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Non-embedding theorems for Y-spaces

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1. *Introduction.* Let X be a compact differentiable manifold of dimension $2m$. A differentiable map from X to euclidean $(2m+t)$-space is an *immersion* if its Jacobian has rank $2m$ at each point of X; it is an *embedding* if it is also one–one. The existence of such an embedding or immersion implies that the characteristic classes of X satisfy certain integrality conditions; these can be used to obtain lower bounds for the integer t. In a similar way many other geometric properties of X can be deduced from a single integrality theorem involving characteristic classes of various vector bundles over X (see for instance (5)).

Now suppose that X has a more complicated structure, and that we are interested in geometric problems involving that structure. It is often possible to find suitable more general integrality theorems, but in many cases these are too complicated to allow explicit calculations. The purpose of this paper is to study one generalization for which explicit calculations are possible. We replace X by a fibre bundle W with fibre X over a $2n$-dimensional differentiable manifold Y, and study problems in which the fibred structure of W is preserved. For example, we consider vector bundles Z over Y with $(2m+t)$-dimensional fibres, and ask whether there exist immersions or embeddings of W in Z which respect the bundle projections onto Y. We show that such information can be deduced from an integrality theorem, and that this theorem can be expressed as a theorem involving cohomology classes of the base manifold Y. These results may be regarded as replacing the original integrality theorem for X by a 'family' of such theorems with 'parameter space' Y. When Y is a point we recover the original theorem.

The layout of the paper is as follows. In section 2 we prove the general integrality theorem, and deduce results concerning tangent vector fields, immersions and embeddings. In section 3 we make a number of calculations involving projective fibre bundles. In section 4 we justify our claim that calculations of the integrality conditions are feasible by obtaining examples of geometric results concerning projective fibre bundles.

We use standard notations concerning vector bundles and characteristic classes with the following modifications. An element of the free group generated by isomorphism classes of vector bundles is called a *virtual vector bundle*; $c(\xi)$ and $\mathrm{ch}(\xi)$ denote respectively the total Chern class and total Chern character of a virtual complex vector bundle ξ; similarly $p(\zeta)$ denotes the total Pontrjagin class of a virtual real vector bundle ζ, and $e(\nu)$ denotes the Euler class of an oriented real vector bundle ν. The methods used in our calculations are closely related to those of Atiyah–Hirzebruch (3),

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Roberts (6), and Sanderson–Schwarzenberger (7). We would also like to acknowledge the opportunity for this research provided by the Warwick Topology Symposium and the Bonn Arbeitstagung.

2. *Integrality theorems for Y-spaces.* Throughout this paragraph we assume that Y is a compact oriented differentiable manifold of dimension $2n$ without boundary, that $d \in H^2(Y, Z)$, that β is a virtual complex vector bundle over Y, and that ζ is a virtual real orientable vector bundle over Y with fibre dimension $t = 2s$ or $2s+1$. Then we define

$$\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) = 2^s e^{\frac{1}{2}d} \operatorname{ch} \beta \prod_j \frac{\frac{1}{2}y_j}{\sinh \frac{1}{2}y_j} \prod_k \cosh \tfrac{1}{2}z_k,$$

$$\hat{A}(Y, \tfrac{1}{2}d, \beta, \zeta) = \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta)[Y],$$

where $$p(Y) = \prod_j (1+y_j^2) \quad \text{and} \quad p(\zeta) = \prod_k (1+z_k^2).$$

With these notations we have the following integrality theorem (see (5) and (6)):

THEOREM 2·1. *If ζ is a vector bundle over Y and $d = w_2(Y) + w_2(\zeta) \bmod 2$ then $\hat{A}(Y, \frac{1}{2}d, \beta, \zeta)$ is an integer. If also $t = 2s$ and $e(\zeta) = 0$ then $\hat{A}(Y, \frac{1}{2}d, \beta, \zeta)$ is an even integer.*

We shall apply Theorem 2·1 in a slightly more complicated situation. Suppose that $\pi: W \to Y$ is a differentiable fibre bundle with fibre X, where W and X are compact oriented differentiable manifolds of dimensions $2(m+n)$ and $2m$ respectively. Let ϕ be the vector bundle over W of tangents along the fibres, and $p(\phi) = \prod_i (1+x_i^2)$. If $f \in H^2(W, Z)$ and γ is a virtual complex vector bundle over W then for each integer r we may define the total class

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, r) = \pi_* \left(2^{rm} e^{\frac{1}{2}f} \operatorname{ch} \gamma \prod_i \frac{x_i (\cosh \frac{1}{2}x_i)^r}{2 \sinh \frac{1}{2}x_i} \right)$$

in $H^*(Y, Q)$. *All these notations will be taken for granted in the rest of this paragraph.*

THEOREM 2·2. *If $\nu = \pi^! \zeta + r\phi$ is a vector bundle over W, and*

$$d \equiv w_2(Y) + w_2(\zeta) \bmod 2, \quad f \equiv (r+1) w_2(\phi) \bmod 2$$

then $$(\mathscr{B}(W, \tfrac{1}{2}f, \gamma, r)\, \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta))[Y]$$

is an integer. If also $t = 2s$ and $e(\nu) = 0$ then $(\mathscr{B}(W, \frac{1}{2}f, \gamma, r)\, \hat{\mathscr{A}}(Y, \frac{1}{2}d, \beta, \zeta))[Y]$ is an even integer.

