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Abstract

We perform a general study of the structure of locally compact modules over compactly generated abelian groups. We obtain a dévissage result for such modules of the form 'compact-by-sheer-by-discrete', and then study more specifically the sheer part. The main typical example of a sheer module is a polycontractible module, that is, a finite direct product of modules, each of which is contracted by some group element. We show that every sheer module has a 'large' polycontractible submodule, in some suitable sense. We apply this to the study of compactly generated metabelian groups. For instance, we prove that they always have a maximal compact normal subgroup, and we extend the Bieri–Strebel characterization of compactly presentable metabelian groups from the discrete case to this more general setting.

1. Introduction

We fix a locally compact abelian (LCA) group A, which will often be assumed to be compactly generated. The general structure of LCA groups ensures that every compactly generated LCA group has a cocompact lattice isomorphic to $\mathbf{Z}^d$ for some d. Therefore, the reader can assume $A = \mathbf{Z}^d$; this is not a significant restriction.

We consider locally compact (LC) A-modules M, that is, M is an LCA group endowed with a continuous homomorphism $A \to \mathrm{Aut}(M)$ (continuity is equivalent to continuity of the action map $A \times M \to M$). We aim to provide general results on the structure of such modules M. They are then illustrated by deducing general structural results on compactly generated LC metabelian groups. One extreme case is when M is discrete. In this case, the study can be handled with classical commutative algebra. The 'opposite' case is when M is compact: such modules then inherit a description, namely as Pontryagin duals of discrete modules. Recall that the Pontryagin dual of an LCA group M is $\hat{M} = \mathrm{Hom}(M, \mathbf{R}/\mathbf{Z})$ (continuous group homomorphisms), and that this defines a contravariant self-equivalence of the category of LCA groups, notably exchanging discrete and compact LCA groups. Functoriality ensures that it is compatible with group actions, and thus it induces a contravariant self-equivalence of the category of A-modules (there is a minor continuity verification for the action; see § 2.1). In particular, denoting by $N^\perp$ the orthogonal

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$\{\chi \in \hat{M} : \chi|_N = 0\}$ for a closed submodule N of M, orthogonality is an order-reversing isomorphism between closed submodules of M and those of $\hat{M}$.

The purpose of the forthcoming definitions is to 'isolate' the 'essential part' of M that is 'in between compact and discrete' (roughly speaking, that has 'few' compact submodules and discrete quotients) and to do it precisely, so as to avoid such quotation marks: this is done in Definition 1.3, and Theorem 1.4. The first definition for this purpose is as follows.

DEFINITION 1.1. Let M be an LC A-module. The polycompact radical $W(M)$ (or $W_A(M)$ in case of ambiguity) of M is the sum of all compact submodules of M. The codiscrete radical $\Omega(M)$ is the intersection of all open submodules.

Note that modding out by the closure of the polycompact radical and taking the codiscrete radical are 'exchanged' by Pontryagin duality (see Proposition 2.2 for details). A first fact is stated in the following theorem.

THEOREM 1.2 (Theorem 2.5). *If A is compactly generated, then $W(M)$ is closed for every LC A-module M, and every compact subset of $W(M)$ is contained in a compact submodule.*

(See Example 7.1 for a counterexample when A is not assumed compactly generated.)

This is easily deduced from a group-theoretic fact, namely that, in a compactly generated LC group, the subgroup generated by all compact normal subgroups is closed. The latter is a nontrivial fact, proved in [Cor15], which relies on work of Trofimov, who essentially proved it for totally disconnected compactly generated LC groups. See § 2.3.

We now alter these definitions because of the following shortcomings:

- $W(M)$ need not be compact;
- $W(M)$ need not be contained in $\Omega(M)$ (e.g., if M is finite, $\Omega(M) = \{0\}$ and $W(M) = M$).

DEFINITION 1.3. A module M is compact parafinite if it is compact with no nonzero finite quotient, that is, M is compact and $M = \Omega(M)$.

A module M is purely discrete if it is discrete with no nonzero finite submodule, that is, M is discrete and $W(M) = \{0\}$.

Define the reduced polycompact radical $W^\flat(M)$ as the sum of all compact parafinite submodules of M. Define the augmented codiscrete radical $\Omega^\sharp(M)$ as the intersection of all open submodules with purely discrete quotient.

A module M is sheer if $W^\flat(M) = \{0\}$ and $\Omega^\sharp(M) = M$.

For instance, an LCA group M is sheer as a module over the trivial group if and only if it has no nontrivial compact connected subgroup and no nontrivial discrete torsion-free quotient.

Clearly, $0 \subseteq W^\flat(M) \subset W(M)$ and $\Omega(M) \subseteq \Omega^\sharp(M) \subseteq M$. Note that the latter are also exchanged by orthogonality in Pontryagin duality: $W(M)^\perp = \Omega(\hat{M})$ and $W^\flat(M)^\perp = \Omega^\sharp(\hat{M})$. In particular, a module is sheer if and only if its Pontryagin dual is sheer. The following theorem is a first observation.

THEOREM 1.4 (Canonical filtration with sheer subquotient, Theorem 2.6). *We have the inclusions $0 \subseteq W^\flat(M) \subseteq \Omega^\sharp(M) \subseteq M$. If A is compactly generated, then $W^\flat(M)$ is compact parafinite, $M/\Omega^\sharp(M)$ is purely discrete, and the subquotient $\Omega^\sharp(M)/W^\flat(M)$ is sheer.*

See § 2.4 for the proof. The first assertion is quite straightforward, while the second one makes use of Theorem 1.2.

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Theorem 1.4 reduces most of the study of modules to that of sheer modules, so we proceed to describe structural results about them. Central in their description is the notion of a contracting automorphism. Recall that an automorphism α of a LC group G is *contracting* if $\alpha^n(x)$ tends to 1 uniformly for x in compact subsets of G.

Definition 1.5. An LC A-module M is said to be contractible if there exists $\alpha \in A$ acting as a contracting automorphism of the LCA group M. It is said to be polycontractible if it splits as a finite topological direct sum of contractible modules (not necessarily with the same contracting element).

An LC A-module is said to be amorphic if it is both sheer and compact-by-discrete. This means that it is profinite-by-(discrete locally finite). (Here both 'locally finite' and 'profinite' refer to the module structure and not just the underlying LCA group.)

An LC A-module is said to be Euclidean if the underlying LCA group is isomorphic to $\mathbf{R}^d$ for some d. It is said to be distal Euclidean if it is Euclidean and the action of A is by matrices with only complex eigenvalues of modulus 1.

All these notions are invariant under Pontryagin duality. For instance, if p is prime and $M = \mathbf{Q}_p$ with some action $i : A \to \mathbf{Q}_p^*$, then M is sheer, and either $i(A)$ consists of elements of norm 1 and M is amorphic, or else M is contractible.

Polycontractible modules are sheer and they constitute the main source of sheer modules, and, in a sense, the most 'interesting' ones. Indeed, it readily follows from the definition that $W(M) = \{0\}$ and $\Omega(M) = M$ when M is contractible. Also, Euclidean modules and amorphic modules are sheer. The following theorem indicates that each sheer module has a canonical 'large' polycontractible submodule.

Theorem 1.6 (Theorem 4.1). *Suppose that A is compactly generated. Every sheer module has a unique decomposition $P \oplus E \oplus V$, with P polycontractible, E amorphic,* and *V distal Euclidean.*

See § 4.1. The uniqueness part is not difficult. The existence part relies on the case $A = \mathbf{Z}$, where it takes a more precise form; see Lemma 3.6. The latter makes use of Willis theory [Wil94], which deals with automorphisms of (possibly nonabelian) totally disconnected locally compact groups. Here we take advantage of the module being an abelian group to obtain these more precise results.

Corollary 1.7 (Corollary 4.3). *Suppose that A is compactly generated. Then for every compactly generated LC A-module M, the polycompact radical $W(M)$ is compact.*

(See Example 7.2 for an example with A not compactly generated, M discrete, and $W(M)$ infinite and thus not compact.)

Example 1.8 (Being sheer does not pass to submodules or quotients). Here are some examples of polycontractible, and hence sheer, modules M over the group $A = \mathbf{Z}$ with a nonsheer closed submodule N, namely a cocompact lattice. In each case, the quotient M/N is compact parafinite and hence is not sheer either. The first example is totally disconnected, and the third is Euclidean.

(1) The module $\mathbf{F}_p(\!(t)\!) \times \mathbf{F}_p(\!(t^{-1})\!)$, where the positive generator of $\mathbf{Z}$ acts on both factors by multiplication by t, is polycontractible. The module $\mathbf{F}_p[t, t^{-1}]$ can be viewed as a discrete cocompact submodule of the former, by the obvious natural diagonal embedding.

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(2) Similarly, the module $\mathbf{R} \times \mathbf{Q}_p$, where the positive generator acts on both factors by multiplication by p, is polycontractible. It admits the diagonal embedding of $\mathbf{Z}[1/p]$ as a discrete submodule. In this case, the quotient, called a 'solenoid', is connected.

(3) Similarly, again with $A = \mathbf{Z}$ acting on $\mathbf{R}^2$ through powers of the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\mathbf{R}^2$ is a sheer module, but its submodule $\mathbf{Z}^2$ is not, since it is purely discrete.

THEOREM 1.9 (Corollary 5.9). *Suppose that A is compactly generated. Let M be a sheer LC A-module. Then M has a canonical decomposition $(\bigoplus_{i=1}^n P_i) \oplus E \oplus V$, with $P_i \neq \{0\}$ contractible monotypic (in the sense that all its simple quotients are isomorphic), the P_i pairwise disjoint (in the sense that for $i \neq j$, simple submodules of P_i and P_j are not isomorphic as LC A-modules), E amorphic, and V distal Euclidean.*

Moreover, every closed sheer submodule N splits with respect to this decomposition, namely $N = (\bigoplus_{i=1}^n N \cap P_i) \oplus (N \cap E) \oplus (N \cap V)$.

Theorem 1.9 gives a reasonable understanding of sheer closed submodules of sheer modules. It mostly reduces to understanding submodules of contractible (monotypic) modules. These ones are tackled by the following theorem.

THEOREM 1.10 (Theorem 5.2). *Let M be a contractible A-module. Let R be the closure of the image of the group ring $\mathbf{Z}[A]$ in $\mathrm{End}_A(M)$. Then R is an artinian commutative ring, M is a module of finite length over R, and the closed A-submodules of M are exactly the R-submodules of M (i.e., these submodules are automatically closed).*

(Here, the group ring is defined regardless of the topology on A.) These modules are thus completely governed by abstract commutative algebra.

While the above describes sheer submodules, it remains to describe arbitrary submodules. Example 1.8 shows that there are nonsheer submodules of sheer modules occurring as cocompact lattices. The following theorem (see Theorem 4.6) shows, informally speaking, that taking cocompact lattices is essentially the only way to produce arbitrary submodules when starting from sheer submodules.

THEOREM 1.11 (Theorem 4.6). *Suppose that A is compactly generated. Let M be a sheer A-module, and N a closed submodule. Then there are unique sheer closed submodules $N^{\boxminus}$, $N^{\boxplus}$ of M such that we have the inclusions $N^{\boxminus} \subseteq N \subseteq N^{\boxplus}$ and $N/N^{\boxminus}$ is a cocompact lattice in $N^{\boxplus}/N^{\boxminus}$.*

Let us now describe applications to metabelian locally compact groups (§ 6.1).

THEOREM 1.12 (Theorem 6.3 and Corollary 6.4). *Let G be a compactly generated metabelian group. Then for every ascending sequence (H_n) of closed normal subgroups, there exists n such that H_n is cocompact in $\overline{\bigcup_k H_k}$. In particular:*

(1) *G has a maximal compact normal subgroup;*

(2) *every closed normal subgroup is compactly generated as normal subgroup.*

Actually (1) is proved directly, making uses of applications in both senses: groups to modules, then modules to groups. Namely, if G is a compactly generated LC group, then $W(G)$, the union of all compact normal subgroups, is closed. This fact is used in the study of modules (Theorem 1.2). Next structural results on LC modules give rise to the compactness result for modules, Corollary 1.7, which in turns implies its analog for groups, (1) above.

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We now describe a more elaborate application. Let G be a compactly generated metabelian group. Let M be a closed normal abelian subgroup such that $Q = G/M$ is abelian. So M is naturally a module over the compactly generated LCA group Q. For a character $\chi \in \mathrm{Hom}(Q, \mathbf{R})$, define $Q_\chi = \{g \in Q : \chi(g) \geq 0\}$. Define the Bieri–Strebel invariant as

$$\Gamma(G) = \{0\} \cup \{\chi \in \mathrm{Hom}(Q, \mathbf{R}) : M \text{ is not comp. gen. as } Q_\chi\text{-module}\}.$$

This was defined by Bieri and Strebel [BS80] when G is discrete. They actually rather defined its complement. But the convention here is much more convenient, for the reason that, after identifying $\mathrm{Hom}(Q, \mathbf{R})$ to a subspace of $\mathrm{Hom}(G, \mathbf{R})$ it does not depend on the choice of M (while the complement does: just consider $G = \mathbf{Z}$ and the choices of $M = \{0\}$ versus $M = \mathbf{Z}$ to see the issue with taking the open complement). The subset $\Gamma(G)$ shares properties with the a spectrum of an operator, or a set of weights. It is invariant under positive homotheties. The following theorem extends the case of discrete metabelian groups from [BS80].

Theorem 1.13 (Theorem 6.25). *Let G be a compactly generated metabelian LC group. Then G is compactly presentable if and only $\Gamma(G)$ contains no line through the origin, or equivalently does not contain any two opposite nonzero characters.*

Example 1.14. The main 'typically nondiscrete' case of Theorem 1.13 is when M (closed normal abelian subgroup of G with abelian quotient $Q = G/M$) is polycontractible, namely $M = \bigoplus_{i \in I} M_i$, with M_i contractible indecomposable (hence either totally disconnected or connected). Let $\theta_i : Q \to \mathbf{R}$ be the logarithm of the homomorphism describing the action of Q on the Haar measure of M_i. Let J be the set of i such that M_i is totally disconnected. Then $\Gamma(M) = \{0\} \cup \bigcup_{j \in J} \mathbf{R}_{\geq 0}\theta_j$.

In the discrete case, Γ is known to be a polyhedral cone, not necessarily reduced to a finite union of half-lines (this is extended to the nondiscrete case in Proposition 6.22). Proving Theorem 1.13 thus has two first special cases: the sheer case which easily reduces to the polycontractible case, and the discrete case due to Bieri and Strebel. Then additional work is needed to 'put things together', relying on the previous structural results.

2. Basic facts

2.1 Pontryagin duality

Continuity of the action defining a LC A-module M is equivalent to continuity of $A \to \mathrm{Aut}_{\mathrm{LCA}}(M)$. Recall that the topology of the latter is defined as follows: the net (α_i) converges to α if the map $(\alpha_i, \alpha_i^{-1}) : M^2 \to M^2$ converges to (α, α^{-1}) uniformly on compact subsets of M. To check that continuity is preserved when passing to Pontryagin duality, we just need to check the following proposition.

