THE STRESS FIELD DUE TO AN EDGE DISLOCATION INTERACTING WITH TWO CIRCULAR INCLUSIONS

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ABSTRACT

A general series solution to the problem of interacting circular inclusions in plane elastostatics is presented in this paper. The analysis is based on the use of the complex stress potentials of Muskhelishvili and the theorem of analytical continuation. The general forms of the complex potentials are derived explicitly for the circular inhomogeneities under arbitrary plane loading. Using the alternation technique, these general expressions were subsequently employed to treat the problem of an infinitely extended matrix containing two arbitrarily located inhomogeneities. The major contribution of the present proposed method is shown to be capable of yielding approximate closed-form solutions for multiple inclusions, thus providing the explicit dependence of the solution on the pertinent parameters. The result shows that the dislocation has a stable equilibrium position at a certain combination of material constants. The case of an inhomogeneity interacting with a circular hole under a remote uniform load is also investigated.

Keywords: Interacting circular inclusions, Analytical continuation, Alternation technique.

1. INTRODUCTION

The existence of inhomogeneities in engineering materials can drastically influence their mechanical behavior. One of the basic problems in solid mechanics is to determine the elastic field induced by the presence of these inhomogeneities. The interaction of dislocations with inhomogeneities thus becomes an important topic to gain a better understanding of certain strengthening and hardening materials. In view of its importance, this problem has received much attention during the last several decades. The first derivation to the force on the dislocation, which is near the straight interface between two dissimilar media, was provided by Head [1]. Depending on the combination of material constants, the dislocation is found to have repelled or attracted by the interface. The interaction between a circular inhomogeneity and an edge dislocation was investigated by Dundurs and Mura [2]. They found that the dislocation may even have a stable equilibrium position at a distance from the interface under certain conditions. This result suggests a so-called trapping mechanism of the motion of dislocations near inhomogeneities. Warren [3] and Stagni and Lizzio [4], respectively provided an exact analytical solution for an elliptical inclusion for a dislocation inside and outside the inclusion. Muskhelishvili [5] and Santare and Keer [6] obtained a closed-form solution for a dislocation near a rigid elliptical inclusion based on the complex variable approach.

So far, most work on the dislocation was concerned with an isolated inhomogeneity. However, most of the materials for engineering applications are multiphase systems. Christenson and Lo [7] proposed a three-phase model which is of great practical and theoretical interest in composite mechanics research. Based on the Laurent series expansion, Luo and Chen [8] gave an exact solution for the stress field due to an edge dislocation embedded in a three-phase composite cylinder. They found that, in comparison with the two-phase model developed by Dundurs and Mura [2], the three-phase model allows the dislocation to have a stable equilibrium position under much less stringent combinations of the material constants. The interaction be-

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tween an edge dislocation and a coated circular inclusion was studied by Xiao and Chen [9] based on the Muskhelishivili complex method. They found that the elastic properties of the inclusion have no significant influence on the force on the dislocation when a coating layer is thick. Based on the analytical continuation and alternation technique, Chao *et al*. [10] gave a series solution to a three-phase composite cylinder under an arbitrary point singularity. Recently, Chen [11] studied the interaction between an edge dislocation and a nonuniformly coated circular inclusion based on the technique of conformal mapping and the method of analytical continuation.

All the above mentioned studies are restricted to a two-phase composite or a three-phase composite cylinder. However, in view of the inherent difficulty involved, only limited attempts have been made to treat the problem of arbitrarily located inhomogeneities. Based on the Laurent series expansion method and the perturbation technique, Gong and Meguid [12] provided a series solution to the problem of interacting circular inhomogeneities in plane elastostatics. The proposed series form of solution was extended to treat the problem of plane thermoelasticity [13]. Since the series form solution is constructed in terms of the relative distance separating the two inhomogeneities, the accuracy of the asymptotic is only preserved when the two inhomogeneities are sufficiently apart. Based on the Cauchy integral formulae and the Laurent series expansion, Wang and Shen [14] provided a series solution to the problem associated with two inclusions. Their solutions call for a large system of algebraic equations and the resulting matrix structures become extremely cumbersome. In this study, we aim to provide a general solution to the two-dimensional problem of the interaction among circular inhomogeneities. The analysis is based on the complex potentials of Muskhelishivili [5] and employs the alternation technique in developing a general series solution to the problem. This method obviously provides a reliable result making the series solution rapidly convergent. The layout of the paper is as follows. Following the brief introduction, the general formulation for plane elasticity and the method of conformal mapping are given in Section 2. The detailed derivation of the general form of the complex potentials for two circular inhomogeneities in an infinite matrix is provided in Section 3. Some numerical examples associated with a remote uniform load and an arbitrarily located edge dislocation are presented in Section 4. Finally, Section 5 concludes this article.

2. PROBLEM FORMULATION

Consider two circular elastic inclusions Ω_a and Ω_c with radius r_1 and r_2, respectively, in an unbounded matrix Ω_b subjected to either a remote uniform load or a dislocation (see Fig. 1). Both the left circular interface Γ_1 formed by Ω_a and Ω_b, and the right circular interface Γ_2 formed by Ω_c and Ω_b are assumed to be perfect, *i.e.* both tractions and displacements are

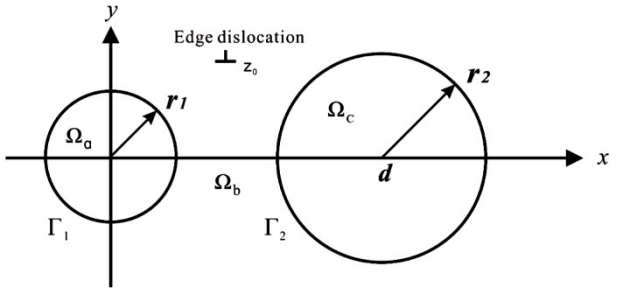


Fig. 1 Two circular inclusions in an infinite body.

continuous across the two interfaces. The origin of the Cartesian coordinate system is chosen to be at the center of the left circular inclusion. The two centers of the two circular inclusions are set apart by the distance d. To solve the problem in terms of complex potentials, the methods of Muskhelishvili are used. The displacements, stresses and the resultant forces are expressed in terms of two complex functions $\phi(z)$ and $\psi(z)$ such that

$$2G(u_x + iu_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{1}$$

$$\sigma_{xx} + \sigma_{yy} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] \tag{2}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right] \tag{3}$$

$$-F_y + iF_x = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{4}$$

where $z = x + iy$, G is the shear modulus, $\kappa = 3 - 4\nu$ for the plane strain problem, $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stress problem, ν is Poisson's ratio. Primes denote differentiation with respect to z and a superimposed bar denotes the complex conjugate. Now we introduce the following mapping function

$$z = m(\zeta) = \frac{x_2\zeta - x_1}{\zeta - 1} \tag{5}$$

where x_1 and x_2 are related by $x_1x_2 = r_1^2$ and $(d - x_1)(d - x_2) = r_2^2$ on obtaining $x_1 = br_1$, $x_2 = r_1/b$ with b being

$$b = \frac{1}{2r_1 d}\left[d^2 + r_1^2 - r_2^2 - \sqrt{(d^2 + r_1^2 - r_2^2)^2 - 4r_1^2 d^2}\right]$$