Proof. If $f \equiv (r+1) w_2(\phi) \bmod 2$ then

$$f + \pi^* d \equiv (r+1) w_2(\phi) + \pi^* w_2(Y) + \pi^* w_2(\zeta) \equiv w_2(W) + w_2(\nu) \bmod 2$$

and therefore Theorem 2·1 implies that $\hat{A}(W, \frac{1}{2}(f + \pi^* d), \gamma + \pi^! \beta, \nu)$ is an integer (and is even if $t = 2s$ and $e(\nu) = 0$). The result now follows from the fact that

$$\begin{aligned}
&\hat{\mathscr{A}}(W, \tfrac{1}{2}(f + \pi^* d), \gamma + \pi^! \beta, \nu)[W] \\
&= \left(2^{rm} e^{\frac{1}{2}f} \operatorname{ch} \gamma \prod_i \frac{x_i (\cosh \frac{1}{2}x_i)^r}{2 \sinh \frac{1}{2}x_i} \pi^* \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) \right)[W] \\
&= \left(\pi_* \left(2^{rm} e^{\frac{1}{2}f} \operatorname{ch} \gamma \prod_i \frac{x_i (\cosh \frac{1}{2}x_i)^r}{2 \sinh \frac{1}{2}x_i} \right) \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) \right)[Y] \\
&= (\mathscr{B}(W, \tfrac{1}{2}f, \gamma, r)\, \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta))[Y].
\end{aligned}$$

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We consider the special cases $r = 1, 0, -1$ one by one.

The case $r = 1$.

The condition that $\nu = \pi^!\zeta + r\phi$ is a vector bundle becomes: there exists a vector bundle ν such that $\phi = \nu + \pi^!(-\zeta)$. We have

$$\begin{aligned}\mathscr{B}(W, \tfrac{1}{2}f, \gamma, 1) &= \pi_*\left(2^m \operatorname{ch}\gamma\, e^{\frac{1}{2}f} \prod_i \frac{\frac{1}{2}x_i}{\tanh \frac{1}{2}x_i}\right)\\ &= 2^m\pi_*\left(\left[\operatorname{ch}\gamma^{(2)} e^f \prod_i \frac{x_i}{\tanh x_i}\right]^{(\frac{1}{2})}\right)\\ &= \left[\pi_*\left(\operatorname{ch}\gamma^{(2)} e^f \prod_i \frac{x_i}{\tanh x_i}\right)\right]^{(\frac{1}{2})}\end{aligned}$$

where, if $t \in Q$ and $z = \sum_i z_i$ is a cohomology class with $z_i \in H^{2i}(Y, Q)$, we write $z^{(t)} = \sum_i t^i z_i$.

For applications of this case to finding the generalized geometric dimension of the tangent bundle along the fibres, we assume that $\rho = -\zeta$ is a real orientable vector bundle with fibre dimension $r = -t$, where $t = 2s$ or $2s+1$. If $\phi - \pi^!\rho$ is a vector bundle we say that W 'admits a stable vector field along the fibres of type ρ'.

THEOREM 2·3. *If* $d \equiv w_2(Y) + w_2(\rho) \bmod 2$, $f \equiv 0 \bmod 2$, *and* W *admits a stable vector field along the fibres of type* ρ *then*

$$(\mathscr{B}(w, \tfrac{1}{2}f, \gamma, 1)\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, -\rho))[Y]$$

is an integer.

As a corollary we have

THEOREM 2·4. *Let* α *be a virtual complex vector bundle over* X, *and*

$$d \equiv w_2(Y) + w_2(\rho) \bmod 2.$$

If $Y \times X$ *admits a vector field along the fibres of type* ρ *then*

$$L(X, \alpha^{(2)})\,\hat{A}(Y, \tfrac{1}{2}d, \beta, -\rho)$$

is an integer, where

$$L(X, \alpha^{(2)}) = \left(\operatorname{ch}\alpha^{(2)} \prod_i \frac{x_i}{\tanh x_i}\right)[X].$$

Proof. Apply Theorem 2·3 to $W = Y \times X$ with $f = 0$ and γ the lift to $Y \times X$ of α. Then

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, 1) = L(X, \alpha^{(2)})\,1,$$

where $1 \in H^0(Y, Q)$ is the identity element.

Remark. If Y is a point then this is a known result. If the stable tangent bundle of X has geometric dimension $\leqslant 2m - r$, where $r = -2s$ or $-2s-1$, then $2^s L(X, \alpha^{(2)})$ is an integer.

The case $r = 0$.

The condition that $\nu = \pi^!\zeta + r\phi$ is a vector bundle becomes: $\pi^!\zeta$ is a vector bundle. Therefore we might expect that integrality conditions on ν can be deduced from integrality conditions on ζ whenever the latter is a vector bundle. This is indeed the case because

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, 0) = \pi_*\left(e^{\frac{1}{2}f} \operatorname{ch}\gamma \prod_i \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i}\right)$$

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and hence the Riemann–Roch Theorem for differentiable manifolds (2) implies that if $f \equiv w_2(\phi) \bmod 2$ then

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, 0)\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) = \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta \otimes \pi_!\gamma, \zeta).$$

Thus in this case the first part of Theorem 2·2 contains no more information than the first part of Theorem 2·1.

