103.13 Squaring the circle by using proportion

Introduction

The task of constructing a square equal in area to a given circle is called squaring the circle. This was one of the three unsolvable problems of antiquity. The geometrical construction of this square must be realised by using only a straight-edge and compass. The construction should be completed in a finite number of steps. It was proved to be an impossible problem as a consequence of the fact that the number π is transcendental. A German mathematician von Lindemann in 1882 [1] has shown that the mathematical constant is not an algebraic number.

The traditional and well-known approach to this problem is to use ruler-and-compass construction to build approximations to $\sqrt{\pi}$. The number can be approximated by various representations, such as $\sqrt{2} + \sqrt{3}$, $\sqrt{10}$, $22/7$, $355/113$ or other higher accuracy rational approximations. In such an approach we construct the value of the approximation of the number with a given accuracy. The constructed value is used in geometrical determination of the side of the square and involves the construction of $\sqrt{\pi}$. This technique resolves the problem of the squaring for a given specific circle.

An extension of Ahmes' method

In this short Note we propose a method based on a proportional division of a radius of a circle. Here we do not construct the values of the approximations of the number π as in the standard approaches. The presented method can be seen as a more accurate extension of the Ahmes' method noted on an ancient Egyptian papyrus. The scribe Ahmes to obtain the area of a circle used the following rule (Problem 50, [2]): "The area of a circle with diameter d is the area of a square with the side $8d/9$." Thus his method divides the diameter in the proportion 8:1. Here we propose to improve the precision of this proportion. For any given accuracy of the number π, it is possible to realise the corresponding squaring the circle using Ahmes' algorithm. We take an estimation of the number π and we perform the quadrature or the rectification of the circle. The presented method bases on Thales' basic proportionality theorem. This theorem provides the evidence that we always can divide a given segment into a predefined proportion. (Euclid, Book VI, Proposition 2: "If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle." [3]). Here we used it to divide the radius into the required proportion.

To simplify the description and to fix our attention we consider a unit circle. Its radius is 1 and the area is π. We can construct a good approximating square with any required level of accuracy. It is easy to determine that the side of the square with the same area as the circle divides

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the radius of the circle in the proportion

$$\frac{\sqrt{\pi}}{2} : 1 - \frac{\sqrt{\pi}}{2}.$$

We denote this proportion by the letter p. After a simple calculation we have

$$p = \frac{\sqrt{\pi}}{2 - \sqrt{\pi}}.$$

To represent the value of the proportion p ($p = 7.78942582838235\ldots$), where Ahmes used 8, we use the continued fraction to generate the best rational approximations. For illustrative purposes we use only 6 terms, and we have $p = [7; 1, 3, 1, 2, 1]$. The corresponding convergents are as follows: numerator = {7, 8, 31, 39, 109, 148} and denominator = {1; 1, 4, 5, 14, 19}. In our example for the last term we have

$$p = \frac{148}{19} = 7.78947368421053\ldots .$$

Using Thales' theorem, we divide the radius of the circle in the proportion 148 : 19 to determine the side of the square. This quadrature results with accuracy of the number $3.14159704543010\ldots$ (compared to the real value $\pi = 3.14159265358979\ldots$). Using more terms in the continued fractions to represent the proportion p we can realise squaring the circle with arbitrary precision.

A similar approach can be applied to rectification of the circle. The rectification of the circle is finding the length of its circumference. Here, we determine the side of a square with the length of the perimeter equal to the circumference. Again, we define the corresponding proportion as

$$\frac{\pi}{4} : 1 - \frac{\pi}{4}$$

and denote it by the letter c. We obtain

$$c = \frac{\pi}{4 - \pi}.$$

In this case we have $c = 3.65979236632549\ldots$. The applied continued fraction approximation generates the following relations:

$$c = [3; 1, 1, 1, 15, 2], \text{ the numerator } \{3, 4, 7, 11, 172, 355\}$$
$$\text{and denominator } \{1, 1, 2, 3, 47, 97\}.$$

Using the last, sixth term of the approximation, we have

$$c = \frac{355}{97} = 3.65979381443299\ldots .$$

The presented material provides the evidence that squaring of the circle using the modified Ahmes' method can be done for any accuracy. We can always divide a radius in a given proportion. Of course, for the large values of the numerators and denominators it is difficult to realise the proposed geometrical construction. This Note mainly shows that the precision of the

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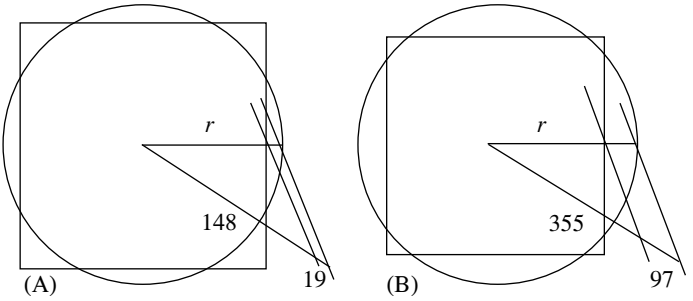


FIGURE 1: Division of the radius r of a circle in proportion to obtain (approximation) the square of the same area (A) and of the same length of the perimeter and circumference (B)

estimation of the number π and its arithmetic square root can be mapped into geometrical construction. The approach proposed here can be performed even using the so called 'rusty compass' method, i.e. the construction can be realised with one opening of the compass. With one opening we can divide any segment into the required proportion. For example, we can mark 148 units followed by 19 on any line from the centre of the circle, and the obtained proportion we can map on the radius. Another approach to improve accuracy of Ahmes' method is based on Bing's set square, [4]. The interested reader may consult another technique to construct regular polygons to approximate the number π [5].

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