Proposition 2.1. *Let M be an LCA group. Then the action of $\mathrm{Aut}(M)$ on the Pontryagin dual $\hat{M} = \mathrm{Hom}(M, \mathbf{R}/\mathbf{Z})$ is continuous.*

Proof. It is enough to prove that if (α_i) is a net in $\mathrm{Aut}(M)$ converging to the identity, then so does $(\hat{\alpha}_i)$. We have, for $\chi \in \hat{M}$ and $x \in M$,

$$\hat{\alpha}_i(\chi)(x) = \chi(\alpha_i^{-1}(x))$$

and thus

$$\hat{\alpha}_i(\chi)(x) - \chi(x) = \chi(\alpha_i^{-1}(x) - x).$$

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Then $\alpha_i^{-1}(x) - x$ converges to 0, uniformly for x in compact subsets. Hence $\chi(\alpha_i^{-1}(x) - x)$ converges to 0 uniformly for (x, χ) in compact subsets. This means that $\hat{\alpha}_i(\chi)$ converges to χ uniformly for χ in compact subsets. In turn, this means that $\hat{\alpha}_i$ converges to the identity uniformly on compact subsets. The same argument with α_i^{-1} yields that $\hat{\alpha}_i^{-1}$ also converges to the identity uniformly on compact subsets. So $\hat{\alpha}_i$ tends to the identity in the topology of $\mathrm{Aut}(\hat{M})$. □

Let us provide, as a sample, the following fact of Pontryagin duality.

PROPOSITION 2.2. *Let M be an A-module. The Pontryagin dual of the inclusion map $\overline{W(M)} \to M$ is (canonically identified with) the quotient map $\hat{M} \to \hat{M}/\Omega(\hat{M})$. In $\hat{M}$, the orthogonal of $\overline{W(M)}$ is $\Omega(\hat{M})$.*

Proof. Consider all pairs (N, f) consisting of an A-module N and a homomorphism $f : M \to N$ such that for every compact A-module K and homomorphism $g : K \to M$, the composition $f \circ g$ is zero. It is understood that homomorphisms in this categories are homomorphisms between the given modules so that the obvious triangle with M commutes. Then the initial object in this category is the quotient map $p : M \to M/\overline{W(M)}$, and its kernel is the inclusion $i : \overline{W(M)} \to M$.

Write $L = M/\overline{W(M)}$. Let $\hat{p} : \hat{L} \to \hat{M}$ be the Pontryagin dual map of p. Consider the category of pairs (N, f) consisting of an A-module N and a homomorphism $N \to \hat{M}$ such that for every discrete A-module D and homomorphism $g : \hat{M} \to D$, we have $g \circ f = 0$. Then, by Pontryagin duality and the property of the previous paragraph, in this category, $\hat{p}$ is a terminal object, and its cokernel is $\hat{i} : \hat{M} \to \widehat{\overline{W(M)}}$.

But it follows from the definition of $\Omega(\cdot)$ that a terminal object in the latter category is also given by the inclusion $\Omega(\hat{M}) \to \hat{M}$, whose cokernel is the projection $\hat{M} \to \hat{M}/\Omega(\hat{M})$. By uniqueness of the terminal object and of taking the cokernel, we deduce that the Pontryagin dual of the inclusion i is naturally the projection $\hat{M} \to \hat{M}/\Omega(\hat{M})$.

Recall that the orthogonal of a submodule N of M is $N^\perp = \{\chi \in \hat{M} : \chi|_N = 0\}$. There is a canonical isomorphism $\hat{N} \simeq \hat{M}/N^\perp$. The second fact therefore follows from the first. Alternatively, a submodule is compact if and only if its orthogonal is open. Orthogonality is an order-reversing isomorphism between the lattices of submodules of M and $\hat{M}$. Then $W(M)$, as supremum (in this lattice) of the set of all compact submodules of M, is mapped to the infimum of the set of all open submodules of $\hat{M}$, and this is $\Omega(\hat{M})$. □

2.2 Lower-sheer and upper-sheer modules

Since 'sheer' involves two conditions, it is useful to separate them.

DEFINITION 2.3. An A-module M is lower-sheer if $W^\flat(M) = \{0\}$, and upper-sheer if $\Omega^\sharp(M) = M$. It is sheer if is both lower-sheer and upper-sheer.

Observe the classes of lower-sheer and upper-sheer A-modules are exchanged under Pontryagin duality. Note that being lower-sheer passes to closed submodules, and upper-sheer passes to quotients. However, being sheer passes neither to submodules nor quotients (Example 1.8).

2.3 The polycompact radical is closed

We use the following theorem.

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Theorem 2.4. *Let G be a compactly generated locally compact group. Then the union $W_{\mathrm{grp}}(G)$ of all compact normal subgroups of G is closed, and every compact subset of $W_{\mathrm{grp}}(G)$ is contained in a compact normal subgroup of G.*

Proof. This is [Cor15, Theorem 2.8]. Let us point out that the proof in [Cor15], published in 2015, has been slightly corrected in the 2017 arXiv version at https://arxiv.org/abs/1306.4194. Let us also mention that the main case, that of G totally disconnected, was essentially due to Trofimov. More precisely, he obtained the case of cocompact subgroups of automorphisms of locally finite connected graphs. The totally disconnected case follows by the 'Cayley–Abels graph' construction [CH16, Proposition 2.E.9]. □

Theorem 2.5. *If A is compactly generated, then $W(M)$ is closed for every LC A-module M, and every compact subset of $W(M)$ is contained in a compact submodule.*

Proof. Let N be the A-submodule generated by some compact subset of M with nonempty interior. Thus N is an open compactly generated submodule of M. So $N \rtimes A$ is a compactly generated LC group, and hence $W_{\mathrm{grp}}(N \rtimes A)$ is closed by Theorem 2.4. We see directly from the definition that $W_{\mathrm{grp}}(N \rtimes A) \cap N$ equals $W(N)$. Thus $W(N)$ is closed in N. It also follows from the definition that $W(M) \cap N = W(N)$. Thus the subgroup $W(M)$ is locally closed, and hence it is closed.

For the additional statement, let K be a compact subset of $W(M)$. Above, we can choose N to contain K. Then the above reference also ensures that K is contained in a compact normal subgroup L of $N \rtimes A$. Then $L \cap N$ is a compact submodule of M containing K. □

2.4 Canonical filtration with sheer subquotient

Theorem 2.6. *We have the inclusions $0 \subseteq W^\flat(M) \subseteq \Omega^\sharp(M) \subseteq M$. If A is compactly generated, then $W^\flat(M)$ is compact parafinite, $M/\Omega^\sharp(M)$ is purely discrete, and the subquotient $\Omega^\sharp(M)/W^\flat(M)$ is sheer.*

(See Example 7.3 for a counterexample when A is not assumed to be compactly generated.)

Proof. We actually check the stronger facts that $W^\flat(M) \subseteq \Omega(M)$ and $W(M) \subseteq \Omega^\sharp(M)$. If N_1 is a compact parafinite submodule and N_2 is an open submodule, the projection of N_1 on M/N_2 is compact discrete, hence finite; since N_1 is parafinite, it is trivial. This shows that $N_1 \subseteq N_2$. Since this holds for all N_1, N_2, we deduce the inclusion $W^\flat(M) \subseteq \Omega^\sharp(M)$. (The other inclusion is similar.)

Now assume that A is compactly generated, and let us show that the reduced polycompact radical is compact for every LC A-module. By Pontryagin duality, openness of the augmented codiscrete radical follows for every LC A-module.

Define N as the closure of $W^\flat(M)$. Let K be the closure of the submodule generated by some compact subset of N with nonempty interior. By Proposition 2.5, K is compact; by construction, K is open in N. Since N is generated by its compact parafinite submodules, this is also true for the discrete module N/K, which is thus generated by its finite parafinite submodules. Since a finite parafinite module is trivial, this means that $N = K$. Thus N is compact, and being topologically generated by parafinite submodules, it is parafinite. Thus $N = W^\flat(M)$. Hence $W^\flat(M)$ is compact; it is clearly parafinite. By Pontryagin duality (applied to $\hat{M}$), we deduce that $\Omega^\sharp(M)$ is open, with purely discrete quotient.

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Given that $W^\flat(M)$ is compact parafinite and the extension of two compact parafinite modules is compact parafinite, we see that $M/W^\flat(M)$ is lower-sheer, and hence so is $\Omega^\sharp(M)/W^\flat(M)$. By Pontryagin duality, the latter is therefore also upper-sheer. □

2.5 Extensions of sheer modules

PROPOSITION 2.7. *An extension of two sheer (respectively, lower-sheer, upper-sheer) modules is also sheer (respectively, lower-sheer, upper-sheer).*

Proof. Upper-sheer means that every homomorphism to a purely discrete module is zero, and this clearly passes to extensions. The lower-sheer case follows by Pontryagin duality. The sheer case follows from the other two cases. □

PROPOSITION 2.8. *Suppose that A is compactly generated. The submodule $\Omega^\sharp(M)$ is the largest upper-sheer submodule of M, and $\Omega^\sharp(\Omega^\sharp(M)) = \Omega^\sharp(M)$.*

The quotient $M/W^\flat(M)$ is the largest lower-sheer quotient module of M, and $W^\flat(M/W^\flat(M)) = \{0\}$.

Proof. Let N be an upper-sheer closed submodule of M. Then the image of N into every purely discrete quotient of M is zero, so $N \subseteq \Omega^\sharp(M)$. Then $M/\Omega^\sharp(\Omega^\sharp(M))$ is an extension of $\Omega^\sharp(M)/\Omega^\sharp(\Omega^\sharp(M))$ by $M/\Omega^\sharp(M)$, both of which are purely discrete (by Theorem 2.6), and hence is purely discrete. So $\Omega^\sharp(\Omega^\sharp(M) \subseteq \Omega^\sharp(M)$, whence the two are equal, which means that $\Omega^\sharp(M)$ itself is upper-sheer.

The second statement follows by Pontryagin duality. □

PROPOSITION 2.9. *Suppose that A is compactly generated. Let M be an LC module and N a submodule. Then $\Omega^\sharp(N) \subseteq \Omega^\sharp(M)$.*

Proof. By Proposition 2.8, $\Omega^\sharp(N)$ is upper-sheer. Hence the image of $\Omega^\sharp(N)$ in the purely discrete module $M/\Omega^\sharp(M)$ is zero, which means that $\Omega^\sharp(N) \subseteq \Omega^\sharp(M)$. □

2.6 Structure of distal modules

PROPOSITION 2.10 (Conze and Guivarc'h [CG74]). *Let G be a group. Let V be a distal Euclidean G-module. If V is irreducible (as real representation), then the image of G in $GL(V)$ has a compact closure. Hence, in general, there is an upper-triangular decomposition such that the image of G in each diagonal block has a compact closure.*

The following consequence is well known.

COROLLARY 2.11. *Let G be a group and V a distal Euclidean G-module. Then no G-orbit $\neq \{0\}$ accumulates at zero.*

Proof. To obtain a contradiction, let $X \neq \{0\}$ be an orbit accumulating at zero. Let W be the span of the given orbit, and W' an irreducible quotient of W. Then the image X' of X in W' is also an orbit accumulating at zero, and spans $W' \neq \{0\}$. Since G has a compact closure in $GL(W')$ by Proposition 2.10, a nonzero orbit cannot accumulate at zero. This is a contradiction. □

2.7 No homomorphism between modules of different kinds

Uniqueness in Theorem 1.6 is based on the following lemma.

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LEMMA 2.12. *Let C be a polycontractible A-module, E an amorphic one,* and *V a distal Euclidean one. Then every homomorphism between any two distinct elements of $\{C, E, V\}$ is zero.*

Proof. It is no restriction to assume that C is contractible.

$\mathrm{Hom}(V, C) = 0$. Let $f : V \to C$ be a homomorphism, and H the closure of the image; it is also contractible. Also, H is connected. So, by the general structure of LCA groups, it has a maximal compact subgroup W and the quotient H/W is isomorphic to $\mathbf{R}^k$ for some k. Then H/W is a quotient module of V and hence is distal as well; since it is contractible we deduce $H/W = \{0\}$, so $H = W$ is compact. Also, W is compact and contractible, hence trivial. So $H = \{0\}$.

$\mathrm{Hom}(V, E) = 0$. This is clear since V is connected and amorphic, which implies it is totally disconnected.

$\mathrm{Hom}(C, V) = 0$, $\mathrm{Hom}(C, E) = 0$. This is clear since no A-orbit in V or E accumulates at zero (for V, this is Corollary 2.11).

$\mathrm{Hom}(E, V) = 0$, $\mathrm{Hom}(E, C) = 0$. This is clear since V and C have no nonzero compact submodule, and E is the union of its compact submodules. □

2.8 Amorphic modules

LEMMA 2.13. *Every closed submodule or quotient of an amorphic module is amorphic.*

Proof. Since being amorphic is invariant under Pontryagin duality, it is enough to consider submodules. Namely, let M be amorphic, Ω a compact open submodule, and N a closed submodule. Since Ω is profinite, so is $N \cap \Omega$, and the discrete quotient $N/(N \cap \Omega)$ being a submodule of the locally finite discrete module M/Ω, it is locally finite as well. Hence N is amorphic. □

2.9 Facts on LCA groups

For a topological group G, we denote by G° the connected component of the unit element. It is said to be *elliptic* if every compact subset is contained in a compact subgroup.

An A-module M can also view it as module over the trivial group $\{0\}$ and we can thus write $W_A(M)$ and $W_{\{0\}}(M)$ to specify the module structure. We always have $W_A(M) \subseteq W_{\{0\}}(M)$, with equality if $W_{\{0\}}(M)$ (which holds, for instance, when M is connected).

Similarly, with self-explanatory notation, we have $W^\flat_{\{0\}}(M) \subseteq W^\flat_A(M)$ and $\Omega^\sharp_A(M) \subseteq \Omega^\sharp_{\{0\}}(M)$. In particular, every sheer A-module is sheer as an LCA group.

LEMMA 2.14. *Let M be an LCA group. The natural homomorphism $M^\circ \times W_{\{0\}}(M) \to M$ defined by $(x, y) \mapsto x - y$, is proper with open image. Its kernel is the diagonal of $W(M^\circ)$. Its discrete cokernel is a torsion-free abelian group, and is the largest such quotient for M.*

Proof. Let us first check that it has an open image, that is, that $M^\circ + W(M)$ is open. This property is trivial when M is connected. It is inherited from open subgroups, and from quotients by compact subgroups. Since M is compact-by-Lie, it is therefore inherited by M.

For properness, first assume that M is σ-compact. The kernel is compact (namely, the diagonal of $W(M^\circ)$) and its image is open by the previous paragraph, hence is closed. So properness follows. In general, properness follows from being proper on every σ-compact open subgroup.

The second assertion (on the kernel) is straightforward.

Let us now prove the third assertion; we can suppose that $W(M^\circ) = \{0\}$. First assume that M/M° is elliptic, in which case we have to show $M = M^\circ + W(M)$. Modding out by a compact

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subgroup, we can suppose that M is Lie. Then the divisible group M° has a complement in M as abstract group. This complement is a discrete subgroup, hence is a complement, and is actually locally finite, thus equals $W(M)$.

Now, let us prove the second assertion in general. If by contradiction $M/M^\circ W(M)$ is not torsion-free, it has a nontrivial finite subgroup. Let N be its inverse in M. Then $N/N^\circ W(N)$ is finite and nontrivial. So N is connected-by-elliptic, implying, by the previous paragraph, $N = N^\circ W(N)$, contradiction.

Finally, clearly every homomorphism to a torsion-free discrete group is trivial on both M° and $W(M)$, whence the 'largest' assertion. □

COROLLARY 2.15. *Let M be a sheer LCA group (i.e., $W(M^\circ) = \{0\}$ and M has no nonzero torsion-free discrete quotient). Then M is the topological direct product of the Euclidean group M° and the totally disconnected group $W_{\{0\}}(M)$. In addition, this is the only decomposition of M as a topological direct product of connected and totally disconnected LCA groups. In particular, whenever M is also an A-module, this is also a decomposition as* an *A-module.*

Proof. This is a particular case of Lemma 2.14, in this case the map $M^\circ \times W_{\{0\}}(M) \to M$ of Lemma 2.14 is an isomorphism: injectivity holding because the kernel $W(M^\circ)$ is trivial, and surjectivity holding because M is upper-sheer. If $M = C \times T$ is another decomposition (C connected and T totally disconnected), clearly $C = M^\circ$. Also T being totally disconnected and upper-sheer, we have $T = W_{\{0\}}(T)$, so $T \subseteq W_{\{0\}}(M)$ and then $M = C + T$ ensures $T = W_{\{0\}}(M)$. □

2.10 Cocompact subgroups

The following is classical; see, for instance, [CH16, Proposition 4.C.11(2)].