The case $r = -1$.

The condition that $\nu = \pi^!\zeta + r\phi$ is a vector bundle becomes: there exists a vector bundle ν such that $\nu + \phi = \pi^!\zeta$. We have

$$\begin{aligned}\mathscr{B}(W, \tfrac{1}{2}f, \gamma, -1) &= \pi_*\left(2^{-m} e^{\frac{1}{2}f} \operatorname{ch}\gamma \prod_i \frac{x_i}{\sinh x_i}\right) \\ &= 2^{-m}\pi_*\left(\left[e^{\frac{1}{2}f} \operatorname{ch}\gamma^{(\frac{1}{2})} \prod_i \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i}\right]^{(2)}\right) \\ &= \left[\pi_*\left(e^{\frac{1}{2}f} \operatorname{ch}\gamma^{(\frac{1}{2})} \prod_i \frac{\frac{1}{2}x_i}{\sinh \frac{1}{2}x_i}\right)\right]^{(2)}.\end{aligned}$$

Thus, if $e^{\frac{1}{2}f} \operatorname{ch}\gamma = e^h \operatorname{ch}\zeta^{(2)}$ where $h \equiv w_2(\phi) \bmod 2$ and ζ is a virtual complex vector bundle over W then by the Riemann–Roch Theorem

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, -1) = [\operatorname{ch}\pi_!\zeta]^{(2)} = \operatorname{ch}(\pi_!\zeta)^{(2)},$$

$$\mathscr{B}(W, \tfrac{1}{2}f, \gamma, -1)\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) = \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta \otimes (\pi_!\zeta)^{(2)}, \zeta).$$

Therefore if $e^{\frac{1}{2}f} \operatorname{ch}\gamma = e^h \operatorname{ch}\gamma^{(2)}$, where $h \equiv w_2(\phi)$, the first part of Theorem 2·2 contains no more information than the first part of Theorem 2·1.

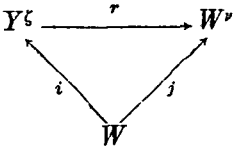
For the applications of this case to immersions and embeddings we assume that ζ is a real orientable vector bundle over Y with total space Z and fibre dimension $t = 2s$ or $2s+1$. We consider a differentiable fibre bundle $\pi\colon W \to Y$ with compact fibre X as above.

THEOREM 2·5. *If $W \to Z$ is a Y-immersion then there is a vector bundle ν over W such that $\phi + \nu = \pi^!\zeta$. If also $2m = \dim X > 0$ and $W \to Z$ is an embedding then $e(\nu) = 0$.*

Proof. The Y-immersion $W \to Z$ induces a monomorphism $\phi \to \pi^!\zeta$ with a quotient bundle ν. Clearly ν is isomorphic to the normal bundle of W in Z. If $W \to Z$ is an embedding then the total space of ν can be identified with a tubular neighbourhood N of W in Z. Let $\bar{N}, \dot{N}$ denote the closure and boundary of N respectively, so that $\bar{N}/\dot{N} = W^\nu$ is the Thom complex of ν. Let $\bar{Z}, \dot{Z}$ denote disc and sphere bundles associated to ζ (with respect to a metric such that $\bar{N} \subset \bar{Z}$). Then $\bar{Z}/\dot{Z} = Y^\zeta$ and there is a map

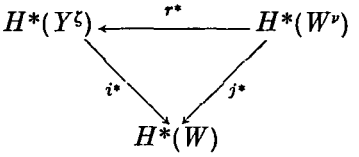
$$r\colon Y^\zeta \to W^\nu$$

obtained by collapsing the complement of N in Z to a point. There are embeddings $i\colon W \to Y^\zeta$ induced by the embedding $W \to Z$, and $j\colon W \to W^\nu$ induced by the zero section of ν, such that the diagram



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is commutative. Consider the corresponding diagram of integer cohomology groups



Let $u \in H^*(W^\nu)$ be such that $j^*u = e(\nu)$. Then $e(\nu) = i^*r^*u = i^*v$, where $v \in H^{l-2m}(Y^\zeta)$. But by the Thom isomorphism theorem $H^{l-2m}(Y^\zeta) \simeq H^{-2m}(Y)$ is zero for $m > 0$. It follows that $e(\nu) = 0$ for $m > 0$.

Remark. It is possible to have an embedding $W \subset Z$ with $e(\nu) \neq 0$ either if $\dim X = 0$ or if X is non-compact. For the first example let $W = Y$ and consider the zero section $Y \to Z$ of some vector bundle ζ over Y with $e(\zeta) \neq 0$. For the second example let W be the total space of a vector bundle ω over Y, and consider a vector bundle ξ over Y with $\omega + \xi = \zeta$ and $e(\xi) \neq 0$.