The mapped ζ plane is shown in Fig. 2. It can be observed that the left inclusion Ω_a is mapped onto the region S_a inside the inner circle of radius $\gamma_1 = \sqrt{x_1 / x_2}$, the right inclusion Ω_c is mapped onto the region S_c outside the outer circle of radius $\gamma_2 = \sqrt{(d - x_1)/(d - x_2)}$ while the matrix Ω_b is mapped onto the annulus S_b. For convenience of calculation, we write $\phi(\zeta) = \phi(m(\zeta))$ and $\psi(\zeta) = \psi(m(\zeta))$ so that in the mapped ζ plane, the displacements, stresses and resultant forces take the form

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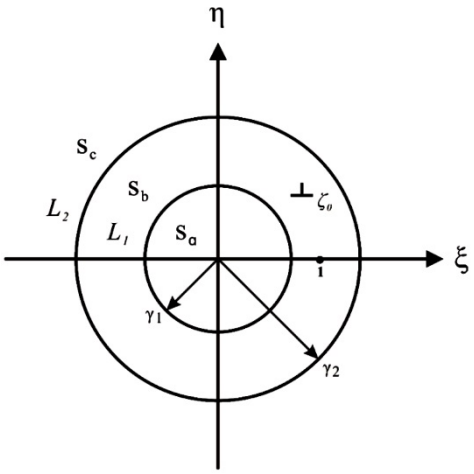


Fig. 2 The conformal transformation.

$$2G(u_x+iu_y)=\kappa\phi(\zeta)-\frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)}-\overline{\psi(\zeta)} \tag{6}$$

$$\sigma_{xx}+\sigma_{yy}=2\left\{\frac{\phi'(\zeta)}{m'(\zeta)}+\frac{\overline{\phi'(\zeta)}}{\overline{m'(\zeta)}}\right\} \tag{7}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2\left[\frac{\overline{m(\zeta)}}{m'(\zeta)}\frac{d}{d\zeta}\left\{\frac{\phi'(\zeta)}{m'(\zeta)}\right\}+\frac{\psi'(\zeta)}{m'(\zeta)}\right] \tag{8}$$

$$-F_y+iF_x=\phi(\zeta)+\frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)}+\overline{\psi(\zeta)} \tag{9}$$

3. STRESS FIELD

The complex potentials for a homogeneous infinite body subjected to a remote uniform tension T perpendicular to the x-axis are

$$\phi_0(\zeta)=\frac{T}{4}\frac{x_2\zeta-x_1}{\zeta-1},\quad \psi_0(\zeta)=\frac{T}{2}\frac{x_2\zeta-x_1}{\zeta-1} \tag{10}$$

while the complex potentials for a homogeneous infinite body subjected to an edge dislocation at $z=z_0$ are given as

$$\phi_0(\zeta)=Q\log\left[\frac{(\zeta-\zeta_0)(x_1-x_2)}{(\zeta-1)(\zeta_0-1)}\right] \tag{11}$$

$$\psi_0(\zeta)=\overline{Q}\log\left[\frac{(\zeta-\zeta_0)(x_1-x_2)}{(\zeta-1)(\zeta_0-1)}\right]-Q\frac{(\zeta-1)(\zeta_0-1)(x_2\overline{\zeta_0}-x_1)}{(\zeta-\zeta_0)(x_1-x_2)(\overline{\zeta_0}-1)} \tag{12}$$

where $Q=\dfrac{G(b_x+ib_y)}{\pi i(1+\kappa)}$, and b_x and b_y are the Cartesian components of the Burgers vector.

For a region bounded by a circle, say $c=|\zeta|$, we introduce an auxiliary stress function $\omega(\zeta)$ such that

$$\omega(\zeta)=\frac{\overline{m}\left(\dfrac{c^2}{\zeta}\right)}{m'(\zeta)}\phi'(\zeta)+\psi(\zeta)$$

Note that unlike the standard Muskhelishvili complex functions $\phi(\zeta)$ and $\psi(\zeta)$, the function $\omega(\zeta)$ is dependent on the radius of any circular interface.

The stress functions associated with two circular inclusions embedded in an infinite body can be assumed as

$$\phi(\zeta)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\phi_{an}(\zeta) & \zeta\in S_a\\ \phi_0(\zeta)+\displaystyle\sum_{n=1}^{\infty}\phi_n(\zeta)+\sum_{n=1}^{\infty}\phi_{bn}(\zeta) & \zeta\in S_b\\ \displaystyle\sum_{n=1}^{\infty}\phi_{cn}(\zeta) & \zeta\in S_c\end{cases} \tag{13}$$

$$\omega(\zeta)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\omega_{an}(\zeta) & \zeta\in S_a\\ \omega_0(\zeta)+\displaystyle\sum_{n=1}^{\infty}\omega_n(\zeta)+\sum_{n=1}^{\infty}\omega_{bn}(\zeta) & \zeta\in S_b\\ \displaystyle\sum_{n=1}^{\infty}\omega_{cn}(\zeta) & \zeta\in S_c\end{cases} \tag{14}$$

Having the stress functions $\phi_0(\zeta)$ and $\omega_0(\zeta)$ in Eqs. (13) and (14), respectively, the stress functions $\phi_n(\zeta)$, $\phi_{an}(\zeta)$, $\phi_{bn}(\zeta)$, $\phi_{cn}(\zeta)$, $\omega_n(\zeta)$, $\omega_{an}(\zeta)$, $\omega_{bn}(\zeta)$ and $\omega_{cn}(\zeta)$ can be obtained by the alternating technique and the analytical continuation method as follows.

Step 1: Analytical continuation across L_1

We first introduce two pairs of stress functions $\phi_{a1}(\zeta)$, $\psi_{a1}(\zeta)$ and $\phi_1(\zeta)$, $\psi_1(\zeta)$, respectively, holomorphic except some singular points in $|\zeta|\le\gamma_1$ and $|\zeta|\ge\gamma_1$ to satisfy the continuity conditions across L_1

$$\phi_{a1}(\sigma)+\overline{\omega_{a1}(\sigma)}=\phi_0(\sigma)+\overline{\omega_0(\sigma)}+\phi_1(\sigma)+\overline{\omega_1(\sigma)}\quad \sigma\in L_1 \tag{15}$$

$$\frac{1}{G_a}\left[\kappa_a\phi_{a1}(\sigma)-\overline{\omega_{a1}(\sigma)}\right]=\frac{1}{G_b}\left[\kappa_b\phi_0(\sigma)-\overline{\omega_0(\sigma)}+\kappa_b\phi_1(\sigma)-\overline{\omega_1(\sigma)}\right]\quad \sigma\in L_1 \tag{16}$$

where $\omega_0(\zeta)=\dfrac{\overline{m}\left(\dfrac{\gamma_1^2}{\zeta}\right)}{m''(\zeta)}\phi_0'(\zeta)+\psi_0(\zeta)$.

By the standard analytical continuation theorem, it follows that

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$$\phi_1(\zeta)+\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)-\overline{\omega_{a1}}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0 -\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a1}=0 \quad |\zeta|\geq\gamma_1 \tag{17}$$

$$\phi_{a1}(\zeta)-\phi_0(\zeta)-\overline{\omega_1}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0 -\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a1}=0 \quad |\zeta|\leq\gamma_1 \tag{18}$$

$$\frac{\kappa_b}{G_b}\phi_1(\zeta)-\frac{1}{G_b}\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{1}{G_a}\overline{\omega_{a1}}\left(\frac{\gamma_1^2}{\zeta}\right) -\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\frac{C_0}{G_b}+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\frac{C_{a1}}{G_a}=0 \quad |\zeta|\geq\gamma_1 \tag{19}$$

$$\frac{\kappa_a}{G_a}\phi_{a1}(\zeta)-\frac{\kappa_b}{G_b}\phi_0(\zeta)+\frac{1}{G_b}\overline{\omega_1}\left(\frac{\gamma_1^2}{\zeta}\right) -\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\frac{C_0}{G_b}+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\frac{C_{a1}}{G_a}=0 \quad |\zeta|\leq\gamma_1 \tag{20}$$

where $C_0=\overline{\phi_0'}(\gamma_1^2)$, $C_{a1}=\overline{\phi_{a1}'}(\gamma_1^2)$.

