Generalisation of an IMO geometry problem

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1. *Introduction and statement of results*

According to Mitchelmore [1], generalisations are the cornerstone of school mathematics, covering various aspects like numerical generalisation in algebra, spatial generalisation in geometry and measurement, as well as logical generalisations in diverse contexts. The process of generalising lies at the heart of mathematical activity, serving as the fundamental method for constructing new knowledge [2, 3]. In this paper we will generalise an interesting geometry problem that appeared in the 1995 edition of the International Mathematical Olympiad (IMO) [4].

Original 36th IMO Geometry Problem: Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent (Figure 1).

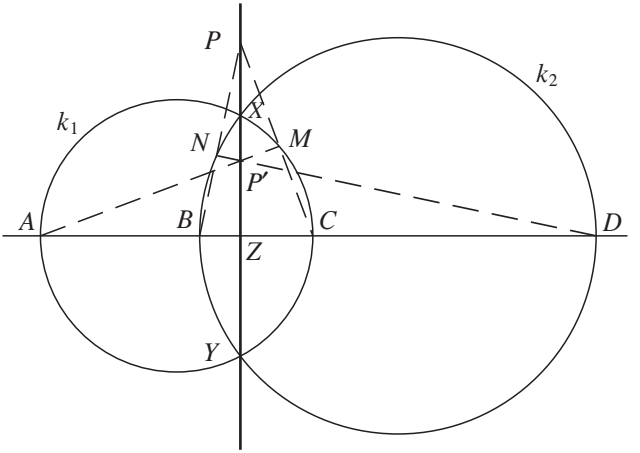


FIGURE 1: 36th IMO geometry problem

Its solution, employing analytic geometry, is presented in [5].

Since the line XY in the original problem represents the radical axis of the two intersecting circles, we shall now expand our investigation to encompass the radical axis in the broader context. Allow us to reiterate that the locus of points possessing equal powers concerning two circles is a line known as the radical axis (or power line) of the circles. Simpler loci problems should be studied meticulously, as they often pose difficulties for an average student (see discussion in [6]). Recent studies have explored

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misconceptions that students might have about analytical geometry and teaching practices in this subject across different educational levels and contexts (see e.g. [7, 8, 9, 10, 11]).

We will now present a generalised statement of the aforementioned problem and present its solution using analytical geometry.

Generalisation of the 36th IMO Geometry Problem: Let k_1 and k_2 be two circles, and let ℓ be the line that contains their centres. The line ℓ and the circle k_1 intersect at points A and C, while line ℓ and the circle k_2 intersect at points B and D. We assume that A, B, C, D appear in that order or in the order A, C, B, D on ℓ. Let p be a line perpendicular to line ℓ, and let P be any point on line p. The line CP intersects the circle k_1 at M (in addition to C), and the line BP intersects the circle k_2 at N (in addition to B). The lines AM and DN intersect at P'. Changing the position of the point P changes the position of the point P'. However, P' will always belong to a line p', parallel to the line p (Figure 2). If p is the radical axis of circles k_1 and k_2, then $p' = p$.

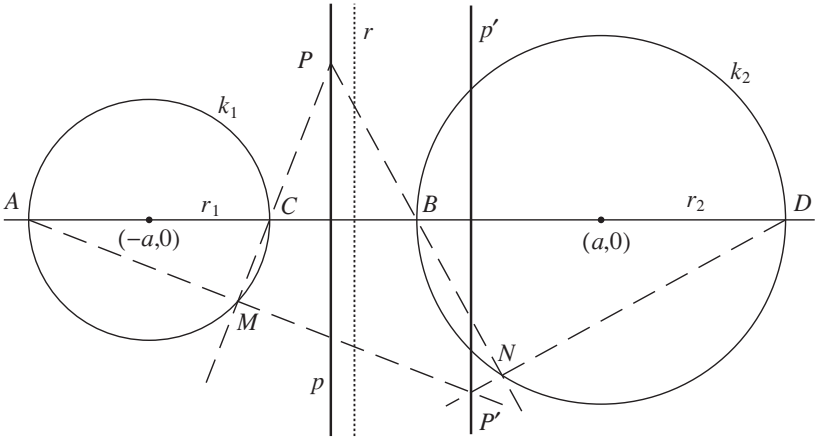


FIGURE 2: 36th IMO geometry problem generalisation

In Figure 2 and all subsequent figures, dotted line r represents radical axis of two circles.