PROPOSITION 2.16. *Let G be an LC group and H a closed cocompact subgroup. Then G is compactly generated if and only H is compactly generated.*

Although we only the next proposition in the abelian case, we write it, for reference, in more generality.

PROPOSITION 2.17. *Let G be a compactly generated LC group. Then G has a finitely generated (f.g.) subgroup with cocompact closure.*

Proof.

(1) If G is connected and Lie, then it has a f.g. dense subgroup.

(2) If G is totally disconnected, let X be Cayley–Abels for G, that is, a nonempty connected graph of finite valency with a vertex-transitive action of G with compact vertex stabilizers. It exists by [CH16, Proposition 2.E.9]. Fix a vertex x_0 of X and let S be a finite subset of G such that the neighbors of x_0 are the sx_0 for $s \in S$. Let Γ be the subgroup generated by S. Then Γ acts transitively on X. Indeed, if $x \in \Gamma x_0$, writing $x = \gamma x_0$, each neighbor of x has the form $\gamma s x_0$ for some $s \in S$, and hence also belongs to Γx_0. If K is the stabilizer of x_0, we then have $G = \Gamma K$, so $\overline{\Gamma}$ is cocompact.

(3) If $W(G^\circ)$ is trivial, then G° is Lie and hence has a dense f.g. subgroup, and since G/G° has a f.g. subgroup with cocompact closure, so does G.

(4) In general, the previous case applies to $G/W(G^\circ)$. Lifting a f.g. subgroup of the latter, we obtain a f.g. subgroup Γ such that $\overline{\Gamma} W(G^\circ)$ is cocompact. Thus $\overline{\Gamma}$ is cocompact. □

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3. One automorphism

We recall that an automorphism ϕ of a locally compact group G is *contracting* if $\lim_{n\to\infty}\phi(x)=1$ for every $x\in G$. In this case, the convergence is uniform on compact subsets of G [Sie86, Lemma 1.4(iv)]. The automorphism is said to be *compacting* if there exists a *compacting subset*, that is, a compact subset $\Omega\subseteq G$ with $\alpha(\Omega)\subseteq\Omega$ and $G=\bigcup_{n\geq 0}\alpha^{-n}(\Omega)$. Note that such Ω has nonempty interior (by Baire's theorem). If M is totally disconnected, Ω can be chosen to be a compact open subgroup. If α is compacting with compacting subset Ω, the intersection $\bigcap_{n\geq 0}\alpha^n(\Omega)$ is a compact subgroup, called the limit subgroup of α; it is independent of the choice of Ω.

Observe that α is contracting if and only if every compact neighborhood of 1 is a compacting subset, if and only α is compacting with limit subgroup reduced to $\{1\}$.

For $\alpha\in A$, we say that M is *α-contractive* (respectively, *α-compactive*) if α acts on M as a contracting (respectively, compacting) automorphism. If M is α-compactive with limit subgroup L, then L is a compact A-submodule and M/L is α-contractive.

We say that M is *contractible* if there exists $\alpha\in A$ such that M is α-contractive.

3.1 Willis' theory

Let G be a totally disconnected locally compact group and α a topological automorphism of G.

If V is a compact open subgroup of G, define $V^{\pm}=\bigcap_{n\geq 0}\alpha^{\pm n}(V)$; these are compact subgroups of V. Note that $\alpha^{\pm 1}(V^{\pm})\supset V^{\pm}$.

Theorem 3.1 (Willis, Theorem 1 in [Wil94]). *For every G and α there exists a compact open subgroup V such that:*

(1) $V=V^+V^-=V^-V^+$;
(2) *the subgroups $V^{++}=\bigcup_{n\geq 0}\alpha^n(V^+)$ and $V^-=\bigcup_{n\geq 0}\alpha^{-n}(V^-)$ are closed subgroups of G. In particular, V^{++} is α^{-1}-compactive and V^- is α-compactive.*

The theorem is stated only for conjugation automorphisms in [Wil94] but, as observed in subsequent papers, the general statement follows by considering the semidirect product $G\rtimes\langle\alpha\rangle$.

3.2 Closedness of the set of contracted points

Now and in what follows, when we consider $\mathbf{Z}[\alpha^{\pm 1}]$-module, it is understood that the acting group is the infinite cyclic group $A=\langle\alpha\rangle$.

Let M be a locally compact $\mathbf{Z}[\alpha^{\pm 1}]$-module. Denote by $C_\alpha(M)$ the set of elements x in M such that $\lim_{n\to+\infty}\alpha^n(x)=0$. This is a possibly nonclosed submodule (see Remark 3.3).

Proposition 3.2. *If M is lower-sheer as $\mathbf{Z}[\alpha^{\pm 1}]$-module (i.e., $W^\flat_{\mathbf{Z}[\alpha^{\pm 1}]}(M)=\{0\}$), then $C_\alpha(M)$ is closed and is an α-contractive module.*

Proof. We can assume that $C_\alpha(M)$ is dense and we have to show that M is α-contractive.

Let V be a compact open subgroup, write $V^-=\bigcap_{n\geq 0}\alpha^{-n}(V)$ (so $\alpha(V^-)\subseteq V^-$) and $V^-=\bigcup_{n\geq 0}\alpha^{-n}(V^-)$. For each $x\in C_\alpha(M)$, the element $\alpha^n(x)$ belongs to V for n large enough, say $n\geq n_0$. Hence $\alpha^{n_0}(x)\in V^-$, and therefore $x\in V^-$. This proves that $C_\alpha(M)\subseteq V^-$.

Willis's theorem (Theorem 3.1) ensures that there exists V such that V^- is closed; we assume V has this property. Then the density of $C_\alpha(M)$ ensures that $V^-=M$. Note that by Baire's theorem, this implies that V^- has nonempty interior, hence is open.

Define $F=\bigcap_{n\geq 0}\alpha^n(V^-)$. Then F is a compact submodule and α induces a contracting automorphism of M/F. To conclude, it remains to show that $F=\{0\}$. Since $W^\flat(M)=0$,

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to show that $F = \{0\}$ it is enough to check that F has no proper open submodule. Otherwise, modding out, we can suppose that F is finite, and let us deduce that $F = \{0\}$.

Let Ω be a compact open subgroup of M with $\Omega \cap F = \{0\}$. Since α is contracting modulo F there exists a compact open subgroup K of Ω with $\alpha(K) \subseteq K + F$, and K can be chosen arbitrary small. In particular, K can be chosen to be contained in $\alpha^{-1}(\Omega)$. So $\alpha(K) \subseteq (K + F) \cap \Omega = K$. Write $F' = F \smallsetminus \{0\}$. Since $\alpha(F') = F'$, we deduce $\alpha(K + F') \subseteq K + F'$. It follows that $K + F' \cap C_\alpha(M) = \emptyset$. By density of $C_\alpha(M)$, we deduce that the open subset $K + F'$ is empty, and hence $F = \{0\}$. □

Remark 3.3. Without lower-sheerness of M, closedness of $C_\alpha(M)$ can fail, even when M is totally disconnected or connected. Let B be a nontrivial finite abelian group, and $M = B^{\mathbf{Z}}$, endowed with the shift automorphism $f \mapsto (n \mapsto f(n+1))$. Then $C_\alpha(M)$ is the set of eventually (at $+\infty$) zero sequences, and thus is not closed. Also, the automorphism of the 2-torus $\mathbf{R}^2/\mathbf{Z}^2$ induced by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ has a nonclosed contraction subgroup. These examples are classical.

LEMMA 3.4. *Let M be a lower-sheer locally compact $\mathbf{Z}[\alpha^{\pm 1}]$-module. Suppose that M is α-compactive. Then M is sheer, $M = C_\alpha(M) \times W(M)$ and $W(M)$ is compact.*

Proof. If N is a discrete quotient of M, then N is also α-compactive, and clearly this implies that N is a locally finite module. Thus M is upper-sheer, and hence sheer. In particular, it is sheer as LCA group (as mentioned at the beginning of § 2.9).

By Theorem 2.5, $W(M)$ is closed, and has a compact open submodule Ω. Then $W(M)/\Omega$ is α-compactive and discrete, hence finite. Hence $W(M)$ is compact.

Now let us prove the decomposition statement. By Corollary 2.15, we can suppose that M is either Euclidean or totally disconnected. The Euclidean case is clear: in this case, since M has no nonzero compact submodule, it is α-contractive.

Assume M is totally disconnected. By Proposition 3.2, $C_\alpha(M)$ is closed. Consider the natural homomorphism $f : C_\alpha(M) \times W(M) \to M$. It is proper, by compactness of $W(M)$. Its kernel is therefore compact, and hence its projection to $C_\alpha(M)$ is a compact α-contractive module, hence trivial. Thus the kernel is contained in $W(M)$ on which the homomorphism is clearly injective. Finally, surjectivity of f is given by [BW04, Corollary 3.17]. □

3.3 A lemma on contraction minus identity

For a $\mathbf{Z}[\alpha^{\pm 1}]$-module, recall that it is understood that $A = \langle \alpha \rangle$. To be polycontractible just means to be the direct sum of an α-contractive and an α^{-1}-contractive module.

LEMMA 3.5. *Let M be a polycontractible locally compact $\mathbf{Z}[\alpha^{\pm 1}]$-module. Then multiplication by $(\alpha - 1)$ is an automorphism of the topological module M.*

Proof. It is enough to assume that M is either α-contractive or α^{-1}-contractive. If the α-contractive case works, the other case follows since $\alpha^{-1} - 1 = -\alpha^{-1}(\alpha - 1)$. So assume that M is α-contractive.

By Siebert's theorem [Sie86, Proposition 4.2] (or Corollary 2.15), we can suppose that M is either Euclidean or totally disconnected. The Euclidean case being straightforward from linear algebra, assume that M is totally disconnected.

Let Ω be a compact open $\mathbf{Z}[\alpha]$-submodule. Observe that for every $v \in M$, we have

$$v \in \Omega \quad \Leftrightarrow \quad \alpha v - v \in \Omega. \tag{3.1}$$

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The implication $\Rightarrow$ is clear. For the sake of a contradiction, let v be a counterexample to $\Leftarrow$, namely $\alpha v - v \in \Omega$ and $v \notin \Omega$. Considering n maximal such that $\alpha^n v \notin \Omega$ (which exists by contractivity), and define $w = \alpha^n v$. Then $w \notin \Omega$ and $\alpha w \in \Omega$. Since $\alpha v - v \in \Omega$ and $\alpha\Omega \subseteq \Omega$, we have $\alpha^n(\alpha v - v) = \alpha w - w \in \Omega$. Hence $w \in \Omega$, a contradiction.

Consider the ultrametric on M defined by $d(x,y) = \exp(\inf\{n : \alpha^n(x-y) \in \Omega\})$. It is complete, addition-invariant and defines the topology of M. Moreover, by (3.1), multiplication by $\alpha - 1$ is an isometric embedding of (M,d) into itself; in particular its image is closed, by completeness. It remains to check that it is surjective: indeed, if we mod out by the image, we obtain a module on which α is contracting but acts as the identity, so we obtain the trivial module. □

3.4 Decomposition under a single automorphism

Lemma 3.6. *Let M be a sheer LC $\mathbf{Z}[\alpha^{\pm 1}]$-module. Then M decomposes as a topological A-module:*

$$M = C_\alpha(M) \oplus C_{\alpha^{-1}}(M) \oplus W(M) \oplus \mathcal{E}(M),$$

where $\mathcal{E}(M)$ is the largest distal subspace of the Euclidean A-module M°.

Proof. For a direct product of an Euclidean and a totally disconnected module, all given subspaces split accordingly. Hence Corollary 2.15 reduces to the cases when M is either Euclidean or totally disconnected (to apply the corollary, note that M being a sheer A-module, it is a sheer LCA group).

The case when M is Euclidean is just plain linear algebra: the subspaces in the decomposition are then as follows. The sum of characteristic subspaces of α with respect to complex eigenvalues of modulus greater than 1 (respectively, less than 1, equal to 1) is $C_\alpha(M)$ (respectively, $C_{\alpha^{-1}}(M)$, $\mathcal{E}(M)$), and $W(M) = \{0\}$.

Now assume that M is totally disconnected. We then have to prove that $M = C_\alpha(M) \oplus C_{\alpha^{-1}}(M) \oplus W(M)$. By Willis's theorem (Theorem 3.1), there are closed submodules N_+, N_- such that N_+ is α-compactive, N_- is α^{-1}-compactive, and $N_+ + N_-$ is open. Therefore, by Lemma 3.4, there are closed submodules C_+, C_-, and a compact submodule Ω such that C_+ is α-contractive, C_- is α^{-1}-contractive, Ω is compact, and $C_+ + C_- + \Omega$ is open. Note that $C_\pm = C_\pm(M)$, and $\Omega \subseteq W(M)$.

Consider the sum homomorphism $f : C_\alpha(M) \oplus C_{\alpha^{-1}}(M) \oplus W(M) \to M$. Let K be its kernel. Let $s = (c_+, c_-, z)$ be an element of K in this decomposition. Then $c_+ + c_- + z = 0$ in M. Thus, for every n, $0 = \alpha^n(c_+) + \alpha^n(c_-) + \alpha^n(z)$. Then, for $n \to \infty$, $\alpha^n(c_+ + z)$ is bounded, while $\alpha^n(c_-)$ tends to infinity unless $c_- = 0$. This forces $c_- = 0$. Similarly, letting n tend to $-\infty$ ensures $c_+ = 0$. Hence $c_+ = c_- = 0$, so $z = -c_+ - c_- = 0$ showing that $K = \{0\}$, that is, f is injective. All this argument showing that f is injective with open image is true for some σ-compact submodule of M. Hence f is proper.

It remains to see that f is surjective. Write $T = C_\alpha(M) + C_{\alpha^{-1}}(M) + W(M)$, which is the image of f and which we already proved to be open; let us show that $M = T$. Since M is sheer, the discrete quotient M/T is locally finite as an A-module.

Take $x \in M$. Since the discrete module M/T is a locally finite module (because M is sheer), for some $k \geq 1$ we have $\alpha^k x = x$ in M/T, that is,$(\alpha^k - 1)x \in T$. Write $P = C_\alpha(M) \oplus C_{\alpha^{-1}}(M)$, so that $T = P \oplus W(M)$. We can decompose $(\alpha^k - 1)x \in T$ according to this decomposition. By Lemma 3.5, we can write $(\alpha^k - 1)x = (\alpha^k - 1)u + v$ with $u \in P$ and $v \in W(T)$. Set $y = x - u$, so that $(\alpha^k - 1)y = v$. Modulo the closed submodule N generated by v (which is compact), we have $\alpha^k y = y$, and therefore, modulo N, the submodule generated by y is actually generated, as a

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group, by the finite set $\{y, \dots, \alpha^{k-1}y\}$. Since M is elliptic as an LCA group, we deduce that the closed submodule of M generated by y is compact, and therefore $y \in W(M)$. So $x = u + y \in P + W(M) \subseteq T$. □

4. Structure of sheer modules

4.1 Decomposition of sheer modules

THEOREM 4.1. *Suppose that A is compactly generated. Every sheer module has a unique decomposition $P \oplus E \oplus V$, with P polycontractible, E amorphic, V distal Euclidean.*

(See Example 7.5 for a counterexample when A is not assumed to be compactly generated.)