Theorems 2·2 and 2·5 now imply

THEOREM 2·6. *If* $d \equiv w_2(Y) + w_2(\zeta) \bmod 2$ *and* $W \to Z$ *is a* Y*-immersion then* $(\mathscr{B}(W, f, \gamma, -1)\hat{\mathscr{A}}(Y, \frac{1}{2}d, \beta, \zeta))[Y]$ *is an integer. If also* $m > 0$, $t = 2s$ *and* $W \to Z$ *is an embedding then* $(\mathscr{B}(W, f, \gamma, -1)\hat{\mathscr{A}}(Y, \frac{1}{2}d, \beta, \zeta))[Y]$ *is an even integer.*

As a corollary we have

THEOREM 2·7. *Let* α *be a virtual complex vector bundle over* X, *and*

$$d \equiv w_2(Y) + w_2(\zeta) \bmod 2.$$

If $Y \times X \to Z$ *is a* Y*-immersion then*

$$\hat{A}(X, 0, \alpha^{(\frac{1}{2})}, 1)\, \hat{A}(Y, \tfrac{1}{2}d, \beta, \zeta)$$

is an integer. If also $m > 0$, $t = 2s$ *and* $Y \times X \to Z$ *is an embedding then*

$$\hat{A}(X, 0, \alpha^{(\frac{1}{2})}, 1)\, \hat{A}(Y, \tfrac{1}{2}d, \beta, \zeta)$$

is an even integer.

Proof. Apply Theorem 2·6 to $W = Y \times X$ with $f = 0$ and γ the lift to $Y \times X$ of α. Then

$$\mathscr{B}(W, f, \gamma, -1) = \hat{A}(X, 0, \alpha^{(\frac{1}{2})}, 1)\, 1,$$

where $1 \in H^0(Y, Q)$ is the identity element.

Remark. If Y is a point then this is essentially Theorem 4·3 of (5).

3. *Projective fibre bundles over* Y. In this paragraph we show how the integrality conditions given by Theorems 2·3 and 2·6 can be calculated explicitly when $\pi: W \to Y$ is a complex or quaternionic projective fibre bundle.

3·1. We first suppose that ξ is a complex vector bundle over Y with fibre C^{m+1}, and that $\pi: W \to Y$ is the associated $P_m(C)$-bundle. In this case there is a complex line bundle θ over W and an exact sequence

$$0 \to \theta \to \pi^!\xi \to \psi \to 0,$$

where ψ is a complex vector bundle, and where the underlying real bundle of $\psi \otimes \theta^*$ is the tangent bundle ϕ along the fibres.

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We make one change to the notations of the previous paragraph: the total Chern class of ξ is written

$$c(\xi) = \prod_{i=1}^{m+1} (1+x_i),$$

so that now we have $c(\theta) = 1+h$ and

$$p(\phi) = \prod_{i=1}^{m+1} (1+(x_i-h)^2).$$

Since ψ is a C^m-bundle there is a relation

$$0 = c_{m+1}(\psi) = \prod_{i=1}^{m+1} (\pi^* x_i - h)$$

in the cohomology of W. It is known that in fact $H^*(W,Z)$ is generated over $\pi^*H^*(Y,Z)$ with this as the only relation, and that π^* is a monomorphism: we therefore regard $H^*(W,Z)$ as an $H^*(Y,Z)$-algebra and omit mention of π^*.

Without loss of generality it is sufficient to consider the case $\gamma = 1$, and to calculate the expression

$$\begin{aligned}\mathscr{B}(W, \tfrac{1}{2}kh, 1, r) &= \pi_* \left(2^{rm} e^{\frac{1}{2}kh} \prod_{i=1}^{m+1} \frac{(x_i-h)\,(\cosh \frac{1}{2}(x_i-h))^r}{2\sinh \frac{1}{2}(x_i-h)} \right) \\ &= \pi_* F(h),\end{aligned}$$

where k is an integer. We use the relation $\prod_{i=1}^{m+1} (x_i-h) = 0$ to write

$$F(h) = t_0 h^m + \ldots + t_m.$$

Then

$$\mathscr{B}(W, \tfrac{1}{2}kh, 1, r) = t_0 = \frac{|F(x_j)\, x_j^{m-1} \ldots x_j\, 1|}{|x_j^m \quad x_j^{m-1} \ldots x_j\, 1|},$$

where $|\ldots|$ denotes a determinant in which only the jth row is written down. Expanding the numerator by the first column we obtain

$$\mathscr{B}(W, \tfrac{1}{2}kh, 1, r) = (-1)^m \sum_j e^{\frac{1}{2}kx_j} \left(\prod_{i \neq j} \frac{(2\cosh \frac{1}{2}(x_i-x_j))^r}{2\sinh \frac{1}{2}(x_i-x_j)} \right).$$

The substitution $a_i = e^{x_i}$ gives

$$\mathscr{B}(W, \tfrac{1}{2}kh, 1, r) = (-1)^m \sum_j a_j^{\frac{1}{2}k} \left(\prod_{i \neq j} (a_i a_j)^{-\frac{1}{2}r+1} \frac{(a_i+a_j)^r}{a_i-a_j} \right).$$