Note that both the terms $\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0$ and $\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a1}$ in Eq. (17) are singular at $\zeta=1$ which must be removed from the functions $\overline{\omega_0}(\gamma_1^2/\zeta)$ and $\overline{\omega_{a1}}(\gamma_1^2/\zeta)$, respectively (see Appendix A).

Solve Eqs. (17) ~ (20) to yield

$$\phi_{a1}(\zeta)=(1+\Lambda_{ab})\phi_0(\zeta)-\Pi_{ba}\frac{(\gamma_1^2-1)^2C_{a1}}{(\zeta-1)} \tag{21}$$

$$\omega_1(\zeta)=\Lambda_{ab}\overline{\phi_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}\overline{C_0}-(1+\Pi_{ba})\frac{(\gamma_1^2-1)^2}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}\overline{C_{a1}} \tag{22}$$

$$\phi_1(\zeta)=\Pi_{ab}\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\Pi_{ab}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0 \tag{23}$$

$$\omega_{a1}(\zeta)=(1+\Pi_{ab})\omega_0(\zeta)+(1+\Pi_{ab})\frac{(\gamma_1^2-1)^2}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}\overline{C_0}-\frac{(\gamma_1^2-1)^2}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}\overline{C_{a1}} \tag{24}$$

where $\Lambda_{ab}=\frac{G_a\kappa_b-G_b\kappa_a}{G_a+G_b\kappa_a}$, $\Pi_{ab}=\frac{G_a-G_b}{G_a\kappa_b+G_b}$.

Step 2: Analytical continuation across L_2

Since $\phi_1(\zeta)$ and $\psi_1(\zeta)$ cannot satisfy the continuity conditions at L_2, two pairs of stress functions $\phi_{b1}(\zeta)$, $\psi_{b1}(\zeta)$ and $\phi_{c1}(\zeta)$, $\psi_{c1}(\zeta)$, respectively, holomorphic in $|\zeta|\leq\gamma_2$ and $|\zeta|\geq\gamma_2$ are introduced to satisfy the continuity conditions along L_2, such that

$$\phi_1(\sigma)+\overline{\omega_1^*(\sigma)}+\phi_0(\sigma)+\overline{\omega_0^*(\sigma)}+\phi_{b1}(\sigma)+\overline{\omega_{b1}(\sigma)}=\phi_{c1}(\sigma)+\overline{\omega_{c1}(\sigma)} \quad \sigma\in L_2 \tag{25}$$

$$\frac{1}{G_b}\left[\kappa_b\phi_0(\sigma)-\overline{\omega_0^*(\sigma)}+\kappa_b\phi_1(\sigma)-\overline{\omega_1^*(\sigma)}+\kappa_b\phi_{b1}(\sigma)-\overline{\omega_{b1}(\sigma)}\right] =\frac{1}{G_c}\left[\kappa_c\phi_{c1}(\sigma)-\overline{\omega_{c1}(\sigma)}\right] \quad \sigma\in L_2 \tag{26}$$

where $\omega_0^*(\zeta)=\omega_0(\zeta)+\frac{\overline{m}\left(\frac{\gamma_2^2}{\zeta}\right)-\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)}{m'(\zeta)}\phi_0'(\zeta)$

$$\omega_1^*(\zeta)=\omega_1(\zeta)+\frac{\overline{m}\left(\frac{\gamma_2^2}{\zeta}\right)-\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)}{m'(\zeta)}\phi_1'(\zeta).$$

By the standard analytical continuation arguments, it follows that

$$\phi_{b1}(\zeta)+\overline{\omega_0^*}\left(\frac{\gamma_2^2}{\zeta}\right)+\overline{\omega_1^*}\left(\frac{\gamma_2^2}{\zeta}\right)-\overline{\omega_{c1}}\left(\frac{\gamma_2^2}{\zeta}\right)+\frac{(\gamma_2^2-1)^2C_0^*}{(\zeta-1)} +\frac{(\gamma_2^2-1)^2C_1^*}{(\zeta-1)}-\frac{(\gamma_2^2-1)^2C_{c1}}{(\zeta-1)}=0 \quad |\zeta|\leq\gamma_2 \tag{27}$$

$$\phi_{c1}(\zeta)-\phi_0(\zeta)-\phi_1(\zeta)-\overline{\omega_{b1}}\left(\frac{\gamma_2^2}{\zeta}\right)+\frac{(\gamma_2^2-1)^2C_0^*}{(\zeta-1)} +\frac{(\gamma_2^2-1)^2C_1^*}{(\zeta-1)}-\frac{(\gamma_2^2-1)^2C_{c1}}{(\zeta-1)}=0 \quad |\zeta|\geq\gamma_2 \tag{28}$$

$$\frac{\kappa_b}{G_b}\phi_{b1}(\zeta)-\frac{1}{G_b}\overline{\omega_0^*}\left(\frac{\gamma_1^2}{\zeta}\right)-\frac{1}{G_b}\overline{\omega_1^*}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{1}{G_c}\overline{\omega_{c1}}\left(\frac{\gamma_2^2}{\zeta}\right) -\frac{(\gamma_2^2-1)^2C_0^*}{G_b(\zeta-1)}-\frac{(\gamma_2^2-1)^2C_1^*}{G_b(\zeta-1)}+\frac{(\gamma_2^2-1)^2C_{c1}}{G_c(\zeta-1)}=0 \quad |\zeta|\leq\gamma_2 \tag{29}$$

$$\frac{\kappa_c}{G_c}\phi_{c1}(\zeta)-\frac{\kappa_b}{G_b}\phi_0(\zeta)-\frac{\kappa_b}{G_b}\phi_1(\zeta)+\frac{1}{G_b}\overline{\omega_{b1}}\left(\frac{\gamma_2^2}{\zeta}\right) -\frac{(\gamma_2^2-1)^2C_0^*}{G_b(\zeta-1)}-\frac{(\gamma_2^2-1)^2C_1^*}{G_b(\zeta-1)}+\frac{(\gamma_2^2-1)^2C_{c1}}{G_c(\zeta-1)}=0 \quad |\zeta|\geq\gamma_2 \tag{30}$$

Solve Eqs. (27) ~ (30) to yield

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$$\phi_{b1}(\zeta)=\Pi_{cb}\left[\overline{\omega_0^*}\left(\frac{\gamma_2^2}{\zeta}\right)+\overline{\omega_1^*}\left(\frac{\gamma_2^2}{\zeta}\right)\right]+\Pi_{cb}\frac{(\gamma_2^2-1)^2(C_0^*+C_1^*)}{(\zeta-1)} \tag{31}$$

$$\omega_{c1}(\zeta)=(1+\Pi_{cb})\left[\omega_0^*(\zeta)+\omega_1^*(\zeta)\right]+(1+\Pi_{cb})(\overline{C_0^*}+\overline{C_1^*})\frac{(\gamma_2^2-1)^2}{\left(\frac{\gamma_2^2}{\zeta}-1\right)}-\frac{(\gamma_2^2-1)^2\overline{C_{c1}}}{\left(\frac{\gamma_2^2}{\zeta}-1\right)} \tag{32}$$

$$\phi_{c1}(\zeta)=(1+\Lambda_{cb})\left[\phi_0(\zeta)+\phi_1(\zeta)\right]-\frac{\Pi_{bc}(\gamma_2^2-1)^2C_{c1}}{\zeta-1} \tag{33}$$

$$\omega_{b1}(\zeta)=\Lambda_{cb}\left[\overline{\phi_0}\left(\frac{\gamma_2^2}{\zeta}\right)+\overline{\phi_1}\left(\frac{\gamma_2^2}{\zeta}\right)\right]+\frac{(\gamma_2^2-1)^2(\overline{C_0^*}+\overline{C_1^*})}{\left(\frac{\gamma_2^2}{\zeta}-1\right)}-\frac{(1+\Pi_{bc})(\gamma_2^2-1)^2\overline{C_{c1}}}{\left(\frac{\gamma_2^2}{\zeta}-1\right)} \tag{34}$$

where $C_0^*=\overline{\phi_0'}(\gamma_2^2)$, $C_1^*=\overline{\phi_1'}(\gamma_2^2)$, $C_{c1}=\overline{\phi_{c1}'}(\gamma_2^2)$.