We need to keep track of the construction in the proof. Therefore we introduce some definitions. For a sheer A-module M and $\alpha \in A$, define $C_\alpha^{\leq 0}(M) = C_{\alpha^{-1}}(M) \oplus W_{\mathbf{Z}[\alpha^{\pm 1}]}(M) \oplus \mathcal{E}_{\mathbf{Z}[\alpha^{\pm 1}]}(M)$. So Lemma 3.6 says that $M = C_\alpha(M) \oplus C_\alpha^{\leq 0}(M)$. If M is a sheer A-module and $\alpha \in A$, this is an A-module decomposition.

Now let $(\alpha_i)_{i \geq 1}$ be a sequence in A. For a sheer LC A-module M, define $B_0 = M$, and by induction $C_i = C_{\alpha_i}(B_{i-1})$ and $B_i = C_{\alpha_i}^{\leq 0}(B_{i-1})$, so that $B_{i-1} = C_i \oplus B_i$. Then for each n, we have $M = C_1 \oplus \cdots \oplus C_n \oplus B_n$.

LEMMA 4.2. *Suppose that for some n, writing $S = \{\alpha_i : i \leq n\}$, we have the following: $S = S^{-1}$, and $\overline{\langle S \rangle}$ is cocompact in A (this exists as soon as A is compactly generated). Then $W(B_n)$ is an amorphic A-module and $\mathcal{E}(B_n) = B_n^\circ$ is a distal Euclidean A-module.*

Proof. Using Corollary 2.15, we can assume that M is either totally disconnected or Euclidean (since the decomposition is preserved at each step).

We start with the easier Euclidean case: we have to prove that B_n is distal. Complexify and triangulate the A-action on the real vector space B_n. Consider a diagonal coefficient, given by a continuous homomorphism $\omega : A \to \mathbf{C}^*$. Then by construction, for each $\alpha \in S$ we have $|\omega(\alpha)| \leq 1$. Since $S = S^{-1}$, we deduce that $S \subseteq \mathrm{Ker}(|\omega|)$. Hence the homomorphism $\log|\omega| : A \to \mathbf{R}$ factors through the compact group $A/\overline{\langle S \rangle}$, and hence it is zero. Thus $|\omega| = 1$, that is, B_n is distal.

Now suppose that M is totally disconnected. For every $\alpha \in S$, the module B_n is amorphic over $\mathbf{Z}[\alpha^{\pm 1}]$. For $T \subseteq S$ symmetric, we prove by induction on $|T|$ that there is a compact open $\langle T \rangle$-submodule; the case T empty is clear. More precisely, suppose that T has maximal cardinal such that this holds, with a compact open $\langle T \rangle$-submodule Ω, and by contradiction choose $\alpha \in S \smallsetminus T$. Since B_n is amorphic over $\mathbf{Z}[\alpha^{\pm 1}]$, the union $\bigcup_{n \in \mathbf{Z}} \alpha^n T$ has a compact closure. Hence the closure of the additive subgroup by this union is compact and invariant under $T \cup \{s, -s\}$, contradiction.

Thus M is an amorphic $\langle S \rangle$-module. Let Ω be a compact open $\langle S \rangle$-submodule. Its stabilizer in A is open and contains the subgroup $\langle S \rangle$ with cocompact closure, and hence is a finite-index subgroup B of A. Therefore the sum $\sum_{a \in A} a\Omega$ is a finite sum, and thus is a compact open A-submodule.

The existence of n and (α_i) is ensured by Proposition 2.17: find a finite family $(\alpha_1, \dots, \alpha_m)$ generating a dense subgroup of a cocompact subgroup. Then set $n = 2m$, $\alpha_{m+i} = \alpha_i^{-1}$ for $1 \leq i \leq m$, and $\alpha_i = 1$ for $i > 2m$. □

Proof of Theorem 4.1. Lemma 2.12 ensures uniqueness: if $P_1 \oplus E_1 \oplus V_1$ is another decomposition, considering projections $P_1 \to E$, $P_1 \to V$, the lemma then implies these are zero, so $P_1 \subseteq P$ and similarly $E_1 \subseteq E$, $V_1 \subseteq V$, whence equality holds.

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For the existence, apply Lemma 4.2 (for a suitable sequence (α_i), which exists since A is compactly generated). Then, in the notation preceding the lemma, $M = C_1 \oplus \cdots \oplus C_n \oplus B_n$ with each C_i contractible, and the lemma says that B_n is the direct sum of an amorphic and a distal Euclidean module. □

COROLLARY 4.3. *Let A be compactly generated. Then for every compactly generated LC A-module M, the submodule $W(M)$ is compact. That is, M has a maximal compact submodule.*

Proof. Modding out by the compact submodule $W^\flat(M)$, we can suppose that M is lower-sheer. By Theorem 4.1, write $N = \Omega^\sharp(M) = P \oplus E \oplus V$ with $E = W(N)$, P polycontractible and V distal Euclidean. Let E' be a compact open submodule of E. Then $Q = M/(P \oplus E' \oplus V)$ is a discrete A-module, with the locally finite submodule E/E'.

Let B be a cocompact lattice in A. Then since $Q \rtimes A$ is a compactly generated LC group, so is its cocompact lattice $Q \rtimes B$ (see Proposition 2.16). Hence Q is a finitely generated B-module. By noetherianity, we deduce that E/E' is also a finitely generated B-module, and since it is also locally finite as a B-module, it is finite. Thus E is compact.

Since the projection of $W(M)$ in the purely discrete group M/N is trivial, we deduce that $W(M) = W(N) = E$ is compact. □

COROLLARY 4.4. *Let $M = P \oplus E \oplus V$ be a sheer module with P polycontractible, E amorphic, and V Euclidean distal. Let M_1 be a sheer submodule of M, and write $P_1 = P \cap M_1$, $E_1 = E \cap M_1$, $V_1 = V \cap M_1$. Then P_1 is polycontractible, E_1 is amorphic, V_1 is distal Euclidean, and $N = P_1 \oplus E_1 \oplus V_1$.*

Proof. Let $N = P_2 \oplus E_2 \oplus V_2$ be the decomposition provided by Theorem 4.1. By Lemma 2.12, $P_2 \subseteq P$, and similarly for E_2 and V_2. So $P_2 \subset P_1$, etc. Thus $N = P_2 \oplus E_2 \oplus V_2 \subseteq P_1 \oplus E_1 \oplus V_1 N$. Thus all inclusions are equalities and the corollary is proved. □

4.2 Sheer submodules and quotients

THEOREM 4.5. *Suppose that A is compactly generated. Let M be a sheer A-module and N a closed submodule. Then N is sheer if and only if M/N is sheer.*

From our understanding, this is not a plain consequence of the definition. The proof uses Theorem 4.1, and, more specifically, its more explicit form given in Lemma 4.2, which outputs a decomposition with some specific properties for the module and its submodule simultaneously. Indeed, one main issue is that a quotient of a polycontractible module need not be polycontractible. However, this is true for a quotient of a contractible module, and we need to be able to exploit this.

Proof. Because of Pontryagin duality, it is enough to prove one direction. Suppose that N is sheer.

We apply Lemma 4.2 simultaneously to M and N. Namely, $M = C_1 \oplus \cdots \oplus C_n \oplus B_n$, $B_n = W(B_n) \oplus B_n^\circ$ with $W(B_n) = W(M)$ amorphic and B_n° distal Euclidean, and $C_i = C_{\alpha_i}(C_i \oplus \cdots \oplus C_n \oplus B_n)$. Consider the same decomposition for N, namely $N = C_1' \oplus \cdots \oplus C_n' \oplus B_n'$. Then from the definition it follows that $C_i' = C_i \cap N$. Also $W(B_n') \subseteq W(M) = B_n$ and $(B_n')^\circ \subseteq B_n^\circ$. So $M/N = C_1'' \oplus \cdots \oplus C_n'' \oplus B_n''$ with $C_i'' = C_i/C_i'$ contracted by α_i and $B_n'' = B_n/B_n'$. Clearly $B_n^\circ/(B_n')^\circ$ is distal Euclidean, and $W(B_n)/W(B_n')$, being a quotient of the amorphic module $W(B_n)$, has to be amorphic as well (Lemma 2.13). Hence M/N is sheer. □

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4.3 Sheer core and envelope

THEOREM 4.6. *Suppose that A is compactly generated. Let M be a fixed sheer LC A-module. For a submodule N of M, write $N^{\boxminus} = \Omega^{\sharp}(N)$ (the* sheer core *of N) and let $N^{\boxplus}$ (the* sheer envelope *of N in M) be the inverse image of $W^{\flat}(M/N) \subseteq M/N$ in M. Then $N^{\boxminus} \subseteq N \subseteq N^{\boxplus}$, $N/N^{\boxminus}$ is purely discrete, and $N^{\boxplus}/N$ is compact parafinite. The submodule $N^{\boxminus}$ is sheer and maximal for this property among submodules of N, and the submodule $N^{\boxplus}$ is sheer and minimal for this property among submodules containing N. If $N_1 \subseteq N_2$ then $N_1^{\boxminus} \subseteq N_2^{\boxminus}$ and $N_1^{\boxplus} \subseteq N_2^{\boxplus}$.*

Remark 4.7. While $N \mapsto N^{\boxminus}$ is intrinsic to N, the assignment $N \mapsto N^{\boxplus}$ definitely depends on M. This may sound strange in view of Pontryagin duality, but rather, the dual statement is that the assignment $M/N \mapsto M/N^{\boxplus}$ only depends on M/N.

Proof of Theorem 4.6. It is enough to prove all assertions about the sheer core, since those about the sheer envelope follow by Pontryagin duality (using Theorem 4.5 in some cases).

The assertion $N^{\boxminus} \subseteq N$ is trivial. $N^{\boxminus}$ is sheer: it is lower-sheer as submodule of the sheer module M, and is upper-sheer by Proposition 2.8. That $N/N^{\boxminus}$ is purely discrete is part of Theorem 2.6. If $N_1 \subseteq N_2$, then by Proposition 2.8 applied successively in N_1 and N_2, the submodule $N_1^{\boxminus}$ is upper-sheer, and hence contained in $N_2^{\boxminus}$. □

4.4 Chains of submodules

We start with the following lemma. Exceptionally, we write it in the noncommutative setting, since the proof is not longer than the case of modules.

LEMMA 4.8. *Let M be a contractible LC A-module. Then there is a bound (depending only on M) on the length of chains of closed submodules of M. More generally, if G is an LC group with a contracting automorphism α, there is a bound on the length of chains of $\langle\alpha\rangle$-invariant subgroups.*

Proof. First case: G is totally disconnected. This case is actually covered by [GW10, Theorem 3.3]; let us provide a much shorter argument. Let α^{-1} multiply the Haar measure of G by n_G. Then n_G is a positive integer, namely, it equals the index $[\Omega : \alpha(\Omega)]$ for some/every compact open subgroup Ω of G such that $\alpha\Omega \subseteq \Omega$. Now let us check that if H is an $\langle\alpha\rangle$-invariant closed subgroup then $n_H \leq n_G$, and if $n_H = n_G$ then $H = G$. Then $n_G = [\Omega : \alpha(\Omega)]$ and $n_H = [\Omega \cap H : \alpha(\Omega \cap H)]$. The inequality immediately follows; if it is an equality we deduce that $\Omega = \alpha(\Omega)(\Omega \cap H)$. It follows that the image of Ω in G/H is a compact $\langle\alpha\rangle$-invariant subset. Since α acts as a contraction (to the base-point) on the coset space G/H, we deduce that this is a point, that is, that $\Omega \subseteq H$. Thus H is open, and since the discrete coset space G/H admits a contraction, we deduce that it is reduced to a point and thus $H = G$. The bound immediately follows (e.g., $\log_2(n_G)$ is such a bound).

Second case: G is connected, namely a d-dimensional simply connected nilpotent Lie group [Sie86, § 2]. Then since every contractible subspace is a real subspace, we have the bound d.

The general case then follows from the Siebert's decomposition [Sie86, § 4]. □

COROLLARY 4.9. *Let M be a polycontractible LC A-module. Then there is a bound (depending only on M) on the length of chains of* sheer *closed submodules of M.*

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Proof. Given Lemma 4.2, this follows from the contractible case, Lemma 4.8. □

Remark 4.10. Corollary 4.9 does not extend to general closed submodules. For instance, we have the bi-infinite sequence of lattices $3^n\mathbf{Z}[1/2]$ ($n \in \mathbf{Z}$) in $\mathbf{R} \times \mathbf{Q}_2$ (viewed as a $\mathbf{Z}[\alpha^{\pm 1}]$-module, α acting by multiplication by 2).

An LC A-module is *topologically characteristically simple* if $M \neq \{0\}$ and the only closed $\mathrm{Aut}(M)$-invariant closed submodules are $\{0\}$ and M ($\mathrm{Aut}(M)$ being the group of topological A-module automorphisms). It is *topologically simple* if $M \neq \{0\}$ and its only closed submodules are $\{0\}$ and M. Of course, topologically simple implies topologically characteristically simple.

THEOREM 4.11. *Suppose that A is compactly generated. Let M be a topologically characteristically simple LC A-module. Then exactly one of the following holds:*

(1) *M is finite;*
(2) *M is compact infinite;*
(3) *M is discrete infinite;*
(4) *M is Euclidean and contractible;*
(5) *M is Euclidean distal;*
(6) *M is contractible and totally disconnected; or*
(7) *M is amorphic, neither discrete nor compact. In this case, M is not topologically simple.*

Proof. By assumption, $M \neq \{0\}$. Clearly, M is either connected or totally disconnected.

If M is connected, let us conclude (we will not need the assumption on A). Then $W(M)$ is a compact characteristic subgroup, hence is either M or trivial. In the first case, M is compact (hence case (2)), and otherwise, M is Euclidean. If M is not distal (case (5)), let us check that M is contractible (case (4)). Indeed, this means that there is some α acting with an eigenvalue z of modulus not equal to 1. If z is real (respectively, nonreal), then $\mathrm{Ker}(\alpha - z)$ (respectively, $\mathrm{Ker}((\alpha - z)(\alpha - \bar{z}))$ is a nontrivial subspace, invariant under $\mathrm{Aut}(M)$, hence equal to M which is then contractible.

Now suppose that M is totally disconnected. Then $\Omega^\sharp(M)$ is either zero or M, and in the first case, Theorem 2.6 (which uses that A is compactly generated) ensures that M is purely discrete (case (3)). Similarly $W^\flat(M)$ is either zero or M, and in the second case, M is compact paracompact (case (2)). Otherwise, M is sheer. Using that A is compactly generated (and that M is totally disconnected), Theorem 4.1 then ensures that M is either contractible (case (6)) or amorphic.

If M is amorphic, then it is compact (case (2)), discrete (case (3)), or neither (case (7)). In the latter case, it has a proper open submodule that is infinite, so is not topologically simple. □

Remark 4.12. By a simple commutative algebra exercise, if M is discrete (case (1) or (3)), the kernel of $A \to \mathrm{Aut}(M)$ is an open subgroup B, and there is a prime ideal P of $\mathbf{Z}[A/B]$ (with A/B acting faithfully on $\mathbf{Z}[A/B]/P$, that is, $A/B \cap (1+P) = \{1\}$) such that M is a nonzero vector space over the field $\mathrm{Frac}(\mathbf{Z}[A/B]/P)$.

