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SEMANTICS FOR PURE THEORIES OF CONNEXIVE IMPLICATION

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Abstract. In this article, I provide Urquhart-style semilattice semantics for three connexive logics in an implication-negation language (I call these "pure theories of connexive implication"). The systems semantically characterized include the implication-negation fragment of a connexive logic of Wansing, a relevant connexive logic recently developed proof-theoretically by Francez, and an intermediate system that is novel to this article. Simple proofs of soundness and completeness are given and the semantics is used to establish various facts about the systems (e.g., that two of the systems have the variable sharing property). I emphasize the intuitive content of the semantics and discuss how natural informational considerations underly each of the examined systems.

§1. Introduction. Connexive logics are characterized by heterodox principles of conditionality and negation such as Aristotle's and Boethius' theses:[1]

$$\neg(\phi \to \neg\phi) \qquad \text{(Aristotle 1)}$$

$$\neg(\neg\phi \to \phi) \qquad \text{(Aristotle 2)}$$

$$(\phi \to \psi) \to \neg(\phi \to \neg\psi) \qquad \text{(Boethius 1)}$$

$$(\phi \to \neg\psi) \to \neg(\phi \to \psi) \qquad \text{(Boethius 2)}$$

Since these characteristic formulae involve only the connectives implication ($\to$) and negation ($\neg$), it is natural to study connexivism in an implication-negation language, and to regard connexive logics formulated in such a language as "pure theories of connexive implication."[2]

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[1] The names come from McCall [11, pp. 415–416]. Strictly speaking, McCall [11] refers to Aristotle 2 as Aristotle's thesis and Boethius 1 as Boethius' thesis.

[2] Wansing and Skurt [23, p. 477] attribute to the Routleys the view that the failure of conjunctive simplification is also a characteristic feature of connexive logic. This position stems from a particular way of motivating connexivism via 'negation as cancellation' (for an accessible discussion of this view, see, e.g., Priest [15]). I will not motivate the systems in this article by appeal to a cancellation view of negation, and I fully agree with Wansing and Skurt's conclusion that "what is characteristic of connexive logic is the interplay between implication and negation" in theses such as Aristotle 1 [23, p. 483].

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The purpose of this article is to present a simple and unified semantic treatment of three pure theories of connexive implication.[3] Two of the systems to be examined are already in the literature, while the third emerges as a semantically natural system intermediate between the other two. The reader will observe that the semantics which I present, which is primarily adapted from the work of Urquhart [18–20] on relevant logic,[4] could be used to characterize a wide variety of pure theories of connexive implication and that the present article only scratches the surface.

The plan of the article is as follows. In §2, I present three pure theories of connexive implication axiomatically and briefly discuss syntactic results concerning them. In §3, I present Urquhart-inspired semantics for these systems and discuss their intuitive interpretation. Proofs of soundness and completeness are given in §4, and other applications of the semantics are examined (e.g., it is proved that two of the systems satisfy the variable sharing property). In §5, I give some concluding remarks and suggest avenues for further investigation.

§2. Pure theories of connexive implication. For concreteness, all the connexive systems examined in this article will be defined in a language $\mathcal{L}$ consisting of a countable set of propositional variables Π and the connectives $\neg$ (unary) and $\rightarrow$ (binary). The formation rules are all standard. I use p, q, etc. for propositional variables and ϕ, ψ, etc. for arbitrary formulae.

DEFINITION 1 (The System $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$). *The system $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ is the smallest set of formulae (in $\mathcal{L}$) containing all instances of the following schemata and closed under modus ponens* (MP):

$$\phi \rightarrow \phi \tag{A1}$$

$$(\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi)) \tag{A2}$$

$$(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi) \tag{A3}$$

$$(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\phi \rightarrow \chi)) \tag{A4}$$

$$\phi \rightarrow \neg\neg\phi \tag{A5}$$

$$\neg\neg\phi \rightarrow \phi \tag{A6}$$

[3] I do intend for the formal model theory I give for these systems to be, in some significant sense, a *semantics*. That is, I do not take it to be merely a tool for proving independence results and the like, but to actually provide a reasonably robust and intuitive interpretation of the systems. I will discuss how the formal semantics is to be understood in §3.

[4] Hitoshi Omori (personal communication) reports independently obtaining several of the results of this paper for the system $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ (see Definition 1) using a semantics based on Urquhart's ideas.

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$$\neg(\phi \to \psi) \to (\phi \to \neg\psi) \tag{A7}$$

$$(\phi \to \neg\psi) \to \neg(\phi \to \psi) \tag{A8}$$

$\mathcal{C}\mathbf{R}^{\neg}_{\to}$ is the system of relevant connexive logic recently developed by Francez [8]. Note that A1–A4 (and modus ponens) axiomatize $\mathbf{R}_{\to}$, the pure implicational fragment of $\mathbf{R}$ (for which, see [2, sec. 3]), and that $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ is a conservative extension of $\mathbf{R}_{\to}$ [8, p. 417].

DEFINITION 2 (The System $\mathcal{C}\mathbf{M}^{\neg}_{\to}$). *The system $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ is $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ extended with all instances of the mingle axiom schema*:

$$\phi \to (\phi \to \phi) \tag{A9}$$

So far as I know, $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ is novel to this article. $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ is a connexive extension of $\mathbf{RM0}_{\to}$, which is to be carefully distinguished from $\mathbf{RM}_{\to}$ (see the discussion in [2, sec. 8.15]). $\mathbf{RM0}_{\to}$ is axiomatized by A1–A4 and A9 (and modus ponens).

DEFINITION 3 (The System $\mathcal{C}\mathbf{C}^{\neg}_{\to}$). *The system $\mathcal{C}\mathbf{C}^{\neg}_{\to}$ is $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ extended with all instances of the weakening* (K) *axiom schema*:

$$\phi \to (\psi \to \phi) \tag{A10}$$

$\mathcal{C}\mathbf{C}^{\neg}_{\to}$ is the implication-negation fragment of Wansing's connexive logic $\mathbf{C}$ (see [21]), which is a connexive variant of a constructive logic, usually called $\mathbf{N4}$, from [1]. That $\mathbf{C}$ is a conservative extension of $\mathcal{C}\mathbf{C}^{\neg}_{\to}$ follows from results in Kamide and Wansing [9]. $\mathcal{C}\mathbf{C}^{\neg}_{\to}$ is a connexive extension of $\mathbf{J}_{\to}$, the pure implicational fragment of intuitionistic logic, which is axiomatized by A1–A4 and A10 (and modus ponens).

Theoremhood with respect to a given system $\mathbf{L}$ ($\vdash_{\mathbf{L}}$) is defined as usual (I will mostly be concerned with theorem schemata and will abuse terminology and notation accordingly).

Although some conservative extension results have already been mentioned, it will be informative (and not too digressive) to quickly show that $\mathcal{C}\mathbf{R}^{\neg}_{\to}$, $\mathcal{C}\mathbf{M}^{\neg}_{\to}$, and $\mathcal{C}\mathbf{C}^{\neg}_{\to}$ conservatively extend $\mathbf{R}_{\to}$, $\mathbf{RM0}_{\to}$, and $\mathbf{J}_{\to}$, respectively. Where $\mathcal{L}_{\to}$ is $\mathcal{L}$ excluding $\neg$ and Φ and $\Phi_{\to}$ are, respectively, the sets of all formulae in $\mathcal{L}$ and $\mathcal{L}_{\to}$, a system $\mathbf{L}$ formulated in $\mathcal{L}$ is said to *conservatively extend* a system $\mathbf{L}'$ formulated in $\mathcal{L}_{\to}$ if, for any $\phi \in \Phi_{\to}$, $\vdash_{\mathbf{L}} \phi$ if and only if $\vdash_{\mathbf{L}'} \phi$.

