\|_\Delta - \frac{\lambda^2}{2}\|\theta_0\|_\Delta^2\|\xi\|_\Delta^{-1} - \frac{1}{2}\|B(\theta,\theta)\|_\Delta^2\|\xi\|_\Delta^{-1}.$$

Since $\|A(\xi)\|_\Delta \geq ((A(\xi),\xi)_\Delta/\|\xi\|_\Delta)$, we infer that

$$\lim_{\|\xi\|_\Delta\to\infty}\|A(\xi)\|_\Delta = \infty$$

for fixed λ. This is the coerciveness of A. For details about these questions for von Kármán equation, see, for example, the paper by Berger [3]. □

LEMMA 4.2. *The operator A is pseudomonotone and bounded.*

PROOF. Since operators L_1, L_2 and C are strongly continuous, thus operator A is a strongly continuous perturbation of the identity. According to Zeidler [40, Proposition 27.6(b)], A is pseudomonotone and bounded. □

We give a general existence result for the reduced Marguerre–von Kármán equation.

THEOREM 4.3. *Under assumptions* (3.4)–(3.6)*, the operator equation* (4.3) *has a solution for every fixed λ.*

PROOF. From Lemmas 4.1 and 4.2, the operator A is coercive, pseudomonotone and bounded. Hence, the main theorem on pseudomonotone operators given by Zeidler [40, Theorem 27.A] (due to Brezis [9]) shows that the operator equation $A(\xi) = Gv + \theta$ has a solution. □

Similar results were obtained for operator equation (4.3): for the case $\lambda > 0$, we refer to the work of Bendob and Ghezal [1, Theorem 3.3]; for the case $\lambda = 1$, we refer to Rao [35, Theorem 2.3], Gratie [21, Theorem 8] and Chueshov and Lasiecka [12, Theorem 1.5.7], with different approaches. For the operator equation of von Kármán equations, see, for example, Berger [3, page 106] and Zeidler [39, Theorem 65.A].

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4.3. Uniqueness result We assume that the linear operators

$$L_1, L_2 \text{ are positive definite,} \tag{4.5}$$

(in the particular case (2.7), L_2 is positive definite [14, Theorem 5.9-1]). We conclude that L_1 and L_2 have an infinite number of distinct eigenvalues $\beta_k > 0$ and $\mu_k > 0$ (each of finite multiplicity), respectively, such that

$$\cdots < \beta_k < \cdots < \beta_2 < \beta_1 \quad \text{with } \beta_k \to 0 \text{ as } k \to \infty,$$

$$\cdots < \mu_k < \cdots < \mu_2 < \mu_1 \quad \text{with } \mu_k \to 0 \text{ as } k \to \infty,$$

where

$$\beta_1 = \sup_{\eta \in H_0^2, \eta \neq 0} \frac{(L_1\eta, \eta)_\Delta}{\|\eta\|_\Delta^2}, \quad \mu_1 = \sup_{\eta \in H_0^2, \eta \neq 0} \frac{(L_2\eta, \eta)_\Delta}{\|\eta\|_\Delta^2}.$$

The smallest characteristic values of the operator of L_1 and L_2 are given respectively by

$$\alpha_1 = \frac{1}{\beta_1} = \inf_{\eta \in H_0^2, \eta \neq 0} \frac{\|\eta\|_\Delta^2}{(L_1\eta, \eta)_\Delta}, \tag{4.6}$$

$$\lambda_1 = \frac{1}{\mu_1} = \inf_{\eta \in H_0^2, \eta \neq 0} \frac{\|\eta\|_\Delta^2}{(L_2\eta, \eta)_\Delta}. \tag{4.7}$$

To give the uniqueness of solutions to reduced Marguerre–von Kármán equations, we need the following additional assumptions:

$$\|Gv + \theta\|_\Delta < \|B\|^{-1}, \tag{4.8}$$

$$\alpha_1 > \alpha^\star, \tag{4.9}$$

$$\lambda < \lambda^\star, \tag{4.10}$$

where

$$\alpha^\star = (1 - \|B\|^{2/3}\|Gv + \theta\|_\Delta^{2/3})^{-1}, \quad \lambda^\star = \lambda_1\Big(1 - \frac{1}{\alpha_1} - \|B\|^{2/3}\|Gv + \theta\|_\Delta^{2/3}\Big).$$

THEOREM 4.4. *Under assumptions* (3.4)–(3.6), (4.5) *and* (4.8)–(4.10), *there exists a unique solution* $\xi \in H_0^2$ *to operator equation* (4.3). *Moreover,* ξ *satisfies the estimate*

$$\|\xi\|_\Delta \le K(\lambda)^{-1}\|Gv + \theta\|_\Delta, \tag{4.11}$$

where

$$K(\lambda) = \min\Big\{1 - \frac{1}{\alpha_1}, 1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1}\Big\} > 0. \tag{4.12}$$

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PROOF.

