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Primitive Points in Rational Polygons

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Abstract. Let $\mathcal{A}$ be a star-shaped polygon in the plane, with rational vertices, containing the origin. The number of primitive lattice points in the dilate $t\mathcal{A}$ is asymptotically $\frac{6}{\pi^2}\operatorname{Area}(t\mathcal{A})$ as $t \to \infty$. We show that the error term is both $\Omega_{\pm}(t\sqrt{\log\log t})$ and $O(t(\log t)^{2/3}(\log\log t)^{4/3})$. Both bounds extend (to the above class of polygons) known results for the isosceles right triangle, which appear in the literature as bounds for the error term in the summatory function for Euler's $\phi(n)$.

1 Introduction

One of the fundamental problems in discrete geometry is to estimate the number of lattice points contained in a polygon. In this paper, we concern ourselves with the set of *primitive lattice points*

$$\mathbb{P} = \{(m,n) \in \mathbb{Z}^2 : \gcd(m,n) = 1\}, \tag{1.1}$$

also known as lattice points visible from the origin. It is a classical result that the number of primitive lattice points in a "reasonable" region in $\mathbb{R}^2$ is approximately $\frac{6}{\pi^2}$ times the area of the region. We are interested in the family $\{t\mathcal{A}\}$ of dilates of a fixed polygon $\mathcal{A}$, for which we define the error term

$$E_{\mathcal{A}}(t) = \#(t\mathcal{A} \cap \mathbb{P}) - \frac{6}{\pi^2}\operatorname{Area}(t\mathcal{A}) = \#(t\mathcal{A} \cap \mathbb{P}) - \frac{6}{\pi^2}t^2\operatorname{Area}(\mathcal{A}). \tag{1.2}$$

The fact that $\#(t\mathcal{A} \cap \mathbb{P}) \sim \frac{6}{\pi^2}t^2\operatorname{Area}(\mathcal{A})$, or equivalently, that $E_{\mathcal{A}}(t) = o(t^2)$, was likely used as far back as Minkowski (see [16, p. 998] or [5, Theorem 459]). Stronger upper bounds than $E_{\mathcal{A}}(t) = o(t^2)$ are relatively easy to obtain; we state the following result, which will be justified in the next section, as a benchmark for comparison.

Proposition 1.1 *If $\mathcal{A} \subset \mathbb{R}^2$ is a polygon, then*

$$E_{\mathcal{A}}(t) \ll t\log t$$

for $t \geq 2$, where the implicit constant may depend on $\mathcal{A}$.

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The purpose of this paper is to improve this upper bound for any *rational polygon* (a polygon all of whose vertices have both coordinates rational), and show that it cannot be improved too much more by providing a strong $\Omega_{\pm}$ result for the error term.

It is instructive to consider the specific example $\mathcal{A} = \Delta$, where Δ is the isosceles right triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$ and thus area $\frac{1}{2}$. Then $\#(t\Delta \cap \mathbb{P})$ is the number of primitive points in the dilate $t\Delta = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le t\}$; that is,

$$\#(t\Delta \cap \mathbb{P}) = \sum_{0 \le m \le t} \sum_{\substack{0 \le n \le m \\ \gcd(m,n)=1}} 1 = 1 + \sum_{1 \le m \le t} \phi(m) \tag{1.3}$$

(where the extra 1 comes from the fact that $\gcd(0,1) = 1$). It is well known that this summatory function of the Euler ϕ-function is asymptotic to $\frac{3}{\pi^2}t^2$, so we define

$$E_{\Delta}(t) = \sum_{1 \le m \le t} \phi(m) - \frac{3}{\pi^2}t^2 + 1 = \#(t\Delta \cap \mathbb{P}) - \frac{3}{\pi^2}t^2. \tag{1.4}$$

Proposition 1.1 implies the estimate $E_{\Delta}(t) \ll t \log t$, which (when phrased in terms of the summatory function of the Euler ϕ-function) is a classical result of Mertens [9]. The best known unconditional upper bound for this error term $E_{\Delta}(t)$ is due to Walfisz [17, p. 144, eq. (3)]: using methods related to exponential sums, he showed that

$$E_{\Delta}(t) \ll t(\log t)^{2/3}(\log\log t)^{4/3}. \tag{1.5}$$

As a consequence of our work, we can extend this bound from the isosceles right triangle Δ to all rational polygons.

Theorem 1.2 *Let $\mathcal{A} \subset \mathbb{R}^2$ be a rational polygon. Then*

$$E_{\mathcal{A}}(t) \ll t(\log t)^{2/3}(\log\log t)^{4/3},$$

where the implicit constant may depend on $\mathcal{A}$.

Prior to Walfisz's result, Chowla and Pillai [14] showed that these upper bounds cannot be significantly improvedby establishing the lower bound

$$\limsup_{t \to \infty} \frac{|E_{\Delta}(t)|}{t \log\log\log t} > 0,$$

that is, $E_{\Delta}(t) = \Omega(t \log\log\log t)$. Later, Erdős and Shapiro [4] obtained a slightly weaker quantitative lower bound but showed that $E_{\Delta}(t)$ oscillates in sign infinitely often. Both results were improved by Montgomery [10].

Proposition 1.3 *We have that $E_{\Delta}(t) = \Omega_{\pm}(t\sqrt{\log\log t})$; in other words,*

$$\limsup_{t \to \infty} \frac{E_{\Delta}(t)}{t\sqrt{\log\log t}} > 0 \quad \textit{and} \quad \liminf_{t \to \infty} \frac{E_{\Delta}(t)}{t\sqrt{\log\log t}} < 0.$$

We use the term *origin-star-shaped* (star-shaped with respect to the origin) to refer to any domain $\mathcal{B}$ for which $\lambda\mathcal{B} \subset \mathcal{B}$ for all $\lambda \in [0,1]$; note that in particular, any origin-star-shaped domain contains the origin. The main result of this paper generalizes this lower bound of Montgomery to this class of polygons.

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Theorem 1.4 *Let $\mathcal{A} \subset \mathbb{R}^2$ be a rational origin-star-shaped polygon. Then*

$$E_{\mathcal{A}}(t) = \Omega_{\pm}(t\sqrt{\log\log t}),$$

where the implicit constant may depend on $\mathcal{A}$.

In the process of applying an adaptation of Montgomery's argument to the general error term $E_{\mathcal{A}}(t)$ for the primitive point counting function for the lattice polygon $\mathcal{A}$, we found that we needed to establish the following "error term independence" result concerning the error term E_Δ for the summatory function of $\phi(n)$, which was defined in equation (1.4); this result may be of independent interest.

Theorem 1.5 *For any positive rational numbers $c_1, \dots, c_k, f_1, \dots, f_k$,*

$$c_1 E_\Delta(f_1 x) + \cdots + c_k E_\Delta(f_k x) = \Omega_{\pm}\big(x\sqrt{\log\log x}\big),$$

where the implied constant may depend upon the c_j and f_j.

In other words, there can be no "magic cancellation" among the terms $E_\Delta(f_j x)$ that makes the oscillation of the sum significantly smaller than that of an individual term. Indeed, Theorem 1.5 holds more generally, when E_Δ is replaced $E_{\mathcal{A}}$ for any rational origin-star-shaped polygon $\mathcal{A}$ (see the remark following the statement of Theorem 2.10).

Furthermore, we conjecture that oscillations of the same size exist even when the c_i are allowed to be negative (where we require the f_i to be distinct in order to avoid trivial cancellations).

Conjecture 1.6 *Theorem 1.5 holds for any rational numbers c_i and any distinct positive rational numbers f_i.*

This conjectural generalization of Theorem 1.5 would imply a stronger version of Theorem 1.4 that holds for any rational polygon $\mathcal{A}$, not necessarily star-shaped or containing the origin.

Conjecture 1.7 *Let $\mathcal{A} \subset \mathbb{R}^2$ be a rational polygon. Then*

$$E_{\mathcal{A}}(t) = \Omega_{\pm}(t\sqrt{\log\log t}),$$

where the implicit constant may depend on $\mathcal{A}$.

The rest of the paper is divided into two sections. In the next section, we show (Theorem 2.10) that the error term $E_{\mathcal{A}}(t)$ can be rewritten as a linear combination of dilates of the totient error function $E_\Delta(t)$, thereby establishing Theorem 1.2 and reducing Theorem 1.4 to Theorem 1.5. We then establish this latter theorem in the final section.

2 Decomposing the Error Term

We begin by giving a proof of Proposition 1.1, not only for the sake of completeness, but also because the structure of the argument foreshadows how we will approach the

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main result of this section, namely, Theorem 2.10. We use the term *pointed triangle* to mean a triangle that has the origin as a vertex.

