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ARTICLE

A stability theorem for multi-partite graphs

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Abstract
The Erdős–Simonovits stability theorem is one of the most widely used theorems in extremal graph theory. We obtain an Erdős–Simonovits type stability theorem in multi-partite graphs. Different from the Erdős–Simonovits stability theorem, our stability theorem in multi-partite graphs says that if the number of edges of an H-free graph G is close to the extremal graphs for H, then G has a well-defined structure but may be far away from the extremal graphs for H. As applications, we strengthen a theorem of Bollobás, Erdős, and Straus and solve a conjecture in a stronger form posed by Han and Zhao concerning the maximum number of edges in multi-partite graphs which does not contain vertex-disjoint copies of a clique.

Keywords: Stability; Turán number; disjoint cliques; multipartite graph
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1. Introduction

Given graphs G and F, if G does not contain a copy of F, then we say that G is F-free. We call an n-vertex F-free graph with maximum number of edges an extremal graph for F. Turán [22] showed that the n-vertex complete $(t-1)$-partite graph with part sizes as equal as possible, denoted by $T(n, t-1)$, is the unique extremal graph for K_t, where K_t is the complete graph on t vertices. Over the following decades, determining the extremal graphs for a given graph (possibly under additional constraints) has developed into one of the central topics in extremal combinatorics.

In this paper we study a multi-partite version of Turán's problem: we consider F-free subgraphs of $K_{n_1,\dots,n_r}$, the complete r-partite graph with parts of sizes $n_1, \dots, n_r$ and $n_1 \geq \cdots \geq n_r$. Denote by $\mathrm{ex}(n_1, \dots, n_r, F)$ the maximum number of edges in an F-free subgraph of $K_{n_1,\dots,n_r}$. Throughout the paper, we write $\mathcal{V} = (V_1, \dots, V_r)$ for the vertex partition of $K_{n_1,\dots,n_r}$, where $|V_i| = n_i$ for each i.

Research on extremal problems in multi-partite graphs can be traced back to the work of Zarankiewicz [23] on $\mathrm{ex}(n, n, K_{2,2})$. Later, Kövári, Sós, and Turán [15] gave an upper bound of $\mathrm{ex}(n, n, K_{s,t})$. The above two results are strongly connected to the Turán numbers of bipartite graphs. Very recently, based a quantitative variant of the random algebraic method, Conlon [5] gave good lower bounds for $\mathrm{ex}(n, m, K_{s,t})$ when n, m satisfy some additional conditions. For related topics of the Zarankiewicz problems, we refer the interested readers to [5] and references therein.

Beyond bipartite settings, extremal problems in multi-partite graphs have been extensively studied from both edge and degree perspectives. Bollobás, Erdős, and Szemerédi [2] showed that if the minimum degree in a 3-partite graph with sizes n, n, n is at least $n+1$, then it contains at least $\min\{4, n\}$ triangles. This problem has since been thoroughly explored, often through its equivalent formulation in terms of *independent transversals*; see [12–14, 21].

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We now turn to edge extremal problems for non-bipartite graphs. Bollobás, Erdős, and Straus [1] determined $\mathrm{ex}(n_1, \dots, n_r; K_t)$ for $r \geq t$. A natural candidate for extremal graphs is to take a partition of the r parts into $(t-1)$ groups and consider the induced complete $(t-1)$-partite subgraph. This motivates the following definition.

Let G be a graph with a vertex partition $\mathcal{P} = (P_1, \dots, P_{t-1})$. We define $G[\mathcal{P}]$ to be the spanning subgraph obtained by removing all edges within the parts of $\mathcal{P}$; that is,

$$E\big(G[\mathcal{P}]\big) = E(G) \setminus \bigcup_{i=1}^{t-1} \binom{P_i}{2}.$$

Equivalently, $G[\mathcal{P}]$ is the intersection of G with the complete $(t-1)$-partite graph on $V(G)$ with parts $\mathcal{P}$.

Now fix integers $r \geq t \geq 3$, and let $\mathcal{V} = (V_1, \dots, V_r)$ be the partition of the vertex set of the complete r-partite graph $K_{n_1,\dots,n_r}$, with $|V_i| = n_i$ and $n_1 \geq \cdots \geq n_r$. Given a $(t-1)$-partition $\mathcal{Q} = (I_1, \dots, I_{t-1})$ of the index set $[r] := \{1, \dots, r\}$, we write $\mathcal{P}_\mathcal{Q} = (V_{I_1}, \dots, V_{I_{t-1}})$, where $V_I := \bigcup_{i \in I} V_i$. Then $K_{n_1,\dots,n_r}[\mathcal{P}_\mathcal{Q}]$ denotes the complete $(t-1)$-partite spanning subgraph formed this way.

We define

$$f(n_1, \dots, n_r, 1, t) := \max_{\mathcal{Q}} e(K_{n_1,\dots,n_r}[\mathcal{P}_\mathcal{Q}]),$$

where the maximum is taken over all partitions $\mathcal{Q}$ of $[r]$ into $(t-1)$ parts. For example, $f(n+1, n, n, 1, 3) = (n+1) \times 2n$. This is achieved by the partition $\mathcal{Q} = \big(\{1\}, \{2, 3\}\big)$, where the subgraph $K_{n+1,n+n}$ has $(n+1) \times 2n$ edges.

Bollobás, Erdős, and Straus [1] showed that these constructions are optimal for avoiding K_t in the multi-partite setting.

Theorem 1.1 (Bollobás, Erdős, and Straus [1]). *Let $r \geq t \geq 3$. Then*

$$ex(n_1, \dots, n_r, K_t) = f(n_1, \dots, n_r, 1, t).$$

Any partition $\mathcal{P}$ of $V(K_{n_1,\dots,n_r})$ (it is possible that $\mathcal{P} \neq \mathcal{P}_\mathcal{Q}$ for any $(t-1)$-partition $\mathcal{Q}$ of $[r]$, see Theorem 1.12) achieving the maximum in Theorem 1.1 is called an *extremal $(t-1)$-partition*.

Our first theorem shows that the extremal graphs in Theorem 1.1 are all $(t-1)$-partite.

Theorem 1.2. *Let $r \geq t$ and $n_1, \dots, n_{t-1}$ be sufficiently large. Each extremal graph in Theorem 1.1 is of form $K_{n_1,\dots,n_r}[\mathcal{P}]$, where $\mathcal{P}$ is an extremal $(t-1)$-partition for $K_{n_1,\dots,n_r}$.*

In Section 1.1, we will provide a precise structural characterization of the extremal graphs in the Bollobás–Erdős–Straus theorem (see Theorem 1.12).

We will also consider the extremal problem for kK_t, the disjoint union of k copies of K_t. Chen, Li, and Tu [4] determined the value of $\mathrm{ex}(n_1, n_2, kK_2)$. Later, De Silva, Heysse, and Young [20] extended this to the general problem of determining $\mathrm{ex}(n_1, \dots, n_r, kK_t)$ for $r > t$. To prove a lower bound on $\mathrm{ex}(n_1, \dots, n_r, kK_t)$ for $r > t$, Han and Zhao [11] constructed a kK_t-free graph by starting from $K_{n_1,\dots,n_r}[\mathcal{P}_\mathcal{Q}]$ and adding all possible edges incident to $k-1$ fixed vertices in some part V_i, where $\mathcal{Q}$ is a $(t-1)$-partition of $[r]$. They showed that this construction is optimal for $\mathrm{ex}(n_1, n_2, n_3, n_4, kK_3)$ when n_4 is sufficiently large.

For any $I \subseteq [r]$, define $m_I = \min_{i \in I} n_i$, $n_I = \sum_{i \in I} n_i$. For a partition $\mathcal{Q}$ of $[r]$, let $n_\mathcal{Q} = \max_{I \in \mathcal{Q}} (n_I - m_I)$. Define

$$f(n_1, \dots, n_r, k, t) := \max_{\mathcal{Q}} \left\{ (k-1)n_\mathcal{Q} + \sum_{I \neq I' \in \mathcal{Q}} n_I n_{I'} \right\},$$

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where the maximum is over all partitions $\mathcal{Q}$ of $[r]$ into $t-1$ parts.

Conjecture 1.3. *Let $r \geq t \geq 3$ and $k \geq 2$, and suppose $n_1, \ldots, n_r$ are sufficiently large. Then*

$$\mathrm{ex}(n_1, \ldots, n_r, kK_t) = f(n_1, \ldots, n_r, k, t).$$

Note that if $n_i < k-1$ for some $i \in [r]$, then it is possible that there is no graph corresponding to $f(n_1, \ldots, n_r, k, t)$. Hence, we introduce the following family of graphs. Let $\mathcal{H}$ be the set of all $(t-1)$-partite subgraphs of $K_{n_1,\ldots,n_r}$. Let $\mathcal{H}^{k-1}$ be the family of graphs obtained from some $H \in \mathcal{H}$ by adding all edges incident to $k-1$ fixed vertices. Define

$$g(n_1, \ldots, n_r, k, t) := \max\{e(F) : F \in \mathcal{H}^{k-1}\}.$$

We will prove Han and Zhao's conjecture in the following stronger[1] form.

Theorem 1.4. *Let $r \geq t \geq 3$ and $k \geq 2$, and $n_1, \ldots, n_{t-1}$ are sufficiently large. Then*

$$\mathrm{ex}(n_1, \ldots, n_r, kK_t) = g(n_1, \ldots, n_r, k, t).$$

Moreover, every extremal graph is obtained from $K_{n_1,\ldots,n_r}[\mathcal{P}]$ by adding all edges incident to $k-1$ fixed vertices for some $(t-1)$-partition $\mathcal{P}$.

Theorems 1.2 and 1.4 follow from our stability theorem (see Theorem 1.17 later).

Remark 1.5. In Theorem 1.4, some extremal graphs for kK_t are not obtained by adding edges to an extremal $(t-1)$-partition. For example, let $K = K_{n_1,\ldots,n_6}$ with parts $V_1, \ldots, V_6$, where $n_1 = n_2 = m+1$ and $n_3 = \cdots = n_6 = m$ for large m. One extremal graph for $3K_3$ is obtained using the partition $\mathcal{P} = (V_1 \cup V_2 \cup V_3, V_4 \cup V_5 \cup V_6)$, which is not an extremal 2-partition.

1.1 Near-extremal configurations

To state Theorem 1.17, we first describe the structure of graphs that are close to extremal.

We begin by presenting an instructive phenomenon: in a multi-partite graph $K_{n_1,\ldots,n_r}$, an H-free subgraph G that is nearly extremal in terms of edge count may still differ significantly from any extremal graph for H. Such a graph G may closely resemble a structured form like $K_{n_1,\ldots,n_r}[\mathcal{P}]$, where $\mathcal{P}$ is a partition with a certain recognisable pattern – but the structure of $\mathcal{P}$ may not close any extremal configuration.

To illustrate this phenomenon, we consider the case $H = K_3$ and use the multi-partite host graph $K_{n+1,n,n}$ as an example; see Example 1.6 and Figure 1.