(i) First, the existence of solutions $\xi \in H_0^2$ to operator equation (4.3) is verified in Theorem 4.3.

(ii) Let $\xi \in H_0^2$ be a solution to (4.3). Then,

$$(\xi - \lambda L_2(\xi) + C(\xi) - L_1(\xi) - Gv - \theta,\ \xi)_\Delta = 0,$$

and hence,

$$\|\xi\|_\Delta^2 - \lambda(L_2(\xi), \xi)_\Delta + (C(\xi), \xi)_\Delta - (L_1(\xi), \xi)_\Delta = (Gv + \theta, \xi)_\Delta.$$

According to Lemma 3.3, we deduce that

$$\|\xi\|_\Delta^2 - \lambda(L_2(\xi), \xi)_\Delta - (L_1(\xi), \xi)_\Delta \le (Gv + \theta, \xi)_\Delta. \tag{4.13}$$

(iii) If $\lambda > 0$, then from the positivity of operators L_1 and L_2, and relations (4.6)–(4.7),

$$-\left(\frac{\lambda}{\lambda_1} + \frac{1}{\alpha_1}\right)\|\xi\|_\Delta^2 \le -\lambda(L_2(\xi), \xi)_\Delta - (L_1(\xi), \xi)_\Delta.$$

We infer that

$$\left(1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1}\right)\|\xi\|_\Delta^2 \le \|Gv + \theta\|_\Delta \|\xi\|_\Delta.$$

For $\lambda < \lambda_1(1 - 1/\alpha_1)$, we get

$$\|\xi\|_\Delta \le \left(1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1}\right)^{-1} \|Gv + \theta\|_\Delta. \tag{4.14}$$

Let ξ_1 and ξ_2 be two solutions of (4.3). Then,

$$(id_{H_0^2} - \lambda L_2 - L_1)(\xi_1 - \xi_2) = C(\xi_2) - C(\xi_1).$$

Hence,

$$((id_{H_0^2} - \lambda L_2 - L_1)(\xi_1 - \xi_2), \xi_1 - \xi_2)_\Delta = (C(\xi_2) - C(\xi_1), \xi_1 - \xi_2)_\Delta.$$

Applying Lemma 3.4,

$$\left(1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1}\right)\|\xi_1 - \xi_2\|_\Delta^2 \le \|B\|^2 \max\{\|\xi_1\|_\Delta^2, \|\xi_2\|_\Delta^2\}\|\xi_2 - \xi_1\|_\Delta^2. \tag{4.15}$$

For $\lambda < \lambda_1(1 - 1/\alpha_1)$ and by (4.14),

$$\|\xi_1 - \xi_2\|_\Delta^2 \le \left(1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1}\right)^{-3} \|B\|^2 \|Gv + \theta\|_\Delta^2 \|\xi_1 - \xi_2\|_\Delta^2.$$

By assumptions (4.9) and (4.10), we have that this inequality can be true only if $\xi_1 = \xi_2$.

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(iv) If $\lambda \leq 0$, then from (4.13) and the positivity of operator L_2,

$$\|\xi\|_\Delta^2 - (L_1(\xi), \xi)_\Delta \leq (Gv + \theta, \xi)_\Delta.$$

From the positivity of operators L_1 and relation (4.6),

$$-\frac{1}{\alpha_1}\|\xi\|_\Delta^2 \leq -(L_1(\xi), \xi)_\Delta.$$

We infer that

$$\left(1 - \frac{1}{\alpha_1}\right)\|\xi\|_\Delta \leq \|Gv + \theta\|_\Delta.$$

For $\alpha_1 > 1$,

$$\|\xi\|_\Delta \leq \left(1 - \frac{1}{\alpha_1}\right)^{-1}\|Gv + \theta\|_\Delta. \tag{4.16}$$

Let ξ_1 and ξ_2 be two solutions of (4.3). Taking into account (4.15), we conclude that

$$\left(1 - \frac{1}{\alpha_1}\right)\|\xi_1 - \xi_2\|_\Delta^2 \leq \|B\|^2 \max\{\|\xi_1\|_\Delta^2, \|\xi_2\|_\Delta^2\}\|\xi_2 - \xi_1\|_\Delta^2.$$

For $\alpha_1 > 1$ and by (4.16),

$$\|\xi_1 - \xi_2\|_\Delta^2 \leq \left(1 - \frac{1}{\alpha_1}\right)^{-3}\|B\|^2\|Gv + \theta\|_\Delta^2\|\xi_1 - \xi_2\|_\Delta^2.$$

By assumptions (4.8) and (4.9), we have that this inequality can be true only if $\xi_1 = \xi_2$.