Case (7) is not much harder to describe: for some prime p, M is a $\mathbf{Z}_p$-module and either M is a finite-dimensional vector space over $\mathbf{Q}_p$, or, for some infinite set I, M has an open subgroup isomorphic to $\mathbf{Z}_p^I$ and pM is dense in M.

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If M is Euclidean, one can check that M is a real or complex vector space with a scalar action, which, in the complex case, is given by a continuous homomorphism $A \to \mathbf{C}^*$ whose image is not contained in $\mathbf{R}^*$.

THEOREM 4.13. *Suppose that A is compactly generated. Let M be a compactly generated A-module. Then:*

- *for every ascending sequence (M_n) of closed submodules of M, for large enough n, M_n is cocompact in $\overline{\bigcup_i M_i}$;*
- *there is a compact submodule W of M, namely $W = W(M)$, such that for every ascending sequence (M_n) of closed submodules of M, the sequence $(W + M_n)$ is stationary.*

Proof. Take the second assertion for granted: write $N = W + M_n$ for $n \geq n_0$. Then, for $n \geq n_0$, M_n is cocompact in $M_n + W = N$, and hence is also cocompact in the smaller submodule $\overline{\bigcup_i M_i}$. The first assertion is therefore proved.

Let us now prove the second assertion. By Corollary 4.3, $W(M)$ is compact. Hence, there is no restriction in assuming that $W(M)$ is trivial.

The module $\Omega^\sharp(M)$ is sheer. Let us use the notation $\boxplus$ (see Theorem 4.6) relative to the ambient module $\Omega^\sharp(M)$. Since $W(M) = \{0\}$, we can write $\Omega^\sharp(M) = P \oplus V$ with P polycontractible and V distal Euclidean. Hence, by Corollary 4.9, the sequence $(M_n \cap \Omega^\sharp(M))^{\boxplus}$ is stationary, say, equal to a certain submodule L. There is no restriction in assuming that $(M_n \cap \Omega^\sharp(M))^{\boxplus} = L$ for all n. By Theorem 4.6, $(M_n \cap \Omega^\sharp(M))^{\boxplus}$ contains $M_n \cap \Omega^\sharp(M)$ as a closed cocompact subgroup. Hence $L \cap M_n$ is closed cocompact in L. It follows that $M'_n = L + M_n$ is closed and contains M_n as a closed cocompact subgroup.

We claim that (M'_n) is stationary. We first observe that $M'_n \cap \Omega^\sharp(M) = L$. The inclusion $\subseteq$ is trivial. If $x \in M'_n \cap \Omega^\sharp(M)$, write it as $y + z$, $y \in L$, $z \in M_n$. Since $x \in \Omega^\sharp(M)$, we have $z \in \Omega^\sharp(M)$, so $z \in M_n \cap \Omega^\sharp(M) \subseteq L$ and hence $x \in L$. The observation ensures that $M'_n \cap \Omega^\sharp(M)$ is stationary. Then the claim follows because the projection of M'_n in the discrete quotient is ascending, hence stationary by noetherianity. □

COROLLARY 4.14. *Let A be compactly generated. Then every closed submodule of a compactly generated LC A-module is compactly generated.*

Proof. Let M be a compactly generated LC module. Let N be a closed submodule. Let N_0 be an open, compactly generated closed submodule of N. Since M is σ-compact, so is N, and hence the discrete quotient N/N_0 is countable. Hence there is an ascending sequence (N_n) of compactly generated submodules of N with N_0 as previously defined, and $\bigcup N_i = N$. By Theorem 4.13, for some n, N_n is cocompact in N. Since N_n is compactly generated and cocompact in N, it follows that N is compactly generated (Proposition 2.16). □

5. Fine structure of contractible modules

LEMMA 5.1. *Let M, M' be LC A-modules. Suppose $M \oplus M'$ is contractible. Then $\mathrm{Hom}_A(M, M')$ (the group of continuous homomorphisms) is locally compact for the compact-open topology.*

Proof. The assumption means that there exists $\alpha \in A$ acting as a contraction on both M and M'; we fix such an α. Since $\mathrm{Hom}_A(M, M')$ is a topological group, it is enough to find a compact neighborhood of 0.

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The zero homomorphism has a basis of neighborhoods of the form $V(K, K')$, where K is a compact symmetric subset of M, and K' is a compact neighborhood of 0 in M', where $V(K, K')$ is the set of continuous homomorphisms $M \to M'$ mapping K into K'.

We first observe that if K generates M as A-module, then $V(K, K')$ is compact has a compact closure V' in M'^M (the set of maps $M \to M'$ endowed with the pointwise convergence topology, ignoring the topology on M). Moreover, the closure of $V(K, K')$ consists of maps that are locally bounded (the image of every compact subset has compact closure). Indeed, if P is a compact subset of M, then $P \subseteq \bigcup_{i=1}^n \alpha_i K$ for some finite family $(\alpha_i)_{1 \leq i \leq n}$, and hence for every $f \in V(K, K')$ we have $f(P) \subseteq \bigcup_{i=1}^n \alpha_i K'$. (The existence of K only makes use of the fact that M is compactly generated, which follows from being contractible.)

Next, we show that every element f of the closure V' is continuous. Namely, we show that, assuming the existence of α as above, every locally bounded homomorphism f is continuous. Indeed, if (x_j) is a net tending to 0 in M, we can write $x_j = \alpha^{k_j}(y_j)$ with (y_j) bounded and $k_j \to +\infty$. So $f(x_j) = \alpha^{k_j} f(y_j) \in \alpha^{k_j}(f(K))$. Since $f(y_j)$ is bounded, while α^{k_j} converges uniformly on bounded subsets to the constant 1. Hence $f(x_j)$ tends to 0.

Thus $V(K, K')$ is compact in the pointwise convergence topology. This implies it is also compact in the compact-open topology, by a general argument only based on the Baire theorem; see [CG70]. □

Theorem 5.2. *Let M be a contractible A-module. Let R be a closed A-subalgebra of $\mathrm{End}_A(M)$. Then every R-submodule of M is closed and, in particular, M has finite length over R. The underlying ring R is an artinian ring (i.e., R has finite length as R-module).*

Proof. Let us first show the closedness statement. Here it is enough to suppose that R is the closure R_0 of the image of $\mathbf{Z}[A]$ and, in particular, is commutative.

Let M be a counterexample, of minimal length as a topological module, and N a nonclosed submodule. By minimality, N is dense, and does not contain any nonzero closed submodule of M.

If $x \in N \smallsetminus \{0\}$ then $N \cap \overline{R_0 x}$ is a nonclosed submodule of M and hence is dense; in other words $\overline{R_0 x} = M$ for every $x \in N \smallsetminus \{0\}$.

If by contradiction M is not simple as a topological module, let L be a closed submodule of M distinct from $\{0\}$ and M. Then by minimality $N \cap L = \{0\}$ and the projection of N on M/L is all of M/L; in particular, $M = N \oplus L$ as an abstract module. The latter holds for every nonzero proper submodule N and nonclosed proper submodule L. This shows that there is no proper inclusion:

- between any two nonclosed submodules of M;
- between any two nonzero proper closed submodules of M. In particular, M has length 2 as a topological module.

It follows that M also has length 2 as an abstract module over R_0. Since it is a faithful R_0-module, we deduce that R_0 is artinian. More precisely, the existence of a faithful module that is the direct sum of two simple modules implies that either R_0 is a field, or the product of two fields. If by contradiction R is the product of two fields, let π and $1 - \pi$ be its nontrivial idempotents. Then in the (then unique) direct product decomposition of M, the summands are $\mathrm{Ker}(\pi)$ and $\mathrm{Ker}(1 - \pi)$, which are closed, a contradiction. Thus R_0 is a field. So R_0 is a locally compact field and M is isomorphic to R_0^2; in this case every R_0-submodule of R_0^2 is closed and we also have a contradiction. Hence M is simple as a topological module.

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Observe that multiplication by α in $\mathrm{End}_A(M)$, and hence in R_0, is contracting. Let us show that for every $m \in M$, the multiplication map $\mathrm{End}(M) \to M$ mapping r to rm is a proper map.

Indeed, by contradiction, let (r_i) be a net tending to infinity in $\mathrm{End}(M)$ such that $(r_i m)$ is bounded, and actually we can suppose that it tends to an element m' of M.

If Ω is a compact neighborhood of 0, we can write $r_i = \alpha^{-k_i}(\phi_i)$, where $\phi_i \in \Omega \setminus \alpha(\Omega)$ and k_i tends to $+\infty$. We can suppose that ϕ_i tends to an element ϕ of $\mathrm{End}(M)$. Define $e_i = r_i m - m'$, which tends to 0. So $\phi_i m - \alpha^{k_i} m' = \alpha^{k_i} e_i$, and hence $\phi_i m$ tends to zero. So $\phi m = 0$. Since the set of m such that $\phi m = 0$ is a closed submodule, we deduce that $\phi = 0$, a contradiction. So the multiplication map $\mathrm{End}(M) \to M$ mapping r to rm is a proper map.

Fix a nonzero $m \in N$. We deduce that $R_0 m$ is closed for every $m \in N$. Since we proved earlier that $R_0 m$ is dense in M, we have $M = R_0 m \subseteq N$, and thus $N = M$, a contradiction. Therefore the closedness assertion is proved. The second one then follows from Lemma 4.8.

(We no longer assume $R_0 = R$.) Since R_0 is commutative and M is a faithful R_0-module of finite length, it follows that R_0 is an artinian ring. Now R is also a contractible module and, applying this to $M = R$, we obtain that R is artinian (i.e., has finite length as left module over itself), since it has finite length over R_0, both as a left or right module. □

Remark 5.3. For $A = \langle \alpha \rangle$ infinite cyclic, in the case of the polycontractible module $M = \mathbf{Z}/p\mathbf{Z}((t)) \times \mathbf{Z}/p\mathbf{Z}((t^{-1}))$ (with α acting by multiplication by t), the closure of the image of $\mathbf{Z}[\alpha^{\pm 1}]$ in $\mathrm{End}(M)$ is reduced to $\mathbf{Z}/p\mathbf{Z}[t^{\pm 1}]$. In particular, M is not finitely generated over this closure.

Let $\mathbf{K}$ be a locally compact field. Let $i : A \to \mathbf{K}^*$ be a continuous homomorphism whose image is not contained in the 1-sphere (the maximal compact subgroup of $\mathbf{K}^*$, which is the group of norm-1 elements for any compatible multiplicative norm), such that the closed additive subgroup generated by $i(A)$ equals $\mathbf{K}$. The following corollary was obtained by Glöckner and Willis for $A = \mathbf{Z}$ (the case of arbitrary A is not more difficult, although we use here a different language even when $A = \mathbf{Z}$).

COROLLARY 5.4. *Each such module is simple contractible, and conversely every simple contractible A-module has this form.*

Proof. Given such data, if $W \subseteq \mathbf{K}$ is a nonzero closed submodule and $w \in W \setminus \{0\}$, then $w^{-1}W$ is also a nonzero submodule and contains 1. Therefore $w^{-1}W$ contains $i(A)$, and hence equals $\mathbf{K}$ by assumption. Hence $W = \mathbf{K}$ and $\mathbf{K}$ is a simple A-module. Since some element of $i(A)$ has norm less than 1, it acts as a contraction on $\mathbf{K}$.

Conversely, let M be a simple contractible module. Then $\mathbf{K} = \mathrm{End}_A(M)$ is a locally compact ring (Lemma 5.1), and is also, by Theorem 5.2, an artinian ring with a faithful simple module, and hence is a field. Let i be the canonical map $A \to \mathbf{K}$. Fixing any nonzero $m \in M$, the map $\mathbf{K} \to M$ mapping $r \mapsto rm$ yields the desired isomorphism. □

PROPOSITION 5.5. *Let M, S be nonzero contractible A-modules, with S simple. Then S is isomorphic to a quotient of M if and only if it is isomorphic to a submodule of M, if and only if it is isomorphic to a subquotient of M.*

Proof. This follows from the analogous statement for finitely generated modules over artinian rings. □

DEFINITION 5.6. The *support* of a polycontractible A-module is the set of its isomorphic classes of simple contractible subquotients (or equivalently, simple contractible submodules, or quotients).

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The polycontractible module is *monotypic* if its support is a singleton; the unique simple module in this singleton is called its *type*. Two polycontractible modules are *disjoint* if their supports are disjoint.

PROPOSITION 5.7. *Two polycontractible modules M, M' are disjoint if and only if* $\mathrm{Hom}_A(M, M') = \{0\}$.

PROPOSITION 5.8. *Every polycontractible A-module M has a unique decomposition $\bigoplus_i M_i$ as a finite direct product of pairwise disjoint monotypic contractible submodules. Every polycontractible submodule N decomposes accordingly, that is, $N = \bigoplus_i (N \cap M_i)$.*

Proof. First suppose that A is compactly generated. Lemma 4.2 reduces to when M is contractible. Then the result follows from Theorem 5.2 and the analogous result for finitely generated modules over artinian rings.

In general, there is a compactly generated open subgroup B of A over which M is polycontractible. The resulting decomposition is A-invariant. Hence we can suppose (for the existence) that M is monotypic over B. Then Lemma 4.8 ensures that the number of summands in a direct decomposition of M is bounded: choose a compactly generated $B' \supseteq B$ such that the monotypic disjoint decomposition has the largest possible number of summands. Then, for every $B'' \supseteq B'$, the monotypic disjoint decomposition is the same over B'' and B'. To show that this is the monotypic decomposition over A, we show the following claim.

CLAIM: Suppose that S, S' are nonisomorphic simple contractible A-modules. Then there is a compactly generated open subgroup B of A such that S, S' are not isomorphic as B-modules.

If there is no common contracting element, the claim is clear. So let $\alpha \in A$ be a common contracting element for S and S', and fix a compactly generated subgroup B_0 containing α. Then, for B containing B_0, we have the submodule $V_B = \mathrm{Hom}_B(S, S')$ of $S \times S$. If $B \subseteq B'$ then $V_B \supseteq V_{B'}$. By Lemma 4.8, there exists a compactly generated subgroup $B \subseteq A$ containing B_0 such that for every compactly generated $B' \subseteq A$ containing B, we have $V_B = V_{B'}$. It follows that $V_A = V_B$. Since $V_A = \{0\}$, we deduce that $V_B = 0$, that is, S and S' are not isomorphic as B-modules. □

COROLLARY 5.9. *Suppose that A is compactly generated. Then every sheer LC A-module M has a canonical decomposition $(\bigoplus_{i=1}^n P_i) \oplus E \oplus V$, with $P_i \neq \{0\}$ contractible monotypic and pairwise disjoint, E amorphic, and V distal Euclidean. (It is canonical up to reindexing the P_i.) Moreover, every closed sheer submodule N splits with respect this decomposition, namely, $N = (\bigoplus_{i=1}^n N \cap P_i) \oplus (N \cap E) \oplus (N \cap V)$.*

Proof. For M itself, Theorem 4.1 reduces to the case of M polycontractible and Theorem 5.8 finishes the job.

For a submodule, Corollary 4.4 reduces to the case of M polycontractible, and again Theorem 5.8 finishes the job. □

Another coarser decomposition is the characteristic one.

PROPOSITION 5.10 (Characteristic decomposition). *Every polycontractible LC A-module uniquely decomposes in a unique way as a topological direct sum of modules,*

$$M = M^\circ \oplus \Big(\bigoplus_p M_p\Big) \oplus \Big(\bigoplus_p N_p\Big),$$

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where p ranges over primes, only finitely many of the summands are nonzero, $p^{n_p}M_p = 0$ for some $n_p \geq 0$, N_p is canonically a finite-dimensional vector space over $\mathbf{Q}_p$, and M° is a Euclidean group.