Example 1.6. *Let n be a positive integer. Consider the host graph $K_{n+1,n,n}$ with parts (V_1, V_2, V_3). By Theorem 1.12, the subgraph $K_{n+1,n,n}[(V_1, V_2 \cup V_3)]$ is the unique edge-maximum K_3-free subgraph of $K_{n+1,n,n}$. However, for any partition $V_3 = V_3^1 \cup V_3^2$, the graph $K_{n+1,n,n}\big[\big(V_1 \cup V_3^1, V_2 \cup V_3^2\big)\big]$ is also K_3-free, and its number of edges is close to that of $K_{n+1,n,n}[(V_1, V_2 \cup V_3)]$.*

In the near-extremal[2] partition $\mathcal{P} = (V_1 \cup V_3^1, V_2 \cup V_3^2)$ from Example 1.6, the set V_3 is split across both parts of $\mathcal{P}$, whereas V_1 and V_2 lie entirely within one part. To describe such behaviour in general, we now introduce the notions of *partial* and *integral* sets with respect to a partition.

We say that a vertex partition of a graph is *proper* if there are no edges within any partite set. Let $\mathcal{V} = (V_1, \ldots, V_r)$ be a fixed proper vertex partition of a graph G, and let $\mathcal{P} = (P_1, \ldots, P_{t-1})$ be a vertex partition of a subset of $V(G)$. Given a pair $(\mathcal{V}, \mathcal{P})$,[3] we make the following definitions:

Definition 1.7 (partial and integral). *We say that a part V_i is integral in P_j (or simply integral) if $V_i \subseteq P_j$, and partial in P_j (or simply partial) if $V_i \cap P_j \neq \emptyset$ but $V_i \not\subseteq P_j$.*

[1] We only require n_{t-1} to be large, and we also characterise all extremal graphs.
[2] In the sense of the number of edges of $K_{n+1,n,n}[\mathcal{P}]$.
[3] Throughout the paper, we always assume that $\mathcal{V}$ is a proper partition.

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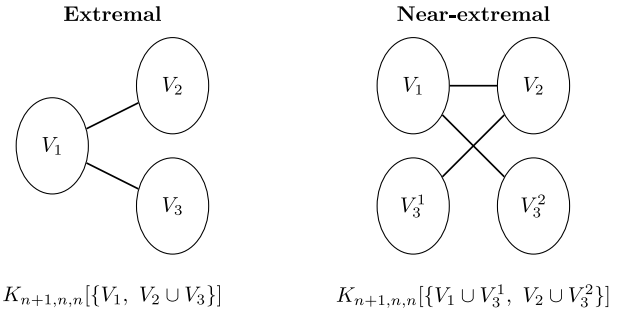


Figure 1. Illustration of Example 1.6.

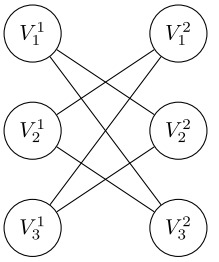


Figure 2. A hypothetical configuration with multiple partial sets.

Definition 1.8 (partial and integral part). *For each $P_j \in \mathcal{P}$, define the integral part of P_j, denoted $I_{\mathcal{V}}(P_j)$, to be the union of all V_i such that $V_i \subseteq P_j$. Define the partial part, denoted $P_{\mathcal{V}}(P_j)$, to be the union of all non-empty intersections $V_i \cap P_j$ for which $V_i \not\subseteq P_j$.*

When the reference partition $\mathcal{V}$ is clear from context, we omit the subscript. We say that P_j is *integral* with respect to $\mathcal{V}$ (or simply *integral*) if $I(P_j) = P_j$, and *partial* if $P(P_j) \neq \emptyset$.

For the pair $(\mathcal{V}, \mathcal{P}) = ((V_1, V_2, V_3), (V_1 \cup V_3^1,\ V_2 \cup V_3^2))$ in Example 1.6, the sets V_1 and V_2 are integral in their respective parts of the partition, while V_3 is partial in both.

This example suggests a typical near-extremal configuration, where most partite sets are entirely contained in one part (integral), and only a small portion (e.g., V_3) is split (partial). A natural question is whether all near-extremal configurations follow this structure, or whether more complex patterns with many partial sets can also occur – such as the one illustrated in Figure 2, where $V_i = V_i^1 \cup V_i^2$ for each $i \in [3]$.

While Figure 2 illustrates a hypothetical configuration with many partial parts, such complexity is rarely seen in actual near-extremal graphs. In practice, even though several partial sets may appear globally, the structure tends to be more controlled: each part typically contains at most one partial V_i. This observation motivates the following definition, which captures the structured nature of near-extremal configurations.

Definition 1.9 (1-partial). *A partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ is said to be 1-partial to $\mathcal{V} = (V_1, \ldots, V_r)$ (or simply 1-partial) if each part P_j contains at most one set V_i that is partial in P_j.*

Beyond this structural simplicity, we also observe a balance in sizes: integral parts tend to be of similar size, and partial sets are generally no larger. In other words, near-extremal partitions are not only simple in structure but also well balanced in size.

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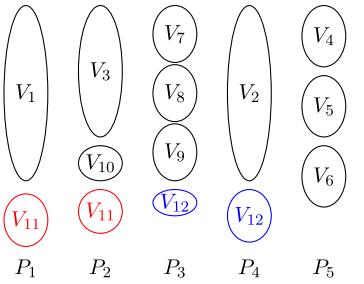


Figure 3. A partition $\mathcal{P} = (P_1, \ldots, P_5)$ that is stable to $\mathcal{V} = (V_1, \ldots, V_{12})$.

We formalise this idea with the following definition of *M-stable* partitions.

Definition 1.10 (M-stable). *Let $r \geq t \geq 3$ be integers. Given a $(t-1)$-partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ and an r-partition $\mathcal{V} = (V_1, \ldots V_r)$. Let $m = \min_{i \in [t-1]} |I(P_i)|$. We say $\mathcal{P}$ is M-stable to $\mathcal{V}$, if the followings hold,*

- *$\mathcal{P}$ is 1-partial to $\mathcal{V}$,*
- *$|I(P_i)| \leq m + M$ if P_i is partial,*
- *$|V_j| \leq m + M$ if V_j is partial,*
- *$|P_j \setminus V_i| \leq m + M$ if V_i is integral in P_j.*

In particular, if $M = 0$, then we simply say that $\mathcal{P}$ is stable to $\mathcal{V}$.

Figure 3 below provides a concrete example of a stable partition.

Remark 1.11. In Figure 3, the partition $\mathcal{P}$ is stable to $\mathcal{V}$. Specifically:

- $V_1, \ldots, V_{10}$ are integral;
- V_{11} and V_{12} are partial;
- $|V_1| = |V_2| = m$, $|V_3| = 0.7m$, $|V_4| = |V_5| = |V_6| = 0.4m$, $|V_7| = |V_8| = |V_9| = m/3$, $|V_{10}| = |V_{11}| = |V_{12}| = 0.3m$.

We now apply the stable structure defined above to describe the extremal configurations in Theorem 1.1. To make this precise, we introduce the notion of *normalisation*: given a partition $\mathcal{P}$, its *normalisation*, denoted $\mathcal{N}(\mathcal{P})$, is the set of partitions obtained by moving all vertices of each partial V_i into a single part P_j that already intersects V_i. The extremal graph structure in Theorem 1.1 is fully characterised by Theorem 1.12.

Theorem 1.12. *Let $r \geq t \geq 3$ and $\mathcal{V}$ be a proper r-partition of $K_{n_1,\ldots,n_r}$. A $(t-1)$-partition $\mathcal{P}$ is an extremal $(t-1)$-partition for $K_{n_1,\ldots,n_r}$ if and only if*

- *$\mathcal{P}$ is stable to $\mathcal{V}$ and*
- *each partition in $\mathcal{N}(\mathcal{P})$ is still an extremal $(t-1)$-partition for $\mathcal{V}$.*

1.2 Stability results

Stability is a central concept in extremal graph theory. A classical theorem of Erdős and Simonovits [6, 7, 19] shows that any graph with nearly extremal edge count and forbidden subgraph F must resemble an extremal configuration. These stability results have been widely applied in various areas, such as determining extremal graphs and addressing supersaturation problems.

For example, Erdős and Simonovits [8] applied their stability theorem to determine the extremal graphs for $K_{2,2,2}$, while Mubayi [16], Pikhurko and Yilma [18] used it to investigate

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supersaturation problems, specifically determining the minimum number of copies of a forbidden graph F in an n-vertex graph H with $\mathrm{ex}(n, F) + q$ edges, where $\mathrm{ex}(n, F)$ represents the extremal number of edges for F. These theorems are particularly useful when considering extremal problems for non-bipartite graphs, extending far beyond Turán-type problems. While we do not aim to list all applications of these results, it is clear that such stability results are essential tools in extremal graph theory.

As a starting point, we recall the classical result by Erdős and Simonovits to motivate our multipartite version of stability (see Theorems 1.15 and 1.17).

Let $K_t^s = T(st, t)$. Erdős and Stone [9] showed that the extremal graphs for K_t^s have $e(T(n, t-1)) + o(n^2)$ edges. In 1968, Erdős and Simonovits [6, 7, 19] proved the following well-known stability theorem.

Theorem 1.13 (Erdős–Simonovits Weak Stability Theorem [6, 7, 19]). *Let F be a graph with chromatic number $t \geq 3$. Then every F-free graph on n vertices with $e(T(n, t-1)) - o(n^2)$ edges contains a $(t-1)$-partite subgraph with $e(T(n, t-1)) - o(n^2)$ edges and can be obtained from an extremal F-free graph by changing only $o(n^2)$ edges.*

Building on the classical weak stability result, we now consider a similar concept for multipartite graphs. The following definitions will be crucial in establishing weak stability results in the multi-partite setting.

For a vertex set $X \subseteq V(G)$ and a partition $\mathcal{P}$ of $V(G)$, we define:

$$\mathcal{P}_X := (P_1 \setminus X, \ldots, P_{t-1} \setminus X).$$

Given $a, b \in (0, 1)$, we use $a \ll b$ to denote that $a < b^N$ for some sufficient large constant N. Let $A \triangle B$ stand for the symmetric difference of the sets A and B. Given two graphs H and F on the same vertex set, we say F is *M-close* to H for some M, if every vertex v satisfies $|N_H(v) \triangle N_F(v)| \leq M$.

Definition 1.14 ((X, ϵ)-stable). *Given integers $t, r, n_1, \ldots, n_r$ with $r \geq t \geq 3$, $n_1 \geq \cdots \geq n_r$. For a given spanning subgraph G of $K_{n_1,\ldots,n_r}$, let $\mathcal{P} = (P_1, \ldots, P_{t-1})$ be a $(t-1)$-partition of $V(G)$ and $\mathcal{V} = (V_1, \ldots, V_r)$ be the proper r-partition of $K_{n_1,\ldots,n_r}$. We say $\mathcal{P}$ is an (X, ϵ)-stable partition for some constant $0 < \epsilon < 1$, if there exists a vertex set $X \subseteq V(G)$ with $|X| \leq \epsilon n_{t-1}$ such that*

- *$G - X$ is ϵn_{t-1}-close to $K_{n_1,\ldots,n_r}[\mathcal{P}] - X$ and*
- *$\mathcal{P}_X$ is stable to $\mathcal{V}_X$.*

We now establish a stability result for near-extremal subgraphs of multi-partite graphs. Theorem 1.15 shows that any such graph G admits a well-structured partition $\mathcal{P}$.