In particular we have the special cases $r = 1$ and $r = -1$; for these we assume $k = 2p$. When $r = 1$

$$\begin{aligned}\mathscr{B}(W, ph, 1, 1) &= (-1)^m \sum_j a_j^p \left(\prod_{i \neq j} \frac{a_i+a_j}{a_i-a_j} \right) \\ &= \frac{|A_j^p\, a_j^{m-1} \ldots a_j\, 1|}{|a_j^m\, a_j^{m-1} \ldots a_j\, 1|},\end{aligned}$$

where $A_j^p = \frac{1}{2} a_j^{p-1} \prod_{i=1}^{m+1} (a_i+a_j)$. If σ_q denotes the qth elementary symmetric function in $a_1, \ldots, a_{m+1}$ then

$$A_j^p = \tfrac{1}{2}[a_j^{m+p} + \sigma_1 a_j^{m+p-1} + \ldots + \sigma_{m+1} a_j^{p-1}].$$

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When $r = -1$ a similar calculation gives

$$\mathscr{B}(W, ph, 1, -1) = (-1)^m \sum_j a_j^p \left(\prod_{i \neq j} \frac{a_i a_j}{a_i^2 - a_j^2} \right)$$
$$= \sigma_{m+1} \frac{\left| a_j^{m+p-1} a_j^{2(m-1)} \dots a_j^2 1 \right|}{\left| a_j^{2m} \quad a_j^{2(m-1)} \dots a_j^2 1 \right|}.$$

3·2. We now consider similarly the case that ξ is a quaternionic vector bundle over Y with fibre H^{d+1}, $m = 2d$, and with symplectic Pontrjagin class $q(\xi) = \prod_{i=1}^{d+1} (1 + x_i^2)$. Let $\pi: W \to Y$ be the associated $P_d(H)$-bundle. Again there is an exact sequence of quaternionic vector bundles

$$0 \to \theta \to \pi^! \xi \to \psi \to 0,$$

where $\phi = \psi \otimes \theta^*$ is the real tangent bundle along the fibres. We write $q(\theta) = 1 + h$ and $h = u^2$. Then

$$p(\phi) = p(\theta \otimes \theta^*)^{-1} p(\pi^! \xi \otimes \theta^*) = (1 + 4u^2)^{-1} \prod_{i=1}^{d+1} (1 + (x_i + u)^2)(1 + (x_i - u)^2),$$

$$w_2(\phi) = 0.$$

Here we regard $H^*(W, Z)$ as an algebra over $H^*(Y, Z)$ with generator $h = u^2$ and with the only relation

$$\prod_{i=1}^{d+1} (x_i^2 - u^2) = 0.$$

Let γ be the complex vector bundle over W with $\operatorname{ch} \gamma = e^u + e^{-u}$. Then

$$\operatorname{ch} \gamma^k = (e^u + e^{-u})^k$$

and it is sufficient to calculate the expression

$$\mathscr{B}(W, 0, \gamma^k, r) = \pi_* \left(2^{rm} (e^u + e^{-u})^k \frac{\sinh u}{u (\cosh u)^r} \prod_{i=1}^{d+1} \frac{(x_i + u)(\cosh \frac{1}{2}(x_i + u))^r}{2 \sinh \frac{1}{2}(x_i + u)} \right.$$
$$\left. \times \prod_{i=1}^{d+1} \frac{(x_i - u)(\cosh \frac{1}{2}(x_i - u))^r}{2 \sinh \frac{1}{2}(x_i - u)} \right) = \pi_* F(u),$$

where k is an integer. We use the relation

$$\prod_{i=1}^{d+1} (x_i^2 - u^2) = 0, \quad m = 2d,$$

to write

$$F(u) = t_0 u^{2d} + \dots + t_{d-1} u^2 + t_d.$$

Then

$$\mathscr{B}(W, 0, \gamma^k, r) = t_0 = \frac{\left| F(x_j) x_j^{2(d-1)} \dots x_j^2 1 \right|}{\left| x_j^{2d} \quad x_j^{2(d-1)} \dots x_j^2 1 \right|}.$$

A calculation similar to that for the complex case gives, when $r = 1$,

$$\mathscr{B}(W, 0, \gamma^k, 1) = \frac{\left| C_j^k \quad (\cosh x_j)^{d-1} \dots \cosh x_j 1 \right|}{\left| (\cosh x_j)^d \, (\cosh x_j)^{d-1} \dots \cosh x_j 1 \right|},$$

where

$$C_j^k = 2^{k-1} [(\cosh x_j)^{d+k} + \tau_1 (\cosh x_j)^{d+k-1} + \dots + \tau_{d+1} (\cosh x_j)^{k-1}]$$

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and τ_q denotes the qth elementary symmetric function in $\cosh x_1, \ldots, \cosh x_{d+1}$. When $r = -1$ we have

$$\mathscr{B}(W, 0, \gamma^k, -1) = 2^{-d+k} \frac{|(\cosh x_j)^k \ (\cosh 2x_j)^{d-1} \ldots \cosh 2x_j \ 1|}{|(\cosh 2x_j)^d (\cosh 2x_j)^{d-1} \ldots \cosh 2x_j \ 1|}.$$

4. *Applications.* 4·1. The determinants of the form occurring in the previous paragraph are well known in the theory of representations (see, for example, Weyl (10), Chapter VIII, and Littlewood (4), Chapter VI). In particular

$$\frac{|y_j^{m+k} y_j^{m-1} \ldots y_j \ 1|}{|y_j^m \quad y_j^{m-1} \ldots y_j \ 1|} = h_k(y_1, \ldots, y_{m+1})$$

is the kth homogeneous product sum in $y_1, \ldots, y_{m+1}$.