To prove the conservative extension results, I will exploit a sort of 'erasure' translation (such translations are useful for obtaining certain results in modal logic [5]). The translation $\varepsilon : \Phi \to \Phi_{\to}$ is given as follows:

1. $\varepsilon(p) = p$
2. $\varepsilon(\neg\phi) = \varepsilon(\phi)$
3. $\varepsilon(\phi \to \psi) = \varepsilon(\phi) \to \varepsilon(\psi)$

LEMMA 1 (Fixed Points). *For any $\phi \in \Phi_{\to}$, $\varepsilon(\phi) = \phi$.*

Proof. Trivial, by induction. □

LEMMA 2. *For any $\phi \in \Phi$:*

1. *If $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\to}} \phi$, then $\vdash_{\mathbf{R}_{\to}} \varepsilon(\phi)$*
2. *If $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\to}} \phi$, then $\vdash_{\mathbf{RM0}_{\to}} \varepsilon(\phi)$*
3. *If $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\to}} \phi$, then $\vdash_{\mathbf{J}_{\to}} \varepsilon(\phi)$*

Proof. By induction on the length of proof. □

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THEOREM 1 (Conservative Extension). $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, *and* $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ *conservatively extend* $\mathbf{R}_{\rightarrow}$, $\mathbf{RM0}_{\rightarrow}$, *and* $\mathbf{J}_{\rightarrow}$, *respectively.*

Proof. The argument is essentially the same in each case. Fix $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ for concreteness and let $\phi \in \Phi_{\rightarrow}$. It is obvious from Definition 2 and the axiomatization of $\mathbf{RM0}_{\rightarrow}$ that if $\vdash_{\mathbf{RM0}_{\rightarrow}} \phi$, then $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$. Conversely, suppose that $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$. By Lemma 2, $\vdash_{\mathbf{RM0}_{\rightarrow}} \varepsilon(\phi)$, hence by Lemma 1, $\vdash_{\mathbf{RM0}_{\rightarrow}} \phi$, which suffices. □

I have described $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, and $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ as connexive extensions of $\mathbf{R}_{\rightarrow}$, $\mathbf{RM0}_{\rightarrow}$, and $\mathbf{J}_{\rightarrow}$, respectively. Theorem 1 precisifies the sense in which they are extensions. I turn now to establishing their connexive credentials.

DEFINITION 4 (Connexivity). *A system of logic* $\mathbf{L}$ *is connexive if each of Aristotle's and Boethius' theses are theorems.*[5]

PROPOSITION 1. $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, *and* $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ *are connexive.*

Proof. It is trivial to show that the system $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ (and hence its extensions) contains each of Aristotle 1, Aristotle 2, and Boethius 2. For Boethius 1, the following axiomatic proof suffices:

1	$(\phi \rightarrow \neg\neg\psi) \rightarrow \neg(\phi \rightarrow \neg\psi)$	A8
2	$((\phi \rightarrow \neg\neg\psi) \rightarrow \neg(\phi \rightarrow \neg\psi)) \rightarrow [((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \neg\neg\psi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg\psi))]$	A2
3	$((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \neg\neg\psi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg\psi))$	MP 1, 2
4	$\psi \rightarrow \neg\neg\psi$	A5
5	$(\psi \rightarrow \neg\neg\psi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \neg\neg\psi))$	A2
6	$(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \neg\neg\psi)$	MP 4, 5
7	$(\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg\psi)$	MP 3, 6

□

Recall that a system of logic $\mathbf{L}$ is said to be *Post consistent* if there is a formula ϕ such that $\nvdash_{\mathbf{L}} \phi$ and *negation consistent* if there is no formula ϕ such that $\vdash_{\mathbf{L}} \phi$ and $\vdash_{\mathbf{L}} \neg\phi$. Then $\mathbf{L}$ is Post (negation) inconsistent if it is not Post (negation) consistent.

It is not hard to see that none of the systems examined here are Post inconsistent.[6] On the other hand, it is well-known that Wansing's $\mathbf{C}$ is negation inconsistent, although, in the literature, examples of its negation inconsistent theorems typically employ conjunction (see, e.g., Priest [16, p. 179]). In fact, it was already observed by McCall in [2, sec. 29.8.1, p. 436] (cf. [11, pp. 416–417]) that, even in an implication-negation language, the addition of connexive theses to very weak theories of implication results in negation inconsistency. Using a pair of formulae identified by McCall, I give a rigorous demonstration that the systems examined in this article are negation inconsistent:

[5] Kapsner [10, p. 143] distinguishes between *weak connexivity* (what I am calling "connexivity") and *strong connexivity*, which involves further semantic criteria. None of the systems investigated in this article are strongly connexive. It is also typically required that $(\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \phi)$ *not* be a theorem for $\mathbf{L}$ to be a connexive logic (see, e.g., [11, p. 417]). In fact, it will be shown that this equivalence thesis fails for all of the systems discussed here (see Proposition 6).

[6] This can just be established as a corollary of Lemma 2, observing that $\varepsilon(p) = p$ is not a theorem of any of the pure implicational systems, and hence cannot be a theorem of any of the connexive systems either.

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PROPOSITION 2. *$\mathcal{CR}^{\neg}_{\rightarrow}$, $\mathcal{CM}^{\neg}_{\rightarrow}$, and $\mathcal{CC}^{\neg}_{\rightarrow}$ are negation inconsistent.*

Proof. It suffices to examine $\mathcal{CR}^{\neg}_{\rightarrow}$. Clearly, by Aristotle 1, $\vdash_{\mathcal{CR}^{\neg}_{\rightarrow}} \neg(((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi))$. The following axiomatic proof suffices to show that $((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi)$ is also provable:

1	$(\neg\phi \rightarrow \phi) \rightarrow (\neg\phi \rightarrow \phi)$	A1
2	$((\neg\phi \rightarrow \phi) \rightarrow (\neg\phi \rightarrow \phi)) \rightarrow (\neg\phi \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi))$	A4
3	$\neg\phi \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi)$	MP 1, 2
4	$(\neg\phi \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi)) \rightarrow [((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi))]$	A2
5	$((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi))$	MP 3, 4
6	$((\neg\phi \rightarrow \phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi)) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi)$	A3
7	$(6) \rightarrow [(5) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi))]$	A2
8	$(5) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi))$	MP 6, 7
9	$((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \phi)$	MP 5, 8
10	$\phi \rightarrow \neg\neg\phi$	A5
11	$(\phi \rightarrow \neg\neg\phi) \rightarrow [((\neg\phi \rightarrow \phi) \rightarrow \phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi)]$	A2
12	$((\neg\phi \rightarrow \phi) \rightarrow \phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi)$	MP 10, 11
13	$(12) \rightarrow [(9) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi))]$	A2
14	$(9) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi))$	MP 12, 13
15	$((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow ((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi)$	MP 9, 14
16	$((\neg\phi \rightarrow \phi) \rightarrow \neg\neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi)$	A8
17	$(16) \rightarrow [(15) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi))]$	A2
18	$(15) \rightarrow (((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi))$	MP 16, 17
19	$((\neg\phi \rightarrow \phi) \rightarrow \neg\phi) \rightarrow \neg((\neg\phi \rightarrow \phi) \rightarrow \neg\phi)$	MP 15, 18

