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Topological structure and entropy of mixing graph maps

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Abstract. Let $\mathcal{P}_G$ be the family of all topologically mixing, but not exact self-maps of a topological graph G. It is proved that the infimum of topological entropies of maps from $\mathcal{P}_G$ is bounded from below by $\log 3/\Lambda(G)$, where $\Lambda(G)$ is a constant depending on the combinatorial structure of G. The exact value of the infimum on $\mathcal{P}_G$ is calculated for some families of graphs. The main tool is a refined version of the structure theorem for mixing graph maps. It also yields new proofs of some known results, including Blokh's theorem (topological mixing implies the specification property for maps on graphs).

1. *Introduction*

It is well known that, in general, there is no connection between topological transitivity and topological entropy for a self-map of an arbitrary compact metric space. A map with positive entropy need not be transitive and a transitive system may have zero entropy. However, there are spaces X such that all topologically transitive self-maps of X have positive topological entropy. For instance, on the compact interval [0, 1] every transitive map has entropy at least $\log\sqrt{2}$ and there is a transitive self-map of [0, 1] with entropy equal to this bound. More generally, if X is a continuum containing an open subset homeomorphic to an open interval other than the circle, then every transitive self-map of X has positive topological entropy, as was recently proved in [**20**]. This suggests a problem: to find best lower bounds for the topological entropy in various classes of transitive self-maps of a fixed space. Let us state it precisely.

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Problem. Let $\mathfrak{F}(X)$ be a subclass of the class of transitive self-maps of a compact metric space X. By $\inf h(\mathfrak{F}(X))$ we mean the best lower bound for the topological entropy of maps from $\mathfrak{F}(X)$, that is, $\inf h(\mathfrak{F}(X)) = \inf\{h(f) : f \in \mathfrak{F}(X)\}$. Moreover, we say that $\inf h(\mathfrak{F}(X))$ is *attainable* if there exists a map $f \in \mathfrak{F}(X)$ such that $h(f) = \inf h(\mathfrak{F}(X))$.

Variants of the above problem have been considered by many authors; see [**1, 4–7, 19, 21, 26, 34, 35, 37**]. For more references and other results of this type (for example, lists of known best lower bounds for the topological entropy of transitive maps on various spaces), see [**5**, p. 341] or [**4, 7, 21, 26**].

The present work is motivated by the results of [**21**], so let us briefly recall them. Let X be the compact interval $[0, 1]$ or the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. The family $\mathcal{M}_X$ of all topologically mixing maps of X is the disjoint union of the family $\mathcal{E}_X$ of all topologically exact maps of X and the family $\mathcal{P}_X = \mathcal{E}_X \setminus \mathcal{M}_X$ of all pure mixing maps. As mentioned above, $\inf h(\mathcal{E}_{[0,1]}) = \log \sqrt{2}$ and it is well known that $\inf h(\mathcal{E}_{\mathbb{S}^1}) = 0$. But it is proved in [**21**] that $\inf h(\mathcal{P}_{[0,1]}) = \log \sqrt{3} = \inf h(\mathcal{P}_{\mathbb{S}^1})$. Hence, if the entropy of a mixing graph map of X is sufficiently small, then the map must be exact, that is, exact maps lower the entropy in the family of mixing maps.

On the other hand, exact maps are regarded as more chaotic than pure mixing maps (see [**10**]). With this interpretation one may reformulate this result: for the compact interval $[0, 1]$ or the circle $\mathbb{S}^1$ adding a *more chaotic* class of exact maps to the *less chaotic* one results in *lowering* topological entropy in the enlarged class of all mixing maps. This contrasts with the common interpretation of the topological entropy as a quantitative measure of chaos present in the system.

It is natural to ask whether a similar phenomenon occurs for maps defined on arbitrary topological graphs, that is, we are motivated by the following problem.

Problem. Let G be a topological graph. Let $\mathcal{P}_G$ denote the family of all pure mixing (that is, topologically mixing, but not topologically exact) self-maps of G. Find the infimum of topological entropies of maps from $\mathcal{P}_G$, that is, find $\inf h(\mathcal{P}_G)$.

The main result we would like to present here is a generalization of results from [**21**]: for a fixed graph G,

$$\frac{\log 3}{\Lambda(G)} \leq \inf h(\mathcal{P}_G),$$

where $\Lambda(G)$ is a constant depending on the combinatorial structure of the graph G. Moreover, we are able to compute $\inf h(\mathcal{P}_G)$ for some graphs and two infinite families of trees (defined in terms of some special structural properties).

Note that for any fixed graph G which is not a tree (containing at least one simple closed curve), the infimum of topological entropies of mixing maps is zero (see [**3**]). Our theorem on the lower bound for the topological entropy of a pure mixing graph map can be rephrased in a way that sounds paradoxical: for any graph G containing a simple closed curve, adding a *more chaotic* class of maps to the *less chaotic* one results in *lowering* topological entropy in the enlarged class.

But the paradox disappears if only we treat the topological entropy as a qualitative indicator of chaos, that is, positive topological entropy is a sign of complex behavior

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present in the system. From this point of view the precise numerical value of the topological entropy is unimportant.

It should also be stressed that the approach in [**21**] cannot be directly adapted to the more general case we consider here. More precisely, methods used in [**21**] can (after some modification) be used to extend the results of [**21**] to trees, but are of no use for graphs containing a circle and at least one branching point. There are two reasons for that. First, it is harder to obtain a covering relation from a containment relation. More precisely, if I and K are two homeomorphic copies of the closed unit interval contained in G and $f\colon G \to G$, then it does not follow from $K \subset f(I)$ that there exists a closed interval $J \subset I$ such that $f(I) = K$ if the graph contains a circle. Second, the interior of a connected set may not be connected if the set in question contains a branching point of the graph. The proofs in [**21**] rely heavily on those two facts.

To solve our problem in the more general context, one must understand the structure of pure mixing graph maps. This is a problem of independent interest, and its solution, which we present as Theorems 4.6 and 6.1, has already found other applications (see [**25**]). Using knowledge on the structure of pure mixing graph maps, we also obtain (with some additional work) new proofs of two known results: Blokh's theorem, stating that all topological mixing graph maps have the specification property (see Theorem 10.8) and [**38**, Theorem 4.2], which is the main result of [**38**] and generalizes [**17**, Theorems B and C] to graph maps (see Corollary 7.5).

Finally, we remark that our version of the structure theorem (Theorems 4.6 and 6.1) for pure mixing graph maps could probably be derived along the lines of Blokh's papers (see also the presentation of Blokh's work in Alsedà *et al* [**3**]), but it is not a simple or direct corollary of any theorem presented in [**12–14**]. To obtain the structure theorem from [**12–14**] or [**3**], one should rather rework the whole proof of Blokh's theorem presented there, and adjust it in many places. We are convinced that this approach to the structure theorem, even if successful, would result in a less transparent and longer proof than ours.

2. *Basic definitions and notation*

2.1. *Notation and terminology.* Let (X, ρ) be a metric space, and let $f\colon X \to X$ be a continuous map. In this paper the letters k, l, m, n, M, N will always denote integers, and by 'a map' we will always mean 'a continuous map'. If $A \subset X$ then we will denote the closure (interior) of A by $\overline{A}$ (int A). An open ball with center at $x \in X$ and radius $\varepsilon > 0$ is denoted by $\mathrm{B}(x, \varepsilon)$. Similarly, if A is a subset of X then $\mathrm{B}(A, \varepsilon) = \bigcup_{x \in A} \mathrm{B}(x, \varepsilon)$.

A *continuum* is a compact, connected metric space. An *arc* is a continuum homeomorphic to the interval $[0, 1]$. If A is an arc and $g\colon [0, 1] \to A$ is a homeomorphism, then the *endpoints* of A are $g(0)$ and $g(1)$. Clearly, the endpoints do not depend on the choice of g.

If X is a continuum and $A \subset X$ is an arc with endpoints x and y, then we say that A is a *free arc* provided $A \setminus \{x, y\}$ is open in X.

2.2. *Topological graphs.* A *topological graph* (or *graph* for short) is a continuum G such that there is a one-dimensional simplicial complex $\mathcal{K}$ whose geometric carrier $|\mathcal{K}|$ is homeomorphic to G; see [**18**, p. 10] (cf. Moise [**29**], who uses the terminology *linear*

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graph). Each such complex is called a *triangulation* of G. Without loss of generality we may assume that each graph we consider is not merely *homeomorphic to* some complex, but it *is* just a simplicial complex embedded in the Euclidean space $\mathbb{R}^3$. In particular, it comes equipped with some fixed triangulation, and all triangulations we consider are just subdivisions of the fixed one. Recall that a triangulation $\mathcal{L}$ of a graph G is a *subdivision* of a triangulation $\mathcal{K}$ if every vertex $\mathcal{L}$ is a vertex for $\mathcal{K}$. Moreover, we assume that G is endowed with the taxicab metric, that is, the distance between any two points of G is equal to the length of the shortest arc in G joining these points. If G is a graph, and $\mathcal{K}$ is a triangulation of G, then every zero-dimensional (one-dimensional) simplex of $\mathcal{K}$ is called a *vertex* (an *edge*) of G with respect to $\mathcal{K}$. The set of all edges with respect to $\mathcal{K}$ is denoted by $\mathcal{K}^e$. Since graphs are defined as simplicial complexes it puts some requirements on the minimal number of vertices and edges, no matter what triangulation we choose. For example, any triangulation of the unit circle always has at least three edges and three vertices. A *tree* is a graph that contains no simple closed curve, in other words, no subset of a tree can be homomorphic to a circle.

The *star* of a vertex v, denoted by $\mathrm{St}(v)$, is the union of all the edges that contain the vertex v. For every $x \in G$ we define the *valence* of x, denoted by $\mathrm{val}(x)$, in the following way: if x is a vertex of G then $\mathrm{val}(x)$ is equal to the number of connected components of $\mathrm{St}(x)\backslash\{x\}$, and $\mathrm{val}(x) = 2$ otherwise. Points $x \in G$ with $\mathrm{val}(x) = 1$ are called *endpoints* of the graph G, and if $\mathrm{val}(x) > 2$ we say that x is a *branching point*. Let $\mathrm{End}(G)$ denote the set of all endpoints of G. Observe that an arc $A \subset G$ is free if and only if A contains no branching point of G.

Note that $\mathrm{val}(x)$ is independent of the choice of triangulation. In particular, every branching point (endpoint) for some triangulation is a branching point (endpoint) for every triangulation.

A tree T is an *n-star* ($n \geq 1$) if $T_n = \{z \in \mathbb{C} : z^n \in [0, 1]\}$, that is, an n-star is n copies of $[0, 1]$ with all 0s identified to a single vertex of valence n. Note that the 1-star T_1 and 2-star T_2 are just the arc, and the star $\mathrm{St}(v)$ of a vertex v of valence n is homeomorphic to an n-star T_n, as well as the closure of any sufficiently small neighborhood of v.

Any subset of G which is a graph itself is called a *subgraph* of G. The family of all subgraphs of G is denoted by $\mathcal{G}(G)$, and coincides with the family of all non-degenerate subcontinua of G. Note that a singleton set is *not* a subgraph of G (graphs are one-dimensional by definition).

Following Nadler [**30**], we define the *disconnecting number* of a graph G as the least $n > 0$ such that every subset D of G of cardinality n *disconnects* G (that is, $G\backslash D$ is not connected). The disconnecting number is well defined and is denoted by $\mathrm{D^s}(G)$ (see [**30**]). Let $\Lambda(G)$ be the maximal disconnecting number among all subgraphs of G. It follows from [**30**] that $\Lambda(G) = \mathrm{D^s}(G) - \chi(G)+1$, where $\chi(G)$ is the Euler characteristic of G.

2.3. *Topological dynamics.* We refer the reader to [**5**] for definitions of basic concepts of the theory of dynamical systems: *(periodic) orbit*, *(semi)conjugacy* etc. We call a map f *transitive* if for every pair of non-empty open subsets U and V of X there is an n such that $f^n(U) \cap V \neq \emptyset$; we say that f is *totally transitive* if all its iterates f^n are transitive; a map f is *weakly mixing* if its Cartesian power $f \times f$ is transitive; a map f is *mixing* if

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for any non-empty sets U and V open in X, there is an $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for $n \geq N$; a map f is *exact* if for any non-empty open subset U of X there is an $n \geq 0$ such that $f^n(U) = X$. It is well known that for any self-map f of a compact metric space X exactness implies mixing, mixing implies weak mixing, weak mixing implies total transitivity, and total transitivity implies transitivity, but the converse implications do not hold. We say that f is *strongly transitive* if for every non-empty open set $U \subset X$ there is $k \geq 0$ such that $X = U \cup f(U) \cup \cdots \cup f^k(U)$. In the special case of non-invertible graph maps total transitivity implies mixing (see [**3, 9**] or [**22**] for a proof). The only examples of transitive graph homeomorphisms are the irrational rotations of the circle, which are totally transitive, but not weakly mixing. Moreover, the transitive graph maps are either totally transitive, or can be decomposed into totally transitive ones.

The following fact is a rewording of [**2**, Corollary 2.7]; it also follows from results of Banks on periodic decompositions [**8**].

