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Covariant kinetic theory for nonlinear plasma waves interaction

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Abstract. A rigorous and most general covariant kinetic formalism is developed to study the nonlinear-wave interaction in relativistic Vlasov plasmas. The typical nonlinear plasma reaction is a nonlinear current measured by the nonlinear plasma conductivity, and these quantities are derived here on the basis of relativistic Vlasov–Maxwell equations. Knowing the nonlinear plasma conductivity allows us to determine all plasma modes nonlinearly excited in plasma. The general covariant form of nonlinear conductivity is provided first for any value of plasma temperature and for the whole complex frequency plane by a correct analytical continuation. The further analysis is restricted to a correct relativistic particle distribution, which is vanishing for particle speeds greater than the speed of light. In the limit of non-relativistic plasma temperatures, the covariant nonlinear conductivity is significantly different from the standard non-covariant non-relativistic results, which are reached only in the formal limit of an infinitely large speed of light $c \to \infty$.

1. Introduction

Under various sources of free energy, plasma becomes a highly nonlinear dispersive medium. Numerous nonlinear wave processes occur in both laboratory and space plasmas [1]: in laser–plasma interaction the high-intensity pump waves stimulate the parametric instabilities and the nonlinear collisionless damping of plasma waves; or the shock structures arising in many astrophysical plasmas feed the well-known two-stream (electrostatic or Weibel) instabilities [2].

Even for relativistic temperatures, the collisionless (Vlasov) plasma condition applies to many astrophysical scenarios where dissipation is dominated by wave–particle interactions rather than binary collisions. In these cases, a fundamental kinetic description is required, but the existing standard non-relativistic results could be improved by the covariant relativistic approaches that reproduce the general and relativistically correct dispersion relations in any frame that is not necessarily inertial.

In the first earlier treatment [3] (hereafter referred to as Paper I) the nonlinear three-wave interaction has been investigated in the case of relativistic Vlasov plasma with an isotropic distribution function of particles. Using the relativistic Vlasov–Maxwell equations, the new relativistic forms of nonlinear current and

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nonlinear plasma conductivity have been derived. The nonlinear current is the typical nonlinear response of a plasma medium where a pump and an idler wave interact exciting a third signal wave.

The nonlinear current also seems to play an important role in the saturation stage of plasma filamentation arising in laser–plasma interaction [4] or in the jet–plasma astrophysical structures [5]: shocks in gamma-ray bursts (GRBs) sources, pulsar outflows or solar winds. The filamentation instability is the final stage of nonlinear-wave interaction when the plasma distribution function is anisotropic and which persists as long as the external perturbation is sufficiently strong. This case should be analyzed using an anisotropic bi-Maxwellian distribution function to calculate the nonlinear conductivity, and it will be discussed in the next papers of this series.

In the present paper we continue to develop a fully relativistic and covariant kinetic theory of nonlinear plasma waves interaction providing an analytical model to estimate the plasma nonlinear response. Starting from the Maxwell equations, we show in Sec. 2 that two-wave coupling leads to a nonlinear component of plasma current, which generates the third wave signal. To find the corresponding nonlinear fluctuation of plasma density as in Paper I we use the relativistic nonlinear Vlasov equation to obtain the general form of the nonlinear current arising from the interaction of two waves with any polarization and any direction of propagation.

Considering only longitudinal waves the general covariant expression of nonlinear plasma conductivity is derived in Sec. 3 for any subluminal or superluminal phase velocity and for the whole complex frequency plane. The nonlinear conductivity is then calculated in Sec. 4 using an appropriate relativistic distribution function, which is vanishing for particle speeds greater than the speed of light. Knowing the nonlinear conductivity, we could find the electric fields of interacting waves as solutions of the nonlinear (coupled) equations system (42)–(44) from Paper I.

For non-relativistic plasma temperatures in Sec. 5 we derive, apparently for the first time, the covariant expression of nonlinear conductivity in terms of the well-documented plasma dispersion function [6]. We show that the covariant form is markedly different from the standard classic expression of nonlinear conductivity provided by the non-covariant non-relativistic theory and which can only be obtained in the unphysical limit of infinite speed of light $c \to \infty$ (because the classical theory assumes all waves as being subluminal).