Step 3: Analytical continuation across L_1

Since $\phi_{b1}(\zeta)$ and $\psi_{b1}(\zeta)$ cannot satisfy the continuity conditions at L_1, two pairs of stress functions $\phi_{a2}(\zeta)$, $\psi_{a2}(\zeta)$ and $\phi_2(\zeta)$, $\psi_2(\zeta)$, respectively, holomorphic in $|\zeta|\le\gamma_1^2$ and $|\zeta|\ge\gamma_1^2$ are introduced to satisfy the continuity conditions along L_1, such that

$$\phi_{a2}(\sigma)+\overline{\omega_{a2}(\sigma)}=\phi_{b1}(\sigma)+\overline{\omega_{b1}^*(\sigma)}+\phi_2(\sigma)+\overline{\omega_2(\sigma)} \quad \sigma\in L_1 \tag{35}$$

$$\frac{\kappa_a}{G_a}\phi_{a2}(\sigma)-\frac{1}{G_a}\overline{\omega_{a2}(\sigma)}=\frac{\kappa_b}{G_b}\phi_{b1}(\sigma)-\frac{1}{G_b}\overline{\omega_{b1}^*(\sigma)}+\frac{\kappa_b}{G_b}\phi_2(\sigma)-\frac{1}{G_b}\overline{\omega_2(\sigma)} \quad \sigma\in L_1 \tag{36}$$

where $\omega_{b1}^*(\zeta)=\omega_{b1}(\zeta)+\dfrac{\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)-\overline{m}\left(\frac{\gamma_2^2}{\zeta}\right)}{m'(\zeta)}\phi_{b1}'(\zeta)$..

By the standard analytical continuation theorem, it follows that

$$\phi_{a2}(\zeta)-\phi_{b1}(\zeta)-\overline{\omega_2}\left(\frac{\gamma_1^2}{\zeta}\right)-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a2}+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{b1}^*=0 \quad |\zeta|\le\gamma_1 \tag{37}$$

$$\phi_2(\zeta)-\overline{\omega_{a2}}\left(\frac{\gamma_1^2}{\zeta}\right)+\overline{\omega_{b1}^*}\left(\frac{\gamma_1^2}{\zeta}\right)-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a2}+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{b1}^*=0 \quad |\zeta|\ge\gamma_1 \tag{38}$$

$$\frac{\kappa_a}{G_a}\phi_{a2}(\zeta)-\frac{\kappa_b}{G_b}\phi_{b1}(\zeta)+\frac{1}{G_b}\overline{\omega_2}\left(\frac{1}{\zeta}\right)+\frac{1}{G_a}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a2}-\frac{1}{G_b}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{b1}^*=0 \quad |\zeta|\le\gamma_1 \tag{39}$$

$$\frac{\kappa_b}{G_b}\phi_2(\zeta)+\frac{1}{G_a}\overline{\omega_{a2}}\left(\frac{1}{\zeta}\right)-\frac{1}{G_b}\overline{\omega_{b1}^*}\left(\frac{1}{\zeta}\right)+\frac{1}{G_a}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a2}-\frac{1}{G_b}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{b1}^*=0 \quad |\zeta|\ge\gamma_1 \tag{40}$$

Solve Eqs. (37) ~ (40) to yield

$$\phi_{a2}(\zeta)=(1+\Lambda_{ab})\,\phi_{b1}(\zeta)-\Pi_{ba}\frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_{a2} \tag{41}$$

$$\omega_2(\zeta)=\Lambda_{ab}\,\overline{\phi_{b1}}\left(\frac{\gamma_1^2}{\zeta}\right)-(1+\Pi_{ba})\frac{(\gamma_1^2-1)^2\overline{C_{a2}}}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}+\frac{(\gamma_1^2-1)^2\overline{C_{b1}^*}}{\left(\frac{\gamma_1^2}{\zeta}-1\right)} \tag{42}$$

$$\phi_2(\zeta)=\Pi_{ab}\,\overline{\omega_{b1}^*}\left(\frac{\gamma_1^2}{\zeta}\right)+\Pi_{ab}\frac{(\gamma_1^2-1)^2C_{b1}^*}{(\zeta-1)} \tag{43}$$

$$\omega_{a2}(\zeta)=(1+\Pi_{ab})\,\omega_{b1}^*(\zeta)-\frac{(\gamma_1^2-1)^2\overline{C_{a2}}}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}+(1+\Pi_{ab})\frac{(\gamma_1^2-1)^2\overline{C_{b1}^*}}{\left(\frac{\gamma_1^2}{\zeta}-1\right)} \tag{44}$$

where $C_{a2}=\overline{\phi_{a2}'}(\gamma_1^2)$, $C_{b1}^*=\overline{\phi_{b1}'}(\gamma_1^2)$.

Repetitions of the second and third step, the analytical continuation method is alternately applied to two interfaces to obtain all the functions $\phi_n(\zeta)$, $\omega_n(\zeta)$, $\phi_{an}(\zeta)$, $\omega_{an}(\zeta)$, $\phi_{bn}(\zeta)$ and $\omega_{bn}(\zeta)(n=1, 2, 3, 4\ldots)$ as follows

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$$\begin{cases} \phi_{a1}(\zeta)=(1+\Lambda_{ab})\,\phi_0(\zeta)-\Pi_{ba}\dfrac{(\gamma_1^2-1)^2 C_{a1}}{(\zeta-1)} \\ \omega_{a1}(\zeta)=(1+\Pi_{ab})\,\omega_0(\zeta)+(1+\Pi_{ab})\dfrac{(\gamma_1^2-1)^2}{\left(\dfrac{\gamma_1^2}{\zeta}-1\right)}\overline{C_0}-\dfrac{(\gamma_1^2-1)^2}{\left(\dfrac{\gamma_1^2}{\zeta}-1\right)}\overline{C_{a1}} \\ \phi_{an}(\zeta)=(1+\Lambda_{ab})\,\phi_{b(n-1)}(\zeta)-\Pi_{ba}\,u(\zeta)C_{an} \\ \omega_{an}(\zeta)=(1+\Pi_{ab})\left[\omega^*_{b(n-1)}(\zeta)+\overline{u}\left(\dfrac{\gamma_1^2}{\zeta}\right)\overline{C^*_{b(n-1)}}\right]-\overline{u}\left(\dfrac{\gamma_1^2}{\zeta}\right)\overline{C_{an}} \end{cases} \quad \zeta\in S_a \tag{45}$$