Proof. Uniqueness is clear, so let us focus on existence. Proposition 5.8 reduces to the case of a monotypic contractible module M, with simple quotient $S \simeq \mathbf{K}$, $\mathbf{K}$ a nondiscrete locally compact field endowed with a homomorphism $A \to \mathbf{K}^*$. Then M has a finite composition series with each subquotient isomorphic to S. If $\mathbf{K}$ has finite characteristic, say characteristic p, then $p^n M = 0$ for some n, and hence the conclusion holds. If $\mathbf{K}$ has characteristic zero, it is a finite extension of the closure of $\mathbf{Q}$, which is isomorphic to either $\mathbf{R}$ or $\mathbf{Q}_p$ for some p. In all cases, the conclusion holds. □

DEFINITION 5.11. We say that a locally compact abelian group V is *characteristically pure* if one of the following assertions holds:

- $p^n V = \{0\}$ for some prime p and $n \geq 1$. We then say that the characteristic is (p, p);
- V is a finite-dimensional vector space over $\mathbf{Q}_p$. We then say that the characteristic is $(0, p)$;
- V is a finite-dimensional real vector space. We then say that the characteristic is $(0, 0)$.

The 'characteristic' is of course unique unless $V = \{0\}$. From Proposition 5.10 it follows that every indecomposable polycontractible module is characteristically pure.

For $A = \langle \alpha \rangle$ infinite cyclic, a contractible module is contracted by either α or α^{-1}. In finite characteristic we have a simple classification result.

THEOREM 5.12. *Suppose that $A = \langle \alpha \rangle$ is infinite cyclic. Let M be an indecomposable module of finite characteristic on which α acts as a contraction. Then there exist a prime p and $n \geq 1$ such that M is isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})(\!(t)\!)$, with α acting by multiplication by t.*

Proof. There exist a prime p and $n \geq 1$ (chosen minimal) such that $p^n A = \{0\}$. Hence we see that M is naturally a module over the ring $(\mathbf{Z}/p^n\mathbf{Z})(\!(t)\!)$, with α acting by multiplication by p. (More precisely, if we consider the unique ring homomorphism $\mathbf{Z}[\alpha^{\pm 1}] \to (\mathbf{Z}/p^n\mathbf{Z})(\!(t)\!)$ mapping $\alpha \mapsto t$, we see that the action $\mathbf{Z}[\alpha^{\pm 1}] \times M \to M$ uniquely factors through a continuous action $(\mathbf{Z}/p^n\mathbf{Z})(\!(t)\!) \times M \to M$.)

Now we observe that $R_n = (\mathbf{Z}/p^n\mathbf{Z})(\!(t)\!)$ is a principal ideal local ring, with its ideals being the ideals (p^k) for $0 \leq k \leq n$. Indeed, first we see that every element of the form $1 + x$ with $x \in t(\mathbf{Z}/p^n\mathbf{Z})[\![t]\!]$ is invertible. Now every element y of $R_n \smallsetminus pR_n$ can be written as $y = t^k \lambda(s + 1 + x)$ with $k \in \mathbf{Z}$, s nilpotent, $\lambda \in (\mathbf{Z}/p^n\mathbf{Z})^\times$ and $x \in t(\mathbf{Z}/p^n\mathbf{Z})[\![t]\!]$. Indeed, k is the smallest degree at which the coefficient of y is not a multiple of p, and λ is the given coefficient. Then $s + (1 + x)$, as sum of a nilpotent element and an invertible element, is invertible.

It follows that every element of R_n can be written as $p^k z$ with $k \geq 0$ and z invertible, showing the desired result.

From the classification of modules over principal ideal rings,[1] we deduce that every finitely generated indecomposable R_n-module has the form $R_n/(p^k)$ for some k. Since n is minimal for the property $p^n M = \{0\}$, we deduce that M is isomorphic to R_n (viewed as the free R_n-module of rank 1). □

[1] We can, for instance, use the fact that every local principal ideal ring is a quotient of a local principal ideal ring. Here this can be done explicitly, namely defining R_∞ as the ring of formal series $\sum_{n \in \mathbf{Z}} a_n t^n$ with $a_n \in \mathbf{Z}_p$ and for which the p-valuation of a_n tends to ∞ when n tends to $-\infty$. Alternatively, the classification of finitely generated modules can be directly established without appealing to the case of domains.

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Remark 5.13. By Theorem 5.12, there are, up to isomorphism, countably many torsion contractible LC modules over an infinite cyclic group. This is in contrast with characteristic zero: indeed, the action on $\mathbf{Q}_p$ with α acting by an element $x \in \mathbf{Q}_p^*$ of modulus not equal to 1 defines a simple contractible module M_x, and the modules M_x are pairwise nonisomorphic, and there are continuum many.

For $A = \mathbf{Z}^2 = \langle \alpha, \beta \rangle$ we also get continuum many simple contractible modules in characteristic p, by letting A act on $\mathbf{K} = (\mathbf{Z}/p\mathbf{Z})(\!(t)\!)$, α acting by multiplication by t and β by a nonzero element $x \in \mathbf{K}^\times$. The resulting modules $M_x = \mathbf{K}$ are pairwise nonisomorphic.

Retrospectively, most of the theory can be reduced to $A = \mathbf{Z}$ by the following proposition.

PROPOSITION 5.14. *Let M be a polycontractible LC A-module. Then there exists $\alpha \in A$ such that M is polycontractible as a $\mathbf{Z}[\alpha^{\pm 1}]$-module. More precisely, there exist n and nonzero continuous homomorphisms $f_1, \dots, f_n : A \to \mathbf{R}$ such that every $\alpha \notin \bigcup_{i=1}^n \mathrm{Ker}(f_i)$ works.*

Proof. Write $M = \bigoplus M_i$ with M_i monotypic. Let S_i be a simple quotient of M_i: it can be identified with a local field, with norm $\|\cdot\|_i$, on which A acts through a homomorphism u_i into the multiplicative group, such that the homomorphism $f_i = \log \circ \|\cdot\|_i \circ u_i$ is nonzero. Then any element of A not in the kernel of f_i acts on S_i, and hence on M_i, as a contraction or the inverse of a contraction. Hence the conclusion. □

Let us now provide a cocompactness criterion.

THEOREM 5.15. *Suppose that A is compactly generated. Let M be a sheer LC module. Write $M = D \oplus \bigoplus_{i=0}^n M_i$ with D distal Euclidean, M_0 amorphic, and each M_i polycontractible with the M_i pairwise disjoint. Let H be a closed submodule of M.*

(1) *Suppose that the sheer envelope $H^{\boxplus}$ of H (see Theorem 4.6) equals M. Then the projection of H on each M_i is dense, and the projection of H on D spans D.*
(2) *Suppose that the projection of H on each M_i is dense, and the projection of H on D spans D. Then H is cocompact in M.*

Proof.

(1) Let H_i be the closure of the projection of H in M_i, and let V be the span of the projection of H in D. Then $V \oplus \bigoplus_{i=0}^n H_i$ is sheer and contains H. The assumption implies that it equals M, hence the conclusion.
(2) Since $H^{\boxplus}$ is sheer, the assumptions ensure that $H^{\boxplus}$ decomposes along the same decomposition: $H^{\boxplus} = (H^{\boxplus} \cap D) \oplus \bigoplus_{i=0}^n (H^{\boxplus} \cap M_i)$. Since the projections of H are dense and $H \subseteq H^{\boxplus}$, we have $H^{\boxplus} \cap M_i = M_i$ for all i. Also, since $H^{\boxplus} \cap D$ spans D and is sheer, we have $H^{\boxplus} \cap D = D$. Hence $H^{\boxplus} = M$. Now Theorem 4.6 says that $H^{\boxplus}/H$ is compact. □

6. Metabelian groups

6.1 Compact generation and ascending sequences

We begin with the following result, which is very classical in the discrete case, and is a particular case of [Abe72, Satz 2.1.b].

LEMMA 6.1. *Let G be a compactly generated metabelian LC group with a closed normal abelian subgroup M such that $A = G/M$ is abelian. Then M is a compactly generated $\mathbf{Z}A$-module.*

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Proof. Since G is compactly generated and G/M is compactly presentable, the kernel M is compactly generated as normal subgroup (see [CH16, Proposition 8.A.10(2)]). □

The following lemma will be superseded by Corollary 6.4, but we need it at this point as it will be used to prove the latter.

LEMMA 6.2. *Let G be a compactly generated metabelian locally compact group. Let N be a closed central subgroup of G. Then N is a compactly generated locally compact group.*

Proof. Let M be an abelian closed normal subgroup such that G/M is abelian. The assumptions imply that $M' = \overline{MN}$ is an abelian normal subgroup. Hence by Lemma 6.1, M' is compactly generated as a $\mathbf{Z}[G/M']$-module. By Corollary 4.14, it follows that its closed submodule N is compactly generated as a $\mathbf{Z}[G/M']$-module. Since it is central, it is thus abstractly compactly generated. □

THEOREM 6.3. *Let G be a compactly generated locally compact metabelian group, and (N_n) an ascending sequence (or more generally, filtering net) of closed normal subgroups of G. Define $N = \overline{\bigcup N_n}$. Then there exists n_0 such that N_n is cocompact in N for all $n \geq n_0$.*

Proof. First, when G is abelian, the result can be viewed as a particular case of Theorem 4.13 (for the action of the trivial group).

Let M be a closed normal abelian subgroup of G such that $A = G/M$ is abelian.

We begin with the particular case when $N_n \cap M$ does not depend on n, say is equal to L. Then modding out by L if necessary, we can suppose that $N_n \cap M = \{1\}$ for all n. Since $[G, N_n] \subseteq N_n \cap M$ for all n, we deduce that N_n is central for all n. Hence N is central in G as well. By Lemma 6.2, it follows that N is compactly generated as well. By the abelian case, there exists n_0 such that N_n is cocompact in N for all $n \geq n_0$.

In general, $M_n = N_n \cap M$ and $V = \overline{\bigcup_n M_n}$. Then (M_n) is an ascending sequence of closed A-submodules of the compactly generated A-module M, so Theorem 4.13 ensures that for large n, M_n is cocompact in V. Write $N = \overline{\bigcup_n VN_n}$. Then VN_n is closed and intersects M in V, and for n large enough, N_n is cocompact in VN_n. Also, the previous paragraph applies: for n large enough, VN_n is cocompact in N. Hence, for n large enough, N_n is cocompact in N. □

Similarly to Corollary 4.14, we deduce the following result.

COROLLARY 6.4. *If G is a compactly generated locally compact metabelian group, then every normal subgroup N is compactly generated as a normal subgroup.*

Proof. Let N_0 be an open subgroup of N, compactly generated as a normal subgroup. Since G and hence N is σ-compact, N/N_0 is countable, hence choose an ascending sequence (N_n) of open subgroups, each open and compactly generated as normal subgroup, whose union is N. By Theorem 6.3, there exists n such that N_n is cocompact in N. Thus the compactly generated submodule N_n has finite index in N, and hence N is compactly generated as well. □

THEOREM 6.5. *Let G be a compactly generated metabelian locally compact group. Then the following assertions hold.*

(1) *G has a maximal compact normal subgroup $W(G)$, and $W(G/W(G)) = \{1\}$.*
(2) *If $W(G) = 1$ and G is totally disconnected, then $[G, G]$ is closed.*

(For counterexamples when some assumptions are removed: for (1), see Example 7.4 providing G metabelian but not compactly generated with $W(G)$ not closed; for (2), see Example 7.5 when

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G is not assumed to be compactly generated, and Example 7.6 when G is not assumed to be totally disconnected.)

Proof. Write $M = \overline{[G,G]}$ and $A = G/M$.

(1) That $W(G)$ compact implies $W(G/W(G))$ trivial is straightforward (for an arbitrary LC group). Hence we need to prove that $W(G)$ is compact. By Corollary 4.3, there is a maximal compact $\mathbf{Z}A$-submodule $W_G(M)$ in M; we can mod out and assume it is trivial, so that $M \cap W(G) = \{0\}$.

Since $[G, W(G)] \subseteq M \cap W(G)$, we deduce that $W(G)$ is central. By Lemma 6.2, the central subgroup $N = \overline{W(G)}$ is a compactly generated LCA group. By density of $W(N)$ in N, we have $\mathrm{Hom}(N, \mathbf{R}) = \{0\}$; N being a compactly generated LCA group with $\mathrm{Hom}(N, \mathbf{R}) = \{0\}$, we see that it is compact, so $N = W(G)$ is compact.

(2) Since $W(G)$ is trivial, $W(M)$ is trivial as well. In particular, M is lower-sheer. Hence the sheer module $\Omega^\sharp(M)$ can be written as $P \times E$ with P polycontractible and $E = W(M)$ amorphic (Theorem 4.1). Since $W(M)$ is trivial, we have $E = \{0\}$, so $M = P$ is polycontractible. It follows from Lemma 3.5 that $M \subseteq [G,G]$. Hence $[G,G]$ is open in $M = \overline{[G,G]}$ and hence $[G,G]$ is closed. □

6.2 Compact presentability

DEFINITION 6.6. Let G be a locally compact group. We denote by $G^* = \mathrm{Hom}(G, \mathbf{R})$ the set of continuous homomorphisms $G \to \mathbf{R}$. For $\chi \in G^*$, define $G_\chi = \{g \in G : \chi(G) \geq 0\}$.

We endow G^* with the compact-open topology. When G is compactly generated, this is a finite-dimensional real vector space, with its usual topology.

LEMMA 6.7. *Let Q be a discrete group, and M a LC group with an action of Q, and fix $\chi \in Q^*$. Suppose that M is compactly generated as a Q_χ-group, and generated as a Q-group by a compact subset S with nonempty interior. Then M is generated by S as a Q_χ-group.*

Proof. If $\chi = 0$ there is nothing to prove; assume otherwise. Let N be the Q_χ-subgroup generated by S; it is open. Fix $u \in Q$ with $\chi(u) < 0$. Then $M = \bigcup_{n \geq 0} u^{-n}(N)$. This ascending union is either constant or strictly increasing, and the latter is excluded since M is compactly generated as Q_χ-group. Hence $N = M$. □

Remark 6.8. The particular case when M is abelian and M is discrete was obtained by Bieri and Strebel [BS80, Proposition 2.1]. The above proof differs from the original proof: indeed, the latter relies on a criterion (written in [BS80, Proposition 2.1]) for M to be compactly generated over Q_v, which is stated as the existence of a certain nonzero element (with suitable properties) in the annihilator in $\mathbf{Z}[Q]$ of M. This criterion does not carry over to the nondiscrete case, since M can be compactly generated over Q_v and have a trivial annihilator: for instance, consider $M = \mathbf{Q}_p$ and $Q = \langle t \rangle$ infinite cyclic, where t acts on $\mathbf{Q}_p$ by multiplication by some transcendental element of positive valuation, and v is defined by $v(t) = -1$. However, we use this criterion in the proof of Lemma 6.15.

Remark 6.9. If we assume that Q is countable, we can, by a simple application of Baire's theorem, remove the assumption that S has nonempty interior. The same holds when Q is a σ-compact locally compact group and the Q-action on M is continuous.

We now need two lemmas of abstract group theory. If G is a group and $u \in G$, define a u^+-subgroup to be a subgroup stable under the left conjugation endomorphism $x \mapsto {}^u x = uxu^{-1}$,

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and similarly define a u^--subgroup; define a u-subgroup as a subgroup stable under conjugation by u and u^{-1}.