Proof of Proposition 1.1 We begin by quoting a reasonably precise estimate [7, special case of Theorem 2.1] for the number of primitive points inside a domain $\mathcal{B}$: if $\mathcal{B}$ is convex and contains the origin, then

$$\#(\mathcal{B}\cap\mathbb{P}) - \frac{6}{\pi^2}\operatorname{Area}(\mathcal{B}) \ll \max\{1, \omega\log\omega\}, \tag{2.1}$$

where ω is the diameter of $\mathcal{B}$. In particular, if $\mathcal{C}$ is a convex polygon containing the origin and $\mathcal{B} = t\mathcal{C}$ is a dilate, then the diameter of $\mathcal{B}$ is a constant multiple of t, and so

$$E_{\mathcal{C}}(t) \ll \max\{1, t\log t\}, \tag{2.2}$$

with the implicit constant depending on $\mathcal{C}$.

However, any polygon $\mathcal{A}$ can be partitioned into sums and differences of finitely many pointed triangles (by which we mean that the indicator function of the set $\mathcal{A}$ can be written as sums and differences of indicator functions of pointed triangles, up to inaccuracies on the edges of these triangles), simply by triangulating $\mathcal{A}$ and noting that any triangle $\mathcal{T}$ can be written as a sum and difference of three pointed triangles defined by the three edges of $\mathcal{T}$. Let $\{\mathcal{T}_j\}_{j=1}^n$ denote the set of these pointed triangles, and let $\epsilon_j \in \{1,-1\}$ indicate whether (the indicator function of) $\mathcal{T}_j$ is added or subtracted, so that $\operatorname{Area}(A) = \sum_{j=1}^n \epsilon_j \operatorname{Area}(\mathcal{T}_j)$. Even though these triangles have sides in common, the number of double-counted lattice points on the t-dilation of each side is at most $O(t)$. Therefore, the bound (2.2) is valid (since pointed triangles are certainly convex polygons containing the origin), which implies

$$\begin{aligned}
E_{\mathcal{A}}(t) &= \#(t\mathcal{A}\cap\mathbb{P}) - \frac{6}{\pi^2}\operatorname{Area}(t\mathcal{A}) \\
&= \Big(\sum_{j=1}^n \epsilon_j \#(t\mathcal{T}_j\cap\mathbb{P}) + O(t)\Big) - \frac{6}{\pi^2}\operatorname{Area}(t\mathcal{A}) \\
&= \sum_{j=1}^n \epsilon_j\Big(\frac{6}{\pi^2}\operatorname{Area}(t\mathcal{T}_j) + E_{\mathcal{T}_j}(t)\Big) + O(t) - \frac{6}{\pi^2}\operatorname{Area}(t\mathcal{A}) \\
&= \sum_{j=1}^n \epsilon_j E_{\mathcal{T}_j}(t) + O(t) \ll t\log t
\end{aligned}$$

for $t \geq 2$, as desired. ∎

Remark In [7], the bound (2.1) is stated without the hypothesis that $\mathcal{B}$ contain the origin; however, this hypothesis is actually necessary. One can easily construct, using the Chinese remainder theorem, a square $\mathcal{B}$ of diameter ω containing no primitive lattice points whatsoever. The area of such a square is a constant times ω^2, and therefore

$$\Big|\#(\mathcal{B}\cap\mathbb{P}) - \frac{6}{\pi^2}\operatorname{Area}(\mathcal{B})\Big| \gg \omega^2,$$

which is incompatible with the claimed bound $\omega\log\omega$.

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The error in the proof of [7, Theorem 2.1] comes when applying [7, Lemma 2.3], which requires $\omega \geq 1$, to a term of the form $\sum_{(\Delta/k)_1} f(kx)$; when $k > \omega$, the diameter of Δ/k is less than 1, and so Lemma 2.3 cannot be applied. However, if Δ is a set containing the origin, then these sets Δ/k are sets with diameter less than 1 that contain the origin, and hence contain no primitive lattice points (indeed, no lattice points at all other than the origin). Therefore, the sums over k in the proof of [7, Theorem 2.1] can be truncated at $k \leq \omega$, and the rest of the argument goes through thereafter.

The remainder of this section is dedicated to proving Theorem 2.10, which asserts that the error term $E_{\mathcal{A}}(t)$ for any rational polygon $\mathcal{A}$ can be rewritten in terms of the totient error function $E_{\Delta}(t)$. We begin with a detailed investigation of this latter function.

Definition 2.1 Throughout this section, we employ the notation $\lfloor x \rfloor$ for the greatest integer not exceeding x and $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x. We also employ the sawtooth function defined as

$$B(x) = \{x\} - \frac{1}{2}$$

for all real numbers x. This function is equal to the first periodic Bernoulli polynomial $\overline{B}_1(x)$ except when x is an integer; more precisely,

$$B(x) = \overline{B}_1(x) - \tfrac{1}{2}\mathbf{1}_{\mathbb{Z}}(x),$$

where $\mathbf{1}_{\mathbb{Z}}$ is the indicator function of the integers.

Next we recall some well-known and useful estimates for sums involving the Möbius μ-function, which satisfies the key identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

For any real numbers $2 \leq x \leq y$, we have

$$\sum_{d \leq x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\Big(\frac{1}{x}\Big); \tag{2.4}$$

$$\sum_{d \leq y} \mu(d) \Big\lfloor \frac{x}{d} \Big\rfloor = 1; \tag{2.5}$$

$$\Big| \sum_{d \leq x} \frac{\mu(d)}{d} \Big| \ll 1. \tag{2.6}$$

The proofs of equations (2.4) and (2.6) are in [6], and the identity (2.5) can be found in [1], for instance. We are now ready to give a more precise formula for the totient error term $E_{\Delta}(t)$.

Proposition 2.2 *For any real numbers $T \geq t \geq 2$,*

$$E_{\Delta}(t) = -t \sum_{d \leq T} \frac{\mu(d)}{d} B\Big(\frac{t}{d}\Big) + O(T).$$

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Proof Starting from equation (1.3) and the key identity (2.3), we have

$$\begin{aligned}\#(t\Delta\cap\mathbb{P}) &= 1+\sum_{1\le m\le t}\sum_{\substack{1\le n\le m\\ \gcd(m,n)=1}}1=\sum_{m\le t}\sum_{\substack{n\le m\\ \gcd(m,n)=1}}\sum_{d\mid\gcd(m,n)}\mu(d)\\ &=\sum_{d\le t}\mu(d)\sum_{\substack{m\le t\\ d\mid m}}\sum_{\substack{n\le m\\ d\mid n}}1=\sum_{d\le t}\mu(d)\sum_{k\le t/d}\sum_{\ell\le k}1\\ &=\sum_{d\le t}\mu(d)\sum_{k\le t/d}k=\sum_{d\le t}\mu(d)\frac{1}{2}\left(\left\lfloor\frac{t}{d}\right\rfloor^2+\left\lfloor\frac{t}{d}\right\rfloor\right)\\ &=\frac{1}{2}\sum_{d\le t}\mu(d)\left\lfloor\frac{t}{d}\right\rfloor^2+\frac{1}{2}\end{aligned}$$

by equation (2.5). The last sum does not change if we extend its range from $d\le t$ to $d\le T$. Expanding $\lfloor\frac{t}{d}\rfloor^2=(\frac{t}{d}-\{\frac{t}{d}\})^2$, we obtain

$$\begin{aligned}\#(t\Delta\cap\mathbb{P}) &= \frac{t^2}{2}\sum_{d\le T}\frac{\mu(d)}{d^2}-t\sum_{d\le T}\frac{\mu(d)}{d}\left\{\frac{t}{d}\right\}+\sum_{d\le T}\mu(d)\left\{\frac{t}{d}\right\}^2+\frac{1}{2}\\ &=\frac{t^2}{2}\left(\frac{6}{\pi^2}+O\left(\frac{1}{T}\right)\right)-t\sum_{d\le T}\frac{\mu(d)}{d}\left\{\frac{t}{d}\right\}+O(T)\end{aligned}$$

by equation (2.4) and a trivial bound. Therefore,

$$\begin{aligned}E_\Delta(t) &= \#(t\Delta\cap\mathbb{P})-\frac{6}{\pi^2}t^2\operatorname{Area}(\Delta)\\ &=\frac{3}{\pi^2}t^2-t\sum_{d\le T}\frac{\mu(d)}{d}\left\{\frac{t}{d}\right\}+O(T)-\frac{3}{\pi^2}t^2\\ &=-t\sum_{d\le T}\frac{\mu(d)}{d}\left(B\left(\frac{t}{d}\right)+\frac{1}{2}\right)+O(T)\\ &=-t\sum_{d\le T}\frac{\mu(d)}{d}B\left(\frac{t}{d}\right)+O(T)\end{aligned}$$

by equation (2.6). ∎

Counting lattice points in the simplest way (each with weight 1) is problematic in our present context. First, we will be decomposing polygons into unions of triangles that share sides, and so double-counting lattice points on these shared boundaries would become an issue; second, any error term as large as the perimeter for counting lattice points in a triangle would result in an unacceptably large error term in the inclusion-exclusion method we use to detect primitive lattice points. Consquently, we employ a more convenient weighting in our lattice-point counting, namely the solid angle sum of a polygon.