$$\begin{cases} \phi_1(\zeta)=\Pi_{ab}\,\overline{\omega_0}\left(\dfrac{\gamma_1^2}{\zeta}\right)+\Pi_{ab}\dfrac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0 \\ \omega_1(\zeta)=\Lambda_{ab}\,\overline{\phi_0}\left(\dfrac{\gamma_1^2}{\zeta}\right)+\dfrac{(\gamma_1^2-1)^2}{\left(\dfrac{\gamma_1^2}{\zeta}-1\right)}\overline{C_0}-(1+\Pi_{ba})\dfrac{(\gamma_1^2-1)^2}{\left(\dfrac{\gamma_1^2}{\zeta}-1\right)}\overline{C_{a1}} \\ \phi_{b1}(\zeta)=\Pi_{cb}\left[\overline{\omega_0^*}\left(\dfrac{\gamma_2^2}{\zeta}\right)+\overline{\omega_1^*}\left(\dfrac{\gamma_2^2}{\zeta}\right)\right]+\Pi_{cb}\dfrac{(\gamma_2^2-1)^2(C_0^*+C_1^*)}{(\zeta-1)} \\ \omega_{b1}(\zeta)=\Lambda_{cb}\left[\overline{\phi_0}\left(\dfrac{\gamma_2^2}{\zeta}\right)+\overline{\phi_1}\left(\dfrac{\gamma_2^2}{\zeta}\right)\right]+\dfrac{(\gamma_2^2-1)^2(\overline{C_0^*}+\overline{C_1^*})}{\left(\dfrac{\gamma_2^2}{\zeta}-1\right)}-\dfrac{(1+\Pi_{bc})(\gamma_2^2-1)^2\overline{C_{c1}}}{\left(\dfrac{\gamma_2^2}{\zeta}-1\right)} \\ \phi_n(\zeta)=\Pi_{ab}\left[\overline{\omega^*_{b(n-1)}}\left(\dfrac{\gamma_1^2}{\zeta}\right)+u(\zeta)C^*_{b(n-1)}\right] \\ \omega_n(\zeta)=\Lambda_{ab}\,\overline{\phi_{b(n-1)}}\left(\dfrac{\gamma_1^2}{\zeta}\right)-(1+\Pi_{ba})\,\overline{u}\left(\dfrac{\gamma_1^2}{\zeta}\right)\overline{C_{an}}+\overline{u}\left(\dfrac{\gamma_1^2}{\zeta}\right)\overline{C^*_{b(n-1)}} \\ \phi_{bn}(\zeta)=\Pi_{cb}\left[\overline{\omega^*_{n-1}}\left(\dfrac{\gamma_2^2}{\zeta}\right)+h(\zeta)\,C^*_{n-1}\right] \\ \omega_{bn}(\zeta)=\Lambda_{cb}\overline{\phi_{n-1}}\left(\dfrac{\gamma_2^2}{\zeta}\right)+\overline{h}\left(\dfrac{\gamma_2^2}{\zeta}\right)\overline{C^*_{n-1}}-(1+\Pi_{bc})\overline{h}\left(\dfrac{\gamma_2^2}{\zeta}\right)\overline{C_{c(n-1)}} \end{cases} \quad \zeta\in S_b \tag{46}$$

$$\begin{cases} \phi_{c1}(\zeta)=(1+\Lambda_{cb})\left[\phi_0(\zeta)+\phi_1(\zeta)\right]-\dfrac{\Pi_{bc}(\gamma_2^2-1)^2 C_{c1}}{\zeta-1} \\ \omega_{c1}(\zeta)=(1+\Pi_{cb})\left[\omega_0^*(\zeta)+\omega_1^*(\zeta)\right]+(1+\Pi_{cb})(\overline{C_0^*}+\overline{C_1^*})\dfrac{(\gamma_2^2-1)^2}{\left(\dfrac{\gamma_2^2}{\zeta}-1\right)}-\dfrac{(\gamma_2^2-1)^2\overline{C_{c1}}}{\left(\dfrac{\gamma_2^2}{\zeta}-1\right)} \\ \phi_{cn}(\zeta)=(1+\Lambda_{cb})\,\phi_{n-1}(\zeta)-\Pi_{bc}\,h(\zeta)\,C_{c(n-1)} \\ \omega_{cn}(\zeta)=(1+\Pi_{cb})\left[\omega^*_{n-1}(\zeta)+\overline{h}\left(\dfrac{\gamma_2^2}{\zeta}\right)\overline{C^*_{n-1}}\right]-\overline{h}\left(\dfrac{\gamma_2^2}{\zeta}\right)\overline{C_{c(n-1)}} \end{cases} \quad \zeta\in S_c \tag{47}$$

for $n = 2, 3, 4, \ldots$.

where $C_n=\overline{\phi_n'}(\gamma_1^2)$, $C_n^*=\overline{\phi_n'}(\gamma_2^2)$, $C_{an}=\overline{\phi_{an}'}(\gamma_1^2)$, $C_{an}^*=\overline{\phi_{an}'}(\gamma_2^2)$, $C_{bn}=\overline{\phi_{bn}'}(\gamma_2^2)$, $C_{bn}^*=\overline{\phi_{bn}'}(\gamma_1^2)$, $C_{cn}=\overline{\phi_{cn}'}(\gamma_2^2)$, $C_{cn}^*=\overline{\phi_{cn}'}(\gamma_1^2)$, $u(\zeta)=\dfrac{(\gamma_1^2-1)^2}{(\zeta-1)}$, $h(\zeta)=\dfrac{(\gamma_2^2-1)^2}{(\zeta-1)}$

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4. RESULTS AND DISCUSSION

The stress functions for two circular inclusions in an infinite solid subjected to an edge dislocation or a remote uniform load are expressed in terms of $\phi_n(\zeta)$ and $\omega_n(\zeta)$ (n = 1, 2, 3…) as indicated in Eqs. (13) and (14), which can be calculated from a homogeneous solution $\phi_0(\zeta)$ and $\omega_0(\zeta)$ by the recurrence formulae. The rate of convergence depends on the ratios $|\phi_{n+1}(\zeta)|/|\phi_n(\zeta)|$ and $|\omega_{n+1}(\zeta)|/|\omega_n(\zeta)|$, which depend on the nondimensional biomaterial constants Λ_{ab} and $\prod_{ab}$ (or Λ_{cb} and $\prod_{cb}$). Note that Λ and $\prod$ are less than 1 and 0.5, respectively for most combinations of materials, which guarantee rapid convergence. As a result, the convergence rate becomes more rapid as the differences of the elastic constants of the neighboring materials get smaller. In the following discussions, all the calculated results sum up the first three terms in a series solution, since they are checked to achieve a good accuracy for the current problem.

4.1 A Remote Uniform Load

As a starting point, we consider two circular inclusions embedded in an infinite body under a remote uniform load T perpendicular to the x-axis. The complex potentials for this problem can be immediately obtained by substituting Eq. (10) into Eqs. (13) and (14). For the limiting case when the region S_a is empty and the regions S_b and S_c are made of the same material for the corresponding circular hole problem, Eqs. (13) and (14) can be simplified to an exact form given by (see Appendix B)

$$\phi(z)=\frac{T}{4}z-\frac{Ta^2}{2z},\ \psi(z)=\frac{T}{2}z-\frac{Ta^2}{2z}-\frac{Ta^4}{2z^3} \tag{48}$$

which are the same as the results provided by England [15].