(v) From assumptions (4.8)–(4.10),

$$\alpha_1 > \alpha^\star > 1 \quad \text{and} \quad \lambda < \lambda^\star < \lambda_1\left(1 - \frac{1}{\alpha_1}\right).$$

Hence,

$$1 - \frac{1}{\alpha_1} > 0 \quad \text{and} \quad 1 - \frac{1}{\alpha_1} - \frac{\lambda}{\lambda_1} > 0.$$

We infer that $K(\lambda) > 0$. Finally, using estimates (4.14) and (4.16) yields the desired result (4.11). □

5. Existence of an optimal control for Marguerre–von Kármán equations

We follow a similar approach to (4.1) and (4.3). By replacing (ξ, Φ) by $(\xi(v), \Phi(v))$, we deduce that the pair $(\xi(v), \Phi(v))$ satisfies the control problem (2.9) if and only if $\xi(v) \in H_0^2$ satisfies the reduced equation:

$$\xi(v) - \lambda L_2(\xi(v)) + C(\xi(v)) - L_1(\xi(v)) = Gv + \theta. \tag{5.1}$$

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The Airy function $\Phi(v) \in H_0^2$ is then given by

$$\Phi(v) = B(\theta, \theta) - B(\xi(v), \xi(v)).$$

Here, $\theta \in H_0^2$ and v belongs to the set of admissible controls $\mathcal{V}_{ad}$, defined as

$$\mathcal{V}_{ad} = \{v \in L^2 | \text{equation (5.1) has a unique solution } \xi(v)\}.$$

We adopt the same general framework of the cost functional as presented by Bock et al. [6] (also see [7]). Therefore, the cost functional corresponding to state equation (5.1) is given by

$$\mathcal{J}(v) = J(\xi(v)) + j(v), \quad v \in \mathcal{V}_{ad}, \tag{5.2}$$

where the functionals, $J : L^2 \longrightarrow \mathbb{R}$ is a state-dependent cost and $j : L^2 \longrightarrow \mathbb{R}$ is a control-dependent cost, are certain functionals.

Let $v^* \in \mathcal{V}_{ad}$ be an optimal control for cost functional (5.2), that is,

$$\mathcal{J}(v^*) = \inf_{v \in \mathcal{V}_{ad}} \mathcal{J}(v).$$

According to Theorem 4.4, there exists a unique solution $\xi(v) \in H_0^2$ to state equation (5.1) for $\lambda < \lambda^\star$. This solution satisfies

$$\|\xi(v)\|_\Delta \leq K(\lambda)^{-1} \|Gv + \theta\|_\Delta.$$

We need the following additional assumption on θ:

$$\theta \in H^4 \cap H_0^2, \tag{5.3}$$

and hence by (3.2), we have $\Delta^2 \theta \in L^2$.

We let

$$\mathcal{V}_{ad} = \left\{ v \in L^2 \middle| \|v + \Delta^2 \theta\|_0 \leq \frac{\overline{\mu}}{\underline{\mu}} \|B\|^{-1} K(\lambda)^{3/2} \right\},$$

where

$$0 < \overline{\mu} < 1 \tag{5.4}$$

is an arbitrary parameter, with $\lambda < \lambda_1(1 - 1/\alpha_1)$ and $\alpha_1 > 1$.

However, by using Lemma 3.1, if $v \in \mathcal{V}_{ad}$, we infer that

$$\|Gv + \theta\|_\Delta = \|G(v + \Delta^2 \theta)\|_\Delta \leq \overline{\mu} \|B\|^{-1} K(\lambda)^{3/2}, \tag{5.5}$$

and thus assumptions (4.8)–(4.10) are valid. Hence, for every $v \in \mathcal{V}_{ad}$, there exists a unique solution $\xi(v) \in H_0^2$ to state equation (5.1).

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Now, we prove the existence theorem for the following optimal control problem.

$$\begin{cases} \text{Find } v^* \in \mathcal{V}_{ad} \text{ such that} \\ \mathcal{J}(v^*) = \inf\limits_{v \in \mathcal{V}_{ad}} \mathcal{J}(v), \\ \xi(v^*) - \lambda L_2(\xi(v^*)) + C(\xi(v^*)) - L_1(\xi(v^*)) = Gv^* + \theta. \end{cases} \tag{5.6}$$

THEOREM 5.1. *Under the assumptions* (3.4)–(3.5), (4.5) *and* (5.3), *let* J *and* j *be weakly lower semicontinuous. Then, there exists at least one optimal control* $v^* \in \mathcal{V}_{ad}$ *for problem* (5.6).