THEOREM 2.1. *Let $f: G \to G$ be a transitive graph map. Then exactly one of the following two statements holds.*

(1) *The map f is totally transitive (and hence mixing).*

(2) *There exist a $k > 1$ and non-degenerate connected subgraphs $G_0, \ldots, G_{k-1}$ of G such that:*
   (a) $G = \bigcup_{i=0}^{k-1} G_i$;
   (b) $G_i \cap G_j = \mathrm{End}(G_i) \cap \mathrm{End}(G_j)$ *for* $i \neq j$;
   (c) $f(G_i) = G_{(i+1 \bmod k)}$ *for* $i = 0, \ldots, k-1$;
   (d) $f^k|_{G_i}$ *is totally transitive (and hence mixing) for* $i = 0, \ldots, k-1$.

We say that a map f is *pure mixing* if f is mixing but not exact. We recommend [**24**] as a source of information on transitivity.

For a definition of the *topological entropy* of f we again refer the reader to [**5**]. Recall that if X is a compact space then the entropy of f is a (possibly infinite) number $h(f) \in [0, +\infty]$. We will use the basic properties of the entropy such as those in [**5**, §4.1] without further reference.

3. *Some properties of graph maps*

In this section we collect some properties of graph maps, which we will use frequently in subsequent sections. The following convention will recur frequently in what follows.

Convention (C). Let J be a free arc (for example, an edge) in a graph G, and let e be one of its endpoints. We identify J with an interval $[0, a] \subset \mathbb{R}$, where $0 < a$ and e is identified with 0. We may also assume that this identification is actually an isometry if necessary, in particular $a = \mathrm{diam}\, J$. Then J could be linearly ordered by the relation $\leq$ induced from $[0, a]$. It allows us to write $x < y$ to denote the relative position of points on J, and use usual interval notation to describe connected subsets of J.

The proof of the next lemma is omitted, as it is straightforward. Alternatively, it can be deduced from [**27**, Lemma 23] or [**23**, Theorem 3.11].

LEMMA 3.1. *For a map f of a graph G the following conditions are equivalent:*

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(1) *f is mixing;*
(2) *for every $\varepsilon > 0$ and $\delta > 0$ there is an integer $N = N(\varepsilon, \delta)$ such that for any subgraph J of G with* diam $J \geq \delta$ *each connected component of the set $G \setminus f^n(J)$ has diameter smaller then ε for every $n \geq N$.*

Definition 3.2. Given a graph map f and free arcs $I, J \subset G$, we say that I *covers* J through f (or f-covers, for short) if there exists a free arc $K \subset I$ such that $f(K) = J$.

Properties of the f-covering relation presented below have elementary proofs. The first five of them are adapted from [**3**, p. 590] (note that closed intervals there are arcs in our terminology).

LEMMA 3.3. *Let $I, J, K, L \subset G$ be free arcs, and let $f, g: G \to G$ be graph maps.*
(1) *If I f-covers I, then there exists $x \in I$ such that $f(x) = x$.*
(2) *If $I \subset K$, $L \subset J$ and J is f-covered by I, then K f-covers L.*
(3) *If I f-covers J and J g-covers K, then I $(g \circ f)$-covers K.*
(4) *If $J \subset f(I)$, and $K_1, K_2 \subset J$ are free arcs such that $K_1 \cap K_2$ is at most one point, then I f-covers K_1, or I f-covers K_2.*
(5) *If $J \subset f(I)$ is a free arc, then there exist free arcs J_1, J_2 such that* int $J_1 \cap$ int $J_2 = \emptyset$, *$J_1 \cup J_2 = J$, and J_1, J_2 are f-covered by I.*
(6) *If $K = J \cap f(I)$ contains at most one endpoint of J, then K is f-covered by I.*
(7) *If $S \subset G$ is a star and $J \subset f(S)$ then there are two free arcs $E_1, E_2 \subset S$ with at most one common point such that J is contained in the sum of their images, or equivalently, $J \subset f(E_1 \cup E_2)$.*

LEMMA 3.4. *Suppose that $Z = C_1 \cup \cdots \cup C_n$, where $n \geq 2$ and $C_1, \ldots, C_n$ are pairwise disjoint free arcs contained in the interior of a free arc $F \subset G$. Let J be a free arc in G such that $f(J)$ intersects the interior of any connected component of $F \setminus Z$. Then J f-covers at least $n - 1$ sets among $C_1, \ldots, C_n$.*

Proof. Note that at most one of the sets $C_1, \ldots, C_n$ is not contained in $f(J)$. Let F' denote the convex hull of $C_1, \ldots, C_n$ in F, that is, the intersection of all compact connected sets containing $C_1 \cup \cdots \cup C_n$. Clearly, F' is a free arc. There are two cases to consider. First, assume that $f(J)$ contains only one endpoint of F'. Then use Lemma 3.3(6) and 3.3(2) to see that all the C_i except at most one are f-covered by J. In the second case there are free arcs K and L such that $K \cup L = F' \cap f(J)$ and K and L have at most one common point. Apply Lemma 3.3(5) to see that K and L must be f-covered by J, and observe that at most one of the sets $C_1, \ldots, C_n$ is not contained in $K \cup L$. □

The proof of the next lemma is left to the reader.

LEMMA 3.5. *For each $\delta > 0$ there is a constant $\xi = \xi(\delta)$ such that if K is a connected subset of G with* diam $K \geq \delta$ *then K contains a free arc J with* diam $J \geq \xi$.

4. *Global behavior of mixing graph maps*

By definition, a mixing map $f: G \to G$ is pure mixing if and only if there exists an open set $U \subset G$ such that $f^n(U) \neq G$ for all $n \geq 0$. We will prove that the graph map is pure

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mixing if and only if there exists a special set of *inaccessible points* $\mathcal{I}(f)$, which are not contained in int $f^n(U)$ for all $n \geq 0$ for any open set $U \subset G$ disjoint from $\mathcal{I}(f)$. The set of inaccessible points has cardinality bounded above by $\mathrm{D}^{\mathrm{s}}(G)$, is forward invariant, $f(\mathcal{I}(f)) \subset \mathcal{I}(f)$, and all its points are periodic points of f. The result is implicit in [**3, 12**]. But our method of proof is new, and we prove in addition that for a given open set $U \subset G$ the sets int $f^n(U)$ grow in G as $n \to \infty$. Moreover, if $x \in$ int $f^n(U)$ for some $n \geq 0$, then eventually $x \in$ int $f^k(U)$ for all sufficiently large k.

Definition 4.1. We say that a free arc $I_U \subset G$ is a *universal arc* for a map f of G if for any $\delta > 0$ there is an integer $M = M(\delta)$ such that I_U is f^m-covered by any free arc J of G with diam $J \geq \delta$ for all $m \geq M$.

LEMMA 4.2. *Let $f: G \to G$ be a mixing map. For every free arc $F \subset G$ there is a universal arc $I_U \subset F$. Moreover:*

(1) *there exists an integer $N_U > 0$ such that I_U is f^n-covered by itself for all $n \geq N_U$, in particular* int $I_U \cup$ int $f(I_U) \cup \cdots \cup$ int $f^k(I_U) \subset$ int $f^{N_U+k}(I_U)$ *for all $k \geq 0$;*

(2) *for any subgraph J of G,*

$$G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(J) \subset G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(I_U).$$

Proof. Let F be any free arc in G. According to our convention *(C)* we may identify F with $[0, 7]$. We define

$$A = [1, 2], \quad B = [3, 4], \quad C = [5, 6] \quad \text{and} \quad I_j = (2j, 2j + 1) \quad \text{for } j = 0, \ldots, 3.$$

Since f is mixing we can find an integer $k > 0$ such that for any $E \in \{A, B, C\}$ we have $f^k(E) \cap I_j \neq \emptyset$ for any $0 \leq j \leq 3$. Using Lemma 3.4 with $Z = A \cup B \cup C$, we deduce that A and C must f^k-cover at least two of the intervals A, B, C. It follows that there is $I_U \in \{A, B, C\}$ which is f^k-covered by both A and C. We will show that I_U is a universal arc. Fix $\delta > 0$ and some closed interval J of G with diam $J \geq \delta$. Let $\varepsilon = \max\{\text{diam } I_j : 0 \leq j \leq 3\}$. Lemma 3.1(2) gives us $N = N(\varepsilon, \delta)$ such that for every $n \geq N$ each connected component of $G \setminus f^n(J)$ has diameter less than ε. We conclude that $f^n(J)$ must intersect every connected component of $F \setminus (A \cup B \cup C)$, hence Lemma 3.4 guarantees that for any $n \geq N$ some $D_n \in \{A, C\}$ must be f^n-covered by J. By the above, and Lemma 3.3(3), the free arc I_U is f^n-covered by J for any $n \geq N + k$. Therefore, we set $M(\delta) = N + k$, and I_U is a universal arc for f as claimed. It follows immediately that (1) holds with $N_U = N(\text{diam } I_U, \varepsilon) + k$. For the proof of (2) we fix a subgraph J of G, and we let $\delta = \text{diam } J$. Lemma 3.5 and the definition of I_U above give us $M = M(\xi(\delta))$ such that $f^j(I_U) \subset f^{M+j}(J)$ for all $j \geq 0$, and so

$$\bigcup_{j=0}^{\infty} \text{int } f^j(I_U) \subset \bigcup_{j=M}^{\infty} \text{int } f^j(J) \subset \bigcup_{j=0}^{\infty} \text{int } f^j(J).$$

Taking complements leads to

$$G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(J) \subset G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(I_U), \tag{4.1}$$

as desired. □

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It can be shown that a transitive graph map is mixing if and only if it has a universal arc, but we will not pursue this here. We are now in a position to define inaccessible points and characterize them with the help of the interiors of images of the universal arc.

Definition 4.3. Let f be a map of G, and let $\mathcal{G}$ denote the family of all subgraphs of G. We define the set of *inaccessible* points of f by

$$\mathcal{I}(f) = G \setminus \bigcap_{J\in\mathcal{G}} \bigcup_{k=0}^{\infty} \operatorname{int} f^k(J) = \bigcup_{J\in\mathcal{G}} \bigcap_{k=0}^{\infty} G \setminus \operatorname{int} f^k(J),$$

where the second equality above follows easily from elementary properties of the operations involved.

LEMMA 4.4. *If f is a mixing graph map and I_U is a universal arc for f then*

$$\mathcal{I}(f) = G \setminus \bigcup_{k=0}^{\infty} \operatorname{int} f^k(I_U).$$

Proof. It is enough to observe that

$$\mathcal{I}(f) = \bigcup_{J\in\mathcal{G}} G \setminus \bigcup_{k=0}^{\infty} \operatorname{int} f^k(J),$$

and apply Lemma 4.2(2). □

LEMMA 4.5. *Let f be a mixing graph map. A free arc is a universal arc for f^l for some $l \geq 1$ if and only if it is a universal arc for f^n for all $n \geq 1$.*

Proof. It is clear that a universal arc for f is also universal for f^l for any $l \geq 2$. It is now enough to show that a universal arc I_U^l for f^l, where $l \geq 2$ is also universal for f. To this end, fix $\delta > 0$, let I_U be a universal arc for f, and let $M = M(\delta)$ be such that every free arc $J \subset G$ with $\operatorname{diam} J \geq \delta$ covers I_U through f^m for all $m \geq M$. We can also find an $L \geq 0$ (a multiple of l) such that I_U^l is f^L-covered by I_U. Now, every free arc $J \subset G$ with $\operatorname{diam} J \geq \delta$ covers I_U^l through f^m for every $m \geq M + L$, and the lemma follows. □

THEOREM 4.6. *For each mixing map f on a graph G the set $\mathcal{I}(f)$ has the following properties.*

(1) *For any $\delta, \varepsilon > 0$ there is an integer $K = K(\varepsilon, \delta)$ such that for any subgraph J of G with $\operatorname{diam} J \geq \delta$ we have $G \setminus \operatorname{int} f^k(J) \subset \mathrm{B}(\mathcal{I}(f), \varepsilon)$ for all $k \geq K$.*
(2) *The set $\mathcal{I}(f)$ has less than $\mathrm{D}^{\mathrm{s}}(G)$ elements. Moreover, $\mathcal{I}(f) \neq \emptyset$ if and only if f is pure mixing.*
(3) *Each point $x \in \mathcal{I}(f)$ is periodic for f and $f(\mathcal{I}(f)) = \mathcal{I}(f)$.*
(4) *For every $n \geq 1$ we have $\mathcal{I}(f) = \mathcal{I}(f^n)$.*

Proof. (1) Let I_U be a universal arc for f. By Lemmas 4.2(1) and 4.4, it is enough to prove that for any $\varepsilon > 0$ there is an integer $n = n(\varepsilon)$ such that

$$G \setminus \operatorname{int} f^n(I_U) \subset \mathrm{B}(\mathcal{I}(f), \varepsilon).$$

Suppose, to the contrary, that there is $\varepsilon > 0$ such that for every $n \geq 0$ we can find $x_n \notin \mathrm{B}(\mathcal{I}(f), \varepsilon) \cup \operatorname{int} f^n(I_U)$. Passing to a subsequence if necessary, we can assume that $\bar{x}$

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is the limit of the sequence $\{x_n\}$. Clearly, $\bar{x} \notin \mathrm{B}(\mathcal{I}(f), \varepsilon)$. If $\bar{x} \in \operatorname{int} f^{\bar{n}}(I_U)$ for some $\bar{n} \geq 0$, then $\bar{x} \in \operatorname{int} f^{\bar{n}}(I_U) \subset \operatorname{int} f^n(I_U)$ for all n large enough, hence $x_n \in \operatorname{int} f^n(I_U)$ for some n, contradicting the definition of the sequence $\{x_n\}$. But then $\bar{x} \in G \backslash \operatorname{int} f^n(I_U)$ for all $n \geq 0$, hence $\bar{x} \in \mathcal{I}(f)$, which is a contradiction.