Consider now the case of an inhomogeneity interacting with a circular hole as depicted in Fig. 3. Let us focus our attention on the change in the hoop stress around the hole due to the presence of the inhomogeneity. The dependence of the normalized hoop stress at point A on the material constants G_c / G_b is shown in Fig. 4. It can be observed that the hoop stress at point a increases (or decreases) with decreasing of the distance d/a if the inhomogeneity is softer (or harder) than the matrix. The two extreme cases of a circular hole (G_c / G_b = 0) and a rigid circular inhomogeneity (G_c / G_b = ∞) provide the maximum increase or decrease in the hoop stress for a given condition.

4.2 An Edge Dislocation

Of particular interest in dislocation interaction problems is the effect of the stress field on the dislocation. A common measure of this effect is the Peach-Koehler force which can be written as [16]

$$F_x+iF_y=(\sigma^*_{xy}b_x+\sigma^*_{yy}b_y)-i(\sigma^*_{xx}b_x+\sigma^*_{xy}b_y) \tag{49}$$

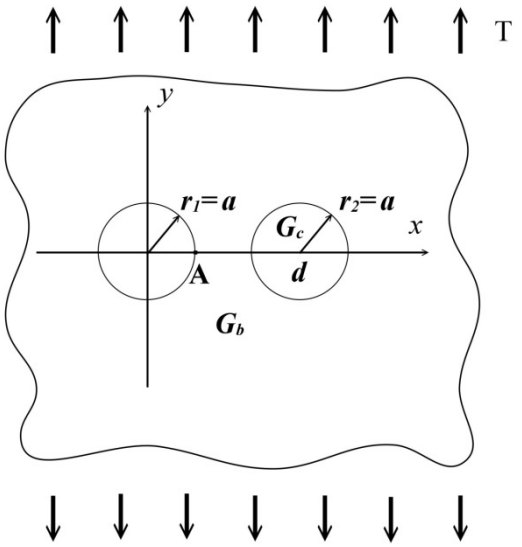


Fig. 3 An inhomogeneity interacting with a circular hole under a remote uniform load.

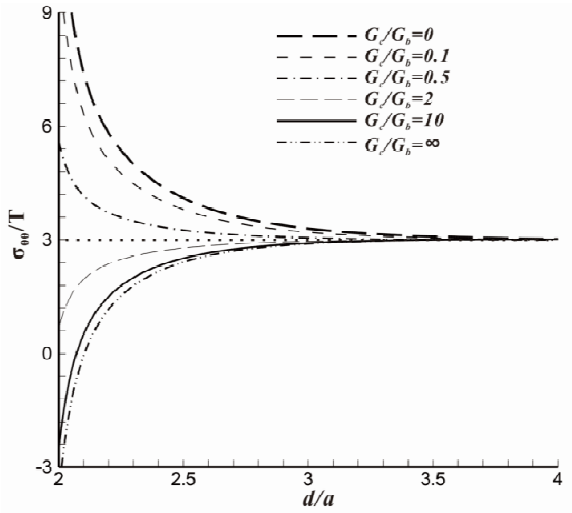


Fig. 4 Variation of the normalized hoop stress at point A for different material constants G_c / G_b.

where F_x and F_y are the Cartesian components of the Peach-Koehler force and σ^*_{xx} 、 σ^*_{yy} 、 σ^*_{xy} are the components of the perturbation stress at the dislocation in the Cartesian coordinates. The normalized forces on the dislocation are defined as

$$F_{x0}=\pi(1+\kappa_b)\rho_bF_1/G_bb_x^2 \tag{50}$$

$$F_{y0}=\pi(1+\kappa_b)\rho_bF_2/G_bb_y^2 \tag{51}$$

where F_1 is the force on the gliding dislocation b_x (Burgers vector (b_x, 0)), and F_2 is the force on the climbing dislocation b_y (Burgers vector (0, b_y)).

In the following examples, we assume the dislocation is located at point z_0 = (x_0, 0), r_1 = a, r_2 = 2a, d = 4a, and v_a = v_b = v_c = 0.3. Figures 5 and 6, respectively, illustrate the variation of F_{x0} and F_{y0} with respect to the dislocation location x_0 / a with G_a < G_b < G_c. This

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configuration represents that the right inclusion is stiffer than the matrix and the left inclusion is softer than the matrix. An interesting result from Fig. 5, there does not exist an equilibrium position that the right inclusion always repels the dislocation and the left inclusion always attracts the dislocation. Similar results can be obtained in Fig. 6 for normalized force F_{y0} on climbing dislocation. Figures 7 and 8, respectively, illustrate the variation of F_{x0} and F_{y0} with respect to the dislocation location x_0 / a with $G_a < G_b$ and $G_c < G_b$. This configuration represents that both the right inclusion and the left inclusion are softer than the matrix. One can also find an interesting phenomenon that there exists an unstable equilibrium position that both the right inclusion and the left inclusion attract the dislocation. It is also shown that the normalized forces on the dislocation increase if the differences of material constants between the matrix and the inclusion get larger. The forces F_{x0} and F_{y0} versus the dislocation location x_0 / a with $G_a > G_b$ and $G_c > G_b$ are plotted in Figs. 9 and 10, respectively. This configuration represents that both the right inclusion and the left inclusion are stiffer than the matrix. It is interesting to see that there exists a stable equilibrium position that both the right inclusion and the left inclusion always repel the dislocation. All the foregoing cases are for the condition when the dislocation is located between two inclusions. Figures 11 and 12, respectively, illustrate the variation of F_{x0} and F_{y0} with respect to the dislocation location x_0 / a for different combinations of material constants when the dislocation is located outside the inclusions. The results show that the right inclusion always repels (or attracts) the dislocation when both the right inclusion and the left inclusion are stiffer (or softer) than the matrix. It is evident to see that the normalized forces on the dislocation tend to vanish as the dislocation location is far away from the inclusion.

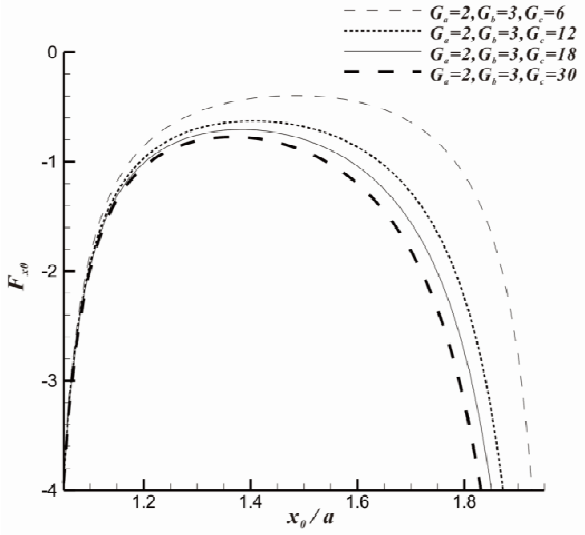


Fig. 5 Normalized force on sliding dislocation F_{x0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a < G_b$, $G_c > G_b$.

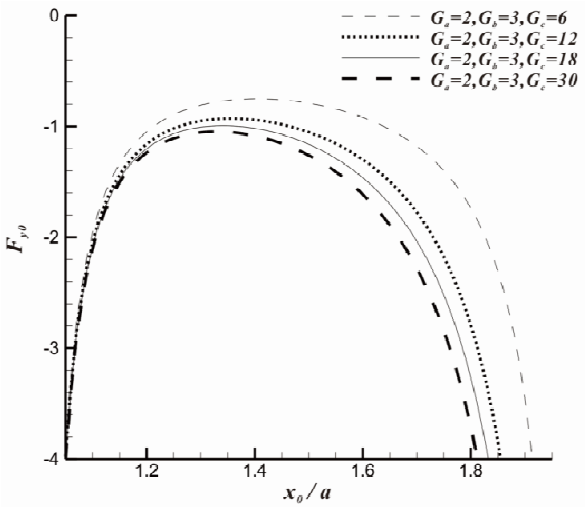


Fig. 6 Normalized force on climbing dislocation F_{y0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a < G_b$, $G_c > G_b$.