PROOF. Using the same idea as in [6, the proof of Theorem 3.1], we deduce the existence of a minimizing sequence $\{v_n\} \subset \mathcal{V}_{ad}$, that is,

$$\lim_{n \to +\infty} \mathcal{J}(v_n) = \inf_{v \in \mathcal{V}_{ad}} \mathcal{J}(v).$$

It is clear that $\mathcal{V}_{ad}$ is bounded in L^2 and that $\mathcal{V}_{ad}$ (closed and convex) is weakly sequentially closed in L^2. Therefore, there exists a subsequence, still denoted by $\{v_n\}$, which converges weakly in L^2:

$$v_n \rightharpoonup v^* \text{ in } L^2 \quad \text{as } n \to \infty, \quad v^* \in \mathcal{V}_{ad}.$$

From (4.11), (5.5), we obtain the following estimate of all states:

$$\|\xi(v)\|_\Delta \le \overline{\mu} \|B\|^{-1} K(\lambda)^{1/2} \quad \text{for all } v \in \mathcal{V}_{ad}. \tag{5.7}$$

This implies that $\{\xi(v_n)\}$ is bounded in H_0^2. We can find a subsequence, still denoted by $\{v_n\}$, such that

$$\xi_n \rightharpoonup \xi^* \text{ in } H_0^2 \quad \text{as } n \to \infty, \quad \xi_n = \xi(v_n),$$

and

$$\xi_n - \lambda L_2(\xi_n) + C(\xi_n) - L_1(\xi_n) = Gv_n + \theta. \tag{5.8}$$

According to Lemmas 3.3 and 3.5, we see that the nonlinear operator C is sequentially compact and the linear operators L_1 and L_2 are compact. Thus, we obtain the following strong convergences:

$$\begin{aligned} C(\xi_n) &\to C(\xi^*) \text{ in } H_0^2 \quad \text{as } n \to \infty, \\ L_1(\xi_n) &\to L_1(\xi^*) \text{ in } H_0^2 \quad \text{as } n \to \infty, \\ L_2(\xi_n) &\to L_2(\xi^*) \text{ in } H_0^2 \quad \text{as } n \to \infty. \end{aligned}$$

Using (3.3) gives

$$Gv_n \rightharpoonup Gv^* \text{ in } H^4 \cap H_0^2 \quad \text{as } n \to \infty.$$

Passing to a weak limit in (5.8),

$$\xi^* - \lambda L_2(\xi^*) + C(\xi^*) - L_1(\xi^*) = Gv^* + \theta.$$

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From the uniqueness of the solution to state equation (5.1),

$$\xi_n \rightharpoonup \xi(v^*) \text{ in } H_0^2 \quad \text{as } n \to \infty, \quad \xi^* = \xi(v^*).$$

The weak lower semicontinuity of J and j gives

$$\begin{aligned} \mathcal{J}(v^*) &= J(\xi(v^*)) + j(v^*) \\ &\leq \liminf_{n\to\infty} J(\xi(v_n)) + \liminf_{n\to\infty} j(v_n) \\ &\leq \liminf_{n\to\infty} \mathcal{J}(v_n) \\ &= \inf_{v\in\mathcal{V}_{ad}} \mathcal{J}(v). \end{aligned}$$

This concludes the proof. □

6. Characterization of the optimal control

Our goal in this section is to prove the uniqueness of the optimal control and derive the necessary optimality condition, by using the Fréchet differentiability.

6.1. Differentiability of the state function Later on (see Theorem 6.2), we show that the mapping

$$\xi(.) : \mathcal{V}_{ad} \longrightarrow H_0^2,$$

defined by state equation (5.1), is Fréchet differentiable.

We also show that the derivative

$$\xi'(v) : L^2 \longrightarrow H_0^2$$

is determined by the solution of the equation:

$$[id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1]\xi'(v)h = Gh, \quad h \in L^2, \tag{6.1}$$

where

$$C'(\xi)\eta = 2B(B(\xi,\eta),\xi) + B(B(\xi,\xi),\eta), \quad \xi,\eta \in H_0^2. \tag{6.2}$$

Here, $C'(\xi) \in \mathcal{L}(H_0^2; H_0^1)$ is the Gâteaux derivative of the operator C at the point $\xi \in H_0^2$ (see, for example, [18, page 90]).

We consider the operator $\mathcal{A}(\xi(v)) : H_0^2 \longrightarrow H_0^2$ defined by

$$\mathcal{A}(\xi(v)) = id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1. \tag{6.3}$$

LEMMA 6.1. *The operator $A(\xi(v))$ is linear, positive definite and symmetric with respect to the inner product $(.,.)_\Delta$.*

PROOF. It is easy to see from the properties of operators B, L_1 and L_2 described in the statement of Lemmas 3.2 and 3.5 and relation (6.2) that the operator $A(\xi(v))$ is linear and symmetric.