Definition 2.3 For a point $p\in\mathbb{R}^2$, let

$$\mathcal{D}(p;r)=\{q\in\mathbb{R}^2\colon d(p-q)<r\}$$

denote the disk of radius r centered at p. Let λ denote Lebesgue measure on $\mathbb{R}^2$, and define for any polygon $\mathcal{A}$ the solid angle at p:

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$$\omega_{\mathcal{A}}(p) = \lim_{r\to 0} \frac{\lambda(\mathcal{D}(p;r)\cap\mathcal{A})}{\lambda(\mathcal{D}(p;r))}.$$

It follows directly from the definition that

$$\omega_{\mathcal{A}}(p) = \begin{cases} 0, & \text{if } p \notin \mathcal{A}, \\ 1, & \text{if } p \text{ is in the interior of } \mathcal{A}, \\ 1/2, & \text{if } p \text{ lies in the interior of an edge of } \mathcal{A}, \\ \theta_p/2\pi, & \text{if } p \text{ is a vertex of } \mathcal{A}, \end{cases}$$

where θ_p is the angle at the vertex p. Using this function, we define the solid angle sum of the t-dilate of the rational polygon $\mathcal{A}$ to be

$$A_{\mathcal{A}}(t) = \sum_{p\in\mathbb{Z}^2} \omega_{t\mathcal{A}}(p).$$

Note that trivially $A_{c\mathcal{A}}(t) = A_{\mathcal{A}}(ct)$ for any constant $c > 0$.

We also define the corresponding sum over primitive lattice points only:

$$\mathbb{P}_{\mathcal{A}}(t) = \sum_{p\in\mathbb{P}} \omega_{t\mathcal{A}}(p).$$

The benefit of this solid angle weighting is that both of these functions are additive, in the sense that $A_{\mathcal{A}\cup\mathcal{B}}(t) = A_{\mathcal{A}}(t) + A_{\mathcal{B}}(t)$ and $\mathbb{P}_{\mathcal{A}\cup\mathcal{B}}(t) = \mathbb{P}_{\mathcal{A}}(t) + \mathbb{P}_{\mathcal{B}}(t)$ for any polygons $\mathcal{A}$ and $\mathcal{B}$ with disjoint interiors.

Given positive rational numbers r_1 and r_2, we let $\mathcal{T}(r_1, r_2)$ denote the right triangle with vertices at $(0,0)$, $(r_1,0)$, and $(0,r_2)$. An exact calculation of the solid angle sum of dilates of $\mathcal{T}(r_1, r_2)$ is available for our use. Recall that $\mathbf{1}_{\mathbb{Z}}$ is the indicator function of the integers.

Proposition 2.4 *For any positive rational numbers r_1 and r_2 and any positive real number t,*

$$\begin{aligned} A_{\mathcal{T}(r_1,r_2)}(t) = {} & \frac{r_1 r_2}{2}t^2 - \overline{B}_1(qr_1r_2t)\frac{t}{q} + \frac{\overline{B}_2(qr_1r_2t)}{2q^2r_1r_2} + \frac{r_1^2+r_2^2}{12r_1r_2} \\ & - s(qr_1, qr_2; r_1t, 0) - s(qr_2, qr_1; r_2t, 0) \\ & - \frac{\arctan(r_1/r_2)}{2\pi}\mathbf{1}_{\mathbb{Z}}(r_1t) - \frac{\arctan(r_2/r_1)}{2\pi}\mathbf{1}_{\mathbb{Z}}(r_2t), \end{aligned} \tag{2.7}$$

where q is the unique positive rational number such that qr_1 and qr_2 are coprime integers. Here, $\overline{B}_1$ and $\overline{B}_2$ are the first and second periodic Bernoulli polynomials, respectively, and

$$s(h,k;y,x) = \sum_{r \pmod k} \overline{B}_1\left(h\frac{r+x}{k} + y\right)\overline{B}_1\left(\frac{r+x}{k}\right)$$

is the Dedekind–Rademacher sum.

While its exact value is unimportant for us, we note that if $r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ in lowest terms, then the rational number q in the above statement is $q = \operatorname{lcm}[b,d]/\gcd(a,c)$, so that $qr_1 = a/\gcd(a,c)\cdot d/\gcd(b,d)$ and $qr_2 = c/\gcd(a,c)\cdot b/\gcd(b,d)$.

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Proof Le Quang and the fourth author [8, Main Theorem] established this result for the case of $A_{\mathcal{T}(h,k)}(t)$ when h and k are positive coprime integers:

$$A_{\mathcal{T}(h,k)}(t) = \frac{hk}{2}t^2 - \overline{B}_1(hkt)t + \frac{\overline{B}_2(hkt)}{2hk} + \frac{h^2+k^2}{12hk} - s(h,k;ht,0) - s(k,h;kt,0) - \frac{\arctan(h/k)}{2\pi}\mathbf{1}_{\mathbb{Z}}(ht) - \frac{\arctan(k/h)}{2\pi}\mathbf{1}_{\mathbb{Z}}(kt).$$

Given positive rational numbers r_1 and r_2, let q be the unique positive rational number such that $h = qr_1$ and $k = qr_2$ are coprime integers. Then

$$A_{\mathcal{T}(r_1,r_2)}(t) = A_{\mathcal{T}(h/q,k/q)}(t) = A_{\mathcal{T}(h,k)/q}(t) = A_{\mathcal{T}(h,k)}(t/q)$$

and therefore

$$\begin{aligned} A_{\mathcal{T}(r_1,r_2)}(t) = {} & \frac{qr_1qr_2}{2}\left(\frac{t}{q}\right)^2 - \overline{B}_1\left(qr_1qr_2\frac{t}{q}\right)\frac{t}{q} \\ & + \frac{\overline{B}_2(qr_1qr_2t/q)}{2qr_1qr_2} + \frac{(qr_1)^2+(qr_2)^2}{12qr_1qr_2} \\ & - s(qr_1,qr_2;qr_1t/q,0) - s(qr_2,qr_1;qr_2t/q,0) \\ & - \frac{\arctan(qr_1/qr_2)}{2\pi}\mathbf{1}_{\mathbb{Z}}(qr_1t/q) - \frac{\arctan(qr_2/qr_1)}{2\pi}\mathbf{1}_{\mathbb{Z}}(qr_2t/q), \end{aligned}$$

which is equivalent to the statement of the proposition. ∎

While we include this exact statement in the hopes that it will be useful in the future, for our present purposes we can use the following approximation.

Corollary 2.5 *For any positive rational numbers r_1 and r_2, there exist constants $c = c(r_1,r_2)$ and $f = f(r_1,r_2)$ such that*

$$A_{\mathcal{T}(r_1,r_2)}(t) = \mathrm{Area}(\mathcal{T}(r_1,r_2))t^2 - 2cB(ft)t + c\mathbf{1}_{\mathbb{Z}}(ft)t + O_{r_1,r_2}(1).$$

Proof In the identity (2.7), the leading coefficient $r_1r_2/2$ is indeed the area of $\mathcal{T}(r_1,r_2)$. Definition 2.1 implies that $-\overline{B}_1(qr_1r_2t)\frac{t}{q} = -2cB(ft)t + c\mathbf{1}_{\mathbb{Z}}(ft)t$ with $c = \frac{1}{2q}$. Finally, $\overline{B}_2$ is a bounded function, and the two arctangent terms in equation (2.7) are bounded in terms of r_1 and r_2; and since $|\overline{B}_1(z)| \le \frac{1}{2}$ we have $|s(h,k;y,x)| \le \frac{k}{4}$ uniformly in h, y, and x, so that the two Dedekind–Rademacher sums in equation (2.7) are also bounded in terms of r_1 and r_2. ∎

Definition 2.6 A *fitting angle* is any angle of the form $\angle POQ$ where $O = (0,0)$, and where $P = (k,\ell)$ and $Q = (m,n)$ are lattice points such that $\det\left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right) = \pm 1$, or equivalently such that $\mathrm{Area}(\triangle POQ) = \frac{1}{2}$. A *fitting triangle* is a rational pointed triangle whose angle at the vertex O is a fitting angle.