Theorem 1.15 (Weak Multi-partite Stability Theorem). *Let F be a graph with chromatic number $t \geq 3$, and let G be an F-free subgraph of $K_{n_1,\ldots,n_r}$, where $n_1 \geq \cdots \geq n_r$ and $r \geq t$. If*

$$e(G) = f(n_1, \ldots, n_r, 1, t) - o\big(n_{t-1}^2\big),$$

then, after removing $o(n_{t-1}^2)$ edges, G admits a $(t-1)$-partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ that is $(X, o(1))$-stable with $|I(P_i \setminus X)| \geq (1 - o(1))n_{t-1}$ for all $i \in [t-1]$.

Remark. Different from Theorem 1.13, our Theorem 1.15 shows that there exist graphs H with $\chi(H) = t$ such that if $e(G) \geq f(n_1, \ldots, n_r, 1, t) - \delta n_{t-1}^2$, then G may be far away from the extremal graph for H, see the Example 1.6.

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Erdős and Simonovits essentially got a more detailed version of Theorem 1.13 when H is an extremal graph for F.

Theorem 1.16 (Erdős–Simonovits Strong Stability Theorem [6, 7, 19]). *Let F be a graph with chromatic number $t \geq 3$. For every $0 < \epsilon < 1$ there exists a constant $\delta > 0$ such that every extremal F-free graph H with $n \geq 1/\delta$ vertices can be partitioned into $t-1$ classes each containing $n/(t-1) + o(n)$ vertices, and, with the exception of at most c_ϵ vertices, each vertex of H is joined to at most ϵn vertices in its own class and to all but ϵn vertices in the other classes. Moreover, we have $\delta(H) = ((t-2)/(t-1))n + o(n)$.*

Following this pattern, we present a stronger structural result for multi-partite graphs, which is particularly useful in the analysis of extremal configurations. Theorem 1.17 shows that all the exceptional vertices in an extremal graph can be confined to a constant-sized set Y, while the remainder of the graph admits a nearly optimal $(t-1)$-partite structure (see (a) and (c) in Theorem 1.17). Moreover, each vertex in Y is highly connected across all partition classes, collectively forming a nearly complete $(t-1)$-partite graph (see (d) in Theorem 1.17). This makes Y not a source of irregularity, but a structured and useful component for embedding the forbidden subgraph F.

Theorem 1.17 (Strong Multi-partite Stability Theorem). *Let F be a graph with chromatic number $t \geq 3$ and $K_{n_1,\ldots,n_r}$ be a complete r-partite graph with a partition $V_1 \cup \ldots \cup V_r$, where $|V_i| = n_i$, $n_1 \geq \ldots \geq n_r$ and $r \geq t$. If $G \subseteq K_{n_1,\ldots,n_r}$ is an extremal graph for F, then for any $\epsilon > 0$ there exist constants $\delta_1 \ll \delta_2 \ll \delta_3 \ll \epsilon$ and $c_{r,\epsilon,F}$ depending on r, ϵ and F such that if $n_{t-1} \geq 1/\delta_1$ then*

- *(a) there exists an (X, δ_2)-stable $(t-1)$-partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ such that $|I(P_i \setminus X)| \geq (1-\delta_2)n_{t-1}$ for any $i \in [t-1]$,*
- *(b) for each vertex v in V_j, $d_G(v) \geq \max\left\{\sum_{i\neq\ell} |P_i \setminus V_j| : 1 \leq \ell \leq t-1\right\} - \epsilon n_{t-1}$,*
- *(c) there exists a set Y with $|Y| \leq c_{r,\epsilon,F}$ such that $G - Y$ is ϵn_{t-1}-close to $K_{n_1,\ldots,n_r}[\mathcal{P}_Y]$, and*
- *(d) every vertex of Y is adjacent to at least $1/\delta_3$ vertices of each class of $\mathcal{P}$ such that they induce a complete $(t-1)$-partite graph.*

The organisation of this paper is as follows. In Section 2, we introduce some lemmas. In Section 3, we present the properties of extremal graphs for K_t^s and prove Theorem 1.12. In Sections 4 and 5, we give the proofs of Theorems 1.15, 1.17, 1.2, and 1.4.

2. Preliminaries

2.1 Definition and notation

Recall that we mostly consider r-partite graphs with a proper partition $\mathcal{V} = (V_1, \ldots, V_r)$. Let $V = \bigcup_{i=1}^r V_i$. For given integer a, let $V_{i\leq a}$ be some V_i with $i \leq a$. Similarly, we define $V_{i\neq a}$, $V_{i\geq a}$. Let x be any vertex of a given graph G and A be a subset of $V(G)$. The *neighbourhood* of x in G is denoted by $N_G(x) = \{y \in V(G) : (x, y) \in E(G)\}$. The *degree* of x in G, denoted by $d_G(x)$, is the size of $N_G(x)$. Let $N_A(x)$ be set of neighbours of x in A and $d_A(x) = |N_A(x)|$. Let $N_G[A] = \bigcap_{x\in A} N_G(x)$. Given two disjoint sets A, B of G, let $G_{A\to B}$ be the graph obtained from G by deleting edges incident with A and adding each edge between A and B. For an independent set A of G, let $G_A = G_{A\to N_G(u)}$ where u is a vertex with maximum degree in A.

Given a partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ of V, we define $\mathcal{P} \wedge \mathcal{V}$ as a refinement of $\mathcal{V}$:

$$\mathcal{P} \wedge \mathcal{V} := (P_i \cap V_j : i \in [t-1], j \in [r]).$$

Graph Removal Lemma is widely used in extremal graph theory and related topics. We will apply the following simple form of Graph Removal Lemma.

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Lemma 2.1 (A simple form of Graph Removal Lemma [10]). *For every $\alpha > 0$, s and $t \geq 3$, there is a δ such that if $n > 1/\delta$ and G is an n-vertex K_t^s-free graph then it contains a K_t-free subgraph H with $e(H) > e(G) - \alpha n^2$.*

The following lemma can be found in the classic book of Bollobás [3].

Lemma 2.2 (Selection lemma [3]). *Let $\epsilon_1, \ldots, \epsilon_k$ be positive numbers, $0 < \alpha < 1$ and t be a natural number. There exists a natural number $N = N(\epsilon_1, \ldots, \epsilon_k; \alpha; t)$ with the following property. For $V_1, \ldots, V_k$ with $|V_i| = n_i$ and $A_{ij} \subset V_i$ with $j \in [N]$, if*

$$\frac{1}{N}\sum_{j=1}^{N} |A_{ij}| \geq \epsilon_i n_i \quad \text{for} \quad i = 1, \ldots, k,$$

then there is a subset $T \subset [N]$ with $|T| = t$, such that if $S \subseteq T$ then

$$\left|\bigcap_{j\in S} A_{ij}\right| \geq (\alpha\epsilon_i)^{|S|} n_i \quad \text{for} \quad i = 1, \ldots, k.$$

The following proposition will be used when the size differences of V_i's are large.

Proposition 2.3. *Let $0 \leq u_1 \leq \ldots \leq u_m$ be integers and $N \leq \sum_{i=1}^{m} u_i$. If $\epsilon < (1/m)^{10^m}$, then there exists an $\eta \in [\epsilon, \sqrt[10^m]{\epsilon}]$ such that either $u_i < \eta N$ or $u_i \geq \sqrt[10]{\eta} N$ for each $i \in [m]$.*

Proof. Let $0 = u_0 \leq u_1 \leq \ldots \leq u_m$. If $u_i \leq \sqrt[10^i]{\epsilon} N$ and $u_{i+1} \geq \sqrt[10^{i+1}]{\epsilon} N$ for some $i \in \{0, 1, \ldots, m-1\}$, then let $\eta = \sqrt[10^i]{\epsilon}$. Otherwise, we have $\sum_{i=1}^{m} u_i \leq m \sqrt[10^m]{\epsilon} N < N$, a contradiction. The proof of Proposition 2.3 is complete. □

Given a graph G with two partitions $\mathcal{V} = (V_1, \ldots, V_r)$ and $\mathcal{P} = (P_1, \ldots, P_{t-1})$. If $\mathcal{P}$ is stable to $\mathcal{V}$, then $\mathcal{P} \wedge \mathcal{V}$ has some good properties. We need the following propositions concerning the properties when $\mathcal{P}$ is M-stable to $\mathcal{V}$.

Proposition 2.4. *Let $\mathcal{P} = (P_1, \ldots, P_{t-1})$ be stable to $\mathcal{V} = (V_1, \ldots, V_r)$ with $r \geq t \geq 3$, where $|V_i| = n_i$ and $n_1 \geq n_2 \geq \ldots \geq n_r$. Then $|I(P_i)| \geq n_{t-1}$ for each $i \in [t-1]$.*

Proof. Let $m = \min_{i\in[t-1]}\{|I(P_i)|\}$. Suppose that $m = |I(P_i)| < n_{t-1}$ for some $i \in [t-1]$. If V_j is partial, then by $\mathcal{P}$ is stable to $\mathcal{V}$, we have $|V_j| \leq m < n_{t-1}$, i.e., $j \geq t$. Thus V_j is integral for $j \leq t-1$. If there are no two of integral V_js in one part of P_i, then $|I(P_i)| \geq n_{t-1} > m$, a contradiction. Thus without loss of generality, we may assume that V_1 and V_2 are integral in P_1, whence $|P_1 \setminus V_1| \geq |V_2| > m$, a contradiction to that $\mathcal{P}$ is stable to $\mathcal{V}$. □

Proposition 2.5. *Let $r \geq t \geq 3$. Let G be an r-partite graph with a proper partition $\mathcal{V} = (V_1, \ldots, V_r)$ with $|V_i| = n_i$ and $n_1 \geq \ldots \geq n_r$, and let $\mathcal{P} = (P_1, \ldots, P_{t-1})$[4] be ϵn_{t-1}-stable to $\mathcal{V}$, where $0 < \epsilon \ll 1$ is a small constant and n_{t-1} is sufficiently large. Suppose that each class of $\mathcal{P} \wedge \mathcal{V}$ is of size at least $\sqrt{\epsilon} n_{t-1}$. If a vertex $v \in P_\alpha \cap V_\beta$ is adjacent to at least $d_{K_{n_1,\ldots,n_r}[\mathcal{P}]}(v) - \epsilon n_{t-1}$ vertices of $\mathcal{V} \setminus V_\beta$, then there exists a $j \in [r]$ such that $d_{I(P_i)}(v) \geq \left(\sqrt{\epsilon}/4\right) n_{t-1}$ for each $i \neq j$.*

Proof. Let $v \in P_\alpha \cap V_\beta$ be adjacent to at least $d_{K_{n_1,\ldots,n_r}[\mathcal{P}]}(v) - \epsilon n_{t-1}$ vertices of $\mathcal{V} \setminus V_\beta$, i.e., v is adjacent to at least $|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)| - \epsilon n_{t-1}$ vertices of $\mathcal{V} \setminus V_\beta$.

Similar to the proof of Proposition 2.4, by $\mathcal{P}$ is ϵn_{t-1}-stable to $\mathcal{V}$, $|I(P_i)| \geq (1-\epsilon) n_{t-1}$. Suppose that $v \in P_\alpha \cap V_\beta$ has less than $\left(\sqrt{\epsilon}/4\right) n_{t-1}$ neighbours in both $I(P_{i_1})$ and $I(P_{i_2})$ with $i_1 \neq i_2$ and $\{i_1, i_2\} \subseteq [t-1]$. We divide our proof into the following cases.