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For all $\eta \in H_0^2$,

$$(\mathcal{A}(\xi(v))\eta, \eta)_\Delta = \|\eta\|_\Delta^2 - \lambda(L_2\eta, \eta)_\Delta + 2(B(B(\xi(v), \eta), \xi(v)), \eta)_\Delta + (B(B(\xi(v), \xi(v)), \eta), \eta)_\Delta - (L_1\eta, \eta)_\Delta.$$

Using Lemma 3.2,

$$\begin{aligned}(\mathcal{A}(\xi(v))\eta, \eta)_\Delta &= \|\eta\|_\Delta^2 - \lambda(L_2\eta, \eta)_\Delta + 2(B(\xi(v), \eta), B(\xi(v), \eta))_\Delta \\ &\quad + (B(\xi(v), \xi(v)), B(\eta, \eta))_\Delta - (L_1\eta, \eta)_\Delta \\ &= \|\eta\|_\Delta^2 - \lambda(L_2\eta, \eta)_\Delta + 2\|B(\xi(v), \eta)\|_\Delta^2 \\ &\quad + (B(\xi(v), \xi(v)), B(\eta, \eta))_\Delta - (L_1\eta, \eta)_\Delta.\end{aligned}$$

Therefore, as in the proof of Theorem 4.4,

$$(\mathcal{A}(\xi(v))\eta, \eta)_\Delta \geq K(\lambda)\|\eta\|_\Delta^2 - \|B\|^2\|\xi(v)\|_\Delta^2\|\eta\|_\Delta^2.$$

Then, using (5.7) gives

$$(A(\xi(v))\eta, \eta)_\Delta \geq (1 - \overline{\mu}^2)K(\lambda)\|\eta\|_\Delta^2.$$

Finally, we set

$$\overline{K}(\lambda) = (1 - \overline{\mu}^2)K(\lambda).$$

Moreover, one can easily see from (4.12) and (5.4) that

$$\overline{K}(\lambda) > 0.$$

Hence,

$$(\mathcal{A}(\xi(v))\eta, \eta)_\Delta \geq \overline{K}(\lambda)\|\eta\|_\Delta^2. \tag{6.4}$$

This proves the positive definiteness of $A(\xi(v))$. □

THEOREM 6.2. *The mapping $\xi(.)$ determined by state equation (5.1) is Fréchet differentiable with respect to $v \in \mathring{\mathcal{V}}_{ad}$ ($\mathring{\mathcal{V}}_{ad}$ denotes the interior of $\mathcal{V}_{ad}$) and $\xi'(v)$ satisfies equation (6.1) for all $h \in L^2$.*

PROOF. As a result of Lemma 6.1, there exists a unique solution $\varphi(h)$ of the equation

$$\mathcal{A}(\xi(v))\varphi(h) = [id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1]\varphi(h) = Gh \quad \text{for all } h \in L^2. \tag{6.5}$$

Let

$$\psi = \psi(h) = \widetilde{\xi} - \varphi(h),$$

where

$$\widetilde{\xi} = \xi(v + h) - \xi(v); \quad v,\ v + h \in \mathcal{V}_{ad}. \tag{6.6}$$

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To establish that $\varphi(h) = \xi'(v)h$ is the differential of ξ in the sense of Fréchet, it is sufficient to verify that

$$\|\psi\|_\Delta = o(h). \tag{6.7}$$

By using state equation (5.1) and (6.5),

$$\begin{aligned}
\mathcal{A}(\xi(v))\psi &= [id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1]\psi \\
&= [id_{H_0^2} - \lambda L_2 - L_1](\xi(v+h) - \xi(v)) + C'(\xi(v))(\xi(v+h) - \xi(v)) \\
&\quad - [id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1]\varphi(h) \\
&= G(v+h) - C(\xi(v+h)) - G(v) + C(\xi(v)) \\
&\quad + C'(\xi(v))(\xi(v+h) - \xi(v)) - Gh \\
&= C'(\xi(v))(\xi(v+h) - \xi(v)) - [C(\xi(v+h)) - C(\xi(v))] \\
&= \int_0^1 [C'(\xi(v)) - C'(\xi(v) + s\widetilde{\xi})]\widetilde{\xi}\, ds.
\end{aligned} \tag{6.8}$$

Moreover, because of the positive definiteness of $\mathcal{A}(\xi(v))$, there exists $C(\lambda) > 0$ such that

$$\|\psi\|_\Delta \le C(\lambda)\|\widetilde{\xi}\|_\Delta^2. \tag{6.9}$$

From (6.5) and (6.8),

$$\begin{aligned}
\mathcal{A}(\xi(v))\varphi(h) &= [id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1](\xi(v+h) - \xi(v) - \psi) \\
&= [id_{H_0^2} - \lambda L_2 - L_1](\xi(v+h) - \xi(v)) + C'(\xi(v))(\xi(v+h) - \xi(v)) \\
&\quad - [id_{H_0^2} - \lambda L_2 + C'(\xi(v)) - L_1]\psi \\
&= [id_{H_0^2} - \lambda L_2 - L_1](\xi(v+h) - \xi(v)) + C'(\xi(v))(\xi(v+h) - \xi(v)) \\
&\quad - C'(\xi(v))(\xi(v+h) - \xi(v)) + [C(\xi(v+h)) - C(\xi(v))] \\
&= [id_{H_0^2} - \lambda L_2 - L_1]\widetilde{\xi} + C(\xi(v+h)) - C(\xi(v)).
\end{aligned}$$