(2) By Lemma 3.1(2), the diameters of components of $\overline{G \backslash f^k(I_U)} = G \backslash \operatorname{int} f^k(J)$ tend to 0 as $k \to \infty$. If there were at least $\mathrm{D}^{\mathrm{s}}(G)$ points in $\mathcal{I}(f)$, then $G \backslash f^k(I_U)$ would have to have at least $\mathrm{D}^{\mathrm{s}}(G)$ components for k large enough. But by [**30**, Lemma 4.2], for each $k \geq 0$ the set $G \backslash f^k(I_U)$ has less than $\mathrm{D}^{\mathrm{s}}(G)$ components, since $f^k(I_U)$ is connected, a contradiction. It is clear from Lemma 4.2(2) and (1) above that $\mathcal{I}(f) \neq \emptyset$ if and only if f is pure mixing.

(3) It is enough to show that for every $x \in \mathcal{I}(f)$ there is $y \in \mathcal{I}(f)$ such that $f(y) = x$. Assume, contrary to our claim, that there exists a point $x \in \mathcal{I}(f)$ such that $f^{-1}(x)$ is disjoint from $\mathcal{I}(f)$. Therefore, we can find an $\varepsilon > 0$ such that $f^{-1}(x) \subset G \backslash B$, where $B = \overline{B(\mathcal{I}(f), \varepsilon)}$. Since $x \notin f(B)$, the set $U = G \backslash f(B)$ is an open neighborhood of x such that

$$f^{-1}(U) = f^{-1}(G \backslash f(B)) \subset G \backslash B.$$

By (1) above, there exists $n > 0$ such that $G \backslash \overline{B(\mathcal{I}(f), \varepsilon)}$ is contained in $f^n(I_U)$. Therefore, $U \subset f^{n+1}(I_U)$, and $x \in \operatorname{int} f^{n+1}(I_U)$, contradicting the assumption $x \in \mathcal{I}(f)$.

(4) Fix $n \geq 1$. It is clear that $\mathcal{I}(f) \subset \mathcal{I}(f^n)$. We will show that the converse inclusion also holds. Let I_U^1 and I_U^n be universal arcs for f and f^n, respectively. Arguing as in the proof of Lemma 4.2(2) and applying (1) of the same lemma, we get N_U^1 such that $\operatorname{int} I_U^1 \cup \operatorname{int} f(I_U^1) \cup \cdots \cup \operatorname{int} f^k(I_U^1) \subset \operatorname{int} f^{N_U^1+k}(I_U^n)$ for $k \geq 0$. In particular, taking $L > 0$ such that $n \cdot L > N_U^1$,

$$\bigcup_{j=0}^{nl-N_U^1} \operatorname{int} f^j(I_U^1) \subset \operatorname{int} (f^n)^l(I_U^n) \quad \text{for every } l \geq L.$$

Summing over all $l \geq L$ and taking complement of both sides finishes the proof. □

5. *Local behavior around inaccessible points—inaccessible sides*

To establish the main theorem of the next section, we need to describe a local behavior of a map f around its inaccessible points. To do so rigorously we have to introduce some technical terminology. Let G be a graph with a fixed triangulation $\mathcal{L}$. A *canonical neighborhood* of a point $p \in G$ is an open set U such that $\overline{U}$ is an n-star, where $n = \operatorname{val}(p)$, and every connected component of $U \backslash \{p\}$ is homeomorphic to $(0, 1)$. Moreover, we require canonical neighborhoods of vertices of G to have disjoint closures. If f is pure mixing map, then without loss of generality we assume that all inaccessible points are vertices. Clearly, any point $p \in G$ has arbitrarily small canonical neighborhoods, and there is $\varepsilon_0 > 0$ such that for every $p \in G$ the open ball $\mathrm{B}(p, \varepsilon_0)$ is a canonical neighborhood of p. We use uniform continuity of f to get an $\varepsilon_c > 0$ such that $f(\mathrm{B}(x, \varepsilon_c)) \subset \mathrm{B}(f(x), \varepsilon_0)$. From now on we assume that with every point $p \in G$ we associate its canonical neighborhood $U_p = \mathrm{B}(p, \varepsilon)$. A pair (p, S_p), where $p \in G$ and S_p is a connected

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component of $U_p \setminus \{p\}$, is called a *side* of p. By a slight abuse of our terminology, we will identify a side with the sole set S_p, where the subscript will remind us which point in G we are using as a base for our side. If $0 < \delta \leq \varepsilon_0$, and S_p is a side, then a *ray* of length δ in the direction S_p is a subset $R(S_p, \delta) = S_p \cap \mathrm{B}(p, \delta)$. Nevertheless, our analysis is local, that is, we are investigating f restricted to $\mathrm{B}(\mathcal{I}(f), \varepsilon)$ for small $\varepsilon > 0$, and our considerations will not depend of the choices we made above. If x, y are two points in a canonical neighborhood U_p of some point $p \in G$ then we let $\langle x, y \rangle$ denote the convex hull of x and y in U_p. It is well defined since U_p is a tree.

Standing assumption. For the rest of this section we fix a graph G and a mixing map $f\colon G \to G$. We also let I_U be a universal arc for f and fix a triangulation $\mathcal{L}$ of G such that all inaccessible points are vertices.

Definition 5.1. We say that a side S_p is *accessible* if there is $n \geq 0$ such that $S_p \subset f^n(I_U)$. A side S_p is an *inaccessible side* if it is not accessible.

LEMMA 5.2. *A point $p \in G$ is inaccessible for f (that is, $p \in \mathcal{I}$) if and only if it has an inaccessible side.*

Proof. By Lemma 4.4 and Theorem 4.6(1), a point p is inaccessible for f if and only if p is not an interior point of $f^n(I_U)$ for every $n \geq 0$. Equivalently, every open neighborhood of p has non-empty intersection with $G \setminus f^n(I_U)$ for every $n \geq 0$. From the above and Lemma 4.2(1) we conclude that $p \in \mathcal{I}(f)$ if and only if there is a side S_p which is not contained in $f^n(I_U)$ for every $n \geq 0$. □

LEMMA 5.3. *The map f has less than $\Lambda(G)$ inaccessible sides.*

Proof. There is n such that all accessible sides and $G \setminus \mathrm{B}(\mathcal{I}(f), \varepsilon_c)$ are contained in $f^n(I_U)$. It follows that every inaccessible side contains exactly one endpoint of the subgraph $f^n(I_U)$. To finish the proof, we conclude from [**30**] that a subgraph of G has less than $\Lambda(G)$ endpoints. □

LEMMA 5.4. *Let $p \in G$. For every side S_p there exist a point $q \in G$ and a side S_q such that $f(q) = p$ and $f(S_q) \cap S_p \neq \emptyset$.*

Proof. Let us choose an infinite sequence $\{y_n\} \subset S_p$ converging to p. Since f is mixing, hence surjective, there is an infinite sequence $\{x_n\}$ such that $f(x_n) = y_n$ for all n. Passing to a subsequence if necessary, we may assume that x_n converges to some $q \in G$, and there is a side S_q such that $\{x_n\} \subset S_q$. By continuity $f(q) = p$, and clearly $f(S_q) \cap S_p \neq \emptyset$ as required. □

LEMMA 5.5. *If $q \in G$ and S_q is a side such that $f(S_q)$ intersects at least two sides of $p = f(q)$, then every side S_p of p such that $f(S_q) \cap S_p \neq \emptyset$ is accessible.*

Proof. Observe that if $x, y \in S_q$ are such that $f(x)$ and $f(y)$ belong to different sides of p then there is a path in $f(S_q)$ joining $f(x)$ with $f(y)$. Since $f(S_q)$ is uniquely arcwise connected this path must contain p. Then there must be a point $q_0 \in S_q$ such that $f(q_0) = p$. Let S_p be a side such that $f(S_q) \cap S_p \neq \emptyset$. We can choose a point $z \in S_q$ such that $f(z) \in S_p$. Then $f(\langle z, q_0 \rangle)$ contains a ray $R(S_p, \delta_0)$ for some $\delta_0 > 0$.

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Clearly, $\langle z, q_0\rangle \subset G\backslash \mathrm{B}(\mathcal{I}(f), \delta_1)$ for some $\delta_1 > 0$. Let $\delta = \frac{1}{2}\min\{\delta_0, \delta_1\}$. By Theorem 4.6(1) there is an integer N such that $f^n(I_U) \supset G\backslash \mathrm{B}(\mathcal{I}(f), \delta)$ for all $n \geq N$. In particular, $f^{N+1}(I_U)$ contains S_p. □

LEMMA 5.6. *If $q \in G$ and S_q is an accessible side then every side S_p of $p = f(q)$ such that $f(S_q) \cap S_p \neq \emptyset$ is accessible.*

Proof. By Lemma 5.5 it is sufficient to consider only the case where $f(S_q)$ intersects only one side S_p. Then $f(S_q)$ contains a ray in the direction S_p and we may proceed as in the proof of Lemma 5.5. □

Let $\mathcal{IS}$ denote the set of all inaccessible sides of points in G.

Definition 5.7. We write $f^*(S_p) = S_q$ if $S_q \in \mathcal{IS}$ is a side of $q = f(p)$ such that $f(S_p) \cap S_q \neq \emptyset$.

THEOREM 5.8. *The formula of Definition 5.7 describes a well-defined bijection $f^*\colon \mathcal{IS} \to \mathcal{IS}$. Moreover, for every $0 < \varepsilon < \varepsilon_c$ there is $\delta > 0$ such that for every $S_p \in \mathcal{IS}$ if $S_q = f^*(S_p)$, then $f(R(S_p, \delta)) \subset R(S_q, \varepsilon)$.*

Proof. Let $S_p \in \mathcal{IS}$. By Lemma 5.4 there exist a point $q \in G$ and its side S_q such that $f(q) = p$ and $f(S_q) \cap S_p \neq \emptyset$. On account of Lemma 5.6, S_q must be inaccessible. It follows from Lemma 5.2 that $q \in \mathcal{I}(f)$. By the above, we may define a function $g^*\colon \mathcal{IS} \to \mathcal{IS}$ such that if $g^*(S_q) = S_p$, then $f(S_p) \cap S_q \neq \emptyset$. By Lemma 5.5, g^* must be injective, and since $\mathcal{IS}$ is finite, g^* is a bijection. We define f^* to be the inverse of g^*. Now, if $f^*(S_p) = S_q$, then $f(S_p)$ is a ray in the direction S_q. Moreover, S_q must be unique. A standard application of uniform continuity finishes the proof. □

COROLLARY 5.9.

(1) *If f^* is a function as above, then the set $\mathcal{IS}$ consists of periodic orbits of f^*.*
(2) *For every $0 < \varepsilon < \varepsilon_c$ there is $\delta > 0$ such that for every side $S_p \in \mathcal{IS}$ there is $1 \leq m < \Lambda(G)$ such that $f^m(p) = p$ and $f^m(R(S_p, \delta)) \subset R(S_p, \varepsilon)$.*
(3) *For every $0 < \varepsilon < \varepsilon_c$ there is $\delta > 0$ such that for every accessible side S_p of some $p \in G$,*

$$f(R(S_p, \delta)) \subset \mathrm{B}(f(p), \varepsilon)\Big\backslash \bigcup_{(f(p),S)\in\mathcal{IS}} R(S, \varepsilon).$$

Proof. All assertions follow from uniform continuity of f, Lemma 5.3, and Theorem 5.8. □

6. *Structure theorem for pure mixing graph maps*

There is a natural way to provide examples of pure mixing maps of a circle. (1) Start with an interval map such that either both endpoints are fixed and at least one of them is inaccessible, or both endpoints are inaccessible and form a single periodic orbit of length two. (2) Identify the endpoints of the interval to obtain a circle. After the identification we still have a well-defined map, with the same number of inaccessible sides as at the beginning. As was noted in [**17**] and elaborated in [**21**], all pure mixing maps of the circle

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may be regarded as a result of applying this procedure to some pure mixing interval map. We will extend this result to a pure mixing map of an arbitrary graph.

From the previous section we see that for a mixing graph map f and a point $x \in G$ either the set of all preimages of x is dense in G, or $f^{-1}(x) = \{y\}$ where x, $y \in \mathcal{I}(f)$, and all sides of points lying on the (finite) orbit of x are inaccessible. Moreover, given a pure mixing graph map f as above, we may construct a new graph G' by detaching inaccessible sides from points of $\mathcal{I}(f)$ (we keep the space compact by adding some additional points). Since inaccessible sides are mapped onto inaccessible sides in a one-to-one way, the map f lifts in a natural way to a new map $g\colon G' \to G'$. Any inaccessible point for g is an endpoint of G' having now only a finite number of inaccessible preimages.