THEOREM 4·1. *Let ξ be a complex vector bundle with $c(\xi) = \prod_{i=1}^{m+1} (1 + x_i)$. Then*

$$\frac{|e^{(m+k)x_j} e^{(m-1)x_j} \ldots e^{x_j} \ 1|}{|e^{mx_j} \quad e^{(m-1)x_j} \ldots e^{x_j} \ 1|} = \operatorname{ch} \sigma^k \xi,$$

where σ^k denotes the k-fold symmetric product.

Proof. In this case

$$h_k(e^{x_1}, \ldots, e^{x_{m+1}}) = \sum_{j_1 \leqslant \ldots \leqslant j_k} e^{x_{j_1} + \cdots + x_{j_k}} = \operatorname{ch} \sigma^k \xi.$$

Remark. It follows similarly that

$$\frac{|e^{2(m+k)x_j} e^{2(m-1)x_j} \ldots e^{2x_j} \ 1|}{|e^{2mx_j} \quad e^{2(m-1)x_j} \ldots e^{2x_j} \ 1|} = \operatorname{ch} \sigma^k \xi^{(2)}$$

is a Chern character, since if η is any virtual complex vector bundle then $\operatorname{ch} \eta^{(2)} = \operatorname{ch} \psi^2 \eta$, where ψ^2 is the operation introduced by Adams (1).

Let ζ be a virtual real vector bundle over Y of dimension $2s$ or $2s+1$. We see from Theorem 2·1 that $\hat{A}(Y, \frac{1}{2}d, \beta, \zeta)$ is connected with the geometric dimension of ζ. Let k be the least integer such that $2^{k-s}\hat{A}(Y, \frac{1}{2}d, \beta, \zeta)$ is an integer for all β, d satisfying the hypotheses of 2·1. Then $0 \leqslant k \leqslant n$, and $2k$ is a lower bound for the geometric dimension of ζ; we call $2k$ the I-dimension of ζ. We can choose a virtual complex line bundle β and $d \equiv w_2(Y) + w_2(\zeta) \bmod 2$ such that $2^{k-s}\hat{A}(Y, \frac{1}{2}d, \beta, \zeta)$ is an odd integer; this will be done in all the following applications.

4·2. We first discuss the case $r = 1$ and W the $P_m(C)$-bundle associated to a complex vector bundle ξ with

$$c(\xi) = \prod_{i=1}^{m+1} (1 + x_i).$$

We choose $p = 1$ in the expression $\mathscr{B}(W, ph, 1, 1)$ of 3·1, so that

$$\mathscr{B}(W, h, 1, 1) = \tfrac{1}{2}(\operatorname{ch} \xi + \operatorname{ch} \xi)$$

and Theorem 2·3 implies

THEOREM 4·2. *Let d, ρ be as in 2·3 and suppose that W admits a stable vector field along the fibres of type ρ. Then*

$$(\operatorname{ch} \xi \,.\, \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, -\rho))[Y]$$

is an integer.

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COROLLARY. *Let* $I\text{-dim}(-\rho) = 2k$. *If* $W = Y \times P_m(C)$ *admits a stable vector field along the fibres of type* ρ *then the fibre dimension of* ρ *is*

$$\leqslant 2v(m+1) - 2k.$$

Here $v(m+1)$ denotes the largest integer v such that 2^v divides $m+1$. To prove the corollary choose ξ trivial so that $\operatorname{ch}\xi = m+1$. As a check on our results we have the known

COROLLARY. *The geometric dimension of the real tangent bundle of* $P_m(C)$ *is*

$$\geqslant 2m - 2v(m+1).$$

This result was proved by Schwarz (8). In fact Steer (9) has shown that the geometric dimension of the real tangent bundle of $P_m(C)$ is equal to $2m - 2v(m+1)$.

4·3. We continue to discuss the case $r = 1$ but let W be the $P_d(H)$-bundle associated to a quaternionic vector bundle ξ with $q(\xi) = \prod_{i=1}^{d+1}(1+x_i^2)$. We choose $k = 1$ in the calculations of 3·2 so that

$$\begin{aligned}\mathscr{B}(W, 0, \gamma, 1) &= 2[\cosh x_1 + \ldots + \cosh x_{d+1}] \\ &= \operatorname{ch}\xi.\end{aligned}$$

Again applying Theorem 2·3 we have

THEOREM 4·3. *Let* d, ρ *be as in* 2·3 *and suppose that* W *admits a stable vector field along the fibres of type* ρ. *Then*

$$(\operatorname{ch}\xi \,.\, \hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, -\rho))[Y]$$

is an integer.

COROLLARY. *Let* $I\text{-dim}(-\rho) = 2k$. *If* $W = Y \times P_d(H)$ *admits a stable vector field along the fibres of type* ρ *then the geometric dimension of* ρ *is*

$$\leqslant 2v(d+1) - 2k + 2.$$

COROLLARY. *The geometric dimension of the real tangent bundle of* $P_d(H)$ *is*

$$\geqslant 4d - 2v(d+1) - 2.$$

The latter result was proved by Schwarz (8). It can be improved by using the theorem of (5), page 305.