Then, the function $\widetilde{\xi}$ satisfies

$$\widetilde{\xi} - \lambda L_2(\widetilde{\xi}) - L_1(\widetilde{\xi}) + C(\xi(v+h)) - C(\xi(v)) = Gh.$$

By virtue of Lemmas 3.1 and 3.4,

$$K(\lambda)\|\widetilde{\xi}\|_\Delta^2 - \|B\|^2 \max\{\|\xi(v)\|_\Delta^2, \|\xi(v+h)\|_\Delta^2\}\|\widetilde{\xi}\|_\Delta^2 \le \underline{\mu}\|h\|_0\|\widetilde{\xi}\|_\Delta.$$

Taking into account estimate (5.7),

$$\|\widetilde{\xi}\|_\Delta \le \underline{K}(\lambda)\|h\|_0, \tag{6.10}$$

where

$$\underline{K}(\lambda) = \underline{\mu}\overline{K}(\lambda)^{-1}.$$

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Using (6.6), (6.9) and (6.10),

$$\|\psi\|_\Delta = \|\xi(v+h) - \xi(v) - \varphi(h)\|_\Delta = o(h),$$

and hence (6.7) is verified. □

6.2. Uniqueness of the optimal control Let us assume that the functional J and j are Fréchet differentiable. Because we have shown in Theorem 6.2 that the mapping $\xi(.)$ determined by state equation (5.1) is Fréchet differentiable at the optimal control v^*, we infer that the functional $\mathcal{J}$ is also Fréchet differentiable at v^* such that

$$\langle \mathcal{J}'(v^*), v\rangle_0 = \langle J'(\xi(v^*)), \xi'(v^*)v\rangle_{-2,2} + \langle j'(v^*), v\rangle_0 \quad \text{for all } v \in \mathcal{V}_{ad}.$$

Similar to the case of optimal control problems governed by von Kármán equations presented by Bock et al. [6] (also see [7]), assume that the derivatives J' and j' satisfy the following conditions:

$$\langle J'(\xi_1) - J'(\xi_2), \xi_1 - \xi_2\rangle_{-2,2} \geq \sigma\|\xi_1 - \xi_2\|_\Delta^2 \quad \text{for all } \xi_1, \xi_2 \in H_0^2, \tag{6.11}$$

$$\langle j'(v_1) - j'(v_2), v_1 - v_2\rangle_0 \geq \rho\|v_1 - v_2\|_0^2 \quad \text{for all } v_1, v_2 \in L^2, \tag{6.12}$$

where σ and ρ are positive constants.

Also, the derivative J' satisfies the following condition:

$$\|J'(\xi)\|_{-2} \leq \kappa_1\|\xi\|_\Delta + \kappa_2, \tag{6.13}$$

where $\kappa_1 > 0$ and κ_2 are some constants.

The technical conditions (6.11)–(6.13) assert that the derivatives J' and j' are strongly monotone and J' grows at most linearly with ξ. The conditions (6.11) and (6.12) are common in monotone operator theory [40], and (6.13) is standard in control theory for PDEs [29, 36]. These conditions play a critical role in the following uniqueness result.

THEOREM 6.3. *Let the assumptions of Theorem 5.1 and (6.11)–(6.13) be valid. If*

$$\rho > \{-\sigma + 6\overline{\mu}\overline{K}(\lambda)^{-1}K(\lambda)^{1/2}\|B\|(\kappa_1\overline{\mu}\|B\|^{-1}K(\lambda)^{1/2} + \kappa_2)\}\underline{K}(\lambda)^2,$$

then there exists a unique solution v^* of problem (5.6).

PROOF. If $v^* \in \mathcal{V}_{ad}$ is a solution to problem (5.6) (v^* is the optimal control), then

$$\langle \mathcal{J}'(v^*), v - v^*\rangle_0 \geq 0 \quad \text{for all } v \in \mathcal{V}_{ad}.$$

Let $v_1^*, v_2^* \in \mathcal{V}_{ad}$ be two solutions to problem (5.6), then

$$\langle \mathcal{J}'(v_1^*), v - v_1^*\rangle_0 = \langle J'(\xi(v_1^*)), \xi'(v_1^*)(v - v_1^*)\rangle_{-2,2} + \langle j'(v_1^*), v - v_1^*\rangle_0 \geq 0, \tag{6.14}$$

$$\langle \mathcal{J}'(v_2^*), v - v_2^*\rangle_0 = \langle J'(\xi(v_2^*)), \xi'(v_2^*)(v - v_2^*)\rangle_{-2,2} + \langle j'(v_2^*), v - v_2^*\rangle_0 \geq 0 \tag{6.15}$$

for all $v \in \mathcal{V}_{ad}$.