The proof is only a formalization of the procedure described above.

THEOREM 6.1. (Structure theorem) *Let $f\colon G \to G$ be a pure mixing graph map. Then there exist a graph G', a pure mixing map $g\colon G' \to G'$, and a continuous surjection $\pi\colon G' \to G$ such that the following statements hold.*

(1) *The map f is a factor of g via π, that is $\pi \circ g = f \circ \pi$. Moreover, π is one-to-one on $G' \setminus \mathcal{I}(g)$ and $\pi(\mathcal{I}(g)) = \mathcal{I}(f)$.*

(2) *If $e \in \mathcal{I}(g)$ then e is an endpoint of G' and $g^{-1}(e) = \{e'\}$ for some $e' \in \mathcal{I}(f)$. Moreover, $\mathcal{I}(g)$ has less than $\Lambda(G)$ elements.*

Proof. Let $\mathcal{L}$ be a triangulation of G. Let X denote the disjoint union of edges of $\mathcal{L}$, so X is homeomorphic to $[0, 1] \times \{1, \ldots, l\}$, for some $l > 0$. Any edge of G can be now identified with a component of X.

There is a unique equivalence relation R on X such that G is the identification space (quotient space) $G = X/R$. The relation R is called the incidence relation and informs us which endpoints of X are to be identified to form G.

Obviously, we can have $x R y$ for $x \neq y$ only if x, y are endpoints of some components of X. We can view equivalence classes with respect to R, denoted by $[x]_R$, as elements of G. Moreover, if the class $[x]_R$ has more than one element then it represents a vertex v of G and consists of $\mathrm{val}(v)$ points of G. The set of all inaccessible sides $\mathcal{IS}$ of points in G is in a one-to-one correspondence with a subset **IS** of endpoints of components of X. Call a point $x \in X$ *regular* if $x \notin$ **IS**.

We define a new relation $R' \subset R$ by declaring $x R' y$ if and only if $x R y$ and either both x and y are regular, or $x = y$. Clearly, R' is an equivalence relation, and $[x]_{R'} \subset [x]_R$ for $x \in X$. Moreover, the space $G' = X/R'$ is easily seen to be homeomorphic to the space obtained by removing from each inaccessible side for f on G an open arc lying on that side. Hence, G' is a graph and a map $\pi\colon G' \to G$ given by $\pi([x]_{R'}) = [x]_R$ is a continuous surjection.

We will now define a map $g\colon G' \to G'$. First, note that if x is a regular point such that $[x]_R \notin \mathcal{I}(f)$ then $[x]_R = [x]_{R'}$. In particular, all sides of $[x]_R$ are accessible. Note also that if $[x]_R = [x]_{R'}$ for some $x \in X$ then there is a neighborhood U of $[x]_R$ in G such that $[z]_R = [z]_{R'}$ for all points $z \in X$ such that $[z]_R \in U$. Let $[y]_R = f([x]_R)$. We have to consider two cases: either $[y]_R$ belongs to $\mathcal{I}(f)$, or not. In the second case, we also have $[y]_R = [y]_{R'}$, and we define $g([x]_{R'}) = f([x]_R) = [y]_{R'}$. In the first case not every side of $[y]_R$ is accessible. But f is mixing so it cannot map any neighborhood of $[x]_R$ to

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a point $[y]_R$; therefore, by Lemma 5.6 there must be an accessible side of $[y]_R$. Hence there must be a regular point $z \in X$ such that $[z]_R = [y]_R$. Clearly, if z' is another regular point such that $[y]_R = [z']_R$ then $zR'z'$ by the definition of R'. Therefore, we may define $g([x]_{R'}) = [z]_{R'}$.

If x is an endpoint representing some inaccessible side S_v of a point $v = [x]_R \in G$, then there is a unique point $y \in X$ representing a side $f^*(S_v)$, where f^* is a map defined in Theorem 5.8. In particular, $y \in \mathcal{IS}$. Then we have $[x]_{R'} = \{x\}$, $[y]_{R'} = \{y\}$ and we may define $g([x]_{R'}) = [y]_{R'}$.

Now, continuity of g easily follows from Lemma 5.6, Theorem 5.8 and Corollary 5.9. To see this, let $w = [x]_{R'} \in G'$ and let $\{w_n\}$ be a sequence in G' converging to w as $n \to \infty$. Let $w_n = [x_n]_{R'}$. If x is a regular point, then without loss of generality we may assume that x_n is also regular point for all n. Moreover, $f([x_n]_R) = g([x_n]_{R'})$, and since f is continuous $g([x_n]_{R'})$ converges to $f(w) = g(w)$. If x is an endpoint representing some inaccessible side S_v of a point $v = [x]_R \in G$, then we may assume that $v_n = [x_n]_R$ is a sequence in G converging to v along S_v. But then $\{g(w_n)\}$ must converge to $g(w)$ along uniquely determined inaccessible side $f^*(S_v)$. Thus, g is continuous.

By definition of g and π, we see that $\pi \circ g = f \circ \pi$. Moreover, $\pi(\mathcal{I}(g)) = \mathcal{I}(f)$ and the set $\mathcal{I}(g)$ may be identified with the set of inaccessible sides of f. It follows from Lemma 5.3 that $\mathcal{I}(g)$ has less than $\Lambda(G)$ elements. If $e \in \mathcal{I}(g)$, then e is an endpoint of G' by definition of R' and $g^{-1}(e) = \{e'\}$ for some uniquely determined point $e' \in \mathcal{I}(f)$ by definition of g and Theorem 5.8. □

Remark 6.2. Note that the factor map π in Theorem 6.1 is one-to-one except on a finite set where it is finite-to-one (the π-preimage of any point in G has at most $\Lambda(G)$ elements). In particular, π is bounded-to-one and therefore $h(f) = h(g)$ (this is a consequence of Bowen's formula; see [**31**, Theorem 5.3], or [**33**] for a proof in the case where π is at-most-countable-to-one).

7. *Transitivity and entropy of pure mixing graph maps*

With the structure theorem at hand we can now study the topological entropy of pure mixing graph maps. In this we will utilize the structure theorem and other tools.

To estimate the topological entropy of pure mixing graph maps we need the notion of a loose horseshoe from [**21**]. Recall that an *s-horseshoe* for f is a free arc J contained in the domain of f, and a collection $\mathcal{C} = \{A_1, \ldots, A_s\}$ of $s \geq 2$ non-empty compact subsets of J fulfilling the following three conditions: (a) each set $A \in \mathcal{C}$ is the union of a finite number of arcs, (b) the interiors of the sets from $\mathcal{C}$ are pairwise disjoint, (c) $J \subset f(A)$ for every $A \in \mathcal{C}$. If the union of elements of $\mathcal{C}$ is a proper subset of J, or J is a proper subset of $f(A)$ for some $A \in \mathcal{C}$, then we say that a horseshoe $(J, \mathcal{C})$ is *loose*. The following lemma is adapted from [**21**] and summarizes results of [**21**, §4.2]. It is easy to see that the assumption that the graph is an interval [0, 1] or a circle was inessential there, and the result holds for arbitrary graphs.

LEMMA 7.1. [**21**] *If a transitive graph map f has a loose s-horseshoe then $h(f) > \log s$.*

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The next two theorems provide lower bounds on the topological entropy of pure mixing graph maps. The first of these facts comes from [**1**, Proposition 4.2] for the tree maps and with the weak inequality. Baldwin [**6**] later observed that the inequality is in fact strict. Here we present a variant of that result which is valid for graph maps.

THEOREM 7.2. *Let f be a transitive map of a graph G. If e is an endpoint of G such that $f^{-1}(e) = \{e\}$, then e is an accumulation point of fixed points of f and $h(f) > \log 3$.*

Proof. We identify the edge containing e with the unit interval $[0, 1]$ with $e = 0$, and we use the induced ordering $<$ and write about intervals, etc. In addition, we agree to write $x < y$ for any $x \in [0, 1]$ and $y \in G\backslash[0, 1]$. We can find an $0 < \varepsilon_0 < 1$ such that $f([0, \varepsilon_0]) \subset [0, 1]$. Observe that for any $0 < \varepsilon \le \varepsilon_0$:

($\star$) if $\varepsilon' = \min f(G\backslash[0, \varepsilon))$, then $0 < \varepsilon' < \varepsilon$ (since $\varepsilon' \ge \varepsilon$ would imply that $G\backslash[0, \varepsilon)$ is a proper invariant set with non-empty interior contradicting transitivity, and $\varepsilon' = 0$ would contradict $f^{-1}(e) = \{e\}$);

($\star\star$) if $\varepsilon'' = \max f([0, \varepsilon])$, then $\varepsilon < \varepsilon''$ (as $\varepsilon'' \le \varepsilon$ would imply that $[0, \varepsilon]$ is invariant, contradicting transitivity).

We will use ($\star$) and ($\star\star$) all the time without further reference. To prove that e is an accumulation point of fixed points of f it is enough to show that for any $0 < \varepsilon < \min f(G\backslash[0, \varepsilon_0))$ we have $f(a) > a$ and $f(b) < b$ for some $0 < a < b < \varepsilon$. To see this, let $\varepsilon' = \min f(G\backslash[0, \varepsilon))$. Then there is $b \in [\varepsilon', \varepsilon)$ such that $f(b) < b$, as otherwise $f(G\backslash[0, \varepsilon')) \subset G\backslash[0, \varepsilon')$, contradicting transitivity. Similarly, there must be $0 < a < b$ such that $f(a) = \max f([0, b]) > b$.

For the proof that $h(f) > \log 3$, let $0 < a < \min f(G\backslash[0, \varepsilon_0))$ be a fixed point of f, and set $b = \min f(G\backslash[0, a))$. We have $0 < b < a$, and since e is an accumulation point of fixed points of f there are fixed points s and t of f such that $0 < s < b$, $s < t$ and $\max f([0, b]) = f(z)$ for some $s < z \le b$. Moreover, we may assume that there is no fixed point in (s, t), so $f(x) > x$ for all $x \in (s, t)$. It follows that $t < a$ and there must be fixed points u and v of f such that $t \le u < v \le a$ and $\min f(G\backslash[0, s)) = f(w)$ for some $u < w < v$. Again we may assume that $f(y) < y$ for all $y \in (u, v)$. There are fixed points p and q such that for some $p < r < q$ we have $f(r) = \max f([0, v]) = \max f([p, q])$. Clearly, we have $s \le p < q \le u$, since $\max f([0, s]) \le \max f([s, t])$ and $\max f([0, v]) \le \max f([0, u])$. Therefore, $[p, r]$, $[r, w]$, $[w, v]$ form a loose 3-horseshoe for f, as $f(r) > v$, and $f(w) < s$. □

THEOREM 7.3. *If f is a pure mixing map of a graph G, then:*

(1) $h(f) > (1/\Lambda(G)) \cdot \log 3$;

(2) *there exists $0 < m < \Lambda(G)$ such that f^m has infinitely many fixed points.*

Proof. By Theorem 6.1, f is a factor of a map $g\colon G' \to G'$ such that $g^{-1}(\mathcal{I}(g)) = \mathcal{I}(g)$. Moreover, $\mathcal{I}(g)$ has less than $\Lambda(G)$ elements, so there is $0 < m < \Lambda(G)$ such that $g^{-m}(e) = \{e\}$ for some endpoint of G'. By Theorem 7.2 we see that $h(g^m) > \log 3$. As g^m is an extension of f^m via bounded-to-one semiconjugacy, we have $h(f^m) = h(g^m) > \log 3$. Moreover, g^m has infinitely many fixed points, and so does f^m. □

As a direct consequence we get the following lower bound for $\inf(h(\mathcal{P}_G))$ for a given graph G.

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COROLLARY 7.4. *If G is a graph, then* $\inf(h(\mathcal{P}_G)) \geq (1/\Lambda(G)) \cdot \log 3$.

As a second corollary we get the following theorem which is contained in [**38**] as Theorem 4.1. Here, $\mathrm{Fix}(f^k)$ denotes the set of all fixed points of f^k.

COROLLARY 7.5. *Let $f: G \to G$ be a graph map with $\#\,\mathrm{Fix}(f^k) < \infty$ for each $k \geq 1$. If f is transitive, then it is strongly transitive, that is, for every non-empty open set $U \subset G$ there exists $n > 0$ such that*

$$G = U \cup f(U) \cup \cdots \cup f^n(U).$$

Proof. If f is an irrational rotation then the result is well known. Assume that f is non-invertible. By Theorem 2.1 it is enough to prove that if f is totally transitive then it must be exact. But it is well known that every totally transitive non-invertible graph map must be mixing (see [**3, 9, 22**] for a proof). By Theorem 7.3 a mixing graph map with $\#\,\mathrm{Fix}(f^k) < \infty$ for each $k \geq 1$ must be exact. □

We recall that a map $f: X \to Y$ is *monotone* if X and Y are topological spaces, f is continuous, and for each point $y \in Y$ its preimage $f^{-1}(y)$ is connected. We say that a graph map $f: G \to G$ is *P-monotone* if there is a finite set $P \subset G$ containing all vertices of some triangulation of G such that for each connected component C of $G \setminus P$ the map $f|_{\overline{C}}: \overline{C} \to G$ is monotone. We say that f is *piecewise monotone* if there exists a finite set $P \subset G$ containing all vertices of some triangulation of G such that f is P-monotone and we need not specify P. If $f: G \to G$ is a P-monotone graph map then the closures of connected components of $G \setminus P$ are called *P-basic intervals* for f. Note that all P-basic intervals are actually arcs since all vertices (in particular, all endpoints and all branching points) are in P. Our definition agrees with the definition of piecewise monotone tree map presented in [**6**, p. 553]. The reader should be aware that there is no standard terminology in the literature, as *piecewise monotone* may refer to a more restrictive (cf. [**37**]), as well as a broader notion (cf. [**36**], [**38**, p. 380]). Note that if f is a piecewise monotone map, then f^k is also piecewise monotone for all $k \geq 2$.