4·4. We now discuss the case $r = -1$ and W the $P_m(C)$-bundle associated to a complex vector bundle ξ with

$$c(\xi) = \prod_{i=1}^{m+1}(1+x_i).$$

Let $2k$ be the I-dimension of ζ. We consider the function

$$P(t) = \left(\frac{\left|e^{2tx_j}\, e^{2(m-1)x_j} \ldots e^{2x_j}\, 1\right|}{\left|e^{2mx_j}\, e^{2(m-1)x_j} \ldots e^{2x_j}\, 1\right|} 2^{k-s}\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta)\right)[Y]$$

for real t. We denote by $P^{(i)}(t)$ the ith difference for $i = 0, 1, \ldots$; this is defined by

$$\begin{aligned}P^{(0)}(t) &= P(t), \\ P^{(i+1)}(t) &= P^{(i)}(t+1) - P^{(i)}(t).\end{aligned}$$

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Then $$P^{(i)}(t) = \left(\frac{\begin{vmatrix} (e^{2x_j}-1)^i e^{2ix_j} e^{2(m-1)x_j} \dots e^{2x_j} 1 \\ e^{2mx_j} \qquad\qquad e^{2(m-1)x_j} \dots e^{2x_j} 1 \end{vmatrix}}{} 2^{k-s}\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta) \right)[Y].$$

It is easy to see from the expansion

$$e^{2ix_j} = 1 + (2x_j)t + (2x_j^2)t^2 + \dots$$

and the fact that Y is finite dimensional, that $P^{(i)}(t)$ is a polynomial in t. The coefficients are rational numbers obtained by evaluating on Y cohomology classes of degree $\geqslant 2(i-m)$. Therefore $P^{(i)}(t)$ is zero for $i > m+n$. The remarks in 4·1 show that the quotient of determinants in the expression for $P^{(i)}(0)$ is a Chern character, and hence that $P^{(i)}(0)$ is an integer for all i. Clearly $P^{(i)}(0)$ is zero for $i < m$. Therefore

$$P(t) = a_{m+n}\binom{t}{m+n} + a_{m+n-1}\binom{t}{m+n-1} + \dots + a_m\binom{t}{m},$$

where the $a_i = P^{(i)}(0)$ are integers obtained by evaluating on Y cohomology classes of degree $\geqslant 2(i-m)$. For example

$$\begin{aligned} a_m &= 2^{k-s}\hat{A}(Y, \tfrac{1}{2}d, \beta, \zeta), \\ a_{m+n} &= 2^{n+k}\left(\frac{\begin{vmatrix} x_i^{m+n} & x_i^{m-1} \dots x_i 1 \end{vmatrix}}{\begin{vmatrix} x_i^m & x_i^{m-1} \dots x_i 1 \end{vmatrix}} \right)[Y] \\ &= 2^{n+k}h_n(x_1, \dots, x_{m+1})[Y]. \end{aligned}$$

We denote $h_n(x_1, \dots, x_{m+1})[Y]$ by $h(\xi)$. If $c_1, \dots, c_n$ are the Chern classes of ξ then

$$h(\xi) = \begin{vmatrix} c_1 & c_2 & \dots & c_n \\ 1 & c_1 & \dots & c_{n-1} \\ 0 & 1 & \dots & c_{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & c_1 \end{vmatrix}[Y].$$

Let $v(N)$ denote the largest integer v such that 2^v divides N, and let $\alpha(N)$ denote the number of terms in the dyadic expansion of N. Then

$$v(N!) = N - \alpha(N)$$

and therefore

$$P(\tfrac{1}{2}) = \frac{1}{2^{2(m+n)-\alpha(m+n)}}[a_{m+n} \cdot (\text{odd integer}) + 2^{v(m+n)+1} \cdot (\text{integer})].$$

If $n+k+v(h(\xi)) \leqslant v(m+n)$ then

$$P(\tfrac{1}{2}) = 2^{v(h(\xi))+k-2m-n+\alpha(m+n)} \cdot (\text{odd integer}).$$

If ξ is trivial then all coefficients of $P(t)$ are zero except a_m and so

$$P(\tfrac{1}{2}) = 2^{-2m+\alpha(m)} \cdot (\text{odd integer}).$$

We now choose $p = -m+2$ in the calculation of 3·1 so that

$$(\mathscr{B}(W, ph, 1, -1)\hat{\mathscr{A}}(Y, \tfrac{1}{2}d, \beta, \zeta))[Y] = 2^{s-k}P(\tfrac{1}{2})$$

and Theorem 2·6 implies

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THEOREM 4·4. (i) *Suppose that* $n+k+v(h(\xi)) \leqslant v(m+n)$. *If there exists a* Y*-immersion* $W \to Z$ *then the fibre dimension of* ζ *is* $> 4m+2n-2\alpha(m+n)-2v(h(\xi))-1$. *If there exists a* Y*-embedding then the fibre dimension is* $> 4m+2n-2\alpha(m+n)-2v(h(\xi))$.

(ii) *If there exists a* Y*-immersion* $Y \times P_m(C) \to Z$, *then the fibre dimension of* ζ *is* $> 4m+2k-2\alpha(m)-1$. *If there exists a* Y*-embedding then the fibre dimension is* $> 4m+2k-2\alpha(m)$.