□

In [2, p. 436], McCall identifies contraction (my A3) as the principal culprit; note that A3 is used in line 6 of the axiomatic derivation given above. Disposing of A3 from $\mathcal{CR}^{\neg}_{\rightarrow}$ does indeed yield negation consistency.[7] A philosophically revealing way to see this is as follows: note that every theorem of $\mathcal{CR}^{\neg}_{\rightarrow}$ *sans* A3 is a theorem of classical propositional logic under the translation τ defined thus (let $\leftrightarrow$ be the material biconditional):[8]

1. $\tau(p) = p$
2. $\tau(\neg\phi) = \neg\tau(\phi)$
3. $\tau(\phi \rightarrow \psi) = \tau(\phi) \leftrightarrow \tau(\psi)$

Since classical propositional logic is negation consistent, $\mathcal{CR}^{\neg}_{\rightarrow}$ *sans* A3 must of course be negation consistent as well.

I think this observation is philosophically revealing because it indicates, in my view, why anyone serious about having a connexive theory of *implication* should bite the bullet on negation inconsistency. The cost of negation consistency is that one ends up axiomatizing what is, in effect, a subtheory of the theory of classical material

[7] However, note that merely deleting A3 from $\mathcal{CC}^{\neg}_{\rightarrow}$ is insufficient to obtain negation consistency. One can derive $(\phi \rightarrow \neg\phi) \rightarrow \neg(\phi \rightarrow \neg\phi)$ and its negation in $\mathcal{CC}^{\neg}_{\rightarrow}$ *sans* A3, making use of A10. That the use of A10 is essential is indicated by Proposition 7.

[8] There is a long tradition of biconditional or biconditional-involving interpretations of certain connexive systems; see, e.g., Meyer [12].

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equivalence; the resulting logic has no uniquely implicational theorems. Accordingly, for connexivism to be an interesting thesis about implication, it should be negation inconsistent.[9]

§3. Semantics. In this section, I present and discuss semantics for $\mathcal{CR}^{\neg}_{\rightarrow}$, $\mathcal{CM}^{\neg}_{\rightarrow}$, and $\mathcal{CC}^{\neg}_{\rightarrow}$. The semantics is directly inspired by the semilattice semantics which Urquhart [18–20] developed for relevant logic.[10] The semantics departs from that of Urquhart in being (implicitly) four-valued and employing a special falsity condition for the conditional which has been studied by Wansing [21], Omori [13], and others.[11]

DEFINITION 5 (Semilattice Frame). *A semilattice frame is a structure* $\mathfrak{F} = \langle S, 0, \cup\rangle$ *where* $\langle S, \cup\rangle$ *is a join-semilattice and* $0 \in S$ *is lattice bottom.*

DEFINITION 6 ($\mathcal{CR}^{\neg}_{\rightarrow}$ Model). *A* $\mathcal{CR}^{\neg}_{\rightarrow}$ *model is a structure* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *where* $\mathfrak{F}$ *is a semilattice frame and* $V : \Pi \rightarrow \mathcal{P}(S) \times \mathcal{P}(S)$. *Where* $V(p) = (X, Y)$, *I write* $V^{+}(p) = X$ *and* $V^{-}(p) = Y$. *Given* $\mathfrak{M}$ *and* $x \in S$, *the relations* $\models^{\mathfrak{M}^{+}}_{x}$ *and* $\models^{\mathfrak{M}^{-}}_{x}$ *are defined as follows*:

1. $\models^{\mathfrak{M}^{+}}_{x} p$ *if and only if* $x \in V^{+}(p)$
2. $\models^{\mathfrak{M}^{-}}_{x} p$ *if and only if* $x \in V^{-}(p)$
3. $\models^{\mathfrak{M}^{+}}_{x} \neg\phi$ *if and only if* $\models^{\mathfrak{M}^{-}}_{x} \phi$
4. $\models^{\mathfrak{M}^{-}}_{x} \neg\phi$ *if and only if* $\models^{\mathfrak{M}^{+}}_{x} \phi$
5. $\models^{\mathfrak{M}^{+}}_{x} \phi \rightarrow \psi$ *if and only if for all* $y \in S$, $\not\models^{\mathfrak{M}^{+}}_{y} \phi$ *or* $\models^{\mathfrak{M}^{+}}_{x\cup y} \psi$
6. $\models^{\mathfrak{M}^{-}}_{x} \phi \rightarrow \psi$ *if and only if for all* $y \in S$, $\not\models^{\mathfrak{M}^{+}}_{y} \phi$ *or* $\models^{\mathfrak{M}^{-}}_{x\cup y} \psi$

Urquhart [18, pp. 159–160] presents an *informational* interpretation of the semilattice semantics that runs essentially as follows: given a semilattice frame $\mathfrak{F} = \langle S, 0, \cup\rangle$, the

[9] This point is orthogonal to the main aims of the article, so I have been brief. However, let me add some important qualifications. I do not intend to deny that there might be negation consistent systems intermediate between $\mathcal{CR}^{\neg}_{\rightarrow}$ and its contractionless variant which have distinctly implicational theses, nor that something other than contraction might be given up with the result that both negation consistency and connexivism can be maintained. Nevertheless, I strongly suspect that any such systems will be thoroughly ad hoc. The challenge for the negation consistent connexivist is to present a system strong enough to really be about implication without its being totally artificial.

[10] Wansing [22] examines connexive systems of categorial grammar that are semantically characterized by certain groupoid (semigroup) structures. Although the intended application of the formalisms are quite different (as are the languages), there are important points of contact between the semantics developed in [22] and that developed here, including how certain connectives are interpreted and the general algebraic approach (semilattices are of course special semigroups). A more thorough examination of the relationship between these semantics is warranted, but beyond the scope of the current article. I thank a referee for bringing this article to my attention.

[11] In being implicitly four-valued (more on this anon), the semantics offered here follows the so-called "American plan" (for a discussion of the American and Australian plans, see [17]). Dunn and Restall [6, sec. 3.6] report that Charlwood and Daniels investigated a four-valued semilattice semantics in [4] (thanks to Mike Dunn for the reference). I have not been able to obtain a copy of their unpublished article.

In tweaking the falsity condition for the conditional (but leaving its truth condition the same) the semantics offered here follows the so-called "Bochum plan" (this terminology is due to Estrada-González [7]). A referee points out that various contra-classical logics (beyond connexive logics) can be modeled on the Bochum plan [7, 14].

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elements of S are to be thought of as pieces of information, 0 is the empty or null piece of information, and, given pieces of information x and y, $\cup$ is an operation for combining them into the piece of information $x \cup y$.[12]

The basic idea of the semantics is that pieces of information support (in the terminology of Urquhart [18], 'determine') statements. In Urquhart's presentation of the semantics, a piece of information can either support p or not support p. To put it differently, $V(p)$ is taken to simply be a subset of S, and those states which are in $V(p)$ support p, and those which are not do not.