In [**38**], Corollary 7.5 was used to obtain the following result, well known in the special case of interval maps [**17, 32**]. We provide here a much shorter proof. Note that Yokoi's definition of piecewise monotone graph map from [**38**, p. 380] allows more than ours, but the kth iterate of the piecewise monotone transitive graph map in Yokoi's sense has at most finitely many fixed points.

COROLLARY 7.6. *Let $f: G \to G$ be a piecewise monotone graph map. If f is transitive then it is strongly transitive; in particular, a non-invertible totally transitive piecewise monotone graph map must be exact.*

Proof. We show that every transitive piecewise monotone map f fulfills the assumptions of Corollary 7.5, that is, $\#\,\mathrm{Fix}(f^k) < \infty$ for each $k \geq 1$. Using Theorem 2.1 again, it is enough to prove that if a non-invertible f is totally transitive then it must be exact. As in the proof of Corollary 7.5, we note that f (so f^k for all $k \geq 1$ as well) is mixing. If f is pure mixing, then by Theorem 7.3 we have $\#\,\mathrm{Fix}(f^k) = \infty$ for some $k \geq 1$. But f^k is piecewise monotone as well so there exists a finite set $P \subset G$ such that f^k is P-monotone and there are two fixed points p, q contained in the interior of some P-basic interval E.

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Let U be the connected component of $E\backslash\{p, q\}$ such that $p, q \in \overline{U}$. Since $f^k(p) = p$, $f^k(q) = q$, and f^k must be monotone on $\overline{U}$ we conclude that $f^k(\overline{U}) = \overline{U}$, contradicting transitivity of f^k.

We have just proved that $\#\operatorname{Fix}(f^k) < \infty$ for every $k > 0$; therefore, the assumptions of Corollary 7.5 are satisfied, which completes the proof. □

8. *Topological entropy of transitive tree maps*

In this section we collect some technical results which we will use in the next section. We will also need some additional terminology, which we recall below.

Let $f: T \to T$ be a tree map. We say that f is *linear* on a set $S \subset T$ if there is a constant α such that $d(f(x), f(y)) = \alpha d(x, y)$ for all $x, y \in S$ (here, as always, d denotes the taxicab metric on T). If a P-monotone map f is linear on every P-basic interval then we call it *P-linear* or *piecewise linear* if there is no need to single out P. A P-monotone map f is a *Markov map* if $f(P) \subset P$. In the above situation we call f a *P-Markov* map for short. If f is a P-Markov map, then the *Markov graph* of f with respect to P (P-Markov graph of f for short) is then defined as a directed graph with the set of P-basic intervals as a set of vertices and with the set of edges defined by the f-covering relation, that is, there is an edge from a P-basic interval I to a P-basic interval J in the Markov graph $(I \to J)$ if and only if J is f-covered through f by I. A *path (of length n)* in a graph $\mathcal{G}$ is any sequence $I_0, I_1, \ldots, I_n$ of vertices of $\mathcal{G}$ such that there exists an edge $I_{i-1} \to I_i$ for each $i = 1, \ldots, n$. A *loop* of length n is any path $I_0, I_1, \ldots, I_n$ of length n such that $I_0 = I_n$. A graph is *strongly connected* if for any pair of its vertices I, J there is a path $I_0, I_1, \ldots, I_n$ in $\mathcal{G}$ with $I = I_0$ and $J = I_n$. If $I_1, \ldots, I_n$ is an enumeration of the set of P-basic intervals then the *incidence matrix* of f with respect to P is a n by n matrix $A = [a_{ij}]$ with $a_{ij} = 1$ if $I_i \to I_j$ and $a_{ij} = 0$ otherwise. A non-negative matrix $A = [a_{ij}]$ is *irreducible* if for each ordered pair of indices i, j, there exists some $k \geq 0$ such that $a_{ij}^k > 0$. The *spectral radius* of a square complex matrix is defined as the largest absolute value of its eigenvalues.

There are also a few results we would like to recall for later reference. The following lemma comes from [**6**, Corollorary 1.11].

LEMMA 8.1. *Suppose that T is a tree and $f: T \to T$ is P-Markov and P-linear with respect to some $P \subset T$. Then f is transitive if and only if the P-Markov graph of f is strongly connected and is not a graph of a cyclic permutation.*

The following lemma is well known. We restate it in a suitable form.

LEMMA 8.2. *Let $f: T \to T$ be a Markov tree map. If $\mathcal{G}$ is any Markov graph of f and $A = [a_{ij}]_{n\times n}$ is the corresponding incidence matrix with a spectral radius $\rho \geq 0$, then $h(f) = \log \rho$ if $\rho > 0$, and $h(f) = 0$ otherwise. Moreover, if there is $s > 0$ such that for every vertex v of $\mathcal{G}$ the number of directed paths of length $n > 0$ starting at v in $\mathcal{G}$ is bounded from above by s^n, then $h(f) \leq \log s$. If in addition $\mathcal{G}$ is strongly connected and there are $t > 0$ and $n > 0$ such that for some vertex v of $\mathcal{G}$ the number of paths of length $n > 0$ starting at v in $\mathcal{G}$ is bounded below by t^n, then $\log t \leq h(f)$.*

Proof. The connection between the spectral radius of the incidence matrix and topological entropy is well known (see [**11**, Lemma 1.5]). It is easy to see that the number of paths

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of length $n > 0$ starting at v in $\mathcal{G}$ is equal to the row sum of row v in A^n. Moreover, the incidence matrix of a graph is irreducible if and only if the graph is strongly connected. The bounds on the spectral radius ρ of an irreducible square matrix $B = [b_{ij}]_{i,j=1}^m$ come from the well-known formula (see, for example, [**28**, Exercise 4.2.3, p. 111]):

$$\min_{i=1,\dots,m} \sum_{j=1}^{m} b_{ij} \le \rho \le \max_{i=1,\dots,m} \sum_{j=1}^{m} b_{ij}. \qquad \square$$

The following theorem is a restatement of [**6**, Theorem 3.2]. We have added the assertion ($\star$), which is not included in the original statement but is a consequence of the proof provided in [**6**].

THEOREM 8.3. *Let T be a tree, and $f : T \to T$ be transitive and Markov. Let z be a fixed point of f, and let T' be the tree which is obtained from T by attaching an arc to z at one of the endpoints of the arc. Then for every $\varepsilon > 0$, there is a transitive Markov map $f' : T' \to T'$ such that $h(f') < h(f) + \varepsilon$ and both ends of the new arc are fixed for f'. Furthermore, f' can be defined so that, in addition, the following statement holds.*

($\star$) *If some subset of the endpoints of T forms a single periodic orbit C, then C is also a periodic orbit of f'.*

Let (X, x_0) and (Y, y_0) be topological pointed spaces (spaces with distinguished base points). The *wedge sum* of (X, x_0) and (Y, y_0) (denoted by $(X, x_0) \wedge (Y, y_0)$) is the quotient of the disjoint union of X and Y by the identification $x_0 \sim y_0$. The mth *wedge power* $(X, x_0)^{\wedge m}$ is defined as wedge sum of m copies of (X, x_0).

LEMMA 8.4. *Let X be a topological space and $f : X \to X$ be a transitive map. If x_0 is a fixed point of f, then for every $m \ge 2$ there is a transitive but not totally transitive map $F : (X, x_0)^{\wedge m} \to (X, x_0)^{\wedge m}$ such that $h(F) = h(f)/m$. Moreover, x_0 is the unique fixed point of F, and all other periodic orbits of f are periodic points of F with m times longer prime periods (formed by the m-times copy of points of the orbit of f). Furthermore, if X is a tree and f is a Markov map, then the same can be said for $(X, x_0)^{\wedge m}$ and F, respectively.*

Proof. Let us identify a disjoint union of m copies of X with $X \times \{0, \dots, m-1\}$. Define a map $F' : X \times \{0, \dots, m-1\} \to X \times \{0, \dots, m-1\}$ by

$$F'((x, i)) = \begin{cases} (x, i+1) & \text{for } i = 0, \dots, m-2, \\ (f(x), 0) & \text{for } i = m-1. \end{cases}$$

It is easy to see that F' induces a quotient map F on $(X, x_0)^{\wedge m}$ with the desired properties. The rest of the proof is now straightforward. $\square$

Note that if $m \ge 2$, then the map constructed in Lemma 8.4 is not totally transitive. To remove this problem we need the following lemma which gives a general method of constructing exact Markov maps from transitive but not totally transitive Markov maps and enables us to control the entropy. The proof of the upper bound for entropy follows the ideas of [**6**].

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LEMMA 8.5. *If f is a transitive P-Markov and P-linear map of a tree T, which is not totally transitive, then for every $\varepsilon > 0$, there is a totally transitive (hence, exact) P'-Markov and P'-linear map $f' : T \to T$ such that $h(f') < h(f) + \varepsilon$. Furthermore, f' can be defined so that, if some subset of the endpoints of T forms a periodic orbit C for f, then C is also a periodic orbit of f'.*

Proof. Let $f : T \to T$ be a transitive but not totally transitive P-Markov and P-linear tree map. We denote the P-Markov graph of f by $\mathcal{G}$. Fix $\varepsilon > 0$ and let $r, s > 1$ be such that $h(f) < \log r < \log s < h(f) + \varepsilon$. By Theorem 2.1, f has the unique fixed point $p \in T$ and we can enumerate the closures of connected components of $T \setminus \{p\}$ by $T_0, T_1, \ldots, T_{n-1}$ with $n \geq 1$ in such a way that $f(T_j) = T_{j+1}$ for $j = 0, \ldots, n-1$, and $f(T_{n-1}) = T_0$ hold. Without loss of generality we may assume that $p \in P$. Moreover, for each $j = 0, 1, \ldots, n-1$ the map $f^n|_{T_j} : T_j \to T_j$ is a totally transitive (thus, exact) piecewise linear Markov map; in particular, there is a point $q \neq p$ such that $f(q) = p$. Without loss of generality we may assume that $q \in T_0$. Let $[x, y]$ be a P-basic interval in T_0 containing q. If $q \notin P$, then either $f|_{[x,y]}$ would not be monotone, or $f(T_0)$ would intersect T_j with $j \neq 1$. Hence $q \in P$, and without loss of generality we may assume that $x = q$. For $j = 0, 1, \ldots, n-1$ let $I_j = [p, z_j]$ denote the P-basic interval in T_j containing p.

Let us choose $\Lambda > 0$ such that $3r^L < s^L$, for all $L \geq \Lambda$, where r and s are as above. Then there exist a point w_0 in the interior of $[p, z_0]$ and $L \geq \Lambda$ such that for $k = 0, \ldots, L-1$ and $j = (k \mod n)$ we have $w_k = f^k(w_0) \in \operatorname{int} I_j$, and $f^L(w_0) \in P$.

Let $P'' = P \cup \{w_j : j = 0, \ldots, L-1\}$, and let $\mathcal{G}''$ be the Markov graph of f with respect to P''. We define $f' : T \to T$ by modifying f on $[x, y]$ only, that is, we put $f'(z) = f(z)$ for all $z \in T \setminus [x, y]$. Next, we choose two points x', y' in $[x, y]$ with $x < y' < x' < y$, then we set $f'(y') = w_0$ and $f'(x') = p$, and extend f' to $[x, x']$ by making it linear on $[x, y']$ and $[y', x']$. We identify $[x', y]$ with $[x, y]$ by a linear homeomorphism ψ such that $\psi(x') = x$, and for $z \in [x', y]$ we define $f'(z) = f(\psi(z))$. Then f' is a continuous map of T, which is P'-Markov and P'-linear with respect to $P' = P'' \cup \{x', y'\}$. Note that if some subset of the endpoints of T forms a single periodic orbit C of f, then C is also a periodic orbit of f' since $f'|_P = f|_P$.

We claim that f' is totally transitive. First, observe that $f'(T_0)$ contains $f'([x, y])$, and hence it intersects the interiors of both T_0 and T_1. As p is the unique fixed point of f', and $T_0, \ldots, T_{n-1}$ are no longer invariant for $(f')^n$, it follows from Theorem 2.1 that f' is totally transitive provided f' is transitive.