Corollary 2.7 *For any fitting triangle $\mathcal{T}$, there exist positive rational numbers $c = c(\mathcal{T})$ and $f = f(\mathcal{T})$ such that*

$$A_{\mathcal{T}}(t) = \mathrm{Area}(\mathcal{T})t^2 - 2cB(ft)t + c\mathbf{1}_{\mathbb{Z}}(ft)t + O_{\mathcal{T}}(1).$$

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Proof First, let Γ be any element of $SL_2(\mathbb{Z})$. Such elements act on the plane, acting as a bijection on the lattice $\mathbb{Z}^2$, sending lines and triangles to lines and triangles, and preserving whether lattice points lie on line segments or inside triangles. Therefore the only disparity between the solid angle sums of $t\mathcal{T}$ and $t\Gamma(\mathcal{T})$ is that the solid angles at the three vertices might be different. Consequently, we certanly have

$$A_{\mathcal{T}}(t) = A_{\Gamma(\mathcal{T})}(t) + O(1)$$

uniformly for all triangles $\mathcal{T}$, all $\Gamma \in SL_2(\mathbb{Z})$, and all $t > 0$.

Given a fitting triangle $\triangle R_1 O R_2$, we can find lattice points $P = (k, \ell)$ and $Q = (m, n)$, on the rays $\overrightarrow{OR_1}$ and $\overrightarrow{OR_2}$, respectively, such that $\det \left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right) = \pm 1$. It follows that $\Gamma = \left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right)^{-1}$ is an element of $SL_2(\mathbb{Z})$ that sends P to $(1, 0)$ and Q to $(0, 1)$, and hence sends R_1 and R_2 to rational points $(r_1, 0)$ and $(0, r_2)$, respectively. Therefore $A_{\mathcal{T}}(t) = A_{\Gamma(\mathcal{T})}(t) + O(1) = A_{\mathcal{T}(r_1, r_2)}(t) + O(1)$, and so this corollary follows directly from Corollary 2.5 (since the areas of $\mathcal{T}$ and $\Gamma(\mathcal{T}) = \mathcal{T}(r_1, r_2)$ are equal). ∎

We say that a polygon $\mathcal{A}$ can be *written as a sum*, or *written as a signed sum*, of finitely many other polygons $\mathcal{B}_j$ if the indicator function of $\mathcal{A}$ can be written as sums, or sums and differences (respectively), of indicator functions of the indicator functions of the $\mathcal{B}_j$, up to inaccuracies on the edges of these polygons.

Lemma 2.8 *Any rational polygon can be written as a signed sum of fitting triangles. If the rational polygon is origin-star-shaped, then it can be written as a sum of fitting triangles.*

Proof First, any rational polygon $\mathcal{P}$ can be written as the signed sum of rational pointed triangles: every edge of $\mathcal{P}$ defines a rational pointed triangle, which is taken with sign +1 if the ray from the origin hits that edge from the interior of the polygon and sign −1 if the ray from the origin hits that edge from the exterior of the polygon. Note that if $\mathcal{P}$ is origin-star-shaped, then all the signs that occur in this decomposition are +1. All that remains is to prove that every rational pointed triangle can be written as a sum of fitting triangles.

Given a rational pointed triangle $\triangle AOB$, consider the convex hull of all the lattice points in the cone bounded by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ other than the origin. The vertices of this convex hull can be written in order as $V_0, \ldots, V_k$ for some $k \geq 1$, where $V_0 \in \overrightarrow{OA}$ and $V_k \in \overrightarrow{OB}$. Note that each $\triangle V_j O V_{j+1}$ contains no lattice points other than its three vertices, by the definition of convex hull. By Pick's theorem, the area of this triangle equals $\frac{1}{2}$, and thus each angle $\angle V_j O V_{j+1}$ is a fitting angle. Let W_j be the intersection of $\overrightarrow{OV_j}$ and $\overline{AB}$, so that $W_0 = A$ and $W_k = B$. Then $\triangle AOB$ is the sum of the fitting triangles $\triangle W_0 O W_1, \ldots, \triangle W_{k-1} O W_k$. (We remark that $V_0, \ldots, V_k$ is a "Hilbert basis" for the lattice points in the cone defined by $\angle AOB$; in that context it is a standard fact that the triangles $\triangle V_j O V_{j+1}$ are "empty" and "unimodular" and thus, in our terminology, have fitting angles $\angle V_j O V_{j+1}$.) ∎

Proposition 2.9 *Let $\mathcal{A}$ be a rational polygon. There exist a positive integer k, rational numbers $c_1, \ldots, c_k$, and positive rational numbers $f_1, \ldots, f_k$ (all depending on $\mathcal{A}$) such*

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that

$$A_{\mathcal{T}}(t) = \mathrm{Area}(\mathcal{A})t^2 - 2t\sum_{j=1}^{k} c_j B(f_j t) + t\sum_{j=1}^{k} c_j \mathbf{1}_{\mathbb{Z}}(f_j t) + O_{\mathcal{A}}(1). \tag{2.8}$$

Furthermore, when $\mathcal{A}$ is an origin-star-shaped polygon, the c_j can be taken to all be positive.

Proof The proposition follows immediately by using Lemma 2.8 to decompose $\mathcal{A}$ as a signed sum of fitting triangles, and then applying Corollary 2.7 to each of those fitting triangles $\mathcal{T}$ (taking $c_j = \pm c(\mathcal{T})$ from Corollary 2.7, depending on the sign attached to $\mathcal{T}$ in the decomposition of $\mathcal{A}$). ∎

The following theorem results from a careful examination of $\mathbb{P}_{\mathcal{A}}(t)$, which can be related to the solid angle sum $A_{\mathcal{A}}(t)$ using the Möbius function as in the proof of Proposition 2.2.

Theorem 2.10 *Let $\mathcal{A}$ be a rational polygon. There exist a positive integer k, rational numbers $r_1, \ldots, r_k$, and positive rational numbers $f_1, \ldots, f_k$ (all depending on $\mathcal{A}$) such that*

$$E_{\mathcal{A}}(t) = \sum_{j=1}^{k} r_j E_{\Delta}(f_j t) + O(t), \tag{2.9}$$

where the implicit constant may depend upon $\mathcal{A}$. Furthermore, if $\mathcal{A}$ is an origin-star-shaped polygon, then the r_j can all be taken to be positive as well.

Remark Theorem 1.2 follows immediately from Theorem 2.10 in light of the known upper bound (1.5). In addition, Theorem 1.4 follows immediately from the combination of Theorem 2.10 and Theorem 1.5; the latter theorem is proved in the next section. Finally, we note that Theorem 2.10 automatically implies the generalization of Theorem 1.5 where the isosceles right triangle Δ is replaced by any rational origin-star-shaped polygon $\mathcal{A}$.

Proof We commence by excluding the (non-primitive) origin and using the Möbius identity (2.3) to write

$$\mathbb{P}_{\mathcal{A}}(t) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \omega_{t\mathcal{A}}\big((m,n)\big) \sum_{d\mid \gcd(m,n)} \mu(d).$$

Interchanging the order of summation (valid since in reality there are only finitely many nonzero terms) and rescaling (which preserves solid angles),

$$\begin{aligned}
\mathbb{P}_{\mathcal{A}}(t) &= \sum_{d=1}^{\infty} \mu(d) \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus(0,0)\\ d\mid m,\, d\mid n}} \omega_{t\mathcal{A}}\big((m,n)\big) \\
&= \sum_{d=1}^{\infty} \mu(d) \sum_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} \omega_{\frac{t}{d}\mathcal{A}}\big((x,y)\big) \\
&= \sum_{d=1}^{\infty} \mu(d)\Big(A_{\mathcal{A}}\Big(\frac{t}{d}\Big) - \omega_{\frac{t}{d}\mathcal{A}}\big((0,0)\big)\Big)
\end{aligned}$$