The semantics for $\mathcal{CR}^{\neg}_{\rightarrow}$ fundamentally breaks with Urquhart's on this point. In a $\mathcal{CR}^{\neg}_{\rightarrow}$ model, V assigns to each variable p two subsets of S. $V^{+}(p)$, the extension of p, is understood to be the set of pieces of information which support the truth of p. $V^{-}(p)$, the antiextension of p, is understood to be the set of pieces of information which support the falsity of p. It is not assumed that $V^{+}(p)$ and $V^{-}(p)$ are disjoint, i.e., that $V^{+}(p) \cap V^{-}(p) = \emptyset$: a contradictory piece of information might support both the truth and falsity of p. It is also not assumed that $V^{+}(p)$ and $V^{-}(p)$ are exhaustive, i.e., that $V^{+}(p) \cup V^{-}(p) = S$: an incomplete piece of information might not support either the truth or falsity of p. The semantics is four-valued in the sense that any piece of information will support the truth of p, the falsity of p, both, or neither.

Once supporting falsity is distinguished from not supporting truth, it is possible to entertain falsity conditions for complex formulae which deviate from simply failing to support truth. The truth condition ($\models^{\mathfrak{M}^{+}}_{x}$) for the conditional given in Definition 6 is exactly as in Urquhart [18, p. 160]. Its intuitive gloss under the informational interpretation is that the truth of a conditional is supported by a piece of information x if, given any piece of information y which supports the truth of the antecedent, the combination of x and y supports the truth of the consequent.

Now, what does it take for a piece of information x to support the falsity ($\models^{\mathfrak{M}^{-}}_{x}$) of a conditional? There are four prima facie reasonable answers:

1. There is some piece of information y which supports the truth of the antecedent and which, when combined with x, does not support the truth of the consequent.
2. There is some piece of information y which supports the truth of the antecedent and which, when combined with x, supports the falsity of the consequent.
3. Given any piece of information y which supports the truth of the antecedent, its combination with x does not support the truth of the consequent.
4. Given any piece of information y which supports the truth of the antecedent, its combination with x supports the falsity of the consequent.

If one is serious about distinguishing falsity and mere untruth, a natural thought is that for a conditional to be *false*, there must be counterexamples. The conditional "if Carol is a swan, then Carol is white" would not be falsified by some incomplete testimony supporting that Carol is a swan but failing to establish the color of Carol's plumage. Therefore, the first option will not do, nor will the third. For it may be that all testimonies as to Carol's swanliness fail to establish the color of her plumage; while

[12] I would be remiss not to point out that Wansing [21, pp. 370–372] has already presented an informational interpretation of **C**. The framework I present here is more general than that of Wansing [21], although it is similar to it both in certain important technical details and broad philosophical motivation.

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this might be a case in which the truth of the conditional is not supported, this would not be a case in which its falsity is supported.[13]

Concerning the remaining options, 2 and 4, both are proper *counterexample conditions*. That is, at least in all nonvacuous cases, for a conditional to be false there must be pieces of information which support the truth of the antecedent and have extensions which support the falsity (not the mere untruth) of the consequent. Where the two options differ is in how many counterexamples are required.

There is at least one compelling reason to think that option 2 is too weak. Under option 2, if inconsistent pieces of information are permitted, as of course they are, these can wreak havoc. To make the point sharply, consider the trivial piece of information, which supports the truth and falsity of everything and contains all other pieces of information. Then every piece of information, no matter how nontrivial, will falsify every conditional: there is a piece of information, the trivial one, which supports the truth of the antecedent and which, when combined with any given piece of information, supports the falsity of the consequent. Therefore, the more stringent condition, which is exactly what is codified in $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ models, is to be preferred.

Further conditions might be imposed on $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ models. Below, I consider two types of heredity condition, both directly inspired by corresponding conditions examined by Urquhart [18].

DEFINITION 7 ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ Model). *A $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ model is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ defined just as in Definition 6 except that $V : \Pi \rightarrow \mathcal{P}(S) \times \mathcal{P}(S)$ is subject to the conditions that*:

1. *$x \in V^{+}(p)$ and $y \in V^{+}(p)$ imply $x \cup y \in V^{+}(p)$*
2. *$x \in V^{-}(p)$ and $y \in V^{-}(p)$ imply $x \cup y \in V^{-}(p)$*

PROPOSITION 3 ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ Heredity). *For any $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ model $\mathfrak{M} = \langle S, 0, \cup, V \rangle$ and any $x, y \in S$*:

1. *$\models^{\mathfrak{M}^{+}}_{x} \phi$ and $\models^{\mathfrak{M}^{+}}_{y} \phi$ imply $\models^{\mathfrak{M}^{+}}_{x \cup y} \phi$*
2. *$\models^{\mathfrak{M}^{-}}_{x} \phi$ and $\models^{\mathfrak{M}^{-}}_{y} \phi$ imply $\models^{\mathfrak{M}^{-}}_{x \cup y} \phi$*

Proof. By routine induction on the complexity of ϕ. I briefly examine the first case for ϕ a conditional $\psi \rightarrow \theta$. Suppose that the result holds for ψ and θ, that $\models^{\mathfrak{M}^{+}}_{x} \psi \rightarrow \theta$ and $\models^{\mathfrak{M}^{+}}_{y} \psi \rightarrow \theta$, and, for contradiction, that $\not\models^{\mathfrak{M}^{+}}_{x \cup y} \psi \rightarrow \theta$. Then, for some $z \in S$, $\models^{\mathfrak{M}^{+}}_{z} \psi$ and $\not\models^{\mathfrak{M}^{+}}_{x \cup y \cup z} \theta$. By the assumptions that $\models^{\mathfrak{M}^{+}}_{x} \psi \rightarrow \theta$ and $\models^{\mathfrak{M}^{+}}_{y} \psi \rightarrow \theta$, it follows that $\models^{\mathfrak{M}^{+}}_{x \cup z} \theta$ and $\models^{\mathfrak{M}^{+}}_{y \cup z} \theta$; hence, by the first induction hypothesis, $\models^{\mathfrak{M}^{+}}_{x \cup y \cup z} \theta$, which is a contradiction. □

DEFINITION 8 ($\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ Model). *A $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ model is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ defined just as in Definition 6 except that $V : \Pi \rightarrow \mathcal{P}(S) \times \mathcal{P}(S)$ is subject to the conditions that*:

1. *$x \in V^{+}(p)$ implies $x \cup y \in V^{+}(p)$*
2. *$x \in V^{-}(p)$ implies $x \cup y \in V^{-}(p)$*

[13] To put the objection slightly more formally, the null piece of information (say) would not support the falsity of the pertinent conditional even if, given any piece of information y which supports that Carol is a swan, the combination of the null piece of information with y (that is, simply, y) failed to support that her plumage is white.