2. *Analytical geometry proof of the generalisation*

Without loss of generality, we establish a coordinate system where ℓ represents the x-axis, and assume circles k_1 and k_2 have centres symmetrically positioned about the y-axis, with k_1 centered at $(-a, 0)$ with radius r_1, and k_2 centered at $(a, 0)$ with radius r_2. The equations of circles k_1 and k_2 are given by $(x + a)^2 + y^2 = r_1^2$ and $(x - a)^2 + y^2 = r_2^2$, respectively. Consequently, we have $A = (-a - r_1, 0)$, $B = (a - r_2, 0)$, $C = (-a + r_1, 0)$ and $D = (a + r_2, 0)$.

As the line p is perpendicular to the x-axis, it is parallel to the y-axis,

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and we can express it using the equation $x = p$. Henceforth the symbol p will serve a dual purpose, representing both the line p and the x-coordinate of the point P. Consequently the point P can be described by $P = (p, q)$, with q changing as point P moves along line p. Let $P' = (p', q')$. Using the tangent-half-parametrisation for the two circles (see e.g. [12, para. 5.13 on p. 91]), let $M = \left(\frac{r_1(1-u^2)}{1+u^2} - a, \frac{2r_1u}{1+u^2}\right)$ and $N = \left(\frac{r_1(1-v^2)}{1+v^2} + a, \frac{2r_2v}{1+v^2}\right)$, for some $u, v \in \mathbb{R} \cup \{\infty\}$.

Next we examine the condition for points P, C and M to be collinear by considering the vanishing of the determinant ([12, para. 1.72 on pp. 14-15]).

$$\begin{vmatrix} p & q & 1 \\ -a + r_1 & 0 & 1 \\ \frac{r_1(1-u^2)}{1+u^2} - a & \frac{2r_1u}{1+u^2} & 1 \end{vmatrix}. \tag{1}$$

Subtracting the second row from the third row, and then taking out the common factors from the third row, we have

$$\frac{2r_1u}{1+u^2} \begin{vmatrix} p & q & 1 \\ -a + r_1 & 0 & 1 \\ -u & 1 & 0 \end{vmatrix} = 0.$$

Expanding the determinant along the third row, we obtain

$$\frac{2r_1u}{1+u^2}\left(-uq - p - a + r_1\right) = 0. \tag{2}$$

Solving for u, and discarding the solution $u = 0$, which would entail $M = C$, we get

$$u = \frac{r_1 - a - p}{q}. \tag{3}$$

The cases $u = 0$ and $q = 0$, where the latter leads to $u = \infty$ and $M = A$, will be addressed later, along with the cases when the denominator becomes zero or tends to infinity.

Similarly, the condition for the collinearity of P', A and M is the vanishing of the determinant

$$\begin{vmatrix} p' & q' & 1 \\ -a - r_1 & 0 & 1 \\ \frac{r_1(1-u^2)}{1+u^2} - a & \frac{2r_1u}{1+u^2} & 1 \end{vmatrix}.$$

The last determinant can be obtained from (1) by replacing p, q, r_1 and u by p', q', $-r_1$ and $-u^{-1}$, respectively. Applying this to (2) leads to

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$$\frac{2r_1}{1+u^2}(q' - up' - ua - ur_1) = 0.$$

Solving for u, and discarding the solution $u = \infty$, which would entail $M = A$, we get

$$u = \frac{q'}{r_1 + a + p'}.$$

From this and (3) we now have

$$\frac{r_1 - a - p}{q} = \frac{q'}{r_1 + a + p'}, \tag{4}$$

Likewise, the conditions for collinearity among points P, B, N, and of P', D, N, are determined by the vanishing of the respective determinants

$$\begin{vmatrix} p & q & 1 \\ a - r_2 & 0 & 1 \\ \dfrac{r_2(1-v^2)}{1+v^2} + a & \dfrac{2r_2v}{1+v^2} & 1 \end{vmatrix} \text{ and } \begin{vmatrix} p' & q' & 1 \\ a + r_2 & 0 & 1 \\ \dfrac{r_2(1-v^2)}{1+v^2} + a & \dfrac{2r_2v}{1+v^2} & 1 \end{vmatrix}.$$