2. Basic framework

The basic model followed in this paper is an infinitely extended collisionless plasma of high-temperature electrons, which requires that the electrons be treated relativistically. For plasmas with low collisionality ($\nu_c/\omega_p \simeq g \ll 1$), the cooperative motion is due to the electromagnetic coupling of the particles, not to collisions. Therefore, the relativistic Vlasov–Maxwell equations are used, and for plasma fluctuations with sufficiently large amplitudes, the nonlinear forms of them are required (see notation and (13)–(17) from Paper I).

The equation governing wave propagation in plasmas is

$$\left[\nabla \times (\nabla \times) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(\mathbf{r},t) = -\frac{4\pi}{c^2}\frac{\partial \mathbf{J}(\mathbf{r},t)}{\partial t}, \tag{2.1}$$

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which follows directly from the Maxwell equations (see [3, (14)–(17)]). Wave interactions give rise to the nonlinear terms in $\mathbf{J}$:

$$\mathbf{J}(\mathbf{r},t) = \mathbf{J}^{\mathrm{L}}(\mathbf{r},t) + \mathbf{J}^{\mathrm{NL}}(\mathbf{r},t), \tag{2.2}$$

with

$$\mathbf{J}^{\mathrm{L}}(\mathbf{r},t) = \sum_i \mathbf{J}_i^{\mathrm{L}}(\mathbf{k}_i,\omega_i) = \sum_i \sigma(\omega_i)\mathbf{E}_i(\mathbf{k}_i,\omega_i), \tag{2.3}$$

and

$$\mathbf{J}^{\mathrm{NL}}(\mathbf{r},t) = \sum_{p\geqslant 2} \mathbf{J}_p^{\mathrm{NL}}(\mathbf{r},t) = \sum_u \mathbf{J}_u^{\mathrm{NL}}(\mathbf{k}_u,\omega_u) = \sum_u \mathbf{J}_u^{0,\mathrm{NL}} e^{-i(\omega_u t - \mathbf{k}_u\cdot\mathbf{r})}. \tag{2.4}$$

We assume that

$$\mathbf{E}(\mathbf{r},t) = \sum_i \mathbf{E}_i(\mathbf{k}_i,\omega_i) = \sum_i \mathbf{E}_i^0 e^{-i(\omega_i t - \mathbf{k}_i\cdot\mathbf{r})}, \tag{2.5}$$

$$\mathbf{D}(\mathbf{r},t) = \epsilon\mathbf{E}(\mathbf{r},t) = \sum_i \epsilon(\omega_i)\mathbf{E}_i(\mathbf{k}_i,\omega_i), \tag{2.6}$$

and $\mathbf{E}_i^0$ is taken as essentially independent of position in space.

For the lowest-order nonlinear phenomena ($p=2$), as a typical three-wave configuration, two waves (indices 1 and 2) give rise to the third wave (index 3) through a first generated nonlinear current:

$$\mathbf{J}_3^{\mathrm{NL}}(\mathbf{r},t) = \mathbf{J}_{12}^{\mathrm{NL}}(\mathbf{r},t) + \mathbf{J}_{21}^{\mathrm{NL}}(\mathbf{r},t). \tag{2.7}$$

With (2.3)–(2.5) and

$$\epsilon(\omega_i) \equiv 1 + \frac{4\pi i}{\omega_i}\sigma(\omega_i), \tag{2.8}$$

(2.1) becomes

$$\left[\nabla\times(\nabla\times) + \epsilon\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(\mathbf{r},t) = -\frac{4\pi}{c^2}\frac{\partial\mathbf{J}^{\mathrm{NL}}(\mathbf{r},t)}{\partial t}, \tag{2.9}$$

clearly indicating (with (2.5) and (2.7)) that wave electric fields $\mathbf{E}_i(\mathbf{k}_i,\omega_i)$ are coupled through nonlinear current $\mathbf{J}^{\mathrm{NL}}$.