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by the definition of $A_{\mathcal{A}}$. We can truncate the outer sum at any real number $T \geq t\,\mathrm{diam}(\mathcal{A})$, since for larger values of d, the diameter of $\frac{t}{d}\mathcal{A}$ is less than 1, and hence $\frac{t}{d}\mathcal{A}$ (which contains the origin) cannot contain any other lattice points. Proposition 2.9 now implies

$$\begin{aligned}
\mathbb{P}_{\mathcal{A}}(t) &= \sum_{d\leq T}\mu(d)\Bigg\{\mathrm{Area}(\mathcal{A})\frac{t^2}{d^2} - 2\frac{t}{d}\sum_{j=1}^{k} c_j B\left(f_j\frac{t}{d}\right) \\
&\qquad + \frac{t}{d}\sum_{j=1}^{k} c_j \mathbf{1}_{\mathbb{Z}}\left(f_j\frac{t}{d}\right) + O_{\mathcal{A}}(1)\Bigg\} \\
&= \mathrm{Area}(\mathcal{A})t^2\sum_{d\leq T}\frac{\mu(d)}{d^2} - 2t\sum_{j=1}^{k} c_j \sum_{d\leq T}\frac{\mu(d)}{d} B\left(f_j\frac{t}{d}\right) \\
&\qquad + t\sum_{j=1}^{k} c_j \sum_{d\leq T}\frac{\mu(d)}{d}\mathbf{1}_{\mathbb{Z}}\left(f_j\frac{t}{d}\right) + O_{\mathcal{A}}\left(\sum_{d\leq T}|\mu(d)|\right) \\
&= \mathrm{Area}(\mathcal{A})t^2\left(\frac{6}{\pi^2} + O\left(\frac{1}{T}\right)\right) - 2t\sum_{j=1}^{k} c_j \sum_{d\leq T}\frac{\mu(d)}{d} B\left(f_j\frac{t}{d}\right) \\
&\qquad + t\sum_{j=1}^{k} c_j \sum_{\substack{d\leq T \\ d\mid f_j t}}\frac{\mu(d)}{d} + O_{\mathcal{A}}(T).
\end{aligned}$$

As long as $T \geq f_j t$, the inner sum on the last line equals 0 if $f_j t \notin \mathbb{Z}$ and $\phi(f_j t)/(f_j t) \leq 1$ if $f_j t \in \mathbb{Z}$, and therefore this simplifies to

$$\mathbb{P}_{\mathcal{A}}(t) = \frac{6}{\pi^2}\mathrm{Area}(\mathcal{A})t^2 - 2t\sum_{j=1}^{k} c_j \sum_{d\leq T}\frac{\mu(d)}{d} B\left(f_j\frac{t}{d}\right) + O_{\mathcal{A}}(T)$$

when $t \leq T$. Using Proposition 2.2, we can rewrite the above in terms of $E_{\Delta}(t)$ to obtain

$$\mathbb{P}_{\mathcal{A}}(t) = \mathrm{Area}(\mathcal{A})\frac{6}{\pi^2}t^2 + \sum_{j=1}^{k}\frac{2c_j}{f_j}E_{\Delta}(f_j t) + O_{\mathcal{A}}(T) \tag{2.10}$$

for any $T \geq t\max\{1, \mathrm{diam}(\mathcal{A}), f_1, \ldots, f_k\}$.

The number of integer points on the boundary of $t\mathcal{A}$ is $O_{\mathcal{A}}(t)$, since $t\mathcal{A}$ has finitely many sides each of which has length $O_{\mathcal{A}}(t)$. Consequently, the difference between the solid angle sum and the ordinary count of lattice points satisfies $\mathbb{P}_{\mathcal{A}}(t) - \#(t\mathcal{A}\cap\mathbb{P}) \ll_{\mathcal{A}} t$, and therefore equation (2.10) implies

$$\begin{aligned}
E_{\mathcal{A}}(t) &= \#(t\mathcal{A}\cap\mathbb{P}) - \mathrm{Area}(\mathcal{A})\frac{6}{\pi^2}t^2 \\
&= \mathbb{P}_{\mathcal{A}}(t) + O_{\mathcal{A}}(t) - \mathrm{Area}(\mathcal{A})\frac{6}{\pi^2}t^2 = \sum_{j=1}^{k}\frac{2c_j}{f_j}E_{\Delta}(f_j t) + O(T).
\end{aligned}$$

Upon setting $T = t\max\{1, \mathrm{diam}(\mathcal{A}), f_1, \ldots, f_k\}$ and $r_j = 2c_j/f_j$ for each $1 \leq j \leq k$, the theorem follows. ∎

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3 Linear Combinations of E_Δ

In this section, we prove Theorem 1.5, showing that positive rational linear combinations of scaled copies of the totient error function E_Δ have oscillations as large as those known for E_Δ itself. We begin by recalling some of the components of Montgomery's argument [10] establishing the oscillations of E_Δ, after which we describe the strategy that leads to our modifications.

Definition 3.1 Define

$$R_0(x) = \sum_{n \le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x.$$

Montgomery [10, Theorem 1] showed that the totient error function is closely connected to the above weighted error.

Lemma 3.2 (Montgomery) $E_\Delta(x) = xR_0(x) + O\big(x \exp(-c\sqrt{\log x})\big)$.

Definition 3.3 Define

$$K(q, \alpha) = -\sum_{d|q} \frac{\mu(d)}{d} B\Big(\frac{\alpha}{d}\Big),$$

where the sawtooth function $B(x)$ was defined in Definition 2.1, and

$$C(q, \alpha) = K(q, \alpha) \frac{6}{\pi^2} \prod_{p|q} \Big(1 - \frac{1}{p^2}\Big)^{-1}.$$

Lemma 3.4 *If b is relatively prime to q, then*

$$K(qb, \alpha b) = \sum_{d_1 d_2 = b} \frac{\mu(d_1)}{d_1} K(q, \alpha d_2).$$

Proof Since every divisor of qb can be written uniquely as a divisor of b times a divisor of q,

$$\begin{aligned} K(qb, \alpha b) &= -\sum_{d|qb} \frac{\mu(d)}{d} B\Big(\frac{\alpha b}{d}\Big) = -\sum_{a|b} \frac{\mu(a)}{a} \sum_{c|q} \frac{\mu(c)}{c} B\Big(\frac{\alpha b}{ac}\Big) \\ &= \sum_{a|b} \frac{\mu(a)}{a} K\Big(q, \frac{\alpha b}{a}\Big), \end{aligned}$$

which is equivalent to the statement of the lemma. ∎

The above quantities appear [10, Lemma 4] in a key part of Montgomery's argument, which displays a bias in the values of R_0 sampled on an arithmetic progression.

Lemma 3.5 (Montgomery) *There exists a positive real number c such that if α is a non-integral real number with $0 < \alpha < q$, then*

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$$\sum_{n=1}^{N} R_0(nq+\alpha) = C(q,\alpha)N + O\big(N\exp(-c\sqrt{\log N})\big)$$

uniformly for $q \le e^{c\sqrt{\log N}}$.

The proof of Lemma 3.5 is based on the following ancient identity due to Raabe [15] for a sum of $B(x)$ over an arithmetic progression of points: for all real numbers x,

$$\sum_{j=1}^{J} B\Big(x+\frac{j}{J}\Big) = B(Jx).$$

The relevance to the problem at hand is due to the fact [10, Lemma 1] that

$$R_0(x) = -\sum_{d\le x} \frac{\mu(d)}{d} B\Big(\frac{x}{d}\Big) + O(1).$$

To exploit Lemma 3.5, Montgomery chose $\alpha = q/4$, so that for all divisors d of q the quantity $B(\alpha/d)$ equals $\pm\frac{1}{4}$, with the sign depending only upon the residue class of d modulo 4. On the other hand, he chose q to be the product of many primes congruent to 3 modulo 4; for any divisor d of this q, the residue class of d (mod 4) depends only on the number of prime factors of d. With these choices, the sign of $B(\alpha/d)$ correlates exactly with the sign of $\mu(d)$, making the quantity $K(q,\alpha)$ large in absolute value. Choosing $\alpha = 3q/4$ instead again makes $K(q,\alpha)$ large but with the opposite sign.

One of our key observations is that we can work modulo a suitably chosen prime P rather than working modulo 4. Instead of choosing q to be the product of many primes congruent to 3 (mod 4), we instead choose q to be the product of many primes that are quadratic nonresidues modulo P. The sign of $B(\alpha/d)$ will not be perfectly correlated with $\mu(d)$, but there will be enough of a systematic bias in the signs of $B(\alpha/d)$ (due to the imperfect distribution of quadratic residues and nonresidues modulo P) that we can still force $K(q,\alpha)$ to be large in absolute value. If we choose the prime P carefully, we can even force all of the different $K(qb_i,\alpha b_i)$ to be large in absolute value and have the same sign.

We introduce the following function, whose oscillations we will want to establish. For the rest of this section, we use boldface variables such as $\mathbf{a}$ to indicate the dependence of various quantities on k-tuples $(a_1,\dots,a_k)$ of variables.

Definition 3.6 Given real numbers $a_1,\dots,a_k$ and $b_1,\dots,b_k$, define

$$G_{\mathbf{a},\mathbf{b}}(x) = a_1R_0(b_1x) + \cdots + a_kR_0(b_kx).$$

The starting point of our modification of Montgomery's method is the following easy consequence of Lemma 3.5.

Lemma 3.7 *Let $a_1,\dots,a_k$ and $b_1,\dots,b_k > 0$ be real numbers. There exists a positive real number c such that if $\alpha \in (0,q)$ is a real number such that none of the αb_j is an*

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integer, then

$$\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nq+\alpha) = N\sum_{j=1}^{k} a_j C(qb_j, \alpha b_j) + O_{\mathbf{a},\mathbf{b}}\big(N\exp(-c\sqrt{\log N})\big)$$

uniformly for $\max\{b_1,\dots,b_k\} < q \le e^{c\sqrt{\log N}}$.