[4]It is possible that $\mathcal{P}$ is not proper.

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Case 1. V_β is partial in both of P_{i_1} and P_{i_2}.

Then $|I(P_{i_1})| \geq |P_\alpha \setminus V_\beta| - \epsilon n_{t-1}$ and $|I(P_{i_2})| \geq |P_\alpha \setminus V_\beta| - \epsilon n_{t-1}$ due to $\mathcal{P}$ is ϵn_{t-1}-stable to $\mathcal{V}$. Therefore, v is adjacent to less than

$$\begin{aligned}
&\left|\bigcup_{i\neq i_1,i_2}\left(P_i \setminus V_\beta\right)\right| + \left(\sqrt{\epsilon}/2\right) n_{t-1}\\
&= \left|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)\right| - |I\left(P_{i_1}\right)| - |I\left(P_{i_2}\right)| + |P_\alpha \setminus V_\beta| + \left(\sqrt{\epsilon}/2\right) n_{t-1}\\
&\leq \left|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)\right| + \epsilon n_{t-1} - |I\left(P_{i_2}\right)| + \left(\sqrt{\epsilon}/2\right) n_{t-1}\\
&< \left|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)\right| - \epsilon n_{t-1}
\end{aligned}$$

vertices of $\mathcal{V} \setminus V_\beta$, where the last strict inequality holds by $|I(P_{i_2})| \geq (1-\epsilon)n_{t-1}$, a contradiction. We finish the proof for this case.

If $P(P_{i_2}) = \emptyset$, then $|P(P_{i_2})| \leq |I(P_{i_1})| - (\sqrt{\epsilon} - \epsilon)n_{t-1}$ by $|I(P_{i_1})| \geq (1-\epsilon)n_{t-1}$. Suppose that $P(P_{i_2}) = P_{i_2} \cap V_\gamma \neq \emptyset$. Since each class of $\mathcal{P} \wedge \mathcal{V}$ is of size at least $\sqrt{\epsilon}n_{t-1}$, $|P(P_{i_2})| \leq |V_\gamma| - \sqrt{\epsilon}n_{t-1} \leq |I(P_{i_1})| + \epsilon n_{t-1} - \sqrt{\epsilon}n_{t-1} \leq |I(P_{i_1})| - (\sqrt{\epsilon} - \epsilon)n_{t-1}$, where the second inequality holds by $\mathcal{P}$ is ϵn_{t-1}-stable to $\mathcal{V}$. From the above two cases, we have $|P(P_{i_1})| + |P(P_{i_2})| \leq |I(P_{i_1})| + |P(P_{i_1})| - (\sqrt{\epsilon} - \epsilon)n_{t-1} = |P_{i_1}| - (\sqrt{\epsilon} - \epsilon)n_{t-1}$. Similarly, $|P(P_{i_1})| + |P(P_{i_2})| \leq |P_{i_2}| - (\sqrt{\epsilon} - \epsilon)n_{t-1}$. Therefore, we have

$$|P(P_{i_1})| + |P(P_{i_2})| + (\sqrt{\epsilon}/2)n_{t-1} < \min\left\{|P_{i_1}|, |P_{i_2}|\right\} - (\sqrt{\epsilon}/4)n_{t-1}. \tag{1}$$

We continue our proof by the following two remainder cases.

Case 2. V_β is integral in one of P_{i_1}, P_{i_2}.

Let V_β be integral in P_{i_1}. Then $\alpha = i_1$ and $|P_{i_2}| = |P_{i_2} \setminus V_\beta|$. By (1), v is adjacent to less than

$$\begin{aligned}
&\left|\bigcup_{i\neq i_1,i_2}\left(P_i \setminus V_\beta\right)\right| + |P\left(P_{i_1}\right)| + |P\left(P_{i_2}\right)| + (\sqrt{\epsilon}/2)n_{t-1}\\
&\leq \left|\bigcup_{i\neq i_1,i_2}\left(P_i \setminus V_\beta\right)\right| + |P_{i_2}| - (\sqrt{\epsilon}/4)n_{t-1}\\
&= \left|\bigcup_{i\neq\alpha,i_2}\left(P_i \setminus V_\beta\right)\right| + |P_{i_2} \setminus V_\beta| - (\sqrt{\epsilon}/4)n_{t-1}\\
&< \left|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)\right| - \epsilon n_{t-1}
\end{aligned}$$

vertices of $\mathcal{V} \setminus V_\beta$, a contradiction. We finish the proof of this case.

Case 3. V_β is not partial in one of P_{i_1} and P_{i_2}, and V_β is not integral in P_{i_1} and P_{i_2}.

Without loss of generality, suppose $V_\beta \cap P_{i_1} = \emptyset$, i.e., $|P_{i_1}| = |P_{i_1} \setminus V_\beta|$, and V_β is not integral in P_{i_2}. Since $\mathcal{P}$ is ϵn_{t-1}-stable to $\mathcal{V}$, we have

$$|P_\alpha \setminus V_\beta| \leq |I(P_{i_2})| + \epsilon n_{t-1} = |P_{i_2} \setminus V_\beta| + \epsilon n_{t-1}.$$

By (1), v is adjacent to at most

$$\begin{aligned}
&\left|\bigcup_{i\neq i_1,i_2}\left(P_i \setminus V_\beta\right)\right| + |P\left(P_{i_1}\right)| + |P\left(P_{i_2}\right)| + \left(\sqrt{\epsilon}/2\right) n_{t-1}\\
&\leq \left|\bigcup_{i\neq i_1,i_2}\left(P_i \setminus V_\beta\right)\right| + |P_{i_1}| - \left(\sqrt{\epsilon}/4\right) n_{t-1}
\end{aligned}$$

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$$= \left|\bigcup_{i=1}^{t-1}\left(P_i \setminus V_\beta\right)\right| - |P_{i_2} \setminus V_\beta| - \left(\sqrt{\epsilon}/4\right) n_{t-1}$$

$$\leq \left|\bigcup_{i=1}^{t-1}\left(P_i \setminus V_\beta\right)\right| - |P_\alpha \setminus V_\beta| - \left(\sqrt{\epsilon}/4 - \epsilon\right) n_{t-1}$$

$$< \left|\bigcup_{i\neq\alpha}\left(P_i \setminus V_\beta\right)\right| - \epsilon n_{t-1}$$

vertices of $\mathcal{V} \setminus V_\beta$, a contradiction. Therefore, there exists a j such that $d_{I(P_i)}(v) \geq (\sqrt{\epsilon}/4)n_{t-1}$ holds for each $i \neq j$. We finish the proof of Proposition 2.5. □

Lemma 2.6. *Let $r \geq t \geq 3$. If the partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ is M-stable to $\mathcal{V} = (V_1, \ldots, V_r)$ and each class of $\mathcal{P} \wedge \mathcal{V}$ is of size at least $4trM$, then there exists a set Z with $|Z| \leq 3trM$ such that $\mathcal{P}_Z$ is stable to $\mathcal{V}_Z$.*

Proof. Let $P_1, P_2, \ldots, P_s$ be the partial classes and $P_{s+1}, P_{s+2}, \ldots, P_{t-1}$ be the integral classes of $\mathcal{P}$. Let $m = \min\{|I(P_i)| : i \in [t-1]\}$. Since $\mathcal{P}$ is M-stable to $\mathcal{V}$, we have (a). $\mathcal{P}$ is 1-partial to $\mathcal{V}$, (b). $|I(P_i)| \leq m + M$ for $i \in [s]$, (c). if V_i is partial, then $|V_i| \leq m + M$ and (d). if V_i is integral in P_j, then $|P_j \setminus V_i| \leq m + M$.

We will find the partition $\mathcal{P}^* = (P_1^*, \ldots, P_{t-1}^*)$ such that $P_i^* \subseteq P_i$ and $\min\{|I(P_i^*)| : i \in [t-1]\} = m$.[5] We first consider partial classes of $\mathcal{P}$. By (b), we can take $X_i^* \subseteq I(P_i)$ with $|X_i^*| = m$ such that $|I(P_i) \setminus X_i^*| \leq M$ for each $i \in [s]$. Let V_j be partial in P_i for each $i \in J \subseteq [s]$. Then by (c) we can take $V_j^* \subseteq V_j$ with $|V_j^*| = \min\{m, |V_j|\}$ such that $|(V_j \setminus V_j^*) \cap P_\ell| = |V_j| - m \leq M$ for some $\ell \in J$. Since each class of $\mathcal{P} \wedge \mathcal{V}$ is of size at least $4trM$, for each $i \in J$ by (d) we can take $Y_i^* \subseteq V_j^* \cap P_i \subseteq P(P_i)$ with $|P(P_i) \setminus Y_i^*| \leq 2M$ such that $|(X_i^* \cup Y_i^*) \setminus V_\ell| \leq m$ for each integral V_ℓ in P_i. Let $P_i^* = X_i^* \cup Y_i^*$ for each $i \in [s]$.

Now we consider the integral classes of $\mathcal{P}$. Let $i \geq s+1$ and $I(P_i) = P_i = V_{i_1} \cup \ldots \cup V_{i_\ell}$. For $\alpha \in [\ell]$, if $|P_i| \geq 4trM + m$, then let $V_{i_\alpha}^* \subseteq V_{i_\alpha}$ such that $|V_{i_\alpha} \setminus V_{i_\alpha}^*| = 2M$; otherwise, let $V_{i_\alpha}^* = V_{i_\alpha}$. Let $P_i^* = V_{i_1}^* \cup \ldots V_{i_\ell}^*$. Since each class of $\mathcal{P} \wedge \mathcal{V}$ is of size at least $4trM$, by (d) it is easy to check that $|P_i^* \setminus V_{i_\alpha}^*| \leq m$ for any $1 \leq \alpha \leq \ell$.

Let $Z = \bigcup_{i=1}^{t-1} P_i \setminus P_i^*$. Then $|Z| \leq 3trM$. By (a), $\mathcal{P}_Z$ is 1-partial to $\mathcal{V}_Z$. From above all, we see that $\mathcal{P}_Z$ is stable to $\mathcal{V}_Z$. We complete the proof of Lemma 2.6. □

3. Properties of extremal graphs in multi-partite graphs

First we present some properties of the extremal graphs for K_t in multi-partite graphs.

Proposition 3.1. *Given integers $n_1 \geq \ldots \geq n_r$ with $r \geq t$. Then $f(n_1, \ldots, n_r, 1, t) \geq f(n_1, \ldots, n_r, 1, t-1) + n_{t-1}^2$.*

Proof. Let $\mathcal{P} = (P_1, \ldots, P_{t-2})$ be a $(t-2)$-partition of $[r]$ which attains $f(n_1, \ldots, n_r, 1, t-1)$. Thus, by the Pigeonhole Principle, one part of $\mathcal{P}$, say P_s, contains at least two integers $i, j \in [t-1]$. Hence $\mathcal{P}^* = (P_1, \ldots, \{i\}, P_s \setminus \{i\}, \ldots, P_{t-2})$ is a $(t-1)$-partition of $[r]$. Note that $n_i \geq n_{t-1}$ and $n_j \geq n_{t-1}$. Therefore, we have $f(n_1, \ldots, n_r, 1, t-1) + n_{t-1}^2 \leq f(n_1, \ldots, n_r, 1, t-1) + n_i n_j \leq f(n_1, \ldots, n_r, 1, t)$, where the last inequality holds by $\mathcal{P}^*$ is a $(t-1)$-partition of $[r]$ and the definition of $f(n_1, \ldots, n_r, 1, t)$. The proof of this proposition is complete. □

Now we present some properties of the extremal graphs for K_t^s.