2.1. Nonlinear current

The nonlinear density fluctuation f^{NL} is solution of the nonlinear Vlasov equation (see [3]) and it will determine the nonlinear current:

$$\begin{aligned}\mathbf{J}_3^{\mathrm{NL}} &= \sigma^{\mathrm{NL}}\mathbf{E}\mathbf{E} = q\int_{-\infty}^{\infty} d^3p\mathbf{v}f^{\mathrm{NL}} \\ &= -iq^2\int_{-\infty}^{\infty} d^3p\frac{\mathbf{p}}{\gamma m\omega_3 - \mathbf{k}_3\cdot\mathbf{p}}\left[\left(\mathbf{E}_1 + \frac{\mathbf{p}\times\mathbf{B}_1}{\gamma mc}\right)\cdot\frac{\partial f_2}{\partial\mathbf{p}} + (1\leftrightarrow 2)\right]\end{aligned} \tag{2.10}$$

where $(1\leftrightarrow 2)$ denotes the interchanging of indices 1 and 2 required by (2.7), and $\gamma = \sqrt{1+p^2/m^2c^2}$ is the Lorentz factor. Otherwise, the linear fluctuations of the distribution function are solutions of the linear Vlasov equation as (see [3])

$$f_i = -i\frac{q}{\omega_i R_i}\left(R_i\mathbf{E}_i + \frac{\mathbf{p}\cdot\mathbf{E}_i}{\gamma m\omega_i}\mathbf{k}_i\right)\cdot\frac{\partial f_0}{\partial\mathbf{p}} = -i\frac{q}{\omega_i R_i}\frac{\partial f_0}{\partial p}\frac{\mathbf{p}\cdot\mathbf{E}_i}{p}, \tag{2.11}$$

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where an isotropic distribution function, $f_0(\mathbf{p}) = f_0(p)$, has been assumed at equilibrium, and the relativistic resonance factors are given by

$$R_i = 1 - (\mathbf{p}\cdot\mathbf{k}_i/\gamma m\omega_i) \quad (i = 1, 2, 3). \tag{2.12}$$

From (2.11)

$$\frac{\partial f_i}{\partial \mathbf{p}} = -i\frac{q}{\omega_i R_i p}\frac{\partial f_0}{\partial p}\left\{\frac{\mathbf{p}\cdot\mathbf{E}_i}{p}\left[\left(p\frac{\partial^2 f_0/\partial p^2}{\partial f_0/\partial p} + \frac{1-R_i-\gamma^2}{\gamma^2 R_i}\right)\frac{\mathbf{p}}{p}\right.\right.$$
$$\left.\left. + \frac{1-R_i}{R_i}\frac{p\mathbf{k}_i}{\mathbf{p}\cdot\mathbf{k}_i}\right] + \mathbf{E}_i\right\}, \tag{2.13}$$

and a new expression for the density current (2.10) is derived

$$\mathbf{J}_3^{\mathrm{NL}} = \sigma^{\mathrm{NL}}\mathbf{E}_1\mathbf{E}_2 = -\frac{2\pi q^3}{m}\int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{\mathbf{p}p_\perp}{\gamma p}\frac{\partial f_0}{\partial p}\frac{\boldsymbol{\Omega}}{\omega_1\omega_2\omega_3 R_1 R_2 R_3}\mathbf{E}_1\mathbf{E}_2, \tag{2.14}$$

with the second-order tensor

$$\boldsymbol{\Omega} = \left[\left(p\frac{\partial^2 f_0/\partial p^2}{\partial f_0/\partial p} - \frac{1}{\gamma^2}\right)(\omega_1 R_1 + \omega_2 R_2)\right.$$
$$+ \left(1-\frac{1}{\gamma^2}\right)\left(\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)c^2}{\omega_1\omega_2} - 1\right)\left(\frac{\omega_1 R_1}{R_2} + \frac{\omega_2 R_2}{R_1}\right)\left.\right]\frac{\mathbf{pp}}{p^2}$$
$$+ (\omega_1 R_1^2 + \omega_2 R_2^2)\left[\mathbf{I} - \left(1-\frac{1}{R_1}\right)\frac{\mathbf{p}\mathbf{k}_1}{\mathbf{p}\cdot\mathbf{k}_1} - \left(1-\frac{1}{R_2}\right)\frac{\mathbf{k}_2\mathbf{p}}{\mathbf{p}\cdot\mathbf{k}_2}\right]. \tag{2.15}$$

Comparing with (25) from Paper I, which describes only the interaction of parallel propagating waves $\mathbf{k}_i \parallel \mathbf{k}_j$ $(i, j = 1, 2, 3)$, the new general expression (2.14) (combined with (2.15)) provides the nonlinear current arising from the interaction of two waves with any polarization and any directions of propagation.