LEMMA 6.10. *Consider a group G with a generating subset $S \cup \{u\}$ with S symmetric. Let $W_0 \subseteq G$ be a subgroup and define $W_n = u^n W_0 u^{-n}$ for $n \in \mathbf{Z}$. Assume that the following conditions hold:*

(1) *for every $n \geq 0$, S normalizes W_n;*
(2) *$uW_0u^{-1} \subseteq W_0$ (i.e., W_0 is a u^+-subgroup);*
(3) *uSu^{-1} is contained in the u^--subgroup generated by S.*

Then the normal subgroup N generated by W_0 is the ascending union $W_\infty = \bigcup_{n\geq 0} u^{-n} W_0 u^n$.

Proof. Observe that because of condition (2), the union $W_\infty = \bigcup_{n\geq 0} W_{-n}$ is indeed ascending, and hence is a subgroup since W_n is a subgroup.

Next, we see by a straightforward induction (based only on (3)) that for every $y \in S$ and $n \geq 0$, the element $u^n y u^{-n}$ belongs to the u^--subgroup generated by S.

Observe that for every $n \geq 0$, $u^{-n} S u^n$ normalizes W_0: indeed, for $s \in S$ and $w \in W_0$, we have

$$^{u^{-n}su^n}W_0 = {}^{u^{-n}s}W_n = {}^{u^{-n}}W_n = W_0.$$

Since by the previous fact, the normal subgroup $\langle\langle S \rangle\rangle$ generated by S is generated by $\bigcup_{n\geq 0} u^{-n} S u^n$, we deduce that $\langle\langle S \rangle\rangle$ normalizes W_0. Since $\langle\langle S \rangle\rangle$ is normal, it therefore normalizes every conjugate of W, hence normalizes each W_n, and hence normalizes W_∞. Since $G = \langle u \rangle \langle\langle S \rangle\rangle$, it therefore normalizes W_∞. □

LEMMA 6.11. *Keep the assumptions of the preceding lemma. Assume that G carries a topology of a locally compact group such that S, W are compact, N is closed, and S has nonempty interior. Then if G/N is compactly presentable, so is G.*

Proof. We can suppose that $W \subseteq S$, and fix $L = S \cup \{u\}$ as generating subset. Consider relators of bounded length for G/N, and lift them to relators of the form $r \equiv u^{-n} w u^n$ with n bounded.

By a compactness argument, there exist n_0 and k_0 such that uSu^{-1} is contained in the set of products of at most n_0 elements of the form $\bigcup_{0\leq k\leq k_0} u^{-k} S u^k$. Prescribe them as relators as well.

Include the relators $xwx^{-1} = w'$ for $(x, w, w') \in L \times W^2$ as well.

Let G' be the group defined by this presentation. By construction, the canonical projection $p : G' \to G$ is injective on each of S and W (which are viewed both as subsets of G' and G, as well as the element u). By Lemma 6.10, the normal subgroup N of G' generated by W is the ascending union $\bigcup_{n\geq 0} u^{-n} W u^n$. It follows that N is mapped injectively onto its image in G, also called N. Since the relators include lifts of relators of G/N, the mapped $G'/N \to G/N$ induced by projection p is an isomorphism. Therefore p is a group isomorphism as well. Since relators have bounded size, this shows that G is compactly presentable. □

6.3 The Bieri–Strebel geometric invariant

Recall from § 6.2 that $G^* = \mathrm{Hom}(G, \mathbf{R})$ and $G_\chi = \{g \in G : \chi(g) \geq 0\}$.

DEFINITION 6.12 (Bieri–Neumann–Strebel invariant in the locally compact setting). Let H be a locally compact group with a continuous action of G by automorphisms. Define

$$\Gamma_G(H) = \{0\} \cup \{\chi \in G^* : H \text{ is not compactly generated over any compactly generated submonoid of } G_\chi\}.$$

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In particular, define

$$\Gamma(G) = \Gamma_G(\overline{[G,G]}).$$

This is introduced in [BNS87] when G, H are discrete, in which case this is a closed subset of G^*, stable under positive homotheties [BNS87, Theorem A].

(The invariant defined in [BS80, BNS87] is rather the quotient of $G^* \smallsetminus \Gamma(G)$ by the group of positive homotheties; while these are obviously equivalent data, it is much more practical not to use the complement, for instance so as to state the next lemma.)

We check (Lemma 6.17) that for G metabelian, this matches with the original definition in [BS80], namely, for a closed abelian normal subgroup M with $Q = G/M$ abelian, we have a natural identification $\Gamma(G) = \Gamma_Q(M)$. (Such an identification would not be possible by working with the open complement in the definition!)

Lemma 6.13. *Let $W, Q' \subseteq Q$ be closed subgroups, with W compact, such that $Q = W + Q'$. Let $\chi : Q \to \mathbf{R}$ be a continuous homomorphism. Then every LC Q-module M that is compactly generated as a Q_χ-module is compactly generated as a Q'_χ-module.*

Proof. Since χ vanishes on W, we have $Q_\chi = Q'_\chi + W$. Let K be a compact subset generating M as a Q_χ-module. Then WK is compact and generates M as Q'-module. □

Lemma 6.14. *Let Q be a σ-compact LC abelian group and M a locally compact group with a continuous action of Q. Consider $\chi \in Q^*$. Suppose that M is compactly generated as a Q_χ-group. Then M is compactly generated over some compactly generated subsemigroup of Q_χ.*

Proof. Let S be a compact generating subset of M as a Q-group, with nonempty interior. By Lemma 6.7, S generates M as a Q_χ-group. Fix $u \in Q$ with $\chi(u) < 0$. Write Q_χ as an ascending union of compactly generated subsemigroups P_k, such that $P_0 \cup \{u\}$ generates Q as a semigroup. Thus if N_k is the P_k-subgroup generated by S, we have $M = \bigcup_{n \geq 0} u^{-n} N_k$ for each k. Besides, we have $\bigcup_{k \geq 0} N_k = M$. In particular, by compactness, $u^{-1}S$ is contained in M_k for some k. By an immediate induction (using that Q is abelian), it follows that $u^{-n}S$ is contained in N_k for all n. We deduce that $N_k = M$. □

The following two lemmas will appear as particular cases of Theorem 6.24, but we need them here for the moment.

Lemma 6.15. *Let Q be a compactly generated abelian group and M a compactly generated $\mathbf{Z}[Q]$-module. Fix $\chi \in Q^*$ and suppose that M is compactly generated as a Q_χ-module. Let N be an open $\mathbf{Z}[Q]$-submodule of M. Then N is compactly generated as a Q_χ-module.*

Proof. Let W be the maximal compact subgroup in Q. Let Q' be a finitely generated subgroup of Q whose projection in Q/W is dense.

Fix $u \in Q'$ with $\chi(u) < 0$. Following [BS80], for $m \in \mathbf{Z}[Q']$, define $\chi(m)$ as the infimum of $\inf\{\chi(q) : q \in Q', m(q) \neq 0\}$. Here m is viewed as a finitely supported function $Q' \to \mathbf{Z}$. (We agree that $\chi(0) = +\infty$.)

By Lemma 6.13, M is compactly generated as a $\overline{Q'}_\chi$-module, and hence as a Q'_χ-module. Hence M/N is finitely generated as a Q'_χ-module. Therefore, by [BS80, Proposition 2.1], there exists $m \in \mathbf{Z}[Q']$ such that m acts on M/N as the identity and $\chi(m) > 0$. Replacing m by a large enough power, we can suppose $\chi(m) > -\chi(u)$.

Let S_0 be a compact neighborhood of 0 in N, generating N as a $\mathbf{Z}[Q]$-submodule. Let T be a finite subset of M such that $S \cup T$ generates M as a Q_χ-module. Write $S = S_0 \cup (1-m)T$, and observe that $S \subseteq N$. Let N' be the Q_χ-submodule of N generated by S. We claim that $N' = N$.

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If $s_0 \in S$, using that $S \cup T$ generates M as a Q'_χ-module, write a finitely supported decomposition

$$us_0 = \sum_{s\in S} a_s s + \sum_{t\in T} b_t t,$$

with $a_s, b_t \in \mathbf{Z}[Q'_\chi]$. Write $(1-m)t = \sum_{s\in S} c_s s$, so that

$$(1-m)\sum_{t\in T} b_t t = \sum_{s\in S}\sum_{t\in T} b_t c_s s.$$

Write $d_s = (1-m)a_s + c_s \sum_{t\in T} b_t$. Then

$$(1-m)ux = \sum_{s\in S} d_s s.$$

Thus

$$us_0 = mus_0 + \sum_{s\in S} d_s s.$$

This shows that $uS \subseteq N' = \mathbf{Z}[Q'_\chi]S$. By induction, $u^n S \subseteq N'$ for all $n \geq 0$. Since every element of Q' can be written as $u^n q$ with $q \in Q'_\chi$ and $n \geq 0$, we deduce that N' is a $\mathbf{Z}[Q']$-submodule, hence equals N. □

LEMMA 6.16. *Let Q be a compactly generated abelian group and M a compactly generated $\mathbf{Z}[Q]$-module. Fix $\chi \in Q^*$ and suppose that M is compactly generated as a $\mathbf{Z}[Q_\chi]$-module. Let N be a closed $\mathbf{Z}[Q]$-submodule of M such that Q acts trivially on M/N. Then N is compactly generated as a $\mathbf{Z}[Q_\chi]$-module.*

Proof. By Corollary 4.14, N is compactly generated as Q-module. By Corollary 4.3, $W(N)$ and $W(M)$ are compact. Modding out if necessary, we can suppose that $W(N) = \{0\}$. Then $N \cap W(M) = \{0\}$, so, modding out again, we can also suppose that $W(M) = \{0\}$. So both N, M are lower-sheer. Using Theorem 4.1, write $\Omega^\sharp(M) = P \times D$ and $\Omega^\sharp(N) = P' \times D'$ with P, P' poly-contractible, D, D' distal Euclidean as Q-modules. Then $P' \subseteq P$ and $D' \subseteq D$. Then P/P' embeds into M/N which is a module with trivial action. Since P is generated by contractible submodules, it has no nonzero homomorphism into M/N, which means that $P \subseteq N$ and thus $P = P'$. By Lemma 6.15, the open submodule $\Omega^\sharp(M)$ is compactly generated as Q_χ-module, and hence so is its quotient P. Hence it is enough to show that N/P is compactly generated as Q_χ-module. Indeed, $\Omega^\sharp(N)/P$ is closed in D and hence compactly generated as abelian group. Finally, the discrete module $N/\Omega^\sharp(N)$ is open in $M/\Omega^\sharp(M)$ and therefore is also, by (the discrete case of) Lemma 6.15, compactly generated. □

LEMMA 6.17. *Let G be a compactly generated metabelian group in an exact sequence*

$$1 \to M \to G \to Q \to 1$$

with M, Q abelian and M closed. Then M is compactly generated as a Q-module, and

$$\Gamma(G) = \Gamma_Q(M)$$

(as subsets of G^*, where Q^* is naturally included in G^*).

Proof. That M is a compactly generated Q-module is already asserted in Lemma 6.1.

Let us first check $\Gamma_Q(M) \subseteq \Gamma(G)$. Let χ belong to $\Gamma_Q(M)$. If by contradiction $\chi \notin \Gamma(G)$, then $\chi \neq 0$ and $\overline{[G,G]}$ is compactly generated over some compactly generated subgroup of G_χ, and hence over Q_χ.

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Since Q is abelian, we have $\overline{[G,G]} \subseteq M$. In particular, M being abelian, it centralizes $\overline{[G,G]}$. Hence $\overline{[G,G]}$ is compactly generated over Q_χ. Since $M/\overline{[G,G]}$ is isomorphic to a closed subgroup of $G/\overline{[G,G]}$, it is compactly generated; hence M is compactly generated over Q_χ. This contradicts the assumption, so $\chi \in \Gamma(G)$.

Conversely, let us show $\Gamma(G) \subseteq \Gamma_Q(M)$. We first check $\Gamma(G) \subseteq Q^*$. Indeed, let χ belong to $G^* \smallsetminus Q^*$. This means that χ is not zero on M. In particular, $MG_\chi = G$. Thus a closed subgroup of M is G_χ-invariant if and only if it is G-invariant. Since M is compactly generated over G, this implies that it is also compactly generated over G_χ, and hence $\chi \notin \Gamma(G)$. We have proved $\Gamma(G) \subseteq Q^*$.

Now let χ belong to $Q^* \smallsetminus \Gamma_Q(M)$. Hence $\chi \neq 0$ and M is compactly generated as a Q_χ-module. Since $\overline{[G,G]}$ is a closed submodule of M and the action on the quotient $M/\overline{[G,G]}$ is trivial, Lemma 6.16 ensures that $\overline{[G,G]}$ is compactly generated as Q_χ-module. Thus $\chi \notin \Gamma(G)$. □

Theorem 6.24 will remove the assumption below that N is open.

Lemma 6.18. *Let Q be a compactly generated abelian group and M a compactly generated $\mathbf{Z}[Q]$-module. Assume that N is a closed submodule. Then*

$$\Gamma_Q(M/N) \subseteq \Gamma_Q(M) \subseteq \Gamma_Q(N) \cup \Gamma_Q(M/N).$$

If N is open in M, then $\Gamma_Q(M) = \Gamma_Q(N) \cup \Gamma_Q(M/N)$.

Proof. Clearly $\Gamma_Q(M/N) \subset \Gamma_Q(M)$. The inclusion $\Gamma_Q(N) \subset \Gamma_Q(M)$ follows from Lemma 6.15 (we only use there that N is open). The reverse inclusion follows from the fact that an extension of compactly generated modules (over Q_χ for given χ) is compactly generated. □

Definition 6.19. Let Q be a compactly generated abelian group. We have an inclusion $\mathrm{Hom}(Q, \mathbf{Q}) \subset \mathrm{Hom}(Q, \mathbf{R}) = Q^*$ (with $\mathbf{Q}$ endowed with the discrete topology), Hom denoting continuous homomorphisms. We call the elements of $\mathrm{Hom}(Q, \mathbf{Q})$ the *rational characters*; if χ is a rational character, then the closed half-line $\mathbf{R}_+\chi$ is called a *rational half-line* in Q^*.

In general, the closure in Q^* of the set of rational characters is the subspace $(Q/Q^\circ)^*$ of Q^* consisting of homomorphisms vanishing on the component Q°. Note that $\chi \in Q^*$ is a scalar multiple of a rational character if and only if it has a discrete image in $\mathbf{R}$.

Lemma 6.20. *Let Q be a compactly generated abelian group. Let M be a monotypic contractible Q-module, with simple quotient $\mathbf{K}$, a nondiscrete locally compact field endowed with a continuous homomorphism $\phi : Q \to \mathbf{K}^*$. Write $\chi_0 = -\log|\phi|$. Then the following assertions hold.*

- *If M is connected then $\Gamma_Q(M) = \{0\}$.*
- *If M is totally disconnected then $\Gamma_Q(M) = \mathbf{R}_+\chi_0$, which is a rational half-line. Moreover, M admits an open compact Q_{χ_0}-submodule.*

Proof. (The description of $\mathbf{K}$ is provided by Corollary 5.4.)

Note that being monotypic contractible, M is either connected or totally disconnected. If M is connected it is compactly generated as a locally compact abelian group and the conclusion is clear, so assume that M is totally disconnected.

Each ball in $\mathbf{K}$ is preserved by Q_{χ_0}, since the latter acts on K by elements of norm at most 1. Hence $\mathbf{K}$ is not a compactly generated Q_{χ_0}-module. Since $\mathbf{K}$ is a quotient of M as a Q_{χ_0}-module, the latter is not compactly generated. So $\chi_0 \in \Gamma_Q(M)$.