Proposition 3.2. *Given an integer t with $t \geq 3$. Let G be an r-partite graph with parts $\mathcal{V} = (V_1, \ldots, V_r)$. Suppose that G is K_t-free with two disjoint sets A, B. If $G[B]$ is $(t-2)$-partite, then $G_{A \to B}$ is also K_t-free.*

[5] $I(P_i^*)$ is $I_{\mathcal{V}^*}(P_i^*)$, where $\mathcal{V}^* = \mathcal{V}_X$ and $X = \bigcup_{i\in[t-1]} P_i \setminus P_i^*$.

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Proof. Suppose for a contradiction that $G_{A\to B}$ contains a copy of K_t. Clearly, $V(K_t)\cap A$ is not empty, as otherwise G contains a copy of K_t, a contradiction. Since A is independent in $G_{A\to B}$, we have $|V(K_t)\cap A|=1$. However, every vertex of A is adjacent to a $(t-2)$-partite graph in $G_{A\to B}$. Thus there is no copy of K_t in $G_{A\to B}$. The result follows. □

Proposition 3.3. *Given a graph F with $\chi(F)\geq 3$. Let G be an r-partite graph with parts $\mathcal{V}=(V_1,\dots,V_r)$. Suppose that G is F-free with two vertex-disjoint independent sets A,B with $|B|\geq|V(F)|$. Let $X=N_G[B]\setminus A$. Then $G_{A\to X}$ is also F-free. In particular, if G is K_t-free for some $t\geq 3$, then G_{V_i} is still K_t-free.*

Proof. Suppose for a contradiction that $G_{A\to X}$ contains a copy of F. Clearly, $V(F)\cap A$ is not empty, as otherwise G contains a copy of F, a contradiction. We can see that $G[X]$ contains a copy of $F-A$, implying $G[X\cup B]$ contains a copy of F (note that A is an independent set in $G_{A\to X}$), a contradiction (G is F-free). The proof of the moreover part is similar and is omitted. We finish the proof of this proposition. □

Now we are ready to prove Theorem 1.12.

Proof of Theorem 1.12. Let $G=K_{n_1,\dots,n_r}[\mathcal{P}]$ and $\mathcal{P}=(P_1,\dots,P_{t-1})$ be an extremal $(t-1)$-partition of $V(G)$. We denote $X_{i,j}=V_i\cap P_j$ for $i\in[r],j\in[t-1]$. We only prove the "only if" part of Theorem 1.12 since the "if" part is trivial.

First, if $V_i\cap P_j\neq\emptyset$, then in any $P_{j'}$ with $j\neq j'$ we have

$$|P_j\setminus V_i|\leq|P_{j'}\setminus V_i|. \tag{2}$$

Otherwise, let F be the graph obtained from G by deleting edges between $X_{i,j}$ and $P_{j'}\setminus V_i$ and adding each edge between $X_{i,j}$ and $P_j\setminus V_i$. Since F is $(t-1)$-partite, F is K_t-free and has more edges than G, a contradiction to the maximality of G. Thus if V_i is partial in $P_j,P_{j'}$ with $j\neq j'$, then

$$|P_j\setminus V_i|=|P_{j'}\setminus V_i|, \tag{3}$$

and if V_i is integral in P_j then for every distinct $P_{j'}$, we have

$$|P_j\setminus V_i|\leq|P_{j'}|. \tag{4}$$

Assume that there exists a part P_{j_1} such that two sets V_{i_1},V_{i_2} with $i_1\neq i_2$ are both partial in it. Let V_{i_1} be partial in P_{j_2} with $j_2\neq j_1$ and V_{i_2} be partial in P_{j_3} with $j_3\neq j_1$ (it is possible that $j_2=j_3$). From (3), we have $|P_{j_1}\setminus V_{i_1}|=|P_{j_2}\setminus V_{i_1}|$ and $|P_{j_1}\setminus V_{i_2}|=|P_{j_3}\setminus V_{i_2}|$. Hence $G^1=G_{X_{i_1,j_2}\to N_G[X_{i_1,j_1}]}=K_{n_1,\dots,n_p}[\mathcal{P}']$ has the same number of edges as G, where $\mathcal{P}'=(P_1,\dots,P_{j_1}\cup X_{i_1,j_2},\dots,P_{j_2}\setminus X_{i_1,j_2},\dots,P_{t-1})$. Let

$$\mathcal{P}^*=(P_1,\dots,(P_{j_1}\cup X_{i_1,j_2})\setminus X_{i_2,j_1},\dots,P_{j_2}\setminus X_{i_1,j_2},\dots,P_{j_3}\cup X_{i_2,j_1},\dots,P_{t-1}).$$

Thus $K_{n_1,\dots,n_p}[\mathcal{P}^*]$ has $e(G^1)+|X_{i_2,j_1}|\cdot|X_{i_1,j_2}|>e(G^1)$ edges, and hence has more edges than G, a contradiction. Therefore, $\mathcal{P}$ is 1-partial.

Let

$$m=\min_{i\in[t-1]}|I(P_i)|.$$

Let i_m be an integer such that $|I(P_{i_m})|=m$. We will show that $|I(P_i)|=m$ for any partial P_i. Without loss of generality, we consider the partial P_1 with V_1 partial in it. By 1-partial we have $I(P_1)=P_1\setminus V_1$, and by definition of m we have $|I(P_1)|\geq m$. Thus we only need to show $|P_1\setminus V_1|\leq m$. If P_{i_m} is integral, then by (2) we have $|P_1\setminus V_1|\leq|P_{i_m}\setminus V_1|=m$, and we are done. Thus P_{i_m} is partial. If V_1 is partial in P_{i_m}, then by (3) we have $|I(P_1)|=I(P_{i_m})|=m$. So we suppose some set V_2 is partial in P_{i_m}. Let P_2 be the other part in which V_2 is partial. Thus $|I(P_2)|=|I(P_{i_m})|=m$. By Proposition 3.3, $H=G_{X_{2,i_m}\to N_G[X_{2,2}]}$ is the K_t-free with same number of edges. Using (2) to the partition $(P_1,P_2\cup X_{2,i_m},\dots,P_{i_m}\setminus X_{2,i_m},\dots,P_{t-1})$ of graph H, we have $|P_1\setminus V_1|\leq|P_{i_m}\setminus V_2|=|I(P_{i_m})|=m$. Therefore, $|I(P_i)|=m$ for any partial P_i.

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Let V_i be partial in P_j and $P_{j'}$. Thus $P_j \setminus V_i$ and $P_{j'} \setminus V_i$ are integral part with $|P_j \setminus V_i| = |P_{j'} \setminus V_i| = m$. In G, we join each vertex of V_i to all other vertices and delete all the edges between $P_j \setminus V_i$ and $P_{j'} \setminus V_i$ and denote the new graph by H'. Note that H' is still a $(t-1)$-partite graph. Thus $e(H) = e(G) - |P_j \setminus V_i||P_{j'} \setminus V_i| + m|V_i| \le e(G)$, implying $|V_i| \le m$.

Let V_i be integral in P_ℓ. If there exists a set V_j which is partial in P_s and $P_{s'}$, then let $H = G_{X_{j,s} \to N_G[X_{j,s'}]}$. Since $|P_{s'} \setminus V_j| = |P_s \setminus V_j|$, by Proposition 3.3 we can easily see that H is still an extremal K_t-free graph. Consider the partition $(P_1, \ldots, P_s \setminus X_{j,s}, \ldots, P_{s'} \cup X_{j,s}, \ldots, P_{t-1})$, by (4) we have $|P_\ell \setminus V_i| \le |P_s \setminus V_j| = m$. It remains to consider the simplest case: there is no partial class in $\mathcal{P}$. Then $|P_\ell \setminus V_i| \le |P_{j \ne \ell}|$ by (4). Above all, we confirm that $\mathcal{P}$ is stable to $\mathcal{V}$.

Since the integral part of partial class equals to each other, it is easy to check that each partition in $\mathcal{N}(\mathcal{P})$ is still an extremal partition. We finish the proof.

Given integers $n_1 \ge \cdots \ge n_r$, when the part sizes are highly unbalanced – in the sense that a few of the largest parts are much bigger than the rest – we can often decompose the extremal number $\mathrm{ex}(n_1, \ldots, n_r, F)$ into two parts: one corresponding to the large parts, and the other to a residual subproblem on the smaller parts. To formalize this idea, we divide the sequence $n_1, \ldots, n_r$ into consecutive segments $B_1, B_2, \ldots$, where the numbers within each segment are of similar size, and there is a noticeable drop between segments. This canonical segmentation enables us to define the parameter $\tau(n_1, \ldots, n_r, t, L)$, which identifies the threshold between dominant and subordinate parts in the extremal analysis. This leads to the following definition.

We say a non-increasing sequence of positive numbers $a_1 \ge a_2 \ge \cdots \ge a_s$ is *L-balance* if it satisfies $a_i \ge a_{i-1}/L^4$ for each $i \in \{2, 3, \ldots, s\}$.

Given a sequence $n_1 \ge \cdots \ge n_r$, we partition it into consecutive maximal L-balance families $B_1, B_2, \ldots, B_k$, such that each B_i is L-balance, and every element of B_i is strictly less than every element of B_{i-1} for $i \ge 2$. This partition is uniquely determined by scanning the sequence from left to right and extracting maximal L-balance segments in order.

We define

$$\tau(n_1, \ldots, n_r, t, L) := \max\left\{ \sum_{i=1}^{x} |B_i| : x \in \mathbb{Z}_{\ge 0},\ \sum_{i=1}^{x} |B_i| \le t-2 \right\}.$$

Clearly, $\tau(n_1, \ldots, n_r, t, L) \le t-2$. In particular, $\tau(n_1, \ldots, n_r, t, L) = 0$ when $|B_1| \ge t-1$.

The following proposition helps us to get the extremal $(t-1)$-partition of a multi-partite complete graph when the difference of its part sizes is large.

Proposition 3.4. *Let $K = K_{n_1,\ldots,n_r}$ with parts $V_1, \ldots, V_r$ of sizes $n_1 \ge \ldots \ge n_r$ and n_{t-1} be sufficiently large. Let $\tau = \tau(n_1, \ldots, n_r, t, r)$. If $\tau \ne 0$, then for each $0 \le s \le \tau$, we have*

$$\mathrm{ex}(n_1, \ldots, n_r, K_t) = \mathrm{ex}(n_{s+1}, \ldots, n_r, K_{t-s}) + \prod\nolimits_{1 \le i < j \le s} n_i n_j + \sum\nolimits_{i=1}^{s} n_i \sum\nolimits_{i=s+1}^{r} n_i.$$

Moreover, each extremal $(t-1)$-partition of K contains V_s for each $1 \le s \le \tau$.