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PROPOSITION 4 ($\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ Heredity). *For any* $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ *model* $\mathfrak{M} = \langle S, 0, \cup, V\rangle$ *and any* $x, y \in S$:

1. $\models^{\mathfrak{M}^+}_{x} \phi$ *implies* $\models^{\mathfrak{M}^+}_{x\cup y} \phi$
2. $\models^{\mathfrak{M}^-}_{x} \phi$ *implies* $\models^{\mathfrak{M}^-}_{x\cup y} \phi$

Proof. Again, by routine induction. I briefly examine the second case for ϕ a conditional $\psi \rightarrow \theta$. Suppose the result holds for ψ and θ, $\models^{\mathfrak{M}^-}_{x} \psi \rightarrow \theta$, and, for contradiction, $\not\models^{\mathfrak{M}^-}_{x\cup y} \psi \rightarrow \theta$. Then there is a z such that $\models^{\mathfrak{M}^+}_{z} \psi$ and $\not\models^{\mathfrak{M}^-}_{x\cup y\cup z} \theta$. By the assumption that $\models^{\mathfrak{M}^-}_{x} \psi \rightarrow \theta$, $\models^{\mathfrak{M}^-}_{x\cup z} \theta$. Hence, by the second induction hypothesis, $\models^{\mathfrak{M}^-}_{x\cup z\cup y} \theta$, which is impossible. □

The philosophical interpretation of the two kinds of heredity condition is clear. The conditions imposed on $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ models in Definition 7 require that if the pieces of information x and y support the truth (falsity) of p, then their combination, $x \cup y$, supports the truth (falsity) of p as well. The conditions imposed on $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ models in Definition 8 are less stringent: if a piece of information x supports the truth (falsity) of p, then any extension of it supports the truth (falsity) of p.

DEFINITION 9 (Validity). *ϕ is valid in a model* $\mathfrak{M} = \langle S, 0, \cup, V\rangle$ ($\models^{\mathfrak{M}} \phi$) *if and only if* $\models^{\mathfrak{M}^+}_{0} \phi$. *$\phi$ is valid in* $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$), *in symbols* $\models_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$ ($\models_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$, $\models_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi$), *if and only if for every* $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$) *model* $\mathfrak{M}$, $\models^{\mathfrak{M}} \phi$.

§4. Semantic results. In §4.1, I show that $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, and $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ are determined by (i.e., sound and complete with respect to) their eponymous classes of models. I examine some applications of the semantics in §4.2.

4.1. Semantic determination results. Proofs of soundness and completeness for each of $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, and $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ follow. The arguments are, for the most part, straightforward adaptations of proofs from Urquhart [18] (see also [3, sec. 47]).

THEOREM 2 ($\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ Soundness). $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$ *implies* $\models_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$.

Proof. By induction on the length of proof. The only cases of interest are A7 and A8; I examine just A8. Suppose, for contradiction, that $\not\models_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} (\psi \rightarrow \neg\theta) \rightarrow \neg(\psi \rightarrow \theta)$. Then there is a $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ model $\mathfrak{M} = \langle S, 0, \cup, V\rangle$ such that $\not\models^{\mathfrak{M}^+}_{0} (\psi \rightarrow \neg\theta) \rightarrow \neg(\psi \rightarrow \theta)$. Then there are $y, z \in S$ such that $\models^{\mathfrak{M}^+}_{y} \psi \rightarrow \neg\theta$, $\models^{\mathfrak{M}^+}_{z} \psi$, and $\not\models^{\mathfrak{M}^-}_{y\cup z} \theta$. But then $\models^{\mathfrak{M}^+}_{y\cup z} \neg\theta$, that is, $\models^{\mathfrak{M}^-}_{y\cup z} \theta$, which is impossible. □

THEOREM 3 ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ Soundness). $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$ *implies* $\models_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$.

Proof. For the mingle axiom (A9), suppose that $\not\models_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \psi \rightarrow (\psi \rightarrow \psi)$. Then there is a $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ model $\mathfrak{M} = \langle S, 0, \cup, V\rangle$ such that $\not\models^{\mathfrak{M}^+}_{0} \psi \rightarrow (\psi \rightarrow \psi)$. Hence, there are $x, y \in S$ such that $\models^{\mathfrak{M}^+}_{x} \psi$, $\models^{\mathfrak{M}^+}_{y} \psi$, and $\not\models^{\mathfrak{M}^+}_{x\cup y} \psi$. This is impossible by Proposition 3. □

THEOREM 4 ($\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ Soundness). $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi$ *implies* $\models_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi$.

Proof. For the weakening axiom (A10), suppose that $\not\models_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \psi \rightarrow (\theta \rightarrow \psi)$. Then there is a $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ model $\mathfrak{M} = \langle S, 0, \cup, V\rangle$ such that $\not\models^{\mathfrak{M}^+}_{0} \psi \rightarrow (\theta \rightarrow \psi)$. Consequently,

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there are $x, y \in S$ such that $\models^{\mathfrak{M}^+}_x \psi$, $\models^{\mathfrak{M}^+}_y \theta$, and $\not\models^{\mathfrak{M}^+}_{x \cup y} \psi$. This is impossible by Proposition 4. □

I adapt Urquhart's completeness proof from Anderson et al. [3, p. 145] for $\mathbf{R}_{\rightarrow}$ to prove completeness here. Where $\overline{\phi} = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$ is a sequence of formulae, then: $\overline{\phi} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \psi$ if and only if $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi_1 \rightarrow \phi_2 \cdots \rightarrow \phi_n \rightarrow \psi$.[14] In particular, note that $\overline{\emptyset} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \psi$ if and only if $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \psi$. I omit the subscript from $\prec$ except where necessary to disambiguate or important to emphasize. Observe the following facts concerning $\prec$ (see [3, p. 145]):

1. If $\overline{\alpha} \prec \phi$, then $\overline{\alpha}' \prec \phi$ where $\overline{\alpha}'$ is any permutation of $\overline{\alpha}$
2. If $\overline{\alpha} \prec \phi \rightarrow \psi$ and $\overline{\beta} \prec \phi$, then $\overline{\alpha}, \overline{\beta} \prec \psi$
3. If $\overline{\alpha}, \gamma, \gamma, \overline{\beta} \prec \phi$, then $\overline{\alpha}, \gamma, \overline{\beta} \prec \phi$
4. If $\overline{\alpha}, \phi \prec \psi$, then $\overline{\alpha}, (\phi \rightarrow \phi) \rightarrow \phi \prec \psi$

Definition 10 (Canonical Model). *The canonical model for* $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ *is the structure* $\mathfrak{M}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} = \langle S_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}, 0_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}, \cup_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}, V_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \rangle$ *defined as follows*:

1. $S_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ *is the set of all finite sets of formulae* (*in the language* $\mathcal{L}$)
2. $0_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ *is* $\emptyset$
3. $\cup_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ *is* $\cup$ (*i.e., set-theoretic union*)
4. $x \in V^{+}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}(p)$ *if and only if* $\overline{x} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} p$ (*for some repetition-free sequence* $\overline{x}$ *of* x)
5. $x \in V^{-}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}(p)$ *if and only if* $\overline{x} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \neg p$ (*for some repetition-free sequence* $\overline{x}$ *of* x)

It is obvious that $\langle S_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}, 0_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}, \cup_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \rangle$ is a semilattice frame (Definition 5) and that the specification of $V_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ determines a $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ model (Definition 6). It will be shown that conditions (4) and (5) in Definition 10 have a suitable generalization for all formulae ϕ (Lemma 3). Before turning to the proof of that, the reader should note the following definition, again from Urquhart:

$$\Box^k \phi = \begin{cases} \phi & k = 0 \\ (\Box^{k-1}\phi \rightarrow \Box^{k-1}\phi) \rightarrow \Box^{k-1}\phi & k > 0 \end{cases}$$

Lemma 3 (Truth Lemma). *For all* $x \in S_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ *and all formulae* ϕ:

1. $\models^{\mathfrak{M}^{+}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \phi$ *if and only if* $\overline{x} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$
2. $\models^{\mathfrak{M}^{-}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \phi$ *if and only if* $\overline{x} \prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \neg\phi$

Proof. By induction on the complexity of ϕ. The basis case is by Definition 10. For negation: $\models^{\mathfrak{M}^{+}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \neg\psi$ if and only if $\models^{\mathfrak{M}^{-}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \psi$ if and only if (by the second induction hypothesis) $\overline{x} \prec \neg\psi$, as desired. On the other hand, $\models^{\mathfrak{M}^{-}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \neg\psi$ if and only if $\models^{\mathfrak{M}^{+}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}}_x \psi$ if and only if (by the first induction hypothesis) $\overline{x} \prec \psi$. Since $\overline{\emptyset} \prec \psi \rightarrow \neg\neg\psi$ and $\overline{\emptyset} \prec \neg\neg\psi \rightarrow \psi$, it follows by the second fact about $\prec$ that $\overline{x} \prec \psi$ if and only if $\overline{x} \prec \neg\neg\psi$, which was to be proved.