To show that f' is transitive we consider the P'-Markov graph of f', denoted by $\mathcal{G}'$. By Lemma 8.1 it is enough to prove that $\mathcal{G}'$ is strongly connected, since it is clear that it is not a graph of a cyclic permutation. We will say that a P'-basic interval J of f' is an 'old' one if either J is a P''-basic interval of f, or $J = [x', y]$. With this nomenclature $\mathcal{G}''$ has two 'new' vertices; that is, two P''-basic intervals $J' = [x, y']$ and $J'' = [y', x']$. Observe that the subgraph of $\mathcal{G}'$ given by the set of old P'-basic intervals together with all edges between them is isomorphic to $\mathcal{G}$. Moreover, if any old P'-basic interval f-covers $[x, y]$, then it also f'-covers $[x', y]$, J', and J'', and the last two intervals f'-cover the old P'-basic interval $[w_0, p]$ since $f'(x) = p$, $f'(y') = w_0$, $f'(x') = p$, and f' is linear

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on $J' = [x, y']$ and $J'' = [y', x']$. It easily follows that $\mathcal{G}'$ is strongly connected and thus f' is transitive. The proof of the claim is completed.

To estimate the topological entropy of f' we fix a vertex v of $\mathcal{G}'$ and provide a bound on the number of paths of length L in $\mathcal{G}'$ starting at v. By the definition of f' and our choice of L every path of length L in $\mathcal{G}'$ can pass at most once through J' or J''. Moreover, to every path α of length L in $\mathcal{G}$ corresponds either exactly one path in $\mathcal{G}'$ if α does not contain $[x, y]$, or exactly three paths in $\mathcal{G}'$ otherwise. By Lemma 8.2, there are at most r^L paths of length L starting at any fixed vertex of G, and we conclude that there are at most $3r^L$ paths of length L in $\mathcal{G}'$ starting at v. Using Lemma 8.2 and by our choice of s, we have that $3r^L < s^L$ and $h((f')^L) \le \log s^L$, hence $h(f') \le \log s < h(f) + \varepsilon$. □

The last lemma shows how to construct pure mixing examples from exact Markov tree maps with the topological entropy as small as possible.

LEMMA 8.6. *Let $f: T \to T$ be an exact Markov tree map. If $\mathcal{O}$ is a periodic orbit for f of length $m > 0$ fully contained in the set of endpoints of T, then for every $\varepsilon > 0$, there is a pure mixing map $f': T \to T$ such that*

$$h(f') < \max\{h(f), (\log 3)/m\} + \varepsilon$$

and $\mathcal{O} = \mathcal{I}(f')$ is a periodic orbit of f'.

Proof. Fix any $\varepsilon > 0$. Let $\mathcal{O}$ be a single periodic orbit for f of length $m > 0$ contained in the set of endpoints of T. Set $\eta = \max\{h(f), (\log 3)/m\}$. Choose $r, s > 1$ such that $\eta < \log r < \log s < \eta + \varepsilon$, and fix $L > 0$ such that $3r^L < s^L$. Take any $P \subset T$ such that f is Markov with respect to P. Choose any $o_0 \in \mathcal{O}$ and for $j = 1, 2, \ldots, m-1$ put $o_j = f^j(o_0)$, and let $I_j = [o_j, z_j]$ denote the P-basic interval containing the endpoint o_j. Reasoning as in the proof of Lemma 8.5, we can find a P-basic interval $[x, y]$ such that $f(x) = o_0$ and $x \ne o_{m-1}$. Since f is exact, it cannot collapse any of the intervals I_j, and thus $[x, y] \ne I_j$ for any j. We order $[x, y]$ such that $x < y$ and we choose two points x', y' in $[x, y]$ with $x < y' < x' < y$. We extend the tree T by attaching an interval $[a_j^\infty, o_j]$ at each $o_j \in \mathcal{O}$ such that for each $j = 0, \ldots, m-1$ there is an isometry $\varphi_j: [a_j^\infty, o_j] \to [a_{j+1}^\infty, o_{j+1}]$ (here we agree that $o_m = o_0$, and $a_m^\infty = a_0^\infty$). Let us order each interval $[a_j^\infty, o_j]$ such that $a_j^\infty < o_j$. We obtain a new tree T', which is homeomorphic with the original T, and $T \subset T'$. Now, to finish the proof it is enough to construct $f': T' \to T'$ such that the entropy bounds hold and $\mathcal{O}' = \{a_0^\infty, \ldots, a_{m-1}^\infty\}$ is a non-accessible periodic orbit of f'. First set $f'(x) = f(x)$ for each $x \in T \setminus [x, y] \subset T'$. Observe that there is $u_0 \in I_0$ such that for $j = 1, \ldots, mL$ and $l = j \mod m$,

$$f'([o_j, (f')^j(u_0)]) \subset I_l \quad \text{and} \quad (f')^{mL}(u_0) = z_0.$$

Next, we choose a sequence $\{a_j\}_{j=0}^\infty \subset [a_{m-1}^\infty, o_{m-1}]$ such that for each $j = 1, 2, \ldots,$

$$a_j < a_{j-1}, \quad a_0 = o_{m-1}, \quad \lim_{j \to +\infty} a_j = a_{m-1}^\infty.$$

Set $\alpha_j = \varphi_{m-1}(a_j)$ for $j = 0, 1, \ldots,$ and find infinite sequences $\{b_j\}_{j=1}^\infty$, $\{c_j\}_{j=1}^\infty$, $\{d_j\}_{j=1}^\infty$, and $\{e_j\}_{j=1}^\infty$ such that for each $j = 1, 2, \ldots,$

$$a_j < c_{2j-1} < b_j < c_{2j} < e_{2j} < d_j < e_{2j-1} < a_{j-1}.$$

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For $k = 0, \ldots, m-2$ and $x \in [a_k^\infty, o_k]$ set $f'(x) = \varphi_k(x)$. For $j = \infty$ put $f'(a_j) = \alpha_j$ and for each $j = 1, 2, \ldots$ set

$$f'(a_j) = \alpha_j, \quad f'(c_{2j-1}) = f'(c_{2j}) = \alpha_{j-1}, \quad f'(e_{2j-1}) = f'(e_{2j}) = \alpha_j,$$

and extend f' linearly to $[a_j, c_{2j-1}]$, $[c_{2j}, e_{2j}]$, and $[e_{2j-1}, a_{j-1}]$. Now for each $j = 1, 2, \ldots$ there are points $u_j \in [\alpha_j, \alpha_{j-1}]$ and $w_{j-1} \in [\alpha_j, \alpha_{j-1}]$ such that the following conditions hold:

($\star$) for each $k = 1, \ldots, L$,

$$(f')^{mk-1}([\alpha_j, u_j]) \subset [a_j, c_{2j-1}] \quad \text{and} \quad (f')^{mk-1}([w_{j-1}, \alpha_{j-1}]) \subset [e_{2j-1}, a_{j-1}],$$

hence $(f')^i$ is well defined on intervals $[\alpha_j, u_j]$, $[w_{j-1}, \alpha_{j-1}]$ for $i = 1, \ldots, mL-1$;

($\star\star$) we have

$$(f')^{mk-1}(u_j) = c_{2j-1} \quad \text{and} \quad (f')^{mk-1}(w_{j-1}) = e_{2j-1}.$$

We have defined two sequences $\{u_j\}_{j=0}^\infty$ and $\{w_j\}_{j=0}^\infty$. For $j = 1, 2, \ldots$ we define

$$f'(b_j) = u_{j-1} \quad \text{and} \quad f'(d_j) = w_j,$$

and extend f' linearly to $[c_{2j-1}, b_j]$, $[b_j, c_{2j}]$, $[e_{2j}, d_j]$, and $[d_j, e_{2j-1}]$. Finally, we set $f'(x') = o_0$, $f'(y') = w_0$, and extend f' linearly to $[x, y']$ and $[y', x']$. We identify $[x', y]$ with $[x, y]$ by a linear homeomorphism ψ such that $\psi(x') = x$ and for $z \in [x', y]$ we define $f'(z) = f(\psi(z))$. Then f' is a continuous map of T'. Reasoning as in [**21**], we get that f' is a pure mixing map, and it is clear that $\mathcal{O}' = \{a_0^\infty, \ldots, a_{m-1}^\infty\}$ is a non-accessible periodic orbit of f'.

To estimate the topological entropy of f', for each $j = 1, 2, \ldots$ we define a sequence of maps $\{f'_j\}_{j=1}^\infty$ such that $h(f'_j) \to h(f')$ as $j \to \infty$, and for each $j = 1, 2, \ldots$ we have $h(f'_j) \le \eta + \varepsilon$. To this end set $Q_0^j = [\alpha_j, \alpha_{j-1}]$, and $Q_k^j = \varphi_{k-1}(Q_{k-1}^j)$. Let

$$\Omega_k^j = \bigcup_{l=1}^{j} Q_k^l \quad \text{and} \quad S_j = \bigcup_{l=1}^{j} \bigcup_{k=0}^{m-1} Q_k^l.$$

Next, we define a sequence of linear Markov maps $f'_j \colon T' \to T'$ by

$$f'_j(x) = \begin{cases} f'(x) & \text{for } x \in T \cup S_j, \\ \varphi_k(x) & \text{if } x \in [a_k^\infty, o_k] \setminus \Omega_k^j \text{ for some } k = 0, \ldots, m-1. \end{cases}$$

Let $X_j = \{x \in T' : (f')^n(x) \in T \cup S_j \text{ for } n = 0, 1, \ldots\}$. Observe that $f'|_{X_j} = f'_j|_{X_j}$ and $h(f'_j) = h(f'_j|_{X_j})$. As $h(f'|_{X_j}) \le h(f')$ and $X_j \subset X_{j+1}$ we also have $h(f'_j) \le h(f'_{j+1})$, and we get

$$\limsup_{j\to\infty} h(f'_j) = \sup h(f'|_{X_j}) \le h(f').$$

Since f'_j converges uniformly to f' on T', and the entropy function is lower semicontinuous, we get

$$h(f') \le \liminf_{j\to\infty} h(f_j).$$

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To finish the proof we need to show that $h(f'_j) \leq \eta + \varepsilon$ for every $j = 1, 2, \ldots$. But by the way f'_j is defined it is a linear Markov map on $T \cup S_j$, so Lemma 8.2 applies, and the upper bound can be obtained by counting paths in the Markov graph of f'_j in a similar way to the proof of Theorem 8.5. The details are left to the reader. □

9. *Examples*

In this section we construct a few examples of pure mixing graph maps which will prove that the lower bounds for $\inf(h(\mathcal{P}_G))$ derived from Corollary 7.4 are in some cases equal to the infimum, hence we solve our main problem in these cases. Recall that an n-star is a tree $T_n = ([0, 1], 0)^{\wedge n}$, where $n \geq 1$.

LEMMA 9.1. *Let $n > 1$ and let T_n be a star with n endpoints. Then for every $\varepsilon > 0$, there is:*

(1) *an exact Markov map $F_\varepsilon : T_n \to T_n$ such that $(\log 3)/n \leq h(F_\varepsilon) < (\log 3)/n + \varepsilon$;*

(2) *a pure mixing map $G_\varepsilon : T_n \to T_n$ such that $(\log 3)/n \leq h(G_\varepsilon) < (\log 3)/n + \varepsilon$.*

Moreover, all endpoints of T_n form a single periodic orbit of F_ε and G_ε.

Proof. First observe that for each $n \geq 2$ there is a transitive Markov map f_n of an n-star T_n such that all endpoints of T_n form a single periodic orbit for f_n and $h(f_n) = (\log 3)/n$. This is a consequence of Lemma 8.4 applied to the 3-horseshoe map given by $f(x) = |1 - |1 - 3x||$ for $0 \leq x \leq 1$. Then we can apply Lemmas 8.5 and 8.6 to finish the proof. □

Let T be a tree. We say that a point $p \in T$ is a *central root* of T if $\mathrm{val}(p) = 2$ and both connected components of $T \setminus \{p\}$ are homeomorphic to each other. The full binary tree B_n with 2^n endpoints can be defined inductively. Let $B_1 = [0, 1]$. Note that $\frac{1}{2} \in [0, 1]$ is the central root of B_1. Assume that we have defined B_n and $z_n \in B_n$ is a central root of B_n. Let $T' = (B_n, z_n) \wedge ([0, 1], 0)$ and let z_{n+1} denote the endpoint $1 \in T'$. We define $B_{n+1} = (T', z_{n+1})^{\wedge 2}$. Clearly, $z_{n+1} \in B_{n+1}$ is a central root of B_{n+1} and thus B_{n+1} has 2^{n+1} endpoints.

LEMMA 9.2. *Let $n \geq 1$ and let B_n be a complete binary tree with 2^n endpoints. Then for every $\varepsilon > 0$, there is:*

(1) *an exact Markov map $F_\varepsilon : B_n \to B_n$ such that $(\log 3)/2^n \leq h(F_\varepsilon) < (\log 3)/2^n + \varepsilon$ and there is a fixed point of F_ε which is a central root for B_n;*

(2) *a pure mixing map $G_\varepsilon : B_n \to B_n$ such that $(\log 3)/2^n \leq h(G_\varepsilon) < (\log 3)/2^n + \varepsilon$.*

Moreover, all endpoints of B_n form a single periodic orbit of F_ε and G_ε.