Proof. The proposition holds trivially for $s = 0$. Let $1 \le s \le \tau$. Take an extremal $(t-1)$-partition $\mathcal{P} = (P_1, \ldots, P_{t-1})$ of K with $|P_1| \ge \ldots \ge |P_{t-1}|$. By Theorem 1.12, $\mathcal{P}$ is stable to $\mathcal{V}$. It is sufficient to show that $P_i = V_i$ for $i = 1, \ldots, \tau$. Theorem 1.1 implies that $H = K_{n_1,\ldots,n_r}[\mathcal{P}]$ is an extremal graph for K_t. By the definition of τ, we have $\sum_{i=\tau+1}^{r} n_i \le r n_\tau / r^4 < n_\tau / r$. We can conclude that each $V_{s \le \tau}$ is integral in $\mathcal{P}$. Otherwise, there is a partial class with size at least n_τ in $\mathcal{V}$, and hence the integral part of each class ($t-1$ classes) of $\mathcal{P}$ is of size at least n_τ (since $\mathcal{P}$ is stable to $\mathcal{V}$), which is impossible since $\tau \le t-2$ and $\sum_{i=\tau+1}^{r} n_i < n_\tau / r$. Thus $|P_{t-1}| \le \sum_{i=\tau+1}^{r} n_i \le n_\tau / r$. Moreover, if P_j contains V_s and some other vertices, then the integral V_s has a size larger than $|P_{t-1}| \ge |I(P_{t-1})|$, which is a contradiction (since $\mathcal{P}$ is stable to $\mathcal{V}$). Therefore, we must have $P_i = V_i$ for $i = 1, \ldots, \tau$. The proof of Proposition 3.4 is complete. □

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4. Proof of Theorems 1.15 and 1.17

Proof outline of Theorem 1.15. All the graphs G appearing in this outline are r-partite F-free graphs with parts $\mathcal{V}=(V_1,\dots,V_r)$ of sizes $n_1\geq\ldots\geq n_r$. Also, we require $e(G)\geq f(n_1,\dots,n_r,1,t)-\delta n_{t-1}^2$.

Base case: $F=K_t$. In order to prove Theorem 1.15, we first prove the stability result for K_t-free graphs in Theorem 4.1. The main tool used to prove Theorem 4.1 is the Zykov's symmetrisation method [24]. By applying Proposition 3.3 recursively to the sets $V_1,\dots,V_r$, we obtain K_t-free graphs $G^1,\dots,G^r$. By symmetrisation operations, G^r has a good structure. Thus it is quite easy to show that G^r has an (X_r,ϵ_r)-stable $(t-1)$-partition $\mathcal{P}_r$ (See Claim 4.2).

Next, we use mathematical induction to G^r. We can show for $k\in[r]$, if G^k has an (X_k,ϵ_r)-stable $(t-1)$-partition $\mathcal{P}_k$, then G^{k-1} also has an (X_{k-1},ϵ_{k-1})-stable $(t-1)$-partition $\mathcal{P}_{k-1}$ (by Claim 4.4,. . .,Claim 4.8). Thus by induction, we obtain $G^{r-1},\dots,G^1$ all have required stable partitions. Eventually, we show G have an (X,ϵ)-stable $(t-1)$-partition.

All cases: Theorem 1.15. We only need to use Graph Removal Lemma to remove ϵn^2 edges in G to obtain a K_t-free graph $\widetilde{G}$ with $e(\widetilde{G})\geq e(G)-\epsilon n^2$. However, we cannot use Theorem 4.1 directly to $\widetilde{G}$ to obtain the required partition $\mathcal{P}$, because it is possible that $n\gg n_{t-1}$. Thus we consider the cases of $n_{t-1}=\theta(n)$ and $n_{t-1}=o(n)$. If $n_{t-1}=\theta(n)$ (see Case 1 in the proof of Theorem 1.15), then we immediately prove it. If $n_{t-1}=o(n)$ (see Case 2 in the proof of Theorem 1.15), we can use Proposition 3.4 to remove all the big V_i of G. The resulting graph $G[S]$ has the vertices number close to n_{t-1}. Thus we can use the Theorem 4.1 to $G[S]$ and obtain a partition $\mathcal{P}^S$. Then we extend $\mathcal{P}^S$ to the desired partition of G.

We begin our proofs with the following theorem.

Theorem 4.1. *Let G be a K_t-free r-partite graph with the proper partition $\mathcal{V}=(V_1,\dots,V_r)$ of sizes $n_1\geq\ldots\geq n_r$. For any $0<\epsilon<1$, there exists $\delta>0$ depending on t, r and ϵ such that if $n_{t-1}\geq 1/\delta$ and $e(G)\geq f(n_1,\dots,n_r,1,t)-\delta n_{t-1}^2$, then G has an (X,ϵ)-stable $(t-1)$-partition $\mathcal{P}$. Moreover, the integral part of each class of $\mathcal{P}_X$ has size larger than $(1-\epsilon)n_{t-1}$.*

Proof. Let

$$\delta\ll\epsilon_r\ll\epsilon_{r-1}\ll\ldots\ll\epsilon_1\ll\epsilon_0=\epsilon\quad\text{and}\quad n_{t-1}\geq 1/\delta.$$

Let G be a K_t-free spanning subgraph of $K_{n_1,\dots,n_r}$ with $e(G)\geq f(n_1,\dots,n_r,1,t)-\delta n_{t-1}^2$. Let $G^0=G$ and define $G^i=G^{i-1}_{V_i}$ recursively. Clearly, by Proposition 3.3 all of G^0, G^1, …, G^r are K_t-free and

$$e(G^r)\geq\ldots\geq e(G^0)=e(G)\geq f(n_1,\dots,n_r,1,t)-\delta n_{t-1}^2. \tag{5}$$

Claim 4.2. *G^r has an (X_r,ϵ_r)-stable $(t-1)$-partition $\mathcal{P}^r$.*

Proof. The hierarchy of constants in the proof satisfies $\delta\ll\xi\ll\epsilon_r$. Since δ is sufficiently small, by Proposition 2.3 for δ there exists a $\xi\in[\delta,\sqrt[10^r]{\delta}]$, such that every class of $\mathcal{V}$ is of size either less than ξn_{t-1} or at least $\sqrt[10]{\xi}n_{t-1}$. Let b be the minimum integer such that $n_b\leq\xi n_{t-1}$. Let $\widetilde{X}$ be the union of sets in $\mathcal{V}$ with size less than ξn_{t-1} and let $|\widetilde{X}|=m_r=\sum_{i=b}^r n_i$. Let $H_0=G^r-\widetilde{X}$.

We next construct the partition $\mathcal{P}^r$. Define H_{i+1} to be the subgraph of H_i induced by the neighbours of a vertex with maximum degree in H_i recursively. Suppose that the above progress stops when $e(H_{s-1})=0$ and let $V(H_s)=\emptyset$. Let $A_i=V(H_{i-1})\setminus V(H_i)$ for $i\in[s]$. Let

$$\mathcal{P}^r=\left(A_1,\dots,A_{s-1},A_s\cup\widetilde{X}\right).$$

Thus $\mathcal{P}^r_{\widetilde{X}}=(A_1,\dots,A_s)$.

We first show $s=t-1$. Clearly, A_s is an independent set in G^r. Since G^r is K_t-free, we have $s\leq t-1$. Otherwise, any vertex in A_s can form a copy of K_t with those vertices picked with maximum degree in H_i's.

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Note that every set of $\mathcal{V}_{\widetilde{X}}$ is integral in $\mathcal{P}^r_{\widetilde{X}}$ (each vertex in V_i has common neighbours in G^r by the construction of G^r). In fact, let x be the vertex with maximum degree in H_0. Then V_i which contains x belongs to A_1 and if x is not adjacent to V_j, then V_j belongs to A_1. Continue this arguments for $A_2, \ldots, A_s$, we can show that every set of $\mathcal{V}_{\widetilde{X}}$ is integral in $\mathcal{P}^r_{\widetilde{X}}$.

Thus

$$e(H_0) \le \sum_{i=1}^{s-1} \left(\sum_{v \in A_i} \frac{d_{A_i}(v)}{2} + d_{H_i}(v) \right) \le \sum_{1 \le i < j \le s} |A_i||A_j| \le f(n_1, \ldots, n_{b-1}, 1, s+1). \tag{6}$$

We will show that

$$f(n_1, \ldots, n_r, 1, t) \ge f(n_1, \ldots, n_{b-1}, 1, t) + m_r \sum_{i=1}^{t-2} n_i. \tag{7}$$

Let $\mathcal{Q} = (Q_1, \ldots, Q_{t-1})$ be a maximal $(t-1)$-partition of $\{n_1, n_2, \ldots, n_{b-1}\}$. Without loss of generality, we may assume that Q_{t-1} does not contain any element in $\{n_1, n_2, \ldots, n_{t-2}\}$. Then $\mathcal{Q}^* = (Q_1, \ldots, Q_{t-1} \cup \{n_b, n_{b+1}, \ldots, n_r\})$ is a $(t-1)$ partition of $\{n_1, n_2, \ldots, n_r\}$. Thus (7) follows by the definition of the function f and $|\bigcup_{i=1}^{t-2} Q_i| \ge \sum_{i=1}^{t-2} n_i$.

If $s < t-1$, then by (6), (7), $b \ge t$ and Proposition 3.1, we have

$$\begin{aligned}
e(G^r) &\overset{(6)}{\le} f(n_1, \ldots, n_{b-1}, 1, s+1) + m_r^2 + m_r \sum\nolimits_{i=1}^{b-1} n_i \\
&\le f(n_1, \ldots, n_{b-1}, 1, t-1) + m_r^2 + m_r \sum\nolimits_{i=1}^{b-1} n_i \\
&\overset{\text{Prop. 3.1}}{\le} f(n_1, \ldots, n_{b-1}, 1, t) - n_{t-1}^2 + m_r^2 + m_r \sum\nolimits_{i=1}^{b-1} n_i \\
&\overset{(7)}{\le} f(n_1, \ldots, n_r, 1, t) + m_r \sum\nolimits_{i=t-1}^{b-1} n_i - n_{t-1}^2 + m_r^2 \\
&\le f(n_1, \ldots, n_r, 1, t) - (1 - \epsilon_r) n_{t-1}^2
\end{aligned}$$

contradicting (5). Therefore, we have $s = t-1$.