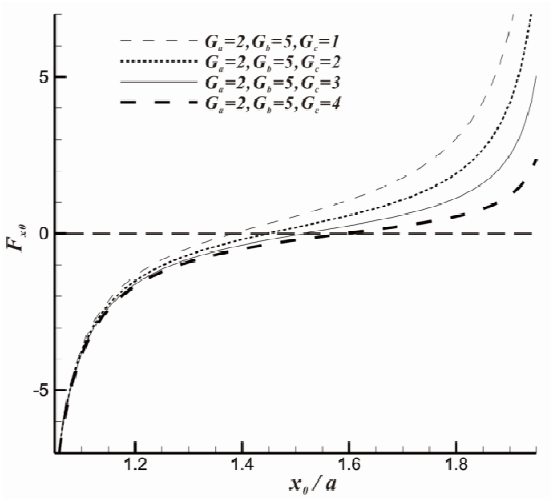


Fig. 7 Normalized force on sliding dislocation F_{x0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a < G_b$, $G_c < G_b$.

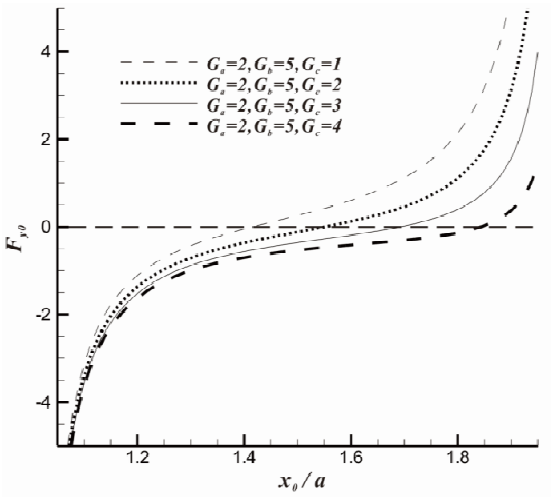


Fig. 8 Normalized force on climbing dislocation F_{y0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a < G_b$, $G_c < G_b$.

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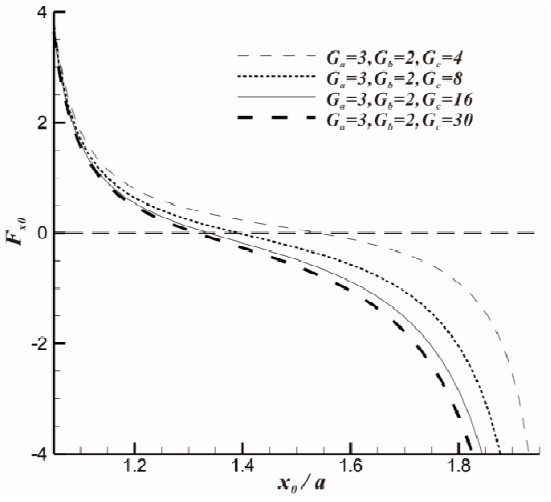


Fig. 9 Normalized force on sliding dislocation F_{x0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a > G_b$, $G_c > G_b$.

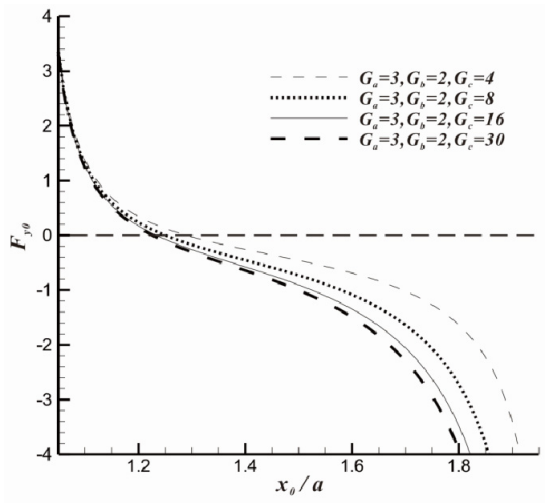


Fig. 10 Normalized force on climbing dislocation F_{y0} versus the dislocation location for $v_a = v_b = v_c = 0.3$, $G_a > G_b$, $G_c > G_b$.

5. CONCLUSIONS

A general solution is presented to the problem of the interaction among two circular inclusions within the framework of plane elastostatics. The analysis is based on the complex variable formulation of Muskhelishvili and the alternation technique. The general forms of complex potentials are derived explicitly for the circular inhomogeneity and the surrounding matrix under either a remote uniform load or an arbitrarily located dislocation. The results of the work should be of interest to those working in the fields of micromechanics and damage theory of composite materials. Furthermore, the present proposed method can be readily extended to treat the problem of multiple inclusions with three or more inhomogeneities.

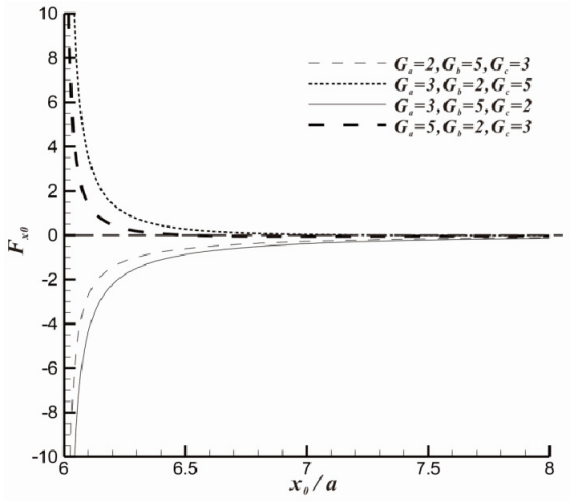


Fig. 11 Normalized force on sliding dislocation F_{x0} versus the dislocation location for $v_a = v_b = v_c = 0.3$.

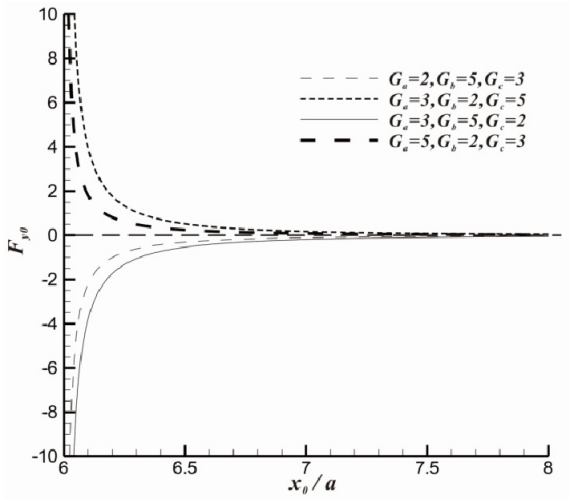


Fig. 12 Normalized force on climbing dislocation F_{y0} versus the dislocation location for $v_a = v_b = v_c = 0.3$.

APPENDIX A

Consider the function $\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)$ in Eq. (17) which is defined as

$$\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right) = \frac{m(\zeta)}{\overline{m'}\left(\frac{\gamma_1^2}{\zeta}\right)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right) + \overline{\psi_0}\left(\frac{\gamma_1^2}{\zeta}\right) \tag{A.1}$$

Now we want to prove Eq. (A.1) contains a singular term $-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)$ which must be removed such that $\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right) + \frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)$ becomes holomorphic in $|\zeta| \geq \gamma_1$.