As a check on both parts of the theorem we have the following known

COROLLARY. $P_m(C)$ *cannot be immersed in real* $(4m-2\alpha(m)-1)$*-space, nor embedded in real* $(4m-2\alpha(m))$*-space.*

As a final example of the non-embedding and non-immersion theorems for complex projective fibre bundles which can be deduced from Theorem 2·6 we consider the case $Y = S^{2n}$. Then the expression for $h(\xi)$ in terms of Chern classes becomes

$$h(\xi) = (-1)^{n-1} c_n(\xi)[S^{2n}].$$

Let W be the $P_m(C)$-bundle associated to a C^{m+1}-bundle ξ over S^{2n} for which $c_n(\xi)[S^{2n}]$ is an odd multiple of $(n-1)!$. Then Theorem 4·4(i) becomes

COROLLARY. *Let* W *be as above and suppose* $2n+k-\alpha(n-1)-1 \leqslant v(m+n)$. *If there exists a* S^{2n}*-immersion* $W \to Z$, *then the fibre dimension of* ζ *is*

$$> 4m-2\alpha(m+n)+2\alpha(n-1)+1.$$

If there exists a S^{2n}*-embedding then the fibre dimension is*

$$> 4m-2\alpha(m+n)+2\alpha(n-1)+2.$$

COROLLARY. *Let* W *be as above with* $m+n = 2^N$, $n > 0$ *and* $N \geqslant 3n-\alpha(n-1)-1$. *Then* W *is a* $P_m(C)$*-bundle which cannot be* S^{2n}*-embedded in any real* $(4m+2\alpha(n-1))$*-plane bundle, even though each fibre* $P_m(C)$ *can be embedded in real* $(4m-1)$*-space.*

4·5. For the case $r = -1$ and W a quaternionic projective bundle over Y we use the fact that $(\cosh x_j)^2 = \frac{1}{2}(\cosh 2x_j + 1)$. Therefore

$$\mathscr{B}(W, 0, \gamma, -1) = 2^{-d+\frac{1}{2}} \frac{\left|(\cosh 2x_j+1)^{\frac{1}{2}} (\cosh 2x_j)^{d-1} \dots 1\right|}{\left|(\cosh 2x_j)^d \quad (\cosh 2x_j)^{d-1} \dots 1\right|}$$

$$= 2^{-d+1} \sum_{i=-}^{n+d} \binom{\frac{1}{2}}{i} \frac{1}{2^i} \frac{\left|(\cosh 2x_j-1)^i (\cosh 2x_j)^{d-1} \dots 1\right|}{\left|(\cosh 2x_j)^d \quad (\cosh 2x_j)^{d-1} \dots 1\right|}.$$

If ξ is trivial and $W = Y \times P_d(H)$ this is equal to $2^{-2d+1}\binom{\frac{1}{2}}{d}$, and we deduce

THEOREM 4·5. *If there exists a* Y*-immersion (resp.* Y*-embedding) of* $Y \times P_d(H)$ *in* Z *then the fibre dimension of* Z *is* $> 8d+2k-2\alpha(d)-3$ *(resp.* $8d+2k-2\alpha(d)-2$).

COROLLARY. $P_d(H)$ *cannot be immersed in real* $(8d-2\alpha(d)-3)$*-space, nor embedded in real* $(8d-2\alpha(d)-2)$*-space.*

4·6. The above applications are intended only as examples of the sort of theorems which can be obtained. It should be noted that in many cases there are dimensions in

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which the integrality theorem can be improved by a factor of 2 and the geometric application improved correspondingly. It is also possible to obtain further applications by explicit calculations: for example, when both X and Y are of low dimension.

REFERENCES

(1) ADAMS, J. F. Vector fields on spheres. *Ann. of Math.* **75** (1962), 603–632.
(2) ATIYAH, M. F. and HIRZEBRUCH, F. Riemann–Roch theorems for differentiable manifolds. *Bull. American Math. Soc.* **65** (1958), 276–281.
(3) ATIYAH, M. F. and HIRZEBRUCH, F. Quelques théorèmes de non-plongement pour les variétés différentiables. *Bull. Soc. Math. France* **87** (1959), 383–396.
(4) LITTLEWOOD, D. E. *The theory of group characters and matrix representations of groups* (Oxford University Press, 1940).
(5) MAYER, K. H. Elliptische Differentialoperatoren und Ganzzahligkeitssätze für charakteristische Zahlen. *Topology* **4** (1965), 295–313.
(6) ROBERTS, R. S. Bundles of Grassmannians and integrality theorems. *Proc. Cambridge Philos. Soc.* **63** (1967), 45–53.
(7) SANDERSON, B. J. and SCHWARZENBERGER, R. L. E. Non-immersion theorems for differentiable manifolds. *Proc. Cambridge Philos. Soc.* **59** (1963), 319–322.
(8) SCHWARZ, W. *Zwei Anwendungen topologischer Ganzzahligkeitssätze.* Diplomarbeit (Bonn, 1965).
(9) STEER, B. Une interpretation géométrique des nombres de Radon–Hurwitz. *Ann. Instit. Fourier Grenoble* (to appear).
(10) WEYL, H. *The classical groups* (Princeton University Press, 1939).

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