Proof. First note that the second part of the theorem follows from the first and Lemma 8.6. We will prove the first part by induction on n. For $n = 1$, note that $B_1 = [0, 1]$, and consider the piecewise linear Markov map $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} 1 - 3x & \text{for } 0 \leq x < 1/6, \\ 3x & \text{for } 1/6 \leq x < 1/3, \\ 2 - 3x & \text{for } 1/3 \leq x < 1/2, \\ 1 - x & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

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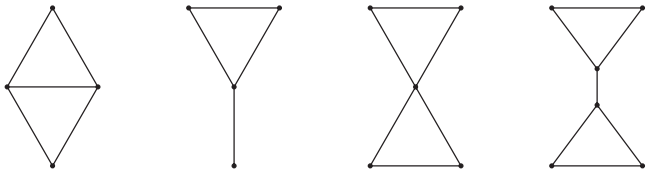


FIGURE 1. From left to right: a possible triangulation of the *theta graph*, *sigma graph*, *figure-eight graph*, and *dumbbell graph*.

It is clear that f is transitive, but not totally transitive, and $h(f) = \log\sqrt{3}$. Given any $\varepsilon > 0$, we can apply Lemma 8.5 to get a piecewise linear Markov and exact map $F_\varepsilon\colon B_1 \to B_1$. Note that $z = \frac{1}{2}$ is a fixed point of F_ε which is a central root for B_2 and the endpoints of $[0, 1]$ form a periodic orbit of length two for F_ε.

Now assume that the theorem holds for $n \geq 1$, fix an $\varepsilon > 0$, and let $G\colon B_n \to B_n$ be an exact map provided by the induction hypothesis for $\varepsilon/3$. Let z be a fixed point of G which is a central root for B_n. Apply Theorem 8.3 to obtain an exact piecewise linear Markov map $G'\colon T' \to T'$ with $h(G') < h(G) + \varepsilon/3$, where T' is B_n with an arc $[z, z_0]$ attached to z, that is, $T' = (B_n, z) \wedge ([z, z_0], z)$. Note that the 2^n endpoints of T' other than z_0 form a periodic orbit for G' and z and z_0 are fixed for G'. Observe that the second wedge power $(T', z_0)^{\wedge 2}$ is just B_{n+1}. Now we can apply Lemma 8.4 to get a transitive map $G''\colon B_{n+1} \to B_{n+1}$ with $h(G'') = h(G')$. Applying Lemma 8.5 to G'' and $\varepsilon/3$, we get the map $F_\varepsilon\colon B_{n+1} \to B_{n+1}$ with all desired properties. □

A σ-graph, θ-graph, and 8-graph are spaces homeomorphic to the symbol representing the Greek letter sigma, theta, and the figure 8, respectively. A dumbbell is a graph homeomorphic to the following subset of a complex plane $\mathbb{C}$: $C_{-2} \cup I \cup C_2$, where $C_\omega = \{z \in \mathbb{C} : |z - \omega| = 1\}$ and I is a line segment joining $z = -1$ with $z = 1$. See Figure 1.

THEOREM 9.3. *Let $\mathcal{P}_G$ denote the family of all pure mixing maps of a given graph G.*

(1) *If T_n is an n-star, $n \geq 2$, then*

$$\inf(h(\mathcal{P}_{T_n})) = (\log 3)/n.$$

(2) *If B_n is a full binary tree with 2^n endpoints, $n \geq 1$, then*

$$\inf(h(\mathcal{P}_{B_n})) = (\log 3)/2^n.$$

(3) *If G_σ is a sigma graph, then*

$$\inf(h(\mathcal{P}_{G_\sigma})) = (\log 3)/2.$$

(4) *If G_θ is a theta graph, then*

$$(\log 3)/4 \leq \inf(h(\mathcal{P}_{G_\theta})) \leq (\log 3)/3.$$

(5) *Let G_8 be a figure-eight graph, and G_d be the dumbbell graph, then*

$$\inf(h(\mathcal{P}_{G_8})) = \inf(h(\mathcal{P}_{G_d})) = (\log 3)/4.$$

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Proof. Parts (1) and (2) follow from Lemmas 9.1 and 9.2, respectively.

To prove (3) we fix $\varepsilon > 0$ and take an exact map of the interval with the endpoints forming a periodic orbit of length two and entropy smaller than $\log \sqrt{3} + \varepsilon/2$. By Theorem 8.3 there is an exact Markov map on the 3-star, with entropy smaller than $\log \sqrt{3} + \varepsilon$, and two of the three endpoints of the 3-star form a periodic orbit for that map. By Lemma 8.6 we can find a pure mixing map of the 3-star with entropy smaller than $\log \sqrt{3} + \varepsilon$, and two of the three endpoints of the 3-star still form a periodic orbit which is inaccessible for that map. Identifying those two endpoints, we get a pure mixing map of the sigma graph with topological entropy at most $\log \sqrt{3} + \varepsilon$. On the other hand, it is easy to see that a pure mixing map of the sigma graph can have at most two inaccessible sides since by Corollary 5.9 they have to form a periodic orbit. Therefore, its topological entropy is greater than $\log \sqrt{3}$.

To see (4) fix $\varepsilon > 0$ and take a pure mixing map of the 3-star T_3 with topological entropy smaller than $(\log \sqrt{3})/3 + \varepsilon$ for which the endpoints of T_3 form an inaccessible periodic orbit of length three. Then we identify these endpoints to get a theta graph, and the proof of the upper bound for the infimum is complete. The lower bound comes from Theorem 7.3.

Finally, (5) follows from Lemma 9.2 and 9.1, respectively. To see this, observe that identifying endpoints in the binary tree B_2 or 4-star in the appropriate way, we get the dumbbell graph and the figure-eight graph, respectively. □

We conjecture that the upper bound in (4) above is actually the infimum, that is, $\inf(h(\mathcal{P}_{G_\theta})) = (\log 3)/3$. Note that the bound from Corollary 7.4 is not always the best possible. It is an interesting question to find the formula for $\inf(h(\mathcal{P}_G))$ depending on the combinatorial structure of G.

10. *Mixing implies the specification property for graph maps*

In this section we present an alternative proof of the fact that every mixing graph map has the specification property. The result was originally proved by Blokh. Our approach extends ideas of the proof presented by Buzzi in [**16**] in the context of a compact interval. But first we recall some terminology.

Definition 10.1. Let $n > 0$ be an integer and let $\varepsilon > 0$. The *closed Bowen ball* is the set

$$B_n(x, \varepsilon) = \{y \in G \,:\, \rho(f^i(x),\, f^i(y)) \le \varepsilon \text{ for } i = 0, \ldots, n\}.$$

By $B'_n(x, \varepsilon)$ we denote the connected component of $B_n(x, \varepsilon)$ containing x.

LEMMA 10.2. *Let f be a mixing map of a graph G. Assume that $\alpha, \varepsilon > 0$ are positive real numbers. Then there are an integer $N = N(\alpha, \varepsilon) > 0$ and positive real number $\delta = \delta(\varepsilon) > 0$ such that for every $y \in G$ and every integer $n \ge 0$ there is a point $z = z(y, \varepsilon, n) \in G$ for which*

$$z \in B_n(y, \varepsilon) \quad \text{and} \quad \mathrm{B}(z, \delta) \subset (f^k(\mathrm{B}(x, \alpha)))\backslash(\mathrm{B}(\mathcal{I}(f), \delta))$$

hold for every $x \in G$ and $k \ge N$.

Proof. Fix $\alpha, \varepsilon > 0$. First, assume that f is exact. It is easy to see that for each $\delta > 0$ there is an integer $N > 0$ such that $f^k(B(x, \alpha)) = G$ for each $x \in G$ and $k \ge N$. Set $z = y$ and the proof for the first case is finished.

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Consider the second case, when f is pure mixing. By the uniform continuity and compactness, it is enough to prove the assertion of the lemma for a map f^m for some $m > 0$. Using Theorem 6.1, we can find the integer m, graph G', and pure mixing map $g\colon G' \to G'$, which is an extension of f^m and $g^{-1}(e) = \{e\}$ for each $e \in \mathcal{I}(g)$. It is sufficient to prove that the lemma holds for g. To simplify further our notation we can assume that g has the unique inaccessible endpoint e_0. The other cases can be handled analogously. Using our convention (C), we isometrically identify the edge containing e_0 with an interval $[0, a]$, with $e_0 = 0$ and $a > 0$. We fix $\varepsilon, \alpha > 0$ and assume that $\varepsilon < a$. By continuity and Theorem 7.2, we can find fixed points $0 < p < p' < \varepsilon$ such that $g([0, p']) \subset [0, \varepsilon]$ and $g([0, p]) \subset [0, p']$. Let $\delta = p/2$. Theorem 4.6(1) allows us to find an integer $N > 0$ such that $G'\backslash[0, \delta] \subset g^k(J)$ for each subgraph J of G' with $\operatorname{diam} J \geq \alpha$ and each $k \geq N$. In particular, $G'\backslash[0, \delta] \subset g^k(B(x, \alpha))$ for every $x \in G'$ and $k \geq N$. Fix $y \in G'$ and let us agree that $\min \emptyset = \infty$. Consider

$$m = \min\{k : g^k(y) \in G'\backslash[0, p]\}.$$

We have the following cases.

Case 1: $m = 0$. Then we set $z = y$.

Case 2: $0 < m < \infty$. Observe that $g^k(y) \in [0, p]$ for $k = 0, 1, \ldots, m-1$. Furthermore, $g^m(y) \in [p, p']$. But $g([p, p']) \subset [0, a]$ and since p, p' are fixed points for g, we see that $[p, p']$ is g^m-covered by itself. Therefore, we can find $z \in [p, p']$ such that $g^m(z) = g^m(y)$.

Case 3: $m = \infty$. Then it is enough to take $z = p$.

Clearly, in any case $z \in B_n(y, \varepsilon)$ for all $n \geq 0$ and $\mathrm{B}(\mathcal{I}(g), \delta) \cap B(z, \delta) = \emptyset$. □

LEMMA 10.3. *Let $\alpha > 0$. Then there is a constant $\gamma = \gamma(\alpha)$ such that for every map $g\colon G \to G$ if $J \subset G$ is a star with $\operatorname{diam} g(J) \geq \alpha$, then there is a free arc $J' \subset J$ which contains two subarcs J_1, J_2 with at most one common point such that $g(J_i)$ is also a free arc g-covered by J_i with $\operatorname{diam} g(J_i) \geq \gamma$ for $i = 1, 2$.*

Proof. By Lemma 3.5 there is a constant $\xi = \xi(\alpha)$ such that $g(J)$ contains a free arc K with $\operatorname{diam} K \geq \xi$. We conclude from Lemma 3.3(7) that K is contained in $f(E_1 \cup E_2)$, where E_1, E_2 are edges of S. Clearly, one of these edges, say E_1, must g-cover a free arc K' with $\operatorname{diam} K' \geq \xi/3$. We set $\gamma = \xi/6$ and write $K' = K_1 \cup K_2$, where K_1, K_2 are free arcs with at most one common point, and $\operatorname{diam} K_i \geq \gamma$. Now an application of Lemma 3.3(2) finishes the proof. □

LEMMA 10.4. *Let $f\colon G \to G$ be a mixing graph map. If $0 < \varepsilon < \frac{1}{2} \operatorname{diam} G$ and $\delta > 0$ then there is an $N = N(\varepsilon, \delta) > 0$ such that $B'_n(x, \varepsilon) \subset B(x, \delta)$ for all $x \in G$ and all $n \geq N$.*

Proof. If the conclusion of the lemma does not hold, then, using compactness of G, we could find a point $x \in G$ and $\varepsilon > 0$ such that the set

$$B = \bigcap_{k=0}^{\infty} B'_k(x, \varepsilon)$$

would have a non-empty interior. Then $\operatorname{diam} f^n(B) \leq 2\varepsilon < \operatorname{diam} G$ for all n, contradicting topological mixing. □

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LEMMA 10.5. *If $f: G \to G$ is a mixing graph then for every $\varepsilon > 0$ there is a constant $\zeta = \zeta(\varepsilon)$ such that*

$$0 < \zeta \leq \operatorname{diam} f^n(\mathrm{B}'_n(x, \varepsilon))$$

for every n and $x \in G$.

Proof. Fix $\varepsilon > 0$. Without loss of generality we may assume that $\varepsilon < \frac{1}{2} \operatorname{diam} G$. Let $\mathcal{K}(\varepsilon)$ be a family of subgraphs of G with diameter at least ε. Then $\mathcal{K}(\varepsilon)$ is a closed subset of a hyperspace of subcontinua of G. Moreover, $\Phi_n = \operatorname{diam} \circ f^n$ is a continuous function on $\mathcal{K}(\varepsilon)$ for all $n \geq 0$. Observe that there is an N such that $\operatorname{diam} f^n(J) \geq \frac{1}{2} \operatorname{diam} G$ for all $J \in \mathcal{K}(\varepsilon)$ and $n \geq N$. Therefore,

$$\beta(\varepsilon) := \inf_{n \geq 0} \{\operatorname{diam} f^n(J) : J \in \mathcal{K}(\varepsilon)\} = \min_{0 \leq j \leq N} \min \Phi_j(\mathcal{K}(\varepsilon)) > 0,$$

as no subgraph of G can be mapped by any f^n onto a point.