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$$\frac{m(\zeta)}{\overline{m'}\left(\frac{\gamma_1^2}{\zeta}\right)}=\frac{\frac{x_2\zeta-x_1}{\zeta-1}}{\frac{(x_1-x_2)}{\left(\frac{\gamma_1^2}{\zeta}-1\right)^2}}=\frac{x_2\zeta-x_1}{\zeta-1}\frac{\left(\frac{\gamma_1^2}{\zeta}-1\right)^2}{(x_1-x_2)}=\frac{(x_2\zeta-x_1)(\gamma_1^2-\zeta)^2}{e\zeta^2(\zeta-1)}=\frac{(e\gamma_1^4-2\gamma_1^2x_1)}{e\zeta}+\frac{x_2}{e}+\frac{x_1\gamma_1^4}{e\zeta^2}-\frac{(\gamma_1^2-1)^2}{(\zeta-1)},\quad e=x_1-x_2 \tag{A.2}$$

The last term $-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}$ in Eq. (A.2) is singular in $|\zeta|\geq\gamma_1$ and the term $\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right)$ from Eq. (A.1) can be decomposed into two parts as

$$\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right)=\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right) \tag{A.3}$$

It is seen that the first part $\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)$ is singular in $|\zeta|\geq\gamma_1$ while the second part $-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right)$ is holormorphic in $|\zeta|\geq\gamma_1$.

Therefore

$$\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)=\left[\frac{(e\gamma_1^4-2\gamma_1^2x_1)}{e\zeta}+\frac{x_2}{e}+\frac{x_1\gamma_1^4}{e\zeta^2}\right]\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)-\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\zeta}\right)+\overline{\psi_0}\left(\frac{\gamma_1^2}{\zeta}\right) \tag{A.4}$$

It can be shown that the function $\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_0'}(\gamma_1^2)$ in Eq. (A.4) is holomorphic in $|\zeta|\geq\gamma_1$.

Similarly, one can also prove that $\overline{\omega_{a1}}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{(\zeta-1)}\overline{\phi_{a1}'}(\gamma_1^2)$ is holomorphic in $|\zeta|\geq\gamma_1$.

APPENDIX B

When the regions S_b and S_c are made of the same material and the region S_a becomes a hole, *i.e.*, $\prod_{ab}=\Lambda_{ab}=-1$ and $\prod_{bc}=\Lambda_{bc}=0$, Eqs (41) and (42) reduce to

$$\phi_n(\zeta)=\omega_n(\zeta)=0 \quad \text{for} \quad n=2,3,4...$$

Since

$$\phi_0(z)=\frac{T}{4}z=\frac{T}{4}\frac{x_2\zeta-x_1}{\zeta-1},\ \psi_0(z)=\frac{T}{2}z=\frac{T}{2}\frac{x_2\zeta-x_1}{\zeta-1}$$
$$\gamma_1^2=\frac{x_1}{x_2},\ a^2=x_1x_2,\ z=m(\zeta)=\frac{x_2\zeta-x_1}{\zeta-1},\ C_0=\overline{\phi_0'(\gamma_1^2)}=\frac{T}{4}\frac{x_2^2}{x_1-x_2} \tag{B.1}$$

we have

$$\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right)=\frac{m(\zeta)}{\overline{m'}\left(\frac{\gamma_1^2}{\zeta}\right)}\overline{\phi_0'}\left(\frac{\gamma_1^2}{\overline{\zeta}}\right)+\overline{\psi_0}\left(\frac{\gamma_1^2}{\overline{\zeta}}\right)=\frac{T}{4}\frac{x_2\zeta-x_1}{\zeta-1}+\frac{Ta^2}{2}\frac{1-\zeta}{x_1-x_2\zeta}$$

$$\omega_1(\zeta)=\Lambda_{ab}\overline{\phi_0}\left(\frac{\gamma_1^2}{\zeta}\right)+\frac{(\gamma_1^2-1)^2}{\left(\frac{\gamma_1^2}{\zeta}-1\right)}\overline{C_0}=-\frac{T}{4}\frac{a^2(1-\zeta)}{x_1-x_2\zeta}+\frac{T}{4}\frac{a^2\zeta-x_2^2\zeta}{x_1-x_2\zeta}=\frac{T}{4}\frac{(2a^2\zeta-a^2-x_2^2\zeta)}{x_1-x_2\zeta} \tag{B.2}$$

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$$\phi_1(\zeta) = -\overline{\omega_0}\left(\frac{\gamma_1^2}{\zeta}\right) - \frac{(\gamma_1^2-1)^2}{(\zeta-1)}C_0 = -\frac{T}{4}\frac{x_2\zeta - x_1}{\zeta-1} - \frac{Ta^2}{2}\frac{1-\zeta}{x_1-x_2\zeta} - \frac{T}{4}\frac{x_1-x_2}{\zeta-1}$$

$$= -\frac{Ta^2}{2}\frac{1-\zeta}{x_1-x_2\zeta} - \frac{T}{4}x_2 = -\frac{Ta^2}{2z} - \frac{T}{4}x_2 \tag{B.3}$$

$$\phi(z) = \phi_0(z) + \phi_1(z) = \frac{T}{4}z - \frac{Ta^2}{2z} - \frac{Tx_2}{4} \tag{B.4}$$

$$\phi_1'(\zeta) = -\frac{Ta^2}{2}\frac{x_2-x_1}{(x_1-x_2\zeta)^2}, \quad \frac{\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)}{m'(\zeta)} = \frac{a^2(1-\zeta)(\zeta-1)^2}{(x_1-x_2\zeta)(x_1-x_2)}$$

$$\frac{\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)}{m'(\zeta)}\overline{\phi_1'}(\zeta) = \frac{Ta^4}{2}\frac{(1-\zeta)^3}{(x_1-x_2\zeta)^3},$$

$$\psi_1(\zeta) = \omega_1(\zeta) - \frac{\overline{m}\left(\frac{\gamma_1^2}{\zeta}\right)}{m'(\zeta)}\overline{\phi_1'}(\zeta)$$

$$= \frac{T}{4}\frac{(2a^2\zeta - a^2 - x_2^2\zeta)}{x_1-x_2\zeta} - \frac{Ta^4}{2}\frac{(1-\zeta)^3}{(x_1-x_2\zeta)^3} = \frac{T}{4}\frac{(2a^2\zeta - 2a^2 + a^2 - x_2^2\zeta)}{x_1-x_2\zeta} - \frac{Ta^4}{2}\frac{(1-\zeta)^3}{(x_1-x_2\zeta)^3}$$

$$= \frac{Ta^2}{2}\frac{\zeta-1}{x_1-x_2\zeta} + \frac{T}{4}\frac{a^2-x_2^2\zeta}{x_1-x_2\zeta} - \frac{Ta^4}{2}\frac{(1-\zeta)^3}{(x_1-x_2\zeta)^3} = -\frac{Ta^2}{2z} + \frac{Tx_2}{4} - \frac{Ta^4}{2z^3} \tag{B.5}$$

$$\psi(z) = \psi_0(z) + \psi_1(z) = \frac{T}{2}z - \frac{Ta^2}{2z} - \frac{Ta^4}{2z^3} + \frac{Tx_2}{4} \tag{B.6}$$

Note that the results in Eqs. (A.8) and (A.10) are only different from the exact solution by a constant term which will not affect the stress field.

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