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Taking $v = v_2^*$ in (6.14), $v = v_1^*$ in (6.15) and adding,

$$\begin{aligned}&\langle J'(\xi(v_1^*)) - J'(\xi(v_2^*)), \xi(v_2^*) - \xi(v_1^*)\rangle_{-2,2} + \langle j'(v_1^*) - j'(v_2^*), v_2^* - v_1^*\rangle_0\\ &\quad - \langle J'(\xi(v_1^*)), \xi(v_2^*) - \xi(v_1^*) - \xi'(v_1^*)(v_2^* - v_1^*)\rangle_{-2,2}\\ &\quad - \langle J'(\xi(v_2^*)), \xi(v_1^*) - \xi(v_2^*) - \xi'(v_2^*)(v_1^* - v_2^*)\rangle_{-2,2} \geq 0.\end{aligned}$$

We set

$$\psi_1 = \xi(v_2^*) - \xi(v_1^*) - \xi'(v_1^*)(v_2^* - v_1^*), \tag{6.16}$$

$$\psi_2 = \xi(v_1^*) - \xi(v_2^*) - \xi'(v_2^*)(v_1^* - v_2^*), \tag{6.17}$$

$$\widehat{\xi} = \xi(v_2^*) - \xi(v_1^*). \tag{6.18}$$

By assumptions (6.11)–(6.12) and relations (6.16)–(6.18),

$$-\sigma\|\widehat{\xi}\|_\Delta^2 - \rho\|v_1^* - v_2^*\|_0^2 - \langle J'(\xi(v_1^*)), \psi_1\rangle_{-2,2} - \langle J'(\xi(v_2^*)), \psi_2\rangle_{-2,2} \geq 0. \tag{6.19}$$

To estimate ψ_1, with similar computations for estimate (6.9), readily

$$\begin{aligned}[id_{H_0^2} - \lambda L_2 + C'(\xi(v_1^*)) - L_1]\psi_1 &= C'(\xi(v_1^*))(\xi(v_2^*) - \xi(v_1^*)) + C(\xi(v_1^*)) - C(\xi(v_2^*))\\ &= \int_0^1 [C'(\xi(v_1^*)) - C'(\xi(v_1^*) + s\widehat{\xi})]\widehat{\xi}\, ds\\ &= \Psi.\end{aligned} \tag{6.20}$$

The mean value theorem implies

$$\begin{aligned}\|\Psi\|_\Delta &= \sup_{\|h\|_\Delta = 1} \left|\left(\int_0^1 [C'(\xi(v_1^*)) - C'(\xi(v_1^*) + s\widehat{\xi})]\widehat{\xi}\, ds, h\right)_\Delta\right|\\ &= \sup_{\|h\|_\Delta = 1} \left|\left(\int_0^1 C''(\xi(v_1^*) + \tau(s)\widehat{\xi})(\widetilde{\xi}, \widetilde{\xi}) s\, ds, h\right)_\Delta\right|,\end{aligned}$$

where $\tau(s) \in (0, s)$ and the second derivative C'' has the form

$$C''(\xi)(\widehat{\xi}, \widehat{\xi}) = 2B(B(\widehat{\xi}, \widehat{\xi}), \xi) + 4B(B(\xi, \widehat{\xi}), \widehat{\xi}) \quad \text{for all } \xi, \widehat{\xi} \in H_0^2.$$

Using estimate (5.7),

$$\|\Psi\|_\Delta \leq 3\overline{\mu}K(\lambda)^{1/2}\|B\|\|\widehat{\xi}\|_\Delta^2. \tag{6.21}$$

From the positive definiteness of $A(\xi(v))$ in (6.4), and relations (6.20) and (6.21),

$$\|\psi_1\|_\Delta \leq 3\overline{\mu}\overline{K}(\lambda)^{-1}K(\lambda)^{1/2}\|B\|\|\widehat{\xi}\|_\Delta^2, \tag{6.22}$$

and similarly for ψ_2.