[14] $\phi_1 \rightarrow \phi_2 \cdots \rightarrow \phi_n \rightarrow \psi$ is to be read: $(\phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi) \cdots))$. Note that, in virtue of A4, the ordering of the ϕ_i is immaterial.

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Urquhart's argument suffices to show that $\models_x^{\mathfrak{M}^+_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \psi \rightarrow \theta$ if and only if $\overline{x} \prec \psi \rightarrow \theta$; some simple tweaks to it will suffice to show that $\models_x^{\mathfrak{M}^-_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \psi \rightarrow \theta$ if and only if $\overline{x} \prec \neg(\psi \rightarrow \theta)$. So suppose, for the right-to-left direction, that $\overline{x} \prec \neg(\psi \rightarrow \theta)$ and, for some arbitrary y, $\models_y^{\mathfrak{M}^+_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \psi$. By the first induction hypothesis, $\overline{y} \prec \psi$ and, since $\overline{\emptyset} \prec \neg(\psi \rightarrow \theta) \rightarrow (\psi \rightarrow \neg\theta)$, by the second fact about $\prec$, it must be that $\overline{x} \prec \psi \rightarrow \neg\theta$. Hence, again by the second fact about $\prec$, it follows that $\overline{x}, \overline{y} \prec \neg\theta$. Permuting and contracting as necessary (licensed by other facts about $\prec$), it is clear that $\overline{x \cup y} \prec \neg\theta$, hence, by the second induction hypothesis, $\models_{x\cup y}^{\mathfrak{M}^-_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \theta$. By the arbitrariness of y, the result is obtained.

Conversely, suppose that $\models_x^{\mathfrak{M}^-_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \psi \rightarrow \theta$. Since x is a finite set of formulae, there is a least k such that $\Box^k\psi \notin x$. By the definition of $\mathfrak{M}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$, $y := \{\Box^k\psi\} \in S_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$. Moreover, since $\psi \prec \psi$, it follows from iteration of the fourth fact about $\prec$ that $\overline{y} \prec \psi$. Hence, by the first induction hypothesis, $\models_y^{\mathfrak{M}^+_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \psi$. By the initial supposition, $\models_{x\cup y}^{\mathfrak{M}^-_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \theta$, and hence by the second induction hypothesis, $\overline{x \cup y} \prec \neg\theta$. Since $\Box^k\psi \notin x$, it is clear that $\overline{x} \prec \Box^k\psi \rightarrow \neg\theta$. By suffixing (a theorem of $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$) and the fact that $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \psi \rightarrow \Box^k\psi$, $\overline{\emptyset} \prec (\Box^k\psi \rightarrow \neg\theta) \rightarrow (\psi \rightarrow \neg\theta)$, hence $\overline{x} \prec \psi \rightarrow \neg\theta$ (by the second fact about $\prec$). Since $\overline{\emptyset} \prec (\psi \rightarrow \neg\theta) \rightarrow \neg(\psi \rightarrow \theta)$, again by the second fact about $\prec$, $\overline{x} \prec \neg(\psi \rightarrow \theta)$, which was to be proved. □

THEOREM 5 ($\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ Completeness). $\models_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$ *implies* $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$.

Proof. If $\models_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$, then, in particular, $\models^{\mathfrak{M}_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \phi$, i.e., $\models_{\emptyset}^{\mathfrak{M}^+_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}} \phi$. Hence, by Lemma 3, $\overline{\emptyset} \prec \phi$, that is, $\vdash_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}} \phi$. □

THEOREM 6 ($\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ Completeness). $\models_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$ *implies* $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} \phi$.

Proof. Define the canonical model for $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$, $\mathfrak{M}_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}$, just as in Definition 10, except replacing all occurrences of $\prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ by $\prec_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}$. Lemma 3 goes through as before. It remains to be shown that $\mathfrak{M}_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}$—in particular, $V_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}$—satisfies the constraints imposed in Definition 7.

So suppose that $x \in V^+_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$ and $y \in V^+_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$; then, by the definition of $\mathfrak{M}_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}$, $\overline{x} \prec p$ and $\overline{y} \prec p$. Since $\vdash_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}} p \rightarrow (p \rightarrow p)$, $\overline{\emptyset} \prec p \rightarrow (p \rightarrow p)$, from which it follows (by the second fact about $\prec$) that $\overline{x} \prec p \rightarrow p$ and then that $\overline{x}, \overline{y} \prec p$. Using contraction and permutation as needed, $\overline{x \cup y} \prec p$, hence $x \cup y \in V^+_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$. The argument to show that $x \in V^-_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$ and $y \in V^-_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$ imply that $x \cup y \in V^-_{\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}}(p)$ is essentially the same. Completeness now follows as in the proof of Theorem 5. □

THEOREM 7 ($\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ Completeness). $\models_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi$ *implies* $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi$.

Proof. $\mathfrak{M}_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}}$, the canonical model for $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$, is again defined as in Definition 10, replacing $\prec_{\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}}$ uniformly by $\prec_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}}$. Again, Lemma 3 goes through. It remains to be shown that $\mathfrak{M}_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}}$ meets the conditions imposed in Definition 8.

Observe that if $\overline{\phi} \prec \psi$, then $\overline{\phi}, \gamma \prec \psi$. For if $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \phi_1 \rightarrow \cdots \rightarrow \phi_n \rightarrow \psi$, since, by A10, $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} (\phi_1 \rightarrow \cdots \rightarrow \phi_n \rightarrow \psi) \rightarrow (\gamma \rightarrow (\phi_1 \rightarrow \cdots \rightarrow \phi_n \rightarrow \psi))$, $\vdash_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}} \gamma \rightarrow (\phi_1 \rightarrow \cdots \rightarrow \phi_n \rightarrow \psi)$, i.e., $\overline{\phi}, \gamma \prec \psi$ (by permuting).