We now state our oscillation result for $G_{\mathbf{a},\mathbf{b}}$, after which we show how Theorem 1.5 is implied by it. Thereafter, our only remaining goal will be to establish this proposition.

Proposition 3.8 *Let $a_1,\dots,a_k$ and $b_1,\dots,b_k$ be fixed positive integers. There exists a constant $\kappa_{\mathbf{a},\mathbf{b}} > 0$, and sequences $Q_{N,\mathbf{b}} \le e^{\sqrt{\log N}}$ and $0 < \alpha_N^+, \alpha_N^- < Q_{N,\mathbf{b}}$ defined for positive integers N, for which $Q_{N,\mathbf{b}}$ tends to infinity with N and*

$$\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^+) = \kappa_{\mathbf{a},\mathbf{b}} N\sqrt{\log\log N} + O_{\mathbf{a},\mathbf{b}}(N),$$
$$\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^-) = -\kappa_{\mathbf{a},\mathbf{b}} N\sqrt{\log\log N} + O_{\mathbf{a},\mathbf{b}}(N).$$

Proof of Theorem 1.5 assuming Proposition 3.8 By Lemma 3.2,

$$\sum_{j=1}^{k} r_j E_\Delta(s_j x) = x\sum_{j=1}^{k} r_j s_j R_0(s_j x) + O_{\mathbf{r},\mathbf{s}}\big(x\exp(-c\sqrt{\log x})\big).$$

Let $D_{\mathbf{s}}$ be the least common denominator of the rational numbers $s_1,\dots,s_k$, and set $a_j = D_{\mathbf{s}} r_j s_j$ and $b_j = D_{\mathbf{s}} s_j$. Replacing x by $D_{\mathbf{s}}x$, we obtain

$$\sum_{j=1}^{k} r_j E_\Delta(s_j D_{\mathbf{s}} x) = x G_{\mathbf{a},\mathbf{b}}(x) + O_{\mathbf{r},\mathbf{s}}\big(x\exp(-c\sqrt{\log x})\big). \tag{3.1}$$

Let $0 < \varepsilon < \kappa_{\mathbf{a},\mathbf{b}}$. If there were to exist a real number x_0 such that $G_{\mathbf{a},\mathbf{b}}(x) < (\kappa_{\mathbf{a},\mathbf{b}} - \varepsilon)\sqrt{\log\log(D_{\mathbf{s}}x)}$ for all $x > x_0$, then we would have

$$\begin{aligned}\sum_{n=1}^{N} & G_{\mathbf{a},\mathbf{b}}(nQ_N + \alpha_N^+) \\ &< (\kappa_{\mathbf{a},\mathbf{b}} - \varepsilon)N\sqrt{\log\log(D_{\mathbf{s}}(NQ_N + \alpha_N^+))} + O(x_0 \max_{1\le t\le x_0} G_{\mathbf{a},\mathbf{b}}(t)) \\ &= (\kappa_{\mathbf{a},\mathbf{b}} - \varepsilon)N\Big(\sqrt{\log\log N} + O_{\mathbf{s}}\Big(\frac{1}{\sqrt{\log N}}\Big)\Big) + O_{\varepsilon,\mathbf{a},\mathbf{b}}(1)\end{aligned}$$

by the bounds $0 < \alpha_N^+ < Q_{N,\mathbf{b}}$; for large N (and hence for large $Q_{N,\mathbf{b}}$), this would contradict Proposition 3.8. Therefore, no such x_0 can exist, in which case equation (3.1) implies that there are arbitrarily large values of x for which

$$\sum_{j=1}^{k} r_j E_\Delta(s_j D_{\mathbf{s}} x) \ge x(\kappa_{\mathbf{a},\mathbf{b}} - \varepsilon)\sqrt{\log\log(D_{\mathbf{s}}x)} + O_{\mathbf{r},\mathbf{s}}\big(x\exp(-c\sqrt{\log x})\big).$$

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In other words,

$$\limsup_{x\to\infty} \frac{r_1 E_\Delta(s_1 x) + \cdots + r_k E_\Delta(s_k x)}{x\sqrt{\log\log x}} \geq \frac{\kappa_{\mathbf{a},\mathbf{b}}}{D_{\mathbf{s}}},$$

and the analogous argument using the values $G(nQ_N + \alpha_N^-)$ gives

$$\liminf_{x\to\infty} \frac{r_1 E_\Delta(s_1 x) + \cdots + r_k E_\Delta(s_k x)}{x\sqrt{\log\log x}} \leq -\frac{\kappa_{\mathbf{a},\mathbf{b}}}{D_{\mathbf{s}}},$$

completing the derivation of Theorem 1.5 from Proposition 3.8. ∎

Before addressing Proposition 3.8 directly, we record some preliminary facts about the distribution of primes in residue classes and the associated L-values.

Lemma 3.9 *Let $P \equiv 3 \pmod 4$ be a prime exceeding 3, and let $\chi_1(\cdot) = \left(\frac{\cdot}{P}\right)$ denote the quadratic character modulo P. Then the class number $h(-P)$ of the field $\mathbb{Q}(\sqrt{-P})$ equals*

$$h(-P) = \frac{\sqrt{P}}{\pi} L(1,\chi_1) = -\frac{1}{P}\sum_{a=1}^{P-1} a\chi_1(a).$$

Proof These results are classical; see, for example, [3, Chapter 6, equations (15) and (19)]. ∎

Lemma 3.10 *Let $P \equiv 3 \pmod 4$ be a prime exceeding 3. If χ_0 denotes the principal character $\pmod P$, then*

$$\prod_{\substack{p\leq y\\ (\frac{p}{P})=-1}} \left(1 + \frac{\chi_0(p)}{p}\right) = c_P\sqrt{\log y} + O_P(1),$$

where

$$c_P = \left(\frac{e^\gamma}{\pi}\frac{P-1}{h(-P)\sqrt{P}} \prod_{(\frac{p}{P})=-1} \left(1-p^{-2}\right)\right)^{1/2}. \tag{3.2}$$

On the other hand, for any nonprincipal character $\chi \pmod P$,

$$\prod_{\substack{p\leq y\\ (\frac{p}{P})=-1}} \left(1 + \frac{\chi(p)}{p}\right) \ll_P 1.$$

Proof Again, let $\chi_1(\cdot) = \left(\frac{\cdot}{P}\right)$ denote the quadratic character $\pmod P$. For any character $\chi \pmod P$, we can write

$$\left(\prod_{\substack{p\leq y\\ (\frac{p}{P})=-1}} \left(1 + \frac{\chi(p)}{p}\right)\right)^2 = \tag{3.3}$$

$$\prod_{p\leq y}\left(1-\frac{\chi(p)}{p}\right)^{-1} \prod_{p\leq y}\left(1-\frac{\chi(p)\chi_1(p)}{p}\right) \prod_{\substack{p\leq y\\ (\frac{p}{P})=-1}} \left(1-\frac{\chi^2(p)}{p^2}\right).$$

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The last product is absolutely convergent uniformly in P; indeed,

$$\Big|\prod_{\substack{p>y\\(\frac{p}{P})=-1}}\Big(1-\frac{\chi^2(p)}{p^2}\Big)\Big| \le \prod_{\substack{p>y\\(\frac{p}{P})=-1}}\Big(1+\frac{1}{p^2}\Big) < \prod_{n>y}\Big(1+\frac{1}{n^2}\Big) < \prod_{n>y}\Big(1-\frac{1}{n^2}\Big)^{-1} = 1+\frac{1}{\lceil y\rceil},$$

and so the last product equals $\prod_{(\frac{p}{P})=-1}(1-\chi^2(p)p^{-2})(1+O(\frac{1}{y}))$ and, in particular, is uniformly bounded. When χ is nonprincipal, we know [11, Theorem 4.11(d)] that

$$\prod_{p\le y}\Big(1-\frac{\chi(p)}{p}\Big)^{-1} = L(1,\chi)+O_\chi\Big(\frac{1}{\log y}\Big) = L(1,\chi)\Big(1+O_P\Big(\frac{1}{\log y}\Big)\Big),$$

since $L(1,\chi)\neq 0$ [11, Theorem 4.9]. (Better error terms are available but are not relevant for us.) In particular, when neither χ nor $\chi\chi_1$ is principal, the first two products on the right-hand side of equation (3.3) are $L(1,\chi)(1+O_P(\frac{1}{\log y}))$ and $L(1,\chi\chi_1)^{-1}(1+O_P(\frac{1}{\log y}))$, respectively; in particular, both are $\ll_P 1$. This estimate establishes the lemma when χ is neither principal nor equal to χ_1.