However, for the sake of simplicity in the following we only consider the case of electrostatic (longitudinal) turbulence ($\mathbf{E}_i \parallel \mathbf{k}_i$, $i = 1, 2$) with interaction of parallel propagating waves (along the same direction) $\mathbf{k}_1 \parallel \mathbf{k}_2$.

3. Covariant case

In the general covariant case, it is convenient to transform to the new variables of integration [7], $y \equiv p_\parallel/(mc)$ and $E \equiv \sqrt{1 + (p_\parallel^2 + p_\perp^2)/m^2c^2}$, which for the corresponding components of nonlinear conductivity tensor in (2.14) yields

$$\sigma_\parallel^{\mathrm{NL}} = -2\pi q^3 mc^2 \int_1^\infty \frac{dE}{E}\frac{\partial f_0}{\partial E}\int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy\, y\frac{\Omega_{\mathrm{LL}}}{\omega_1\omega_2\omega_3 R_1 R_2 R_3}, \tag{3.1}$$

$$\sigma_\perp^{\mathrm{NL}} = -2\pi q^3 mc^2 \int_1^\infty \frac{dE}{E}\frac{\partial f_0}{\partial E}\int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \sqrt{E^2-1-y^2}\frac{\Omega_{\mathrm{LL}}}{\omega_1\omega_2\omega_3 R_1 R_2 R_3}, \tag{3.2}$$

where Ω_{LL} is the component of second-order tensor $\boldsymbol{\Omega}$ corresponding to the interaction of two longitudinal waves ($\mathbf{E}_i \parallel \mathbf{k}_i$, $i = 1, 2$):

$$\Omega_{\mathrm{LL}} = \frac{y^2}{E}\left(\frac{\partial^2 f_0}{\partial E^2}\Big/\frac{\partial f_0}{\partial E}\right)(\omega_1 R_1 + \omega_2 R_2) + \left(1-\frac{y^2}{E^2}\right)\left(\frac{\omega_1 R_1}{R_2} + \frac{\omega_2 R_2}{R_1}\right) \tag{3.3}$$

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and

$$R_i = 1 - \frac{yk_i c}{\omega_i E} = 1 - \frac{y}{z_i E} \quad (i = 1, 2, 3). \tag{3.4}$$

In (3.4) the inverse index of refraction $z = \omega/kc$ is introduced.

3.1. Reduction of $\sigma_{\parallel}^{\mathrm{NL}}$

Introducing (3.3) in (3.1) we obtain

$$\sigma_{\parallel}^{\mathrm{NL}} = -2\pi q^3 mc^2 (I_1 + I_2) \tag{3.5}$$

with

$$I_1 = \int_1^{\infty} \frac{dE}{E^2} \frac{\partial^2 f_0}{\partial E^2} \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy\, y^3 \frac{1}{\omega_1 \omega_3 R_1 R_3} + (1 \leftrightarrow 2) \tag{3.6}$$

$$I_2 = \int_1^{\infty} \frac{dE}{E} \frac{\partial f_0}{\partial E} \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy\, y \left(1 - \frac{y^2}{E^2}\right) \frac{1}{\omega_1 \omega_3 R_1^2 R_3} + (1 \leftrightarrow 2) \tag{3.7}$$

where $(1 \leftrightarrow 2)$ denotes the interchanging of indices 1 and 2. In terms of the functions $u = u(E)$ and $v = v(E)$ defined by

$$\frac{\partial u}{\partial E} = -E^2 \frac{\partial^2 f_0}{\partial E^2}, \quad \frac{\partial v}{\partial E} = -E \frac{\partial f_0}{\partial E} \tag{3.8}$$

and using the substitution $t = y/E$ the integrals (3.6) and (3.7) from above read

$$\begin{aligned} I_1 &= -\frac{z_1}{\omega_1 \omega_3} \int_1^{\infty} dE \frac{\partial u}{\partial E} \int_{-\sqrt{1-E^{-2}}}^{\sqrt{1-E^{-2}}} dt \frac{t^3}{(z_1 - t)(z_3 - t)} + (1 \leftrightarrow 2) \\ &= \frac{2z_1}{\omega_1 \omega_3 (z_3 - z_1)} \int_1^{\infty} dE \frac{u\sqrt{E^2-1}}{E^2} \left[\frac{z_1^2}{(z_1^2 - 1)E^2 + 1} - \frac{z_3^2}{(z_3^2 - 1)E^2 + 1} \right] \\ &\quad + (1 \leftrightarrow 2) \end{aligned} \tag{3.9}$$