So suppose that $x \in V^+_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}}(p)$. Then $\overline{x} \prec p$ and so, by repeated application of the observation, $\overline{x}, \overline{y} \prec p$, for any $y \in S_{\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}}$. Permuting and contracting as needed,

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$\overline{x \cup y} \prec p$, i.e., $x \cup y \in V^{+}_{\mathcal{CC}^{\neg}_{\rightarrow}}(p)$. A similar argument establishes that $x \in V^{-}_{\mathcal{CC}^{\neg}_{\rightarrow}}(p)$ implies $x \cup y \in V^{-}_{\mathcal{CC}^{\neg}_{\rightarrow}}(p)$. Hence, $\mathfrak{M}_{\mathcal{CC}^{\neg}_{\rightarrow}}$ is indeed a $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ model, and completeness follows as before. □

4.2. Applications of the semantics. Below, I consider some technical applications of the semantics. The principal result of this subsection is that $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ satisfy the variable sharing property:

DEFINITION 11 (Variable Sharing Property). *A logic* **L** *has the variable sharing property if* $\vdash_{\mathbf{L}} \phi \rightarrow \psi$ *only if* ϕ *and* ψ *share a propositional variable.*

Proofs of the variable sharing property typically make use of ad hoc matrices. I will instead prove that $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ have the variable sharing property by adapting a semantic technique recently developed in [24].[15] The argument given below follows the structure of the central argument of [24] quite closely, though there are some small but significant differences.

DEFINITION 12 (ω Finite-Cofinite Semilattice Frame). *The* ω *finite-cofinite semilattice frame is the structure* $\mathfrak{F}_{fc} = \langle S_{fc}, 0_{fc}, \cup_{fc} \rangle$ *specified as follows*:

1. S_{fc} *is the set of all finite or cofinite subsets of* ω
2. $0_{fc} = \emptyset$
3. $\cup_{fc}$ *is* $\cup$ (*i.e., set-theoretic union*)

In what follows, I write $\mathbb{FIN}$ for the set of all finite subsets of ω and $\mathbb{COF}$ for the set of all cofinite subsets of ω.

LEMMA 4 *Let* $P = \{p_i, \ldots, p_m\}$ *be a set of propositional variables and let* $\mathfrak{M} = \langle \mathfrak{F}_{fc}, V \rangle$ *be such that, for each* $p \in P$, $V^{+}(p) = V^{-}(p) = \mathbb{FIN}$. *Then for any* ϕ *in the language of* P:

1. $\models^{\mathfrak{M}^{+}}_{x} \phi$ *if and only if* $x \in \mathbb{FIN}$
2. $\models^{\mathfrak{M}^{-}}_{x} \phi$ *if and only if* $x \in \mathbb{FIN}$

Proof. The basis case is by the specification of $\mathfrak{M}$. For $\neg$, $\models^{\mathfrak{M}^{+}}_{x} \neg\psi$ if and only if $\models^{\mathfrak{M}^{-}}_{x} \psi$ if and only if (by the induction hypothesis) $x \in \mathbb{FIN}$. Similarly, $\models^{\mathfrak{M}^{-}}_{x} \neg\psi$ if and only if $\models^{\mathfrak{M}^{+}}_{x} \psi$ if and only if (by the induction hypothesis) $x \in \mathbb{FIN}$.

For $\rightarrow$ (part 1), suppose that $x \in \mathbb{FIN}$ and consider an arbitrary $y \in S_{fc}$ such that $\models^{\mathfrak{M}^{+}}_{y} \psi$. By the first induction hypothesis, $y \in \mathbb{FIN}$. Since it then follows that $x \cup y \in \mathbb{FIN}$, by the first induction hypothesis, $\models^{\mathfrak{M}^{+}}_{x \cup y} \theta$. By the arbitrariness of y, $\models^{\mathfrak{M}^{+}}_{x} \psi \rightarrow \theta$. Conversely, suppose $x \notin \mathbb{FIN}$ (i.e., that $x \in \mathbb{COF}$); then, picking any $y \in \mathbb{FIN}$, it is clear that $\models^{\mathfrak{M}^{+}}_{y} \psi$ (by the first induction hypothesis) and $\not\models^{\mathfrak{M}^{+}}_{x \cup y} \theta$ (by the first induction hypothesis and fact that $x \cup y \notin \mathbb{FIN}$). Hence, $\not\models^{\mathfrak{M}^{+}}_{x} \psi \rightarrow \theta$, as desired.

[15] Note that the variable sharing property for both of these logics can also be proved using Lemma 2 and the (external) results that $\mathbf{R}_{\rightarrow}$ and $\mathbf{RM0}_{\rightarrow}$ have the variable sharing property (see Anderson and Belnap, Jr. [2, secs. 5.1.2 and 8.15]). For if ϕ and ψ have no variables in common, then $\varepsilon(\phi)$ and $\varepsilon(\psi)$ do not either, and so $\varepsilon(\phi \rightarrow \psi) = \varepsilon(\phi) \rightarrow \varepsilon(\psi)$ is a nontheorem of either pure implicational system, from which the result follows. The semantic method is preferable insofar as it does not rely on external results concerning the pure implicational systems and can easily be generalized if further connectives are added.

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For $\rightarrow$ (part 2), suppose that $x \in \mathbb{FIN}$ and consider $y \in S_{fc}$ such that $\models_y^{\mathfrak{M}^+} \psi$. As before, $y \in \mathbb{FIN}$, hence, since $x \cup y \in \mathbb{FIN}$, $\models_{x \cup y}^{\mathfrak{M}^-} \theta$ (by the second induction hypothesis). Thus, $\models_x^{\mathfrak{M}^-} \psi \rightarrow \theta$. Conversely, if $x \notin \mathbb{FIN}$, then, taking any $y \in \mathbb{FIN}$, it is the case that $\models_y^{\mathfrak{M}^+} \psi$ (first induction hypothesis) and $\not\models_{x \cup y}^{\mathfrak{M}^-} \theta$ (second induction hypothesis). Therefore, $\not\models_x^{\mathfrak{M}^-} \psi \rightarrow \theta$, as desired. □

LEMMA 5. *Let $P = \{p_i, \ldots, p_m\}$ be a set of propositional variables and let $\mathfrak{M} = \langle \mathfrak{F}_{fc}, V \rangle$ be such that, for each $p \in P$, $V^+(p) = V^-(p) = \mathbb{COF}$. Then for any ϕ in the language of P:*

1. *If $x \in \mathbb{COF}$, then $\models_x^{\mathfrak{M}^+} \phi$*
2. *If $x \in \mathbb{COF}$, then $\models_x^{\mathfrak{M}^-} \phi$*

Proof. The basis case is by the specification of $\mathfrak{M}$. For $\neg$, suppose that $x \in \mathbb{COF}$. By the induction hypotheses, $\models_x^{\mathfrak{M}^+} \psi$ and $\models_x^{\mathfrak{M}^-} \psi$. Hence, both $\models_x^{\mathfrak{M}^-} \neg\psi$ and $\models_x^{\mathfrak{M}^+} \neg\psi$, as required.

For $\rightarrow$, suppose that $x \in \mathbb{COF}$ and take $y \in S_{fc}$ arbitrary such that $\models_y^{\mathfrak{M}^+} \psi$. Since $x \cup y \in \mathbb{COF}$ (regardless of whether y is itself finite or cofinite), by the induction hypotheses, $\models_{x \cup y}^{\mathfrak{M}^+} \theta$ and $\models_{x \cup y}^{\mathfrak{M}^-} \theta$. Hence, by the arbitrariness of y, $\models_x^{\mathfrak{M}^+} \psi \rightarrow \theta$ and $\models_x^{\mathfrak{M}^-} \psi \rightarrow \theta$, which was to be proved. □

THEOREM 8 (Variable Sharing Property for $\mathcal{CM}^{\neg}_{\rightarrow}$). *$\mathcal{CM}^{\neg}_{\rightarrow}$ has the variable sharing property.*

Proof. Suppose that ϕ and ψ have *no* variables in common. Define $\mathfrak{M} = \langle \mathfrak{F}_{fc}, V \rangle$ as follows. For every variable p in ϕ, put $V^+(p) = V^-(p) = \mathbb{COF}$; for every variable p in ψ, put $V^+(p) = V^-(p) = \mathbb{FIN}$; for any variable p in neither, put $V^+(p) = V^-(p) = \emptyset$. It must be verified that $\mathfrak{M}$ is a $\mathcal{CM}^{\neg}_{\rightarrow}$ model; in particular, it must be shown that it satisfies the heredity conditions imposed in Definition 7.