By assumption (6.13), relations (5.7), (6.19) and (6.22),

$$\{-\sigma + 6\overline{\mu}\overline{K}(\lambda)^{-1}K(\lambda)^{1/2}\|B\|(\kappa_1\overline{\mu}\|B\|^{-1}K(\lambda)^{1/2} + \kappa_2)\}\|\widehat{\xi}\|_\Delta^2 - \rho\|v_1^* - v_2^*\|_0^2 \geq 0. \tag{6.23}$$

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We set $h = v_1^* - v_2^*$ in (6.10), then,

$$\|\widehat{\xi}\|_\Delta \le \underline{K}(\lambda)\|v_1^* - v_2^*\|_0.$$

Combining this estimate with (6.23),

$$[\{-\sigma + 6\overline{\mu}\overline{K}(\lambda)^{-1}K(\lambda)^{1/2}\|B\|(\kappa_1\overline{\mu}\|B\|)^{-1}K(\lambda)^{1/2} + \kappa_2)\}\underline{K}(\lambda)^2 - \rho]\|v_1^* - v_2^*\|_0^2 \ge 0.$$

It is easy to infer the following sufficient condition for the uniqueness of the optimal control:

$$\begin{aligned}\rho &> \{-\sigma + 6\overline{\mu}\overline{K}(\lambda)^{-1}K(\lambda)^{1/2}\|B\|(\kappa_1\overline{\mu}\|B\|)^{-1}K(\lambda)^{1/2} + \kappa_2)\}\underline{K}(\lambda)^2 \\ &= \{-\sigma + 6\overline{\mu}(1-\overline{\mu}^2)^{-1}K(\lambda)^{-1/2}\|B\|(\kappa_1\overline{\mu}\|B\|)^{-1}K(\lambda)^{1/2} + \kappa_2)\}\underline{\mu}^2(1-\overline{\mu}^2)^{-2}K(\lambda)^{-2}. \quad \square\end{aligned}$$

6.3. Necessary optimality condition If $v^* \in \mathcal{V}_{ad}$ is the optimal control, then we have the following system:

$$\begin{cases}\langle J'(\xi(v^*)), \xi'(v^*)(v - v^*)\rangle_{-2,2} + \langle j'(v^*), v - v^*\rangle_0 \ge 0 \quad \text{for all } v \in \mathcal{V}_{ad}, \\ \mathcal{A}(\xi(v^*))\xi'(v^*)h = Gh \quad \text{for all } h \in L^2.\end{cases} \tag{6.24}$$

We recall that the linear operator $\mathcal{A}$ defined in (6.3) is symmetric due to Lemma 6.1. Then, the system (6.24) can be rewritten as

$$\begin{cases}(p + \mathcal{R}_0 j'(v^*), v - v^*)_0 \ge 0 \quad \text{for all } v \in \mathcal{V}_{ad}, \\ \mathcal{A}(\xi(v^*))p = \mathcal{R}J'(\xi(v^*)),\end{cases}$$

where $\mathcal{R}_0 : (L^2)^\star \longrightarrow L^2$ and $\mathcal{R} : H^{-2} \longrightarrow H_0^2$ are Riesz-representative operators, and p is the adjoint state.

Taking into account (3.8),

$$\begin{aligned}\langle J'(\xi(v^*)), \xi'(v^*)(v - v^*)\rangle_{-2,2} &= (\mathcal{R}J'(\xi(v^*)), \xi'(v^*)(v - v^*))_\Delta \\ &= (A(\xi(v^*))p, \xi'(v^*)(v - v^*))_\Delta \\ &= (p, A(\xi(v^*))\xi'(v^*)(v - v^*))_\Delta \\ &= (p, G(v - v^*))_\Delta \\ &= (p, v - v^*)_0.\end{aligned}$$

From this, we infer that $p + \mathcal{R}_0 j'(v^*)$ represents $\mathcal{J}'(v^*)$.

Finally, we conclude that the optimal control v^* for problem (5.6) is characterized by

$$\begin{cases}(p + \mathcal{R}_0 j'(v^*), v - v^*)_0 \ge 0 \quad \text{for all } v \in \mathcal{V}_{ad}, \\ \mathcal{A}(\xi(v^*))p = \mathcal{R}J'(\xi(v^*)), \\ \xi(v^*) - \lambda L_2(\xi(v^*)) + C(\xi(v^*)) - L_1(\xi(v^*)) = Gv^* + \theta.\end{cases}$$

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7. Conclusions and commentary

In this work, our interest was focused on studying the optimal control of Marguerre–von Kármán equations for a thin nonlinearly elastic shallow shell. More precisely, we developed the ideas originally presented by Bock et al. [6] for plates to a more general geometry of shallow shells. In particular, we obtained the desired characterization for the optimal control under specific regularity assumptions on the function θ that defined the middle surface of the shell and the traces ϕ_0, ϕ_1 that were calculated through the von Kármán forces.

Several interesting questions remain to be answered. While this work is devoted to the classical Marguerre–von Kármán equations, we are interested in extending our methods to the generalized Marguerre–von Kármán equations. We also intend to study the numerical approximation of the optimal control of Marguerre–von Kármán equations.

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