We next show $G^r - \widetilde{X}$ is 0-close to $K_{n_1,\ldots,n_r}[\mathcal{P}^r] - \widetilde{X}$, i.e., every vertex in $V(H_i)$ for $i \in [s-1]$ is incident with all the vertices of A_i in graph H_{i-1}. Otherwise, suppose that $v_{j_1} \in V(H_i) \cap V_{j_1}$ is not incident with some vertex $v_{j_2} \in A_i \cap V_{j_2}$. Then there is no edge between V_{j_1} and V_{j_2} in G^r by our construction. Therefore, we have

$$\sum_{v \in A_i} \frac{d_{A_i}(v)}{2} + d_{H_i}(v) \le |A_i||V(H_i)| - \frac{|V_{j_1}||V_{j_2}|}{2} \le |A_i||V(H_i)| - (\sqrt[10]{\xi} n_{t-1})^2.$$

It follows from (6) that $e(H_0) \le f(n_1, \ldots, n_{b-1}, 1, t) - \sqrt[5]{\xi} n_{t-1}^2$. Since ξ is sufficiently small, similar as the proof of $s = t-1$, by (7), we have $e(G^r) \le e(H_0) + m_r^2 + m_r \sum_{i=1}^{b-1} n_i \le f(n_1, \ldots, n_r, 1, t) - \sqrt[4]{\xi} n_{t-1}^2$, a contradiction to (5). Note that every vertex in A_i has at most $|V(H_i)|$ neighbours in H_{i-1}. Thus A_i is an independent set in H_{i-1}. Therefore, $\mathcal{P}^r_{\widetilde{X}}$ induces a complete $(t-1)$-partite subgraph in H_0.

Note that each V_i either belongs to $\widetilde{X}$ or is integral in $\mathcal{P}^r_{\widetilde{X}}$. Given A_i and distinct $A_{i'}$, let V_j be an integral set in A_i. Let $S := \bigcup_{\iota \ne i'} A_\iota$. Since each vertex in $A_{i'}$ is adjacent to each vertex in S and $A_{i'}$ is an independent set with size at least ξn_{t-1}, it follows from Proposition 3.3 that $G^r_{V_j \to S}$ is K_t-free and hence $e(G^r_{V_j \to S}) \le f(n_1, \ldots, n_r, 1, t)$. Combining with (5), we have $|A_i \setminus V_j| - |A_{i'}| \le \delta n_{t-1}^2 / |V_j| \le \delta n_{t-1} / \xi \le \epsilon_r^2 n_{t-1}$. Thus $\mathcal{P}^r_{\widetilde{X}}$ is $\epsilon_r^2 n_{t-1}$-stable to $\mathcal{V}_{\widetilde{X}}$. Applying Lemma 2.6, we obtain the required X_r such that $\mathcal{P}^r_{X_r}$ is stable to $\mathcal{V}_{X_r}$. By above, $\mathcal{P}^r$ is the (X_r, ϵ_r)-stable $(t-1)$-partition of G^r. □

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Note in the proof of Claim 4.2 we showed that $V_i \setminus X_r$ for $i \in [k]$ is integral in $\mathcal{P}^r_{X_r}$ and $G^r - X_r$ is $(t-1)$-partite. That is, we have proved the case $k = r$ of the following mathematical induction.

Induction hypothesis.

Suppose that G^k has an (X_k, ϵ_k)-stable $(t-1)$-partition $\mathcal{P}^k$; each $V_{j\le k} \setminus X_k$ is integral in $\mathcal{P}^k_{X_k}$ and $G^k - X_k$ is $(t-1)$-partite.

We will show that

(a) G^{k-1} has an $(X_{k-1}, \epsilon_{k-1})$-stable $(t-1)$-partition $\mathcal{P}^{k-1}$;

(b) each $V_{j\le k-1} \setminus X_{k-1}$ is integral in $\mathcal{P}^{k-1}_{X_{k-1}}$;

(c) $G^{k-1} - X_{k-1}$ is $(t-1)$-partite.

We first do some pre-processes. For $i \in [t-1]$, let P^k_i denote the set of i^{th} part of $\mathcal{P}^k$. By induction hypothesis, $\mathcal{P}^k$ is (X_k, ϵ_k)-stable and $V_k \setminus X_k \subseteq P^k_{i_k}$. Then by $|X_k| \le \epsilon_k n_{t-1}$ each $v \in V_k \setminus X_k$ satisfies

$$d_{G^k}(v) \ge \sum\nolimits_{i\ne i_k} |P^k_i \setminus X_k| - \epsilon_k n_{t-1} \ge \sum\nolimits_{i\ne i_k} |P^k_i| - 2\epsilon_k n_{t-1}. \tag{8}$$

Let

$$Z_1 = \left\{ v \in V_k \colon d_{G^{k-1}}(v) \le \sum_{i\ne i_k} |P^k_i| - \sqrt{\epsilon_k} n_{t-1} \right\}.$$

By (5), (8) and our construction, we have

$$\begin{aligned}
\delta n^2_{t-1} &\ge e(G^k) - e(G^{k-1}) \\
&\ge |Z_1| \left(\left(\sum\nolimits_{i\ne i_k} |P^k_i| - 2\epsilon_k n_{t-1} \right) - \left(\sum\nolimits_{i\ne i_k} |P^k_i| - \sqrt{\epsilon_k} n_{t-1} \right) \right) \\
&\ge |Z_1| \left(\sqrt{\epsilon_k} - 2\epsilon_{k-1} \right) n_{t-1},
\end{aligned}$$

implying $|Z_1| \le \sqrt{\epsilon_k} n_{t-1}$. Thus every vertex v of $V_k \setminus Z_1$ satisfies

$$d_{G^{k-1}}(v) \ge \sum\nolimits_{i\ne i_k} |P^k_i| - \sqrt{\epsilon_k} n_{t-1}. \tag{9}$$

Now we try to construct the desired $(t-1)$-partition of G^{k-1} by partition vertices in $V'_k = V_k \setminus (X_k \cup Z_1)$ into $B_1 \cup \ldots \cup B_{t-1}$ such that if $x \in B_i$, then x has minimum number of neighbours in $P^k_i \setminus (X_k \cup Z_1)$. Let $P^{k-1}_j = (P^k_j \setminus V'_k) \cup B_j$ for $j \in [t-1]$. We construct the $(t-1)$-partition

$$\mathcal{P}^{k-1} = \left(P^{k-1}_1, \ldots, P^{k-1}_{t-1} \right).$$

By Proposition 2.3, for $\sqrt[4]{\epsilon_k}$ there exists a $\xi \in [\sqrt[4]{\epsilon_k}, \sqrt[4\times 10^{(t-1)r}]{\epsilon_k}]$ such that every set of $(\mathcal{P}^{k-1} \wedge \mathcal{V})_{X_k \cup Z_1}$ is of size either less than ξn_{t-1} or more than $\sqrt[10]{\xi} n_{t-1}$. Let Z_2 be the union of sets in $(\mathcal{P}^{k-1} \wedge \mathcal{V})_{X_k \cup Z_1}$ with size less than ξn_{t-1}. Let $\overline{X} = X_k \cup Z_1 \cup Z_2$. Thus $|\overline{X}| \le 4tr\xi n_{t-1}$.

For $i \in [r], j \in [t-1]$, let

$$\overline{V}_i = V_i \setminus \overline{X},\ \overline{P}_j = P^{k-1}_j \setminus \overline{X} \text{ and } X_{i,j} = \overline{V}_i \cap \overline{P}_j.$$

By $Z_2 \subseteq \overline{X}$, every non-empty $X_{i,j}$ satisfies

$$|X_{i,j}| \ge \sqrt[10]{\xi} n_{t-1}. \tag{10}$$

We first show that such $\mathcal{P}^{k-1}$ and $\overline{X}$ satisfy item (*c*).

Claim 4.3. *$G^{k-1} - \overline{X}$ is $(t-1)$-partite.*

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Proof. Let $\widetilde{P}_i = P_i \setminus (\overline{X} \cup V_k)$. By induction hypothesis, $G^k - X_k$ is edgeless in each part of $\mathcal{P}^k_{X_k}$. By our construction, $G^k - V_k = G^{k-1} - V_k$. Thus, we only need to show that G^{k-1} is edgeless between $\widetilde{P}_i$ and $X_{k,i}$ for $i \in [t-1]$.

Clearly, we may suppose that $|\widetilde{P}_i| \geq \sqrt[10]{\xi} n_{t-1}$. Otherwise, by the definition of Z_2, we have $\widetilde{P}_i = \emptyset$. We immediately finish the proof. Let $a \in X_{k,i}$. By $|\overline{X}| \leq 4tr\xi n_{t-1}$ and $\mathcal{P}^k_{X_k}$ is stable to $\mathcal{V}_{X_k}$ ($\mathcal{P}^k$ is (X_k, ϵ_k)-stable), $\mathcal{P}^k_{\overline{X}}$ is $\sqrt{\xi} n_{t-1}$-stable to $\mathcal{V}_{\overline{X}}$. Since $V_k \setminus X_k \subseteq P^k_{i_k}$, by (10) each part of $(\mathcal{P}^k \wedge \mathcal{V})_{\overline{X}}$ has size at least $\sqrt[10]{\xi} n_{t-1}$. By (9) we have

$$\begin{aligned} d_{G^{k-1}}(a) &\geq \sum\nolimits_{\ell \neq i_k} |P^k_\ell| - \sqrt{\epsilon_k} n_{t-1} \\ &\geq \sum\nolimits_{\ell \neq i_k} |P^k_\ell| - \sqrt{\xi} n_{t-1} \\ &\geq \sum\nolimits_{\ell \neq i_k} |\overline{P}_\ell \setminus \overline{V}_k| - |\overline{X}| - \sqrt{\xi} n_{t-1} \\ &\geq \sum\nolimits_{\ell \neq i_k} |\overline{P}_\ell \setminus \overline{V}_k| - \sqrt[5]{\xi} n_{t-1}. \end{aligned}$$

Thus applying Proposition 2.5 (here we consider the edges in G^{k-1} and the partition $(\mathcal{P}^k \wedge \mathcal{V})_{\overline{X}}$), there exists an $m(a)$ such that a has more than $(\sqrt[10]{\xi}/4) n_{t-1}$ neighbours in the integral part of every $P^k_{i \neq m(a)} \setminus \overline{X}$. If there is an edge between a and $\widetilde{P}_i$, then by the definition of $X_{k,i} = B_i$, there is an edge ab with $b \in \widetilde{P}_{m(a)}$. Since $\mathcal{P}^k_{X_k}$ is (X_k, ϵ_k)-stable, we can construct a copy of K_t in G^{k-1} with vertices a, b and their common neighbours in the integral parts of $\widetilde{P}_{i \neq m(a)}$'s, a contradiction. Hence we finish the proof of Claim 4.3. □

To prove item (a), we will first show that $\mathcal{P}^{k-1}_{\overline{X}}$ is $\sqrt[8]{\xi} n_{t-1}$-stable to $\mathcal{V}_{\overline{X}}$. The proofs of the following claims are quite similar to the proof in Theorem 1.12.