$$\begin{aligned} I_2 &= -\frac{z_1^2 z_3}{\omega_1 \omega_3} \int_1^{\infty} dE \frac{\partial v}{\partial E} \int_{-\sqrt{1-E^{-2}}}^{\sqrt{1-E^{-2}}} dt \frac{t(1 - t^2)}{(z_1 - t)^2 (z_3 - t)} + (1 \leftrightarrow 2) \\ &= \frac{2z_1^2 z_3}{\omega_1 \omega_3 (z_3 - z_1)^2} \int_1^{\infty} dE \frac{v\sqrt{E^2-1}}{E^4} \\ &\quad \times \left\{ \frac{2z_1 (z_3 - z_1) E^2}{[(z_1^2 - 1)E^2 + 1]^2} - \frac{1}{(z_1^2 - 1)E^2 + 1} + \frac{1}{(z_3^2 - 1)E^2 + 1} \right\} + (1 \leftrightarrow 2). \end{aligned} \tag{3.10}$$

From (3.8)

$$u(E) = E \frac{\partial v(E)}{\partial E} - 2v(E), \tag{3.11}$$

which is used in (3.9) to find

$$\begin{aligned} I_1 = \frac{2z_1}{\omega_1 \omega_3 (z_3 - z_1)} \Bigg\{ & \int_1^{\infty} dE \frac{v}{E^2 \sqrt{E^2-1}} \left[\frac{z_3^2}{(z_3^2 - 1)E^2 + 1} - \frac{z_1^2}{(z_1^2 - 1)E^2 + 1} \right] \\ & + 2 \int_1^{\infty} dE \frac{v\sqrt{E^2-1}}{E^2} \left[\frac{z_3^2}{[(z_3^2 - 1)E^2 + 1]^2} - \frac{z_1^2}{[(z_1^2 - 1)E^2 + 1]^2} \right] \Bigg\} + (1 \leftrightarrow 2) \end{aligned} \tag{3.12}$$

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and collecting (3.12) and (3.10), the component (3.5) of nonlinear conductivity finally reads

$$\sigma_{\parallel}^{+,\mathrm{NL}} = \frac{4\pi q^3 mc^2}{\omega_1\omega_3}\frac{z_1}{z_3-z_1}\left\{\left(1+\frac{z_1z_3}{z_3-z_1}\right)\int_1^\infty dE\frac{v(E)\sqrt{E^2-1}}{E^4}\right.$$
$$\times\left[\frac{1}{(1-z_3^2)E^2-1}-\frac{1}{(1-z_1^2)E^2-1}\right]$$
$$\left.-2\int_1^\infty dE\frac{v(E)\sqrt{E^2-1}}{E^2}\left[\frac{z_3^2}{[(1-z_3^2)E^2-1]^2}-\frac{z_1^2(1-z_3)}{[(1-z_1^2)E^2-1]^2}\right]\right\}$$
$$+(1\leftrightarrow 2) \tag{3.13}$$

with $v(E)$ defined by the isotropic distribution function $f_0(E)$ in (3.8).

Relation (3.13) is the general covariant form of longitudinal conductivity plasma response to the interaction of two (longitudinal) plasma waves. We can use it to find the expressions of plasma waves solving the nonlinear equation system (42)–(44) from Paper I. However, relation (3.13) is not valid for the whole complex frequency plane because it has been obtained with the assumption of a positive imaginary part, $\Gamma > 0$, of frequency $\omega = \omega_r + \imath\Gamma$ (see also Paper I). The index '+' in (3.13) indicates this condition.

3.2. Analytic continuation for subluminal waves

As we noted above, the relation (3.13) holds for positive values of the imaginary part of the frequency $\Gamma = \Im\omega > 0$, corresponding to $\Im z > 0$. In order to derive the corresponding dispersion relations for negative values of the imaginary part of frequency, $\Gamma < 0$ (i.e. $\Im z < 0$), we have to analytically continue the integrals in (3.13) into the negative imaginary plane of the complex variable z.

For superluminal waves generally defined in unmagnetized plasma by $|z_r| = |v_{\mathrm{phase}}/c| > 1$, all the integrals in (3.13) admit no pole inside the integration interval, and we simply have

$$\sigma_{\parallel}^{-,\mathrm{NL}} = \sigma_{\parallel}^{+,\mathrm{NL}}, \tag{3.14}$$

which also holds for plasma waves with subluminal phase velocities $|z_r| < 1$, but with susceptible growing amplitudes $\Gamma > 0$.