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Conversely, let χ be an element in $Q^* \smallsetminus \mathbf{R}_+\chi_0$. Then there exists $\alpha \in Q$ with $\chi(\alpha) > 0$ (i.e., $\alpha \in Q_\chi$) and $\chi_0(\alpha) < 0$ (i.e., $|\phi(\alpha)| > 1$). Then for every compact neighborhood V of 0 in $\mathbf{K}$ we have $\mathbf{K} = \bigcup_{n\geq 0} \phi(\alpha^n)V$. In particular, $\mathbf{K}$ is compactly generated over Q_χ. Since M is an iterated extension of copies of the Q-module $\mathbf{K}$, we deduce that M is compactly generated over Q_χ as well, that is, $\chi \notin \Gamma_Q(M)$.

Since χ_0 has a discrete image, $\mathbf{R}_+\chi_0$ is a rational half-line.

Let us check the additional assertion of existence of an open Q_{χ_0}-submodule. Choose $u \in Q$ such that $\chi_0(u)$ is the positive generator of $\chi_0(Q)$. Then Q_{χ_0} is generated by $\mathrm{Ker}(\chi_0) \cup \{u\}$ as a semigroup. As a $\mathrm{Ker}(\chi_0)$-module, $\mathbf{K}$ and hence M is amorphic. So there is a compact open $\mathrm{Ker}(\chi_0)$-submodule $L \subseteq M$. Since u acts as a contraction on $\mathbf{K}$ and hence on M, for large enough n, the submodule $u^n L$ is contained in L. Hence $\sum_{n\geq 0} u^n L$ is a compact open Q_{χ_0}-submodule of M. □

PROPOSITION 6.21. *Let Q be a compactly generated abelian group. Let M be an upper-sheer compactly generated locally compact Q-module. Then $\Gamma_Q(M)$ is a finite union of rational half-lines (emanating from 0). More precisely, for $\chi \in Q^* \smallsetminus \{0\}$, $\chi \in \Gamma_Q(M)$ if and only if there is a totally disconnected simple contractible closed submodule $\mathbf{K}$ of M with a normed field structure, such that the action is given by some homomorphism $\phi : Q \to \mathbf{K}^*$, and such that χ is positively proportional to $-\log|\phi|$.*

Proof. If M is polycontractible, it is a finite direct product of contractible monotypic modules and the result follows from Lemma 6.20.

In general, modding out by $W(M)$ (which is compact, by Corollary 4.3), we can suppose that $W(M) = \{0\}$. So M is sheer and by Theorem 4.1 it can be written as $P \times D$, with P polycontractible and D distal Euclidean. Since $\Gamma_Q(M) = \Gamma_Q(M/D) = \Gamma_Q(P)$, we are done. □

PROPOSITION 6.22. *Let Q be a compactly generated abelian locally compact group. Let M be a compactly generated locally compact Q-module. Then $\Gamma_Q(M)$ is a rational polyhedral cone, that is, it is the union of finitely many convex polyhedral cones (based at one) whose faces have rational equations.*

Proof. Let N be an upper-sheer open submodule of M. Then $\Gamma_Q(M/N)$ is rational polyhedral by Bieri and Groves's theorem [BG84], and $\Gamma_Q(N)$ is a finite union of rational half-lines by Proposition 6.21. Then Lemma 6.18 ensures that $\Gamma(M)$ is the union of $\Gamma_Q(M/N)$ with a finite union of rational half-lines. □

LEMMA 6.23. *Let Q be a compactly generated abelian group. Let M be a sheer module and N a closed submodule. Then $\Gamma_A(N) \subseteq \Gamma_A(M)$.*

Proof. The first case is when N is polycontractible. This case directly follows from Proposition 6.21.

In general, let $N^{\boxplus}$ be the sheer envelope, as in Theorem 4.6. By the polycontractible case, $\Gamma_A(N^{\boxplus}) \subseteq \Gamma_A(M)$. This thus reduces to the case when $N^{\boxplus} = M$, which we now assume.

Let χ be an element of $\Gamma_Q(N)$, and write $Q' = \mathrm{Ker}(\chi)$. Since M is a sheer Q-module, it is a sheer Q'-module. Using Theorem 4.1, write $N = D \oplus M_0 \oplus P$ with D a distal Euclidean Q'-module, M_0 an amorphic Q'-module, and P a polycontractible Q'-module. In turn, write the monotypic decomposition (Proposition 5.8) of the Q'-module P as $P = \bigoplus_{i=1}^n M_i$. Write $D = M_{-1}$.

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Let u be an element of Q with $\chi(u) < 0$. Then there is a proper Q_χ-submodule L of N such that $N = \bigcup_{n\geq 0} u^{-n}L$ (i.e., one can take a Q_χ-submodule of N generated by any compact generating subset of N as a Q-module with nonempty interior in N). For $i \geq 0$, let L_i (respectively, N_i) be the closure of the projection of L (respectively, N) on M_i. For $i = -1$, we define L_{-1} (respectively, N_{-1}) as the span of the projection of L (respectively, N) on M_{-1}. By Theorem 5.15(1), $N_i = M_i$ for all $i \geq -1$.

Then $\bigcup_{n\geq 0} u^{-n}L_i = N_i = M_i$, and thus L_i is an open Q_χ-submodule of M_i.

Suppose by contradiction that for all i, we have $L_i = M_i$. By Theorem 5.15(2), L is cocompact in M. But L is open of infinite index in N. We reach a contradiction.

Hence, there exists $i \geq -1$ such that $L_i \neq M_i$. Then M_i, being the strictly increasing union of Q_χ-submodules $\bigcup_{n\geq 0} u^{-n}L_i$, is not compactly generated as a Q_χ-module. We deduce $\chi \in \Gamma_Q(M_i) \subseteq \Gamma_Q(M)$ (because M_i is a quotient of M), and we are done. □

THEOREM 6.24. *Let Q be a compactly generated abelian group. Let M be a LC module and N a closed submodule. Then $\Gamma_Q(N) \subseteq \Gamma_Q(M)$.*

Proof. The proof relies on two main particular cases: when N is open (Lemma 6.15), and when M is sheer (Lemma 6.23).

The result is trivial if M is not compactly generated, since then $\Gamma_Q(M) = Q^*$. Assume that M is compactly generated. By Corollary 4.3, $W(M)$ is compact. Note that $W(N) \subseteq W(M)$. We have the closed embedding $N/W(N) \subseteq M/W(M)$, and since $\Gamma_Q(-)$ does not change when modding out by a compact submodule, we can suppose that $W(N) = W(M) = \{0\}$. Thus M is lower-sheer.

Write $\Omega = \Omega^\sharp(M)$, so Ω is sheer. Then $\Gamma_Q(N) \subseteq \Gamma_Q(\Omega \cap N) \cup \Gamma_Q(N/(\Omega \cap N))$ by Lemma 6.18. So it is enough to show that the latter two are contained in $\Gamma_Q(M)$. Indeed, first, we have $\Gamma_Q(\Omega \cap N) \subseteq \Gamma_Q(\Omega) \subseteq \Gamma_Q(M)$: here the first inclusion follows from Lemma 6.23, and the second from Lemma 6.15. Second, we have $\Gamma_Q(N/N \cap \Omega) = \Gamma_Q(N + \Omega/\Omega) \subseteq \Gamma_Q(M/\Omega) \subseteq \Gamma_Q(M)$. Here the first inclusion follows from Lemma 6.15, and the second one is trivial. □

6.4 Characterization of compact presentability

The following generalizes the Bieri–Strebel theorem, which Bieri and Strebel proved in the discrete case [BS80].

THEOREM 6.25. *Let G be a compactly generated locally compact metabelian group. Then G is compactly presentable if and only if $\Gamma(G)$ contains no line (i.e., $\Gamma(G) \cap -\Gamma(G) = \{0\}$).*

Proof. We can suppose that G is totally disconnected, since replacing G by G/G° does not affect being compactly presentable, and does not modify $\Gamma(G)$ (it would modify its open complement!). Let M be a closed abelian normal subgroup of G such that $Q = G/N$ is abelian.

By Proposition 6.22, $\Gamma(G) \cap -\Gamma(G)$ is a rational polyhedral cone. Hence if not reduced to $\{0\}$, it contains a rational half-line; some element in this line is a homomorphism χ onto $\mathbf{Z}$; both χ and $-\chi$ belong to $\Gamma(G)$. Then $\Gamma(G) = \Gamma_Q(M)$ by Lemma 6.17, and hence $\pm\chi \notin \Gamma_Q(M)$. The first Bieri–Strebel theorem [BS78], in its locally compact version [CH16, Corollary 8.C.4] implies that G is not compactly presentable.

Conversely, suppose that $\Gamma(G) \cap -\Gamma(G) = \{0\}$ and let us show that G is compactly presentable. So M is a compactly generated Q-module, and, by Corollary 4.3, $W(M)$ is compact. We can mod out and suppose that $W(M)$ is reduced to zero. Since M is totally disconnected and $W(M)$ is zero, Theorem 4.1 ensures that the open submodule $N = \Omega^\sharp(M)$ is polycontractible.

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By Proposition 5.8, there exists $k \geq 0$ such that N is the direct product of k nonzero monotypic contractible modules. We prove that G is compactly presentable by induction on k.

If $k = 0$, we have $N = \{0\}$ and thus G is discrete. By Bieri and Strebel's characterization of finitely presentable metabelian groups [BS80], G is finitely presentable.

If $k \geq 1$, let V be one of these monotypic submodules. By induction, G/V is compactly presentable. Let $\mathbf{K}$ be the simple quotient of V. By Corollary 5.4, $\mathbf{K}$ has a structure of a nondiscrete totally disconnected locally compact field, such that the Q-action is given by a homomorphism $\phi : G \to \mathbf{K}^\times$ (factoring through $Q = G/M$); define $\chi = -\log|\phi|$. Thus $\Gamma_Q(V)$ is the half-line generated by χ (Lemma 6.20). Fix $u \in G$ with $\chi(u) > 0$ equal to the positive generator of $\chi(G)$; thus $0 < |\phi(u)| < 1$.

By Lemma 6.20, there is a compact open Q_χ-submodule W of N. Thus $uW \subseteq W$ and $\bigcup_{n\geq 0} u^{-n}W = N$. Consider a symmetric subset $S \subset \mathrm{Ker}(\chi)$ with nonempty interior such that $S \cup \{u\}$ generates G. Let us check the assumptions of Lemma 6.11: those written in Lemma 6.11 are fulfilled; among those in Lemma 6.10, (1) and (2) hold, and let us check (3), namely that uSu^{-1} is contained in the u^--subgroup H generated by S. Indeed, clearly H is open; we have $\mathrm{Ker}\,\chi = \bigcup_{n\geq 0} u^n H u^{-n}$ (ascending union). Since $-\chi \notin \Gamma(G)$, the normal subgroup $\overline{[G,G]}$ is compactly generated as a $G_{-\chi}$-group, and hence so is $\mathrm{Ker}\,\chi$ (since $\mathrm{Ker}\,\chi/\overline{[G,G]}$ is a compactly generated abelian group). Therefore the above ascending union stabilizes, which means that $uHu^{-1} = H$, and hence that $G = H$. Therefore Lemma 6.11 implies that G is compactly presentable. □

7. A few counterexamples

This section gathers 'counterexamples' to some of the above results, when one assumption is relaxed. In particular, we consider cases when the acting group A is not compactly generated.

Here we fix A to be discrete free abelian of countable rank with basis $(e_i)_{i\geq 1}$. In each case, p is a fixed prime (and A could also be replaced with A/pA).

Example 7.1 (Theorem 1.2 fails without compact generation of A). Consider the additive group M of $\mathbf{F}_p(\!(t)\!)$. Let A act as follows:

$$e_i \cdot \sum_n a_n t^n = \sum_{n \neq \pm i} a_n t^n + a_i t^{-i} + a_{-i} t^i.$$

In other words, e_i exchanges t^i and t^{-i}, fixes other monomials, and is extended by continuity. Then $W_A(M) = \mathbf{F}_p[t, t^{-1}]$ is not closed.

Example 7.2 (Corollary 1.7 fails without compact generation of A). Let I be the ideal of $\mathbf{Z}/p\mathbf{Z}[A]$ generated by all $e_i - 1$ ('augmentation ideal'). Consider the discrete module $M = \mathbf{Z}/p\mathbf{Z}[A]/I^2$. It is finitely generated, that is, compactly generated. Then $W_A(M) = I/I^2$ is infinite, that is, not compact.

Example 7.3 (Theorem 2.6 fails without compact generation of A). Let $(X_n)_{n\geq 1}$ be a partition of $\mathbf{Z}$ such that for each n, X_n is infinite and $X_n \cap \mathbf{N}_{\geq 1} = \{n\}$. Let σ_n be a permutation of $\mathbf{Z}$ acting as a single infinite cycle on X_n and identity elsewhere. Let M be the additive group of $\mathbf{F}_p(\!(t)\!)$. Let A act as follows:

$$e_i \cdot \sum_n a_n t^n = \sum_n a_n t^{\sigma_i(n)}.$$

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This is well defined because σ_i maps every lower-bounded subset to a lower-bounded subset, and continuous because σ_i is identity near $+\infty$. Since the σ_i have pairwise disjoint supports, they commute and thus this defines an action of A. Then, as an A-module, M is residually purely discrete: indeed, it embeds into the product $\prod_n \mathbf{F}_p^{(X_n)}$. Hence $\Omega^\sharp(M) = \{0\}$. But M is not discrete. Thus $\Omega^\sharp(M)$ is not open.

If N is the Pontryagin dual, then N is the closure of $W^\flat(N)$ but N is not compact.

Example 7.4 (Noncompactly generated metabelian group G in which $W(G)$ is not closed, cf. Theorem 6.5(1)). This is a classical example [WY72, § 6, Example 1]. Let (p_i) be a sequence of odd primes (it can be constant or injective), and $G = \bigoplus_i \mathbf{F}_{p_i} \rtimes \prod_i \mathbf{F}_{p_i}^*$ (as an abstract group, this can be viewed as a subgroup of $\prod_i \mathbf{F}_{p_i} \rtimes \mathbf{F}_{p_i}^*$, namely consisting of those sequences for which the additive term is eventually zero). Then G is metabelian and $W(G)$ is not closed and actually dense, namely equal to $\bigoplus_i \mathbf{F}_{p_i} \rtimes \mathbf{F}_{p_i}^*$.

Example 7.5 (Theorem 6.5(2) and Theorem 4.1 fail without compact generation of A). Let M be the group of adeles, namely the sequences (n_p), with $n_p \in \mathbf{Q}_p$ and $n_p \in \mathbf{Z}_p$ for all but finitely many p, with compact open subgroup $\prod_p \mathbf{Z}_p$. Define $G = M \rtimes \prod_p \langle p \rangle$ with the natural action. Then $[G, G]$ equals the dense subgroup $\bigoplus_p \mathbf{Q}_p$ of M, hence is not closed, although $W(G)$ is trivial (thus Theorem 6.5(2) 'fails' here).

Also, this module is sheer, has no nonzero Euclidean or amorphic submodule, but is not polycontractible; hence Theorem 4.1 'fails' too.

Example 7.6. Theorem 6.5(2) is not true without the 'totally disconnected' assumption. Indeed, fix a dense proper subgroup Λ of $\mathbf{R}$, for example, $\mathbf{Z}[\sqrt{2}]$. Consider the semidirect product $G = (\mathbf{R} \times \Lambda) \rtimes \mathbf{Z}$ where the action is by powers of the automorphism $(x, t) \mapsto (x + t, t)$. Then $[G, G]$ is the nonclosed subgroup $\Lambda \times \{0\}$ of $\mathbf{R} \times \Lambda$.

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Conflicts of interest

None.

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