When $\chi=\chi_1$, the first and third factors are still bounded, while the second factor now actually diverges to 0, and hence, in particular, is still bounded. Finally, when $\chi=\chi_0$ is principal, the second factor converges to $1/L(1,\chi_1)=\sqrt{P}/\pi h(-P)$ by Lemma 3.9, while the first factor on the left-hand side of equation (3.3) is

$$\prod_{p\le y}\Big(1-\frac{\chi_0(p)}{p}\Big)^{-1} = \prod_{\substack{p\le y\\p\neq P}}\Big(1-\frac{1}{p}\Big)^{-1} = \frac{P-1}{P}(e^\gamma\log y)\Big(1+O\Big(\frac{1}{\log y}\Big)\Big)$$

by Mertens's formula [11, Theorem 2.7(e)]. This asymptotic evaluation of the right-hand side of equation (3.3) establishes the lemma when χ is principal, indeed with the stronger error term $O_P(\frac{1}{\sqrt{\log y}})$. ∎

We can now define the modulus $Q_{N,\mathbf{b}}$ appearing in the statement of Proposition 3.8, in terms of a companion prime $P_\mathbf{b}$.

Definition 3.11 Let $P_\mathbf{b}$ be the smallest prime in the residue class $P_\mathbf{b}\equiv -1 \pmod{8b_1\dots b_k}$. Note that $P_\mathbf{b}\equiv 7 \pmod 8$, and so -1 is a quadratic nonresidue (mod $P_\mathbf{b}$), while 2 is a quadratic residue (mod $P_\mathbf{b}$). Furthermore, if p is any odd prime dividing one of the b_j, then by quadratic reciprocity [12, Theorem 3.1], we have, since $P_\mathbf{b}\equiv 3 \pmod 4$ and $P_\mathbf{b}\equiv -1 \pmod p$,

$$\Big(\frac{p}{P_\mathbf{b}}\Big) = (-1)^{(p-1)/2}\Big(\frac{P_\mathbf{b}}{p}\Big) = (-1)^{(p-1)/2}\Big(\frac{-1}{p}\Big) = 1.$$

Therefore each prime dividing every b_j is a quadratic residue (mod $P_\mathbf{b}$).

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Now define, for any integer $N \geq 1$,

$$Q_{N,\mathbf{b}} = \prod_{\substack{p \leq c\sqrt{\log N} \\ \left(\frac{p}{P_\mathbf{b}}\right)=-1}} p,$$

where c is the constant from Lemma 3.7. Since

$$\log Q_{N,\mathbf{b}} = \sum_{\substack{1 \leq a \leq P_\mathbf{b} \\ \left(\frac{a}{P_\mathbf{b}}\right)=-1}} \sum_{\substack{p \leq c\sqrt{\log N} \\ p \equiv a \pmod{P_\mathbf{b}}}} \log p = \sum_{\substack{1 \leq a \leq P_\mathbf{b} \\ \left(\frac{a}{P_\mathbf{b}}\right)=-1}} \theta\big(c\sqrt{\log N}; P_\mathbf{b}, a\big),$$

where the sum on the right-hand side is over $\phi(P_\mathbf{b})/2$ reduced residue classes modulo $P_\mathbf{b}$, the prime number theorem for arithmetic progressions to a fixed modulus (see [11, Corollary 11.21]) implies that

$$\log Q_{N,\mathbf{b}} \sim \frac{1}{2} c\sqrt{\log N}.$$

In particular, $Q_{N,\mathbf{b}}$ tends to infinity with N, and $Q_{N,\mathbf{b}} < e^{c\sqrt{\log N}}$ when N is large enough.

Note also that $Q_{N,\mathbf{b}}$ is squarefree and relatively prime to $P_\mathbf{b}$ and to each b_j, since every prime $p \mid b_j$ satisfies $\left(\frac{p}{P_\mathbf{b}}\right) = 1$. Finally, note that any divisor d of $Q_{N,\mathbf{b}}$ has the convenient (and, for us, crucial) property that $\left(\frac{d}{P_\mathbf{b}}\right) = \mu(d)$, since both quantities equal $(-1)^{\#\{p \mid d\}}$.

Lemma 3.12 *For any $1 \leq b \leq P-1$, we have*

$$\sum_{\substack{d \mid Q_{N,\mathbf{b}} \\ d \equiv b \pmod{P_\mathbf{b}}}} \frac{1}{d} = \frac{c_{P_\mathbf{b}}}{\sqrt{2}(P_\mathbf{b}-1)} \sqrt{\log\log N} + O_\mathbf{b}(1),$$

where c_P was defined in equation (3.2).

Proof From the orthogonality of the characters modulo $P_\mathbf{b}$,

$$\begin{aligned}
\sum_{\substack{d \mid Q_{N,\mathbf{b}} \\ d \equiv b \pmod{P_\mathbf{b}}}} \frac{1}{d} &= \frac{1}{P_\mathbf{b}-1} \sum_{\chi \pmod{P_\mathbf{b}}} \bar{\chi}(b) \sum_{d \mid Q_{N,\mathbf{b}}} \frac{\chi(d)}{d} \\
&= \frac{1}{P_\mathbf{b}-1} \sum_{\chi \pmod{P_\mathbf{b}}} \bar{\chi}(b) \prod_{p \mid Q_{N,\mathbf{b}}} \Big(1 + \frac{\chi(p)}{p}\Big) \\
&= \frac{1}{P_\mathbf{b}-1} \sum_{\chi \pmod{P_\mathbf{b}}} \bar{\chi}(b) \prod_{\substack{p \leq c\sqrt{\log N} \\ \left(\frac{p}{P_\mathbf{b}}\right)=-1}} \Big(1 + \frac{\chi(p)}{p}\Big) \\
&= \frac{1}{P_\mathbf{b}-1} c_{P_\mathbf{b}} \sqrt{\log(c\sqrt{\log N})} + O_{P_\mathbf{b}}(1)
\end{aligned}$$

by Lemma 3.10. The statement of the lemma follows upon noting that $\log(c\sqrt{\log N}) = \frac{1}{2}\log\log N + O(1)$. ∎

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Lemma 3.13 *For any integer m that is not a multiple of $P_{\mathbf{b}}$ and for any integer $N \geq 3$,*

$$K\Big(Q_{N,\mathbf{b}}, \frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big) = \Big(\frac{m}{P_{\mathbf{b}}}\Big)\Big(\frac{Q_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\frac{c_{P_{\mathbf{b}}}h(-P_{\mathbf{b}})}{\sqrt{2}(P_{\mathbf{b}}-1)}\sqrt{\log\log N} + O_{\mathbf{b}}(1).$$

Remark The exact value of the leading constant is not as important for us as the fact that its dependence on m is only in the term $\left(\frac{m}{P_{\mathbf{b}}}\right)$, so that the sign of the leading constant depends on whether m is a quadratic residue or nonresidue modulo $P_{\mathbf{b}}$.

Proof By Definition 3.3 and the coincidence between the Legendre symbol and the Möbius function on divisors of $Q_{N,\mathbf{b}}$ (as noted at the end of Definition 3.11), we have

$$\begin{aligned}
-K\Big(Q_{N,\mathbf{b}}, \frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big) &= \sum_{d|Q_{N,\mathbf{b}}} \frac{\mu(d)}{d} B\Big(\frac{mQ_{N,\mathbf{b}}}{dP_{\mathbf{b}}}\Big) \\
&= \sum_{d|Q_{N,\mathbf{b}}} \Big(\frac{d}{P_{\mathbf{b}}}\Big)\frac{1}{d} B\Big(\frac{mQ_{N,\mathbf{b}}}{dP_{\mathbf{b}}}\Big) \\
&= \sum_{d|Q_{N,\mathbf{b}}} \Big(\frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\Big(\frac{mQ_{N,\mathbf{b}}/d}{P_{\mathbf{b}}}\Big)\frac{1}{d} B\Big(\frac{mQ_{N,\mathbf{b}}}{dP_{\mathbf{b}}}\Big) \\
&= \Big(\frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\sum_{a=1}^{P_{\mathbf{b}}-1}\Big(\frac{a}{P_{\mathbf{b}}}\Big)B\Big(\frac{a}{P_{\mathbf{b}}}\Big) \sum_{\substack{d|Q_{N,\mathbf{b}} \\ mQ_{N,\mathbf{b}}d^{-1}\equiv a \pmod{P_{\mathbf{b}}}}} \frac{1}{d}.
\end{aligned}$$