For subluminal damped waves with $|z_r| < 1$ and $\Gamma < 0$ each of the integrals in (3.13) admits the pole $E_{ci} = 1/\sqrt{1-z_i^2} > 1$ inside the integration interval $[1,\infty]$, and the analytical continuation of (3.13) into negative imaginary plane (negative index) reads as follows

$$\sigma_{\parallel}^{-,\mathrm{NL}} = \sigma_{\parallel}^{+,\mathrm{NL}} + \imath\frac{4\pi^2 q^3 mc^2}{\omega_1\omega_3}\frac{z_1}{z_3-z_1}$$
$$\times\left\{v(E_{c3})z_3^3\sqrt{1-z_3^2}\left[(1-z_3^2)\left(1+\frac{z_1z_3}{z_3-z_1}\right)-1\right]v(E_{c1})z_1^3\sqrt{1-z_1^2}\right.$$
$$\left.\times\left[(1-z_1^2)\left(1+\frac{z_1z_3}{z_3-z_1}\right)-1+z_3\right]\right\}+(1\leftrightarrow 2). \tag{3.15}$$

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3.3. Interlude

Combining now (3.13) and (3.15) we derive the general form of the longitudinal conductivity

$$\sigma_{\parallel}^{\mathrm{NL}} = \sigma_{\parallel}^{+,\mathrm{NL}} + \imath\frac{4\pi^2 q^3 mc^2}{\omega_1\omega_3}\frac{z_1}{z_3 - z_1}$$
$$\times\left\{H(1-|\Re z_3|)H(-\Im z_3)v(E_{c3})z_3^3\sqrt{1-z_3^2}\left[(1-z_3^2)\left(1+\frac{z_1 z_3}{z_3-z_1}\right)-1\right]\right.$$
$$-H(1-|\Re z_1|)H(-\Im z_1)v(E_{c1})z_1^3\sqrt{1-z_1^2}$$
$$\left.\times\left[(1-z_1^2)\left(1+\frac{z_1 z_3}{z_3-z_1}\right)-1+z_3\right]\right\}+(1\leftrightarrow 2). \tag{3.16}$$

holding for the whole complex frequency plane, i.e. for interaction of any growing or damped waves. The frequency and the wavenumber of signal wave (noted here with index 3) are unknown in the conductivity expression, but they are usually provided by the conservation laws

$$\omega_3 = \omega_1 \pm \omega_2, \quad \mathbf{k}_3 = \mathbf{k}_1 \pm \mathbf{k}_2. \tag{3.17}$$

4. Relativistic Maxwellian plasma

The isotropic equilibrium distribution function f_0 is assumed to be Maxwellian. However, a rigorous relativistic analysis requires an appropriate relativistic distribution function, which is vanishing for particle speeds greater than the speed of light. We therefore consider the Maxwell–Boltzmann–Jüttner distribution function

$$f_0(p) = Ce^{-\mu E}, \tag{4.1}$$

with a normalization constant

$$C = \frac{n_0\mu}{4\pi(mc)^3 K_2(\mu)}, \quad \mu = \frac{mc^2}{k_{\mathrm{B}}T}, \tag{4.2}$$

so that $\int d^3p\, f_0 = n_0$. $K_\nu(\mu)$ denotes the modified Bessel function. We use (4.1) in the second equation of (3.8) to find

$$v = -\left(E + a\frac{1}{\mu}\right)Ce^{-\mu E}, \tag{4.3}$$

which can be introduced in (3.13)

$$\sigma_{\parallel}^{+,\mathrm{NL}} = \frac{4\pi q^3 mc^2}{\omega_1\omega_3}\frac{z_1}{z_3-z_1}\left[\left(1+\frac{z_1 z_3}{z_3-z_1}\right)J_1 - 2J_2\right] + (1\leftrightarrow 2) \tag{4.4}$$

to calculate the integrals:

$$J_1^+ = -C\int_1^\infty dE\frac{e^{-\mu E}\sqrt{E^2-1}}{E^4}$$
$$\times\left(E+\frac{1}{\mu}\right)\left[\frac{1}{(1-z_3^2)E^2-1} - \frac{1}{(1-z_1^2)E^2-1}\right] \tag{4.5}$$