Fix $x \in G$. We claim that $\operatorname{diam} f^n(\mathrm{B}'_n(x, \varepsilon)) \geq \beta(\varepsilon)$ defined above. We have $f^{n+1}(\mathrm{B}'_{n+1}(x, \varepsilon)) \subset f(f^n(\mathrm{B}'_n(x, \varepsilon)))$ for all n. Furthermore, note that:

($\star$) if $f^{n+1}(\mathrm{B}'_{n+1}(x, \varepsilon)) \neq f(f^n(\mathrm{B}'_n(x, \varepsilon)))$ then there is $y \in B'_n(x, \varepsilon)$ with

$$\rho(f^{n+1}(x), f^{n+1}(y)) \geq \varepsilon,$$

hence $\operatorname{diam} f^{n+1}(B'_{n+1}(x, \varepsilon)) \geq \varepsilon$;

($\star\star$) if $f^{n+1}(\mathrm{B}'_{n+1}(x, \varepsilon)) = f(f^n(\mathrm{B}'_n(x, \varepsilon)))$ and $\operatorname{diam} f^n(B'_n(x, \varepsilon)) \geq \varepsilon$ then

$$\operatorname{diam} f^{n+1}(\mathrm{B}'_{n+1}(x, \varepsilon)) \geq \beta(\varepsilon)$$

by the definition of $\beta(\varepsilon)$.

Applying ($\star$) or ($\star\star$) accordingly, we get:

($\star\star\star$) if $\operatorname{diam} f^n(B'_n(x, \varepsilon)) \geq \varepsilon$ for some $n \geq 0$ then $\operatorname{diam} f^{n+k}(\mathrm{B}'_n(x, \varepsilon))) \geq \beta(\varepsilon)$ for every $k \geq 0$ by the definition of $\beta(\varepsilon)$.

Now, $\mathrm{B}(x, \varepsilon) = \mathrm{B}'_0(x, \varepsilon)$, and we proceed by induction. □

The following definition was introduced by Bowen in [**15**].

Definition 10.6. We say that a continuous map $f: X \to X$ acting on a compact metric space (X, d) has the *specification property* if for every $\varepsilon > 0$ there exists an integer $M = M(\varepsilon)$ such that for any $s > 1$, for any s points $x_1, x_2, \ldots, x_s \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ with $a_i - b_{i-1} \geq M$, for $2 \leq i \leq s$ and for any integer p with $p \geq M + b_s - a_1$ there exists a point $x \in X$ with $f^p(x) = x$ such that $d(f^n(x), f^n(x_i)) < \varepsilon$ for $a_i \leq n \leq b_i$, and $1 \leq i \leq s$.

Remark 10.7. Without loss of generality we can take $a_1 = 0$ and $p = M + b_s$ in the above definition.

THEOREM 10.8. (Blokh) *Every mixing graph map has the specification property.*

Proof. Let $f: G \to G$ be a mixing graph map and fix $\varepsilon > 0$. Our first task is to find a suitable integer $N > 0$ as in the definition of specification. Without loss of generality we may assume that $\varepsilon < \frac{1}{2} \operatorname{diam} G$. Set $\alpha = \operatorname{diam} I_U$, where I_U is a universal arc for f. Let $N_1 = N(\varepsilon/2, \alpha/2)$ and $\delta = \delta(\varepsilon/2)$ be given by Lemma 10.2 for $\varepsilon/2$ and α as

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above. We may also assume that the open ball $\mathrm{B}(x, \delta)$ is a canonical neighborhood of x for each $x \in G$. We plug δ and $\varepsilon/2$ into Lemma 10.4 to get an $N_2 = N(\varepsilon/2, \delta)$ such that $B'_n(x, \varepsilon/2) \subset \mathrm{B}(x, \delta)$ for all $x \in G$ and all $n \geq N_2$. As a direct consequence of Lemmas 10.5 and 10.3 we can find a constant $\beta > 0$ such that for every $x \in G$ and every $n > 0$ the following condition holds.

($\star$) There is a free arc $J' \subset B'_n(x, \varepsilon/2)$ containing two free arcs J_1 and J_2 with at most one common point and two free arcs K_1 and K_2 with $\operatorname{diam} K_i \geq \beta$ for $i = 1, 2$ and such that K_i is f^n-covered by J_i for $i = 1, 2$.

For $\beta > 0$ as defined above we can find an $N_3 > 0$ such that if $n \geq N_3$ then I_U is f^n-covered by each closed interval K with $\operatorname{diam} K > \beta$.

We claim that $N = N_1 + N_2 + N_3$ will fulfill the definition of the specification property. For the proof of our claim we assume, to simplify the notation, that $s = 2$ and we choose any $x_1, x_2 \in G$ and integers $0 = a_1 \leq b_1 < a_2 \leq b_2$, with $a_2 - b_1 \geq N$. Finally, we fix any $p \geq b_2 + N$. For $i = 1, 2$ let $y_i = f^{a_i}(x_i)$ and $n(i) = b_i - a_i + N_2$. Let $z_i = z(y_i, \varepsilon/2, n(i)) \in B_{n(i)}(y_i, \varepsilon/2)$ be provided by Lemma 10.2. By our choice of N_2 we conclude that $B_i = B'_{n(i)}(z_i, \varepsilon/2) \subset \mathrm{B}(z_i, \delta)$ for $i = 1, 2$. Let us denote the free arcs constructed in B_i in ($\star$) above by J_1^i and J_2^i. By Lemma 10.2 we have $\mathrm{B}(z_i, \delta) \subset f^n(I_U)$ for each $i = 1, 2$ and $k \geq N_1$. Applying Lemma 3.3(3), we see that for $i = 1, 2$ and k as above I_U must f^k-cover at least one free arc $I_i(k) \in \{J_1^i, J_2^i\}$. Let $I_1 = I_1(k(1))$, where $k(1) = a_2 - b_1 - N_2 + N_3$, and $I_2 = I_2(k(2))$, where $k(2) = p - (b_2 + N_2 + N_3)$. Appealing again to the condition ($\star$), we see that each I_i covers through $f^{n(i)}$ an interval K_i with $\operatorname{diam} K_i \geq \beta$. This in turn implies that each K_i must f^{N_3}-cover I_U. In conclusion, we get

$$I_1 \overset{f^{n(1)}}{\Longrightarrow} K_1 \overset{f^{N_3}}{\Longrightarrow} I_U \overset{f^{k(1)}}{\Longrightarrow} I_2 \overset{f^{n(2)}}{\Longrightarrow} K_2 \overset{f^{N_3}}{\Longrightarrow} I_U \overset{f^{k(2)}}{\Longrightarrow} I_1,$$

where $I \overset{f}{\Longrightarrow} K$ denotes that I f-covers K. It follows that I_1 is f^p-covered by itself, since $p = n(1) + k(1) + n(2) + k(2) + 2N_3$. Therefore, there is a p-periodic point $q \in I_1$ such that $q \in B'_{n(1)}(z_1, \varepsilon/2)$ and $z_1 \in B_{n(1)}(y_1, \varepsilon/2)$. Moreover, $r = f^{a_2}(q) \in I_2$, hence $r \in B'_{n(2)}(z_2, \varepsilon/2)$ and $z_2 \in B_{n(2)}(y_2, \varepsilon/2)$. This finishes the proof. □

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References

[1] Ll. Alsedà, S. Baldwin, J. Llibre and M. Misiurewicz. Entropy of transitive tree maps. *Topology* **36**(2) (1997), 519–532.

[2] Ll. Alsedà, M. A. del Río and J. A. Rodríguez. A splitting theorem for transitive maps. *J. Math. Anal. Appl.* **232**(2) (1999), 359–375.

[3] Ll. Alsedà, M. A. del Río and J. A. Rodríguez. Transitivity and dense periodicity for graph maps. *J. Difference Equ. Appl.* **9**(6) (2003), 577–598.

[4] Ll. Alsedà, S. Kolyada, J. Llibre and L'. Snoha. Entropy and periodic points for transitive maps. *Trans. Amer. Math. Soc.* **351**(4) (1999), 1551–1573.

https://doi.org/10.1017/etds.2013.6 Published online by Cambridge University Press

[5] Ll. Alsedà, J. Llibre and M. Misiurewicz. *Combinatorial Dynamics and Entropy in Dimension One (Advanced Series in Nonlinear Dynamics, 5)*, 2nd edn. World Scientific, River Edge, NJ, 2000.
[6] S. Baldwin. Entropy estimates for transitive maps on trees. *Topology* **40**(3) (2001), 551–569.
[7] F. Balibrea and L'. Snoha. Topological entropy of Devaney chaotic maps. *Topology Appl.* **133**(3) (2003), 225–239.
[8] J. Banks. Regular periodic decompositions for topologically transitive maps. *Ergod. Th. & Dynam. Sys.* **17**(3) (1997), 505–529.
[9] J. Banks and B. Trotta. Weak mixing implies mixing for maps on topological graphs. *J. Difference Equ. Appl.* **11**(12) (2005), 1071–1080.
[10] F. Blanchard. Topological chaos: What may this mean? *J. Difference Equ. Appl.* **15**(1) (2009), 23–46.
[11] L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young. Periodic points and topological entropy of one dimensional maps. *Global Theory of Dynamical Systems (Lecture Notes in Mathematics, 819)*. Springer, Berlin, 1980, pp. 18–34.
[12] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. I. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* (46) (1986), 8–18.
[13] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. II. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* (47) (1987), 67–77.
[14] A. M. Blokh. Dynamical systems on one-dimensional branched manifolds. III. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* (48) (1987), 32–46.
[15] R. Bowen. Periodic points and measures for axiom A-diffeomorphisms. *Trans. Amer. Math. Soc.* **154** (1971), 377–397.
[16] J. Buzzi. Specification on the interval. *Trans. Amer. Math. Soc.* **349**(7) (1997), 2737–2754.
[17] E. M. Coven and I. Mulvey. Transitivity and the centre for maps of the circle. *Ergod. Th. & Dynam. Sys.* **6**(1) (1986), 1–8.
[18] F. H. Croom. *Basic Concepts of Algebraic Topology (Undergraduate Texts in Mathematics, 51)*. Springer, New York, 1978.
[19] M. Dirbák. Extensions of dynamical systems without increasing the entropy. *Nonlinearity* **21**(11) (2008), 2693–2713.
[20] M. Dirbák, L'. Snoha and V. Špitalský. Minimality, transitivity, mixing and topological entropy on spaces with a free interval. *Ergod. Th. & Dynam. Sys.*, to appear, doi 10.1017/S0143385712000442, published online 21 August 2012.
[21] G. Harańczyk and D. Kwietniak. When lower entropy implies stronger Devaney chaos. *Proc. Amer. Math. Soc.* **137**(6) (2009), 2063–2073.
[22] G. Harańczyk, D. Kwietniak and P. Oprocha. A note on transitivity, sensitivity and chaos for graph maps. *J. Difference Equ. Appl.* **17**(10) (2011), 1549–1553.
[23] H. Kato. Concerning continuum-wise fully expansive homeomorphisms of continua. *Topology Appl.* **53**(3) (1993), 239–258.
[24] S. Kolyada and L'. Snoha. Some aspects of topological transitivity—a survey. *Grazer Math. Ber.* **334** (1997), 3–35.
[25] P. Kościelniak, P. Oprocha and M. Tuncali. Hereditary indecomposable inverse limits of graphs: shadowing, mixing and exactness. *Proc. Amer. Math. Soc.*, to appear.
[26] D. Kwietniak and M. Misiurewicz. Exact Devaney chaos and entropy. *Qual. Theory Dyn. Syst.* **6**(1) (2005), 169–179.
[27] D. Kwietniak and P. Oprocha. Topological entropy and chaos for maps induced on hyperspaces. *Chaos Solitons Fractals* **33** (2007), 76–86.
[28] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[29] E. E. Moise. *Geometric Topology in Dimensions 2 and 3 (Graduate Texts in Mathematics, 47)*. Springer, New York, 1977.
[30] S. B. Nadler Jr. Continuum theory and graph theory: disconnection numbers. *J. Lond. Math. Soc.* (2) **47**(1) (1993), 167–181.
[31] P. Oprocha and G. Zhang. Dimensional entropy over sets and fibres. *Nonlinearity* **24**(8) (2011), 2325–2346.
[32] W. Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.* **122** (1966), 368–378.
[33] C. Robinson. *Dynamical Systems (Studies in Advanced Mathematics, 28)*, 2nd edn. CRC Press, Boca Raton, FL, 1999.

https://doi.org/10.1017/etds.2013.6 Published online by Cambridge University Press

[34] Y. Su and X. Ye. Topological entropy of a class of transitive maps of a tree. *Topology Proc.* **24**(Summer) (1999), 597–609.
[35] V. Špitalský. Entropy and exact Devaney chaos on totally regular continua. *Discrete Contin. Dyn. Syst. A* **33**(7) (2013), 3135–3152.
[36] X. Ye. Non-wandering points and the depth of a graph map. *J. Aust. Math. Soc. A* **69**(2) (2000), 143–152.
[37] X. Ye. Topological entropy of transitive maps of a tree. *Ergod. Th. & Dynam. Sys.* **20**(1) (2000), 289–314.
[38] K. Yokoi. Strong transitivity and graph maps. *Bull. Pol. Acad. Sci. Math.* **53** (2005), 377–388.

https://doi.org/10.1017/etds.2013.6 Published online by Cambridge University Press