The first one can be obtained from (1) by replacing a, r_1 and u by $-a$, $-r_2$ and $-v^{-1}$, respectively; and the second one by replacing p, q, a, r_1 and u by p', q', $-a$, r_2 and v, respectively. Applying the same changes to (2), we obtain $v = \dfrac{q}{r_2 - a + p}$ and $v = \dfrac{r_2 + a - p'}{q'}$, respectively. Hence

$$\frac{q}{r_2 - a + p} = \frac{r_2 + a - p'}{q'}. \tag{5}$$

Multiplying (4) and (5) yields the following relationship:

$$\frac{r_1 - a - p}{r_2 - a - p} = \frac{r_2 + a - p'}{r_1 + a + p'}, \tag{6}$$

which indicates that p' depends on p (as well as on a, r_1 and r_2) but is independent of q. This results in P' lying on a line perpendicular to the x-axis and thus parallel to line p, which was what we wanted to prove.

To get an explicit value for p', we perform crosswise multiplication in (6), and obtain

$$(r_1 - a - p)(r_1 + a + p') = (r_2 - a + p)(r_2 + a - p').$$

This yields

$$\begin{aligned} &(r_1 - a - p)p' + (r_2 - a + p)p' \\ &= (r_2 - a + p)(r_2 + a) - (r_1 - a - p)(r_1 + a) \\ &= (r_2 - a)(r_2 + a) + p(r_2 + a) - (r_1 - a)(r_1 + a) + p(r_1 + a). \end{aligned}$$

From this we obtain

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$$(r_1 + r_2 - 2a)p' = r_2^2 - r_1^2 + p(r_1 + r_2 + 2a),$$

and so

$$p' = \frac{r_2^2 - r_1^2 + p(r_1 + r_2 + 2a)}{r_1 + r_2 - 2a}. \tag{7}$$

To get q', we take the reciprocal of (4) to obtain $\dfrac{q}{r_1 - a - p} = \dfrac{r_1 + a + p'}{q'}$, and upon adding this to (5) we arrive at

$$q\left(\frac{1}{r_1 - a - p} + \frac{1}{r_2 - a + p}\right) = \frac{r_1 + r_2 + 2a}{q'}.$$

Now

$$q' = \frac{(r_1 + r_2 + 2a)(a - r_1 + p)(a - r_2 - p)}{q(r_1 + r_2 - 2a)}. \tag{8}$$

From (7) and (8) we see that

$$P' = \left(\frac{r_2^2 - r_1^2 + p(r_1 + r_2 + 2a)}{r_1 + r_2 - 2a}, \frac{(r_1 + r_2 + 2a)(a - r_1 + p)(a - r_2 - p)}{q(r_1 + r_2 - 2a)}\right).$$

Now we will consider special cases.

If P is on the x-axis, i.e. if $q = 0$, then $q' = \infty$, and P' is the point at infinity. Then, by the equations $u = \dfrac{r_1 - a - p}{q}$ and $v = \dfrac{q}{r_2 - a + p}$, we have $u = \infty$ and $v = 0$, i.e., $M = A$ and $N = D$. The tangent lines of the circles k_1 and k_2 at points A and D, respectively, represent the limiting position of the lines AM and DN in the previous case (see Figure 3). Their intersection point disappears (or we can say that they intersect at the infinite point P_∞' on the line p').

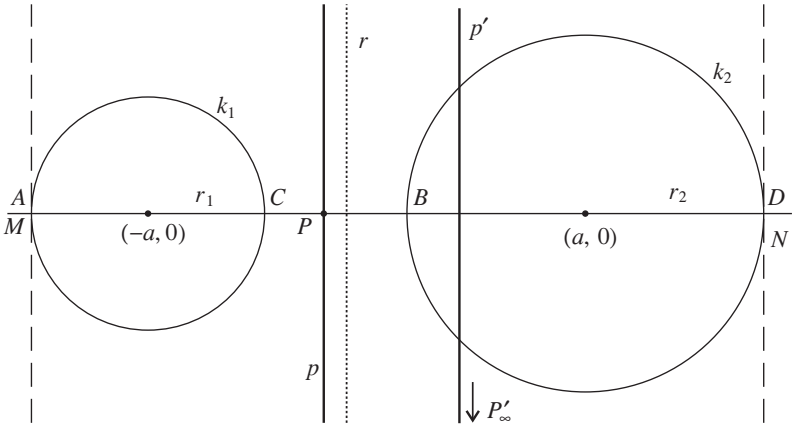


FIGURE 3: P' is at infinity

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If the line $x = p$ passes through the points B or C, then $p = a - r_2$ or $r_1 - a$, respectively, and by (6), $p' = -a - r_1$ or $a + r_2$, respectively. Then, by (8), $q' = 0$ in either case, so $P' = A$ or D, respectively, and the loci of point P' consist of a single point (see Figure 4). Within lines passing through the point A, respectively D, there is obviously the line p' tangent to the circle k_1, respectively k_2, and parallel to p.