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$$J_2^+ = -C\int_1^\infty dE \frac{e^{-\mu E}\sqrt{E^2-1}}{E^2} \times \left(E+\frac{1}{\mu}\right)\left[\frac{z_3^2}{[(1-z_3^2)E^2-1]^2} - \frac{z_1^2(1-z_3)}{[(1-z_1^2)E^2-1]^2}\right]. \tag{4.6}$$

5. Non-relativistic thermal plasmas ($\mu \gg 1$)

We now consider the limit of non-relativistic plasma temperatures $\mu \gg 1$, and to lowest order in $\mu^{-1} \ll 1$

$$K_2(\mu) \simeq \sqrt{\frac{\pi}{2(\mu)}}e^{-(\mu)}, \quad C = \left(\frac{\mu}{2\pi}\right)^{3/2}\frac{n_0}{(mc)^3}e^{\mu} \tag{5.1}$$

and with the substitution $E = \sqrt{1+s^2}$ the first integral (4.5) becomes

$$J_1^+ = -\left(\frac{\mu}{2\pi}\right)^{3/2}\frac{n_0}{(mc)^3}e^{\mu}\int_0^\infty ds \frac{s^2 e^{-\mu\sqrt{1+s^2}}}{(1+s^2)^2}\left[\frac{1}{(1-z_3^2)s^2-z_3^2} - \frac{1}{(1-z_1^2)s^2-z_1^2}\right]. \tag{5.2}$$

Owing to the exponential function the main contribution to the integral for large values of $\mu_a \gg 1$ comes from small values of $s \ll 1$, so that we may approximate $\sqrt{1+s^2}-1 \simeq s^2/2$ yielding

$$\begin{aligned} J_1^+ &= -\left(\frac{\mu}{2\pi}\right)^{3/2}\frac{n_0}{(mc)^3} \\ &\quad\times\left\{\frac{1}{1-z_3^2}\left[\int_0^\infty ds\, e^{-\mu s^2/2} + \frac{z_3^2}{1-z_3^2}\int_0^\infty ds \frac{e^{-\mu s^2/2}}{s^2 - z_3^2/(1-z_3^2)}\right] - (3 \leftrightarrow 1)\right\} \\ &= -\frac{n_0\mu}{2^{3/2}\pi(mc)^3}\left[\frac{f_3 Z^+(f_3)}{1-z_3^2} - \frac{f_1 Z^+(f_1)}{1-z_1^2} + \frac{z_3^2-z_1^2}{(1-z_1^2)(1-z_3^2)}\right], \end{aligned} \tag{5.3}$$

where we have used (in the positive imaginary frequency plane) the plasma dispersion function of Fried and Conte [6]

$$Z^+(f) = \pi^{-1/2}\int_{-\infty}^\infty dx \frac{e^{-x^2}}{x-f}, \tag{5.4}$$

of the argument

$$f_i = \sqrt{\frac{\mu}{2}}\frac{z_i}{\sqrt{1-z_i^2}}. \tag{5.5}$$

The second integral (4.6) is calculated in the same manner as

$$J_2^+ = \frac{n_0\mu}{2^{3/2}\pi(mc)^3}\left[\frac{f_3 Z^+(f_3/2)}{4(1-z_3^2)} - \frac{(1-z_3)f_1 Z^+(f_1/2)}{4(1-z_1^2)} + \frac{z_3^2-z_1^2+z_3(1-z_3^2)}{2(1-z_1^2)(1-z_3^2)}\right]. \tag{5.6}$$

For non-relativistic plasma temperatures ($\mu \gg 1$) we can assume that $|f| \gg 1$ for which (see the asymptotic approximation of plasma dispersion function [6]) $Z^+(f/2) \simeq 2Z^+(f)$ and after summation of the integrals (5.3) and (5.6) in (4.4)

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we obtain

$$\sigma_{\parallel}^{+,\mathrm{NL}} \simeq -\frac{n_0 q^3 \mu}{\sqrt{2} m^2 c \omega_1 \omega_3} \frac{z_1}{z_3 - z_1} \left(1 + \frac{z_1 z_3}{z_3 - z_1}\right) J^+ + (1 \leftrightarrow 2) \tag{5.7}$$

with J^+ given by

$$J^+ = \frac{f_3 Z^+(f_3)}{1 - z_3^2} - \frac{(1 + z_3) f_1 Z^+(f_1)}{1 - z_1^2} + \frac{z_3^2 - z_1^2 - z_3(1 - z_3^2)}{(1 - z_1^2)(1 - z_3^2)}. \tag{5.8}$$