There are fundamentally three cases to consider: p is a variable in ϕ, p is a variable in ψ, or p is a variable in neither (by hypothesis, p cannot be a variable in both). In the first case, the required conditions hold vacuously unless $x \in \mathbb{COF}$ and $y \in \mathbb{COF}$, in which case $x \cup y \in \mathbb{COF} = V^+(p) = V^-(p)$, as desired. In the second case, again, the required conditions hold vacuously unless $x \in \mathbb{FIN}$ and $y \in \mathbb{FIN}$, in which case $x \cup y \in \mathbb{FIN} = V^+(p) = V^-(p)$, as needed. In the third case, the required conditions simply hold vacuously. Therefore, $\mathfrak{M}$ is a $\mathcal{CM}^{\neg}_{\rightarrow}$ model.

Now, for the proof of the theorem proper, observe that by Lemma 5, $\models_\omega^{\mathfrak{M}^+} \phi$, and by Lemma 4, $\not\models_{\emptyset \cup \omega}^{\mathfrak{M}^+} \psi$. Hence, $\not\models_\emptyset^{\mathfrak{M}^+} \phi \rightarrow \psi$, as desired. □

COROLLARY 1 (Variable Sharing Property for $\mathcal{CR}^{\neg}_{\rightarrow}$). *$\mathcal{CR}^{\neg}_{\rightarrow}$ has the variable sharing property.*

Proof. $\mathcal{CR}^{\neg}_{\rightarrow}$ is a subsystem of $\mathcal{CM}^{\neg}_{\rightarrow}$. Hence, the result is immediate from Theorem 8. □

The following proposition addresses the inclusion relations among the three systems examined in this article.

PROPOSITION 5. $\mathcal{CR}^{\neg}_{\rightarrow} \subset \mathcal{CM}^{\neg}_{\rightarrow} \subset \mathcal{CC}^{\neg}_{\rightarrow}$.

Proof. That $\mathcal{CR}^{\neg}_{\rightarrow} \subseteq \mathcal{CM}^{\neg}_{\rightarrow} \subseteq \mathcal{CC}^{\neg}_{\rightarrow}$ is immediate from their axiomatic specification. By Theorem 8 and the fact that $\vdash_{\mathcal{CC}^{\neg}_{\rightarrow}} p \rightarrow (q \rightarrow q)$, $\mathcal{CM}^{\neg}_{\rightarrow} \neq \mathcal{CC}^{\neg}_{\rightarrow}$. To show that

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$\mathcal{CR}^{\neg}_{\rightarrow} \neq \mathcal{CM}^{\neg}_{\rightarrow}$, it suffices to construct a $\mathcal{CR}^{\neg}_{\rightarrow}$ countermodel to an instance of A9. The interested reader can verify that $p \rightarrow (p \rightarrow p)$ fails to be true in the following model: $\mathfrak{M} = \langle \{1, 2, 3, 6\}, 1, \text{lcm}, V \rangle$ where $V^+(p) = \{2, 3\}$ and otherwise V^+ and V^- are empty. □

The next proposition addresses a further requirement sometimes imposed on connexive logics (see Footnote 5 above).

PROPOSITION 6. *For* $\mathbf{L} \in \{\mathcal{CR}^{\neg}_{\rightarrow}, \mathcal{CM}^{\neg}_{\rightarrow}, \mathcal{CC}^{\neg}_{\rightarrow}\}, \nvdash_{\mathbf{L}} (\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \phi)$.

Proof. It suffices to find a $\mathcal{CC}^{\neg}_{\rightarrow}$ countermodel to the instance $(p \rightarrow q) \rightarrow (q \rightarrow p)$. Consider the structure $\mathfrak{M} = \langle \{0, 1, 2\}, 0, \max, V \rangle$ where $V^+(q) = \{2\}$ and otherwise V^+ and V^- are empty. It is not hard to see that this model meets the conditions imposed in Definition 8. That $\not\models^{\mathfrak{M}} (p \rightarrow q) \rightarrow (q \rightarrow p)$ is easily checked. □

This last proposition substantiates a claim made in Footnote 7 about the independence of $(\phi \rightarrow \neg\phi) \rightarrow \neg(\phi \rightarrow \neg\phi)$ from $\mathcal{CR}^{\neg}_{\rightarrow}$ and $\mathcal{CM}^{\neg}_{\rightarrow}$ (that is, systems lacking A10).

PROPOSITION 7. *For* $\mathbf{L} \in \{\mathcal{CR}^{\neg}_{\rightarrow}, \mathcal{CM}^{\neg}_{\rightarrow}\}, \nvdash_{\mathbf{L}} (\phi \rightarrow \neg\phi) \rightarrow \neg(\phi \rightarrow \neg\phi)$.

Proof. It suffices to find a $\mathcal{CM}^{\neg}_{\rightarrow}$ countermodel to the instance $(p \rightarrow \neg p) \rightarrow \neg(p \rightarrow \neg p)$. Consider the structure $\mathfrak{M} = \langle \{1, 2, 3, 6\}, 1, \text{lcm}, V \rangle$ where $V^+(p) = \{3\}$, $V^-(p) = \{6\}$, and otherwise V^+ and V^- are empty. Observe that this structure meets the heredity conditions imposed in Definition 7 (though, n.b., not those imposed in Definition 8). It is left to the reader to verify that $\not\models^{\mathfrak{M}} (p \rightarrow \neg p) \rightarrow \neg(p \rightarrow \neg p)$. □

§5. Concluding remarks. In this article, I developed a simple semantics for three pure theories of connexive implication. The semantics, which takes its primary inspiration from the work of Urquhart, shows that these three systems arise from natural variations of constraints on models. All of the systems examined are demonstrably connexive, and two of them have the variable sharing property (a standard necessary condition on a system's being a relevant logic). $\mathcal{CR}^{\neg}_{\rightarrow}$ and $\mathcal{CM}^{\neg}_{\rightarrow}$ are both quite plausible pure theories of relevant connexive implication and merit further investigation.

This article has left many avenues open for further exploration. I note just a couple of these here. A wide array of relevant implicational logics have been examined using the semilattice semantics in [3, 18]. Adding further structure to semilattice frames and modifying the truth conditions accordingly allows for the implicational fragments of (for example) **T** and **E** to be characterized. It should be possible to develop natural theories of connexive *entailment* by modifying Urquhart's semantics for those systems.

Another avenue to explore concerns augmentations of these pure theories in fuller languages. It was observed at the beginning of the article that Wansing's **C** is a conservative extension of $\mathcal{CC}^{\neg}_{\rightarrow}$. What do some of the logics which conservatively extend $\mathcal{CR}^{\neg}_{\rightarrow}$ and $\mathcal{CM}^{\neg}_{\rightarrow}$ with axioms for extensional connectives like $\wedge$ and $\vee$ look like? It would be interesting to find out.

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