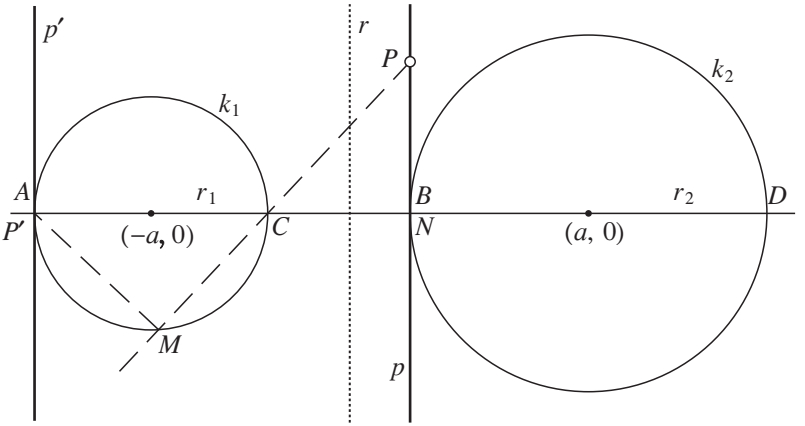


FIGURE 4: p touching k_2 at point B

Next, if $r_1 + r_2 = 2a$, the circles k_1 and k_2 touch externally (in which case $B = C$), and by (7) and (8) we have $p' = \infty$ and $q' = \infty$; therefore, the point P' is at infinity. Thus lines AM and ND are parallel (see Figure 5), resulting in the vanishing of line p'. Notice that one may conclude the same from the fact that the points P, C, B, M, N are collinear, and $\angle AMC = \angle DNB = 90°$ as angles subtended by the respective diameters.

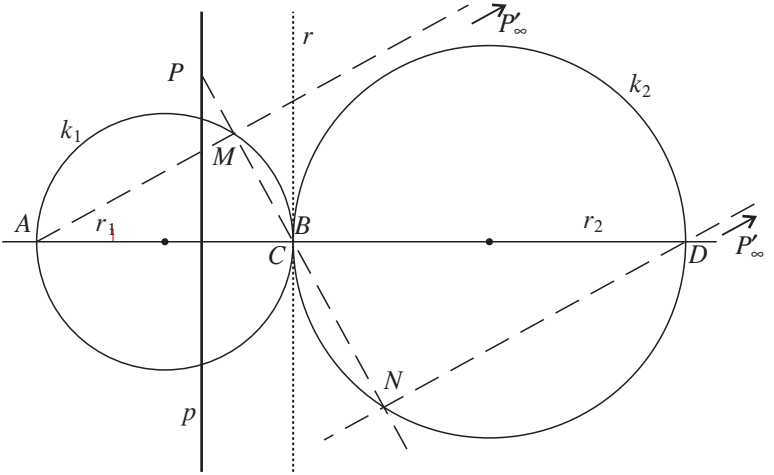


FIGURE 5: Touching circles

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Finally, the equation defining the radical axis for circles k_1 and k_2 gives us (see [12, para. 5.51 on p. 99])

$$(x + a)^2 + y^2 - r_1^2 = (x - a)^2 + y^2 - r_2^2,$$

which simplifies to

$$x = \frac{r_1^2 - r_2^2}{4a}.$$

From (7), $p = p'$ if, and only if, $p = \frac{r_1^2 - r_2^2}{4a}$, which means that if p is the radical axis of the two circles k_1 and k_2, then $p' = p$.

Remark: In the case that points on line ℓ appear in order A, B, C, D and line p passes through the intersection points of the two circles (in which case line p is the radical axis of the circles) we obtain the original IMO problem.

Note that more cases can be considered by changing the roles of points A and C, and/or B and D, placing one circle inside the other, or having one touch the other internally. We invite interested readers to investigate these other cases.

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