For negative imaginary frequencies we can find $\sigma_{\parallel}^{-,\mathrm{NL}}$ substituting (4.3) in (3.15), or it is easier to use in (5.8) the well-known analytic continuation of the plasma dispersion function into the negative imaginary frequency plane ($\Gamma < 0$) [8]

$$Z^-(f) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - f} + 2\imath\sqrt{\pi} e^{-f^2}, \quad \Im(f) < 0, \tag{5.9}$$

and the plasma waves are covariantly described now for all complex frequencies by

$$\sigma_{\parallel}^{\mathrm{NL}} = -\frac{n_0 q^3 \mu}{\sqrt{2} m^2 c \omega_1 \omega_3} \frac{z_1}{z_3 - z_1} \left(1 + \frac{z_1 z_3}{z_3 - z_1}\right) J + (1 \leftrightarrow 2) \tag{5.10}$$

where J is given by

$$J = J^+ + 2\imath\sqrt{\pi} \left[H(-\Im z_3) \frac{f_3}{1 - z_3^2} e^{-f_3^2} - H(-\Im z_1) \frac{(1 + z_3) f_1}{1 - z_1^2} e^{-f_1^2} \right]. \tag{5.11}$$

5.1. Limit infinite speed of light $c \to \infty$

We finally consider the plasma-wave interaction in the formal limit of an infinitely large speed of light, $c \to \infty$, which corresponds to the classical non-covariant non-relativistic theory and where all waves have subluminal phase velocities. From (5.7)–(5.11) we obtain in this limit

$$\begin{aligned} \sigma_{\parallel}^{\mathrm{NL},\infty} = & -\sqrt{2} \frac{n_0 q^3 c}{m^2 v_{\mathrm{th}}^2} \frac{1}{\omega_1 \omega_3} \left(\frac{k_1}{\omega_1} \frac{\omega_3}{k_3} - 1 \right)^{-1} \\ & \times [f_3^{\infty} (Z^+(f_3^{\infty}) + 2\imath\sqrt{\pi} H(-\Im\omega_3) e^{-(f_3^{\infty})^2}) \\ & - f_1^{\infty} (Z^+(f_1^{\infty}) + 2\imath\sqrt{\pi} H(-\Im\omega_1) e^{-(f_1^{\infty})^2})] + (1 \leftrightarrow 2) \end{aligned} \tag{5.12}$$

with

$$v_{\mathrm{th}}^2 = \frac{2 k_{\mathrm{B}} T}{m}, \quad f_i^{\infty} = \frac{\omega_i}{k_i v_{\mathrm{th}}}. \tag{5.13}$$

and which holds for all complex frequencies and for all wavenumbers.

Assuming again $|f^{\infty}| \gg 1$, the exponential functions from the residues in (5.12) are very small ($e^{-(f^{\infty})^2} \ll 1$) and the asymptotic approximation of plasma dispersion function in (5.12) leads to

$$\sigma_{\parallel}^{\mathrm{NL},\infty} \simeq -\frac{n_0 q^3 N_3 (N_1 + N_3)}{\sqrt{2} m^2 c \omega_1 \omega_3} + (1 \leftrightarrow 2), \tag{5.14}$$

which agrees with the standard results [9] and where $N = kc/\omega$ is the index of refraction.

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6. Conclusion

On the basis of relativistic Vlasov–Maxwell equations, we have developed a covariant kinetic formalism to determine the nonlinear plasma conductivity that allows us to find all the plasma waves nonlinearly excited in plasma. The general covariant form of nonlinear conductivity is provided first for any value of plasma temperature and for the whole complex frequency plane by a correct analytical continuation. Then, we have restricted the analysis to an appropriate relativistic particle distribution that is vanishing for particle speeds greater than the speed of light. In the limit of non-relativistic plasma temperatures, we have derived (apparently for the first time) the covariant nonlinear conductivity, which is significantly different from the standard non-covariant non-relativistic results. Only in the strictly unphysical formal limit of an infinitely large speed of light $c \to \infty$ do the covariant forms reduce to the standard expressions.

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