The last congruence is equivalent to d being in the reduced residue class $mQ_{N,\mathbf{b}}a^{-1} \pmod{P_{\mathbf{b}}}$, and so Lemma 3.12 applies:

$$\begin{aligned}
&-K\Big(Q_{N,\mathbf{b}}, \frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big) \\
&\quad= \Big(\frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\sum_{a=1}^{P_{\mathbf{b}}-1}\Big(\frac{a}{P_{\mathbf{b}}}\Big)B\Big(\frac{a}{P_{\mathbf{b}}}\Big)\Big(\frac{c_{P_{\mathbf{b}}}}{\sqrt{2}(P_{\mathbf{b}}-1)}\sqrt{\log\log N} + O_{\mathbf{b}}(1)\Big) \\
&\quad= \Big(\frac{mQ_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\frac{c_{P_{\mathbf{b}}}}{\sqrt{2}(P_{\mathbf{b}}-1)}\sqrt{\log\log N}\sum_{a=1}^{P_{\mathbf{b}}-1}\Big(\frac{a}{P_{\mathbf{b}}}\Big)B\Big(\frac{a}{P_{\mathbf{b}}}\Big) + O_{\mathbf{b}}(1).
\end{aligned}$$

By the definition of the Bernoulli polynomial B, this sum equals

$$\sum_{a=1}^{P_{\mathbf{b}}-1}\Big(\frac{a}{P_{\mathbf{b}}}\Big)B\Big(\frac{a}{P_{\mathbf{b}}}\Big) = \frac{1}{P_{\mathbf{b}}}\sum_{a=1}^{P_{\mathbf{b}}-1} a\Big(\frac{a}{P_{\mathbf{b}}}\Big) - \frac{1}{2}\sum_{a=1}^{P_{\mathbf{b}}-1}\Big(\frac{a}{P_{\mathbf{b}}}\Big) = -h(-P_{\mathbf{b}}) - 0$$

by Lemma 3.9, which completes the proof of the lemma. ∎

We now have all the tools we need to establish Proposition 3.8 and hence our main theorems.

Proof of Proposition 3.8 Given a sufficiently large integer N, define two numbers

$$\alpha_N^{\pm} = \frac{m^{\pm}Q_{N,\mathbf{b}}}{P_{\mathbf{b}}},$$

where the integers $1 \leq m^{\pm} \leq P_{\mathbf{b}}-1$ satisfy $m^{\pm} \equiv \pm Q_{N,\mathbf{b}} \pmod{P_{\mathbf{b}}}$; note that none of the numbers $\alpha_N^{\pm}b_j$ is an integer, since neither $Q_{N,\mathbf{b}}$ nor any of the b_j is a multiple of the

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prime $P_{\mathbf{b}}$. Since we confirmed in Definition 3.11 that $Q_{N,\mathbf{b}} < e^{c\sqrt{\log N}}$, we can invoke Lemma 3.7:

$$\begin{aligned}\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^+) &= N\sum_{j=1}^{k} a_j C(Q_{N,\mathbf{b}}b_j, \alpha_N^+ b_j) + O\big(N\exp(-c\sqrt{\log N})\big)\\ &= N\sum_{j=1}^{k} a_j K(Q_{N,\mathbf{b}}b_j, \alpha_N^+ b_j)\frac{6}{\pi^2}\prod_{p\mid Q_{N,\mathbf{b}}b_j}\Big(1-\frac{1}{p^2}\Big)^{-1} + O(N).\end{aligned}$$

As noted in Definition 3.11, each b_j is relatively prime to $Q_{N,\mathbf{b}}$. Thus, by Lemma 3.4 with our choice of $\alpha_N^{\pm}$,

$$\begin{aligned}\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^{\pm}) = O(N) + \frac{6}{\pi^2}N\prod_{p\mid Q_{N,\mathbf{b}}}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{j=1}^{k} a_j\\ \times\prod_{p\mid b_j}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{d_1d_2=b_j}\frac{\mu(d_1)}{d_1}K\Big(Q_{N,\mathbf{b}}, \frac{m^{\pm}d_2Q_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big).\end{aligned}$$

By Lemma 3.13,

$$\begin{aligned}\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^{\pm}) &= O(N) + \frac{6}{\pi^2}N\prod_{p\mid Q_{N,\mathbf{b}}}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{j=1}^{k} a_j\prod_{p\mid b_j}\Big(1-\frac{1}{p^2}\Big)^{-1}\\ &\times\sum_{d_1d_2=b_j}\frac{\mu(d_1)}{d_1}\Big(\Big(\frac{m^{\pm}d_2}{P_{\mathbf{b}}}\Big)\Big(\frac{Q_{N,\mathbf{b}}}{P_{\mathbf{b}}}\Big)\frac{c_{P_{\mathbf{b}}}h(-P_{\mathbf{b}})}{\sqrt{2}(P_{\mathbf{b}}-1)}\sqrt{\log\log N} + O_{\mathbf{b}}(1)\Big).\end{aligned}$$

As noted in Definition 3.11, every prime dividing b_j is a quadratic residue modulo $P_{\mathbf{b}}$, which implies that $\big(\frac{d_2}{P_{\mathbf{b}}}\big) = 1$ always. Moreover, $m^+ \equiv Q_{N,\mathbf{b}} \pmod{P_{\mathbf{b}}}$, so the product of Legendre symbols $\big(\frac{m^+}{P_{\mathbf{b}}}\big)\big(\frac{Q_{N,\mathbf{b}}}{P_{\mathbf{b}}}\big)$ equals 1; on the other hand, since -1 is a quadratic nonresidue modulo $P_{\mathbf{b}}$, the product of Legendre symbols $\big(\frac{m^-}{P_{\mathbf{b}}}\big)\big(\frac{Q_{N,\mathbf{b}}}{P_{\mathbf{b}}}\big)$ equals -1. Consequently,

$$\begin{aligned}\sum_{n=1}^{N} G_{\mathbf{a},\mathbf{b}}(nQ_{N,\mathbf{b}} + \alpha_N^{\pm}) &= \pm\frac{6}{\pi^2}\frac{c_{P_{\mathbf{b}}}h(-P_{\mathbf{b}})}{\sqrt{2}(P_{\mathbf{b}}-1)}N\sqrt{\log\log N}\prod_{p\mid Q_{N,\mathbf{b}}}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{j=1}^{k} a_j\\ &\times\prod_{p\mid b_j}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{d_1d_2=b_j}\frac{\mu(d_1)}{d_1} + O(N).\end{aligned}$$

The last sum equals $\sum_{d\mid b_j}\frac{\mu(d)}{d} = \frac{\phi(b_j)}{b_j}$, which when multiplied by the preceding product becomes $\prod_{p\mid b_j}(1+\frac{1}{p})^{-1}$. In addition, by the same argument as in the proof of

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Lemma 3.10,

$$\prod_{p|Q_{N,\mathbf{b}}}\Big(1-\frac{1}{p^2}\Big)^{-1}=\prod_{\left(\frac{p}{P_{\mathbf{b}}}\right)=-1}\Big(1-\frac{1}{p^2}\Big)^{-1}+O\Big(\frac{1}{\sqrt{\log N}}\Big).$$

We therefore see that we have established the proposition with

$$\begin{aligned}\kappa_{\mathbf{a},\mathbf{b}} &= \frac{6}{\pi^2}\frac{c_{P_{\mathbf{b}}}h(-P_{\mathbf{b}})}{\sqrt{2}(P_{\mathbf{b}}-1)}\prod_{\left(\frac{p}{P_{\mathbf{b}}}\right)=-1}\Big(1-\frac{1}{p^2}\Big)^{-1}\sum_{j=1}^{k}a_j\prod_{p|b_j}\Big(1+\frac{1}{p}\Big)^{-1}\\ &= \frac{e^{\gamma/2}3\sqrt{2}}{\pi^{5/2}}\frac{h(-P_{\mathbf{b}})^{1/2}}{P^{1/4}\sqrt{P_{\mathbf{b}}-1}}\prod_{\left(\frac{p}{P_{\mathbf{b}}}\right)=-1}\Big(1-\frac{1}{p^2}\Big)^{-1/2}\sum_{j=1}^{k}a_j\prod_{p|b_j}\Big(1+\frac{1}{p}\Big)^{-1}\end{aligned}\tag{3.4}$$

by the definition (3.2) of c_P. ∎

Remark The condition that all of the a_j are positive is used here only to ensure that this last line is non-zero. In the case where the a_j may be arbitrary, and thus where the innermost sum in equation (3.4) may equal 0, it is possible that modifying $Q_{N,\boldsymbol{b}}$ and the above argument could prove the totient independence for any coefficients; however, we succeeded in getting this to work only when $k \le 3$.

It is interesting to note that the methods of [2], which handle primitive points in planar convex regions that have nowhere-vanishing Gaussian curvature, involve the analysis of zeta functions and thus are quite different from the present methods.

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