Claim 4.4. *If $\overline{V}_i$ is partial in distinct $\overline{P}_s, \overline{P}_\ell$, then $\big||\overline{P}_s \setminus \overline{V}_i| - |\overline{P}_\ell \setminus \overline{V}_i|\big| \leq \sqrt{\xi} n_{t-1}$.*

Proof. Without loss of generality, suppose for a contradiction that $\overline{V}_1$ (we do not consider the size of each $\overline{V}_i$) is partial in parts $\overline{P}_{i_1}, \overline{P}_{i_2}$ with $|\overline{P}_{i_2} \setminus \overline{V}_1| - |\overline{P}_{i_1} \setminus \overline{V}_1| > \sqrt{\xi} n_{t-1}$. Let $S = \bigcup_{i \neq i_1} (\overline{P}_i \setminus \overline{V}_1)$. Note that $|X_{1,i_2}| \geq \sqrt[10]{\xi} n_{t-1}$ by (10). By Claim 4.3, $G^{k-1}[S]$ is $(t-2)$-partite. Hence by (5) and Proposition 3.2, $H = G^{k-1}_{X_{1,i_2} \to S}$ is a K_t-free graph with

$$\begin{aligned} e(H) &\geq e(G^{k-1}) + \left(\sqrt{\xi} n_{t-1} - |\overline{X}|\right) |X_{1,i_2}| \\ &\geq f(n_1, \ldots, n_r, 1, t) - \delta n^2_{t-1} + \left(\sqrt{\xi} n_{t-1} - 4tr\xi n_{t-1}\right) \sqrt[10]{\xi} n_{t-1} \\ &> f(n_1, \ldots, n_r, 1, t), \end{aligned}$$

a contradiction. We finish the proof of Claim 4.4. □

Now we start to show the correctness of item (a). From Claims 4.5-4.8, we verify that $\mathcal{P}^{k-1}_{\overline{X}}$ is $\sqrt[8]{\xi} n_{t-1}$-stable to $\mathcal{V}_{\overline{X}}$.

Claim 4.5. $\mathcal{P}^{k-1}_{\overline{X}}$ *is 1-partial to* $\mathcal{V}_{\overline{X}}$.

Proof. Without loss of generality, suppose for a contradiction that both $\overline{V}_1, \overline{V}_2$ are partial in $\overline{P}_1$ (as in Claim 3, we do not consider the size of each $\overline{V}_i$). Let $\overline{V}_1$ be partial in $\overline{P}_s$ and $\overline{V}_2$ be partial in $\overline{P}_\ell$ (it is possible that $s = \ell$). Since $\overline{V}_2$ is partial in $\overline{P}_1, \overline{P}_\ell$, it follows from Claim 4.4 that $|\overline{P}_1 \setminus \overline{V}_2| - |\overline{P}_\ell \setminus \overline{V}_2| \geq -\sqrt{\xi} n_{t-1}$. Let $S_1 = \bigcup_{i \neq \ell} (\overline{P}_i \setminus \overline{V}_2)$. By Claim 4.3, $G^{k-1}[S_1]$ is $(t-2)$-partite. Thus by Proposition 3.2, $H^1 = G^{k-1}_{X_{2,1} \to S_1}$ is a K_t-free graph with

$$e(H^1) \geq e(G^{k-1}) + (|\overline{P}_1 \setminus \overline{V}_2| - |\overline{P}_\ell \setminus \overline{V}_2| - |\overline{X}|)|X_{2,1}| \geq e(G^{k-1}) - 2\sqrt{\xi} n_{t-1} |X_{2,1}|.$$

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Let $S_2 = \bigcup_{i=2}^{t-2} (\overline{P}_i \setminus \overline{V}_1) \cup X_{2,1}$. Then by Claim 4.3, $H^1[S_2]$ is $(t-2)$-partite. Hence by Proposition 3.2, $H^2 = H^1_{X_{1,s} \to S_2}$ is a K_t-free graph. Since $\overline{V}_1$ is partial in $\overline{P}_1, \overline{P}_s$, Claim 4.3 implies $|\overline{P}_1 \setminus X_{1,1}| - |\overline{P}_s \setminus X_{1,s}| \le \sqrt{\xi} n_{t-1}$. By (10), we have $|X_{2,1}|, |X_{1,s}| \ge \sqrt[10]{\xi} n_{t-1}$. Thus we have

$$\begin{aligned} e(H^2) &\ge e(H^1) + \left(|X_{2,1}| - \sqrt{\xi} n_{t-1} - |\overline{X}|\right) |X_{1,s}| \\ &\ge e(G^{k-1}) - 2\sqrt{\xi} n_{t-1} |X_{2,1}| + \frac{1}{2} |X_{2,1}||X_{1,s}| \\ &\ge e(G^{k-1}) + \frac{1}{3} \sqrt[5]{\xi} n_{t-1}^2 \\ &> f(n_1, \dots, n_r, 1, t), \end{aligned}$$

a contradiction. Thus $\mathcal{P}^{k-1}_{\overline{X}}$ is 1-partial to $\mathcal{V}_{\overline{X}}$. □

By Claim 4.5 we know that if $\overline{V}_i$ is partial in $\overline{P}_j$, then $I(\overline{P}_j) = \overline{P}_j \setminus \overline{V}_i$. Let

$$m = \min_{i \in [t-1]} \left|I(\overline{P}_i)\right| \text{ and } i_m \text{ be an integer such that } \left|I(\overline{P}_{i_m})\right| = m.$$

Claim 4.6. *If $\overline{P}_i$ is partial, then $\left|I(\overline{P}_i)\right| \le m + \sqrt[5]{\xi} n_{t-1}$.*

Proof. Suppose that $\overline{V}_1$ is partial in $\overline{P}_{i_1}$ and $\left|I(\overline{P}_{i_1})\right| \ge m + \sqrt[5]{\xi} n_{t-1}$. If $\overline{P}_{i_m}$ is integral, then let $S = \bigcup_{i \ne i_m} (\overline{P}_i \setminus \overline{V}_1)$. By Claim 4.3, $G^{k-1}[S]$ is $(t-2)$-partite. Thus by Proposition 3.2, $G^{k-1}_{X_{1,i_1} \to S}$ is a K_t-free graph with $e(G^{k-1}) + \left(\sqrt[5]{\xi} n_{t-1} - \overline{X}\right) \sqrt[10]{\xi} n_{t-1} > f(n_1, \dots, n_r, 1, t)$ edges, a contradiction. Thus $\overline{P}_{i_m}$ must be partial. By Claims 4.4 and 4, $\overline{V}_1$ cannot be partial in $\overline{P}_{i_m}$. Let $\overline{V}_2$ be the set that is partial in $\overline{P}_{i_m}$. Let $a \in \overline{V}_2$ be a vertex with maximum number of neighbours in G^{k-1}. Without loss of generality, let $a \in P_{\ell_1}$ and $\overline{V}_2 \cap P_{\ell_2} \ne \emptyset$, where $\ell_1 \ne \ell_2$ (it is possible that $\ell_1 = i_m$ or $\ell_2 = i_m$). Clearly, $H = G^{k-1}_{\overline{V}_2 \to N_{G^{k-1}}(a)}$ is K_t-free by Proposition 3.3 and has more number of edges than G^{k-1}. By Claim 4.4, $|I(\overline{P}_{\ell_2})| \le |I(\overline{P}_{i_m})| + \sqrt{\xi} n_{t-1} = m + \sqrt{\xi} n_{t-1}$. Let $S' = (\bigcup_{i \ne \ell_2} \overline{P}_i) \cup \overline{V}_2$. Then $H[S']$ is $(t-2)$-partite by Claim 4.3. By Proposition 3.2, $H_{X_{1,i_1} \to S'}$ is a K_t-free graph with more than

$$\begin{aligned} &e(H) + \left(|S'| - \left|\bigcup\nolimits_{i \ne i_1} \overline{P}_i \setminus \overline{V}_1\right| - |\overline{X}|\right) \cdot |X_{1,i_1}| \\ &\quad \ge e(H) + \left(|I(\overline{P}_{i_1})| - |I(\overline{P}_{\ell_2})| - |\overline{X}|\right) \cdot \sqrt[10]{\xi} n_{t-1} \\ &\quad \ge e(G^{k-1}) + \sqrt{\xi} n_{t-1}^2 > f(n_1, \dots, n_r, 1, t) \end{aligned}$$

edges, a contradiction. The proof of Claim 4.6 is complete. □

Claim 4.7. *If $\overline{V}_i$ is partial, then $|\overline{V}_i| \le m + \sqrt[7]{\xi} n_{t-1}$.*

Proof. Without out loss of generality, suppose that $\overline{V}_1$ is partial in $\overline{P}_{i_1}, \overline{P}_{i_2}$. Suppose otherwise that $|\overline{V}_1| \ge m + \sqrt[7]{\xi} n_{t-1}$. By Claims 4.5 and 4.6, we have

$$|\overline{P}_{i_1} \setminus \overline{V}_1| = I(\overline{P}_{i_1}) \le m + \sqrt[5]{\xi} n_{t-1} \le |\overline{V}_1| - \sqrt[7]{\xi} n_{t-1} + \sqrt[5]{\xi} n_{t-1} \le |\overline{V}_1| - \sqrt[6]{\xi} n_{t-1}.$$

In G^{k-1}, we delete the edges incident with $\overline{V}_1$ first and then join every vertex of $\overline{V}_1$ to each vertex of $\bigcup_{j=1}^{t-1} (\overline{P}_j \setminus \overline{V}_1)$ and delete the edges between $\overline{P}_{i_1} \setminus \overline{V}_1$ and $\overline{P}_{i_2} \setminus \overline{V}_1$. Denote the resulting graph by J. Since the common neighbours of each added edge induces a $(t-3)$-partite subgraph graph in J, J is a K_t-free graph. By Claim 4.6, we have

$$|\overline{P}_j \setminus \overline{V}_1| - |\overline{P}_{i_1} \setminus \overline{V}_1| \ge -\sqrt[5]{\xi} n_{t-1} \text{ for any } j \ne i_1 \tag{11}$$

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and by Claim 4.4 we have

$$|\overline{P}_{i_1} \setminus \overline{V}_1| - |\overline{P}_{i_2} \setminus \overline{V}_1| \geq -\sqrt[5]{\xi} n_{t-1}. \tag{12}$$

Set $a = |\overline{V}_1| - |\overline{P}_{i_1} \setminus \overline{V}_1|$. Then $a \geq \sqrt[6]{\xi} n_{t-1}$. Thus

$$\begin{aligned}
e(J) &\geq e(G^{k-1}) + \sum_{j=1}^{t-1} \left(|\overline{P}_j \setminus \overline{V}_1||X_{1,j}|\right) - |\overline{V}_1||\overline{X}| - |\overline{P}_{i_1} \setminus \overline{V}_1||\overline{P}_{i_2} \setminus \overline{V}_1| \\
&\overset{(11)}{\geq} e(G^{k-1}) + \left(|\overline{P}_{i_1} \setminus \overline{V}_1| - \sqrt[5]{\xi} n_{t-1} - 4tr\xi n_{t-1}\right) |\overline{V}_1| - |\overline{P}_{i_1} \setminus \overline{V}_1||\overline{P}_{i_2} \setminus \overline{V}_1| \\
&\geq e(G^{k-1}) + |\overline{P}_{i_1} \setminus \overline{V}_1| \left(|\overline{V}_1| - |\overline{P}_{i_2} \setminus \overline{V}_1|\right) - 2\sqrt[5]{\xi} n_{t-1} |\overline{V}_1| \\
&\overset{(12)}{\geq} e(G^{k-1}) + |\overline{P}_{i_1} \setminus \overline{V}_1| \left(a - \sqrt[5]{\xi} n_{t-1}\right) - 2\sqrt[5]{\xi} n_{t-1} \left(|\overline{P}_{i_1} \setminus \overline{V}_1| + a\right) \\
&\geq e(G^{k-1}) + |\overline{P}_{i_1} \setminus \overline{V}_1| \left(a - 3\sqrt[5]{\xi} n_{t-1}\right) - 2\sqrt[5]{\xi} n_{t-1} a \\
&\geq e(G^{k-1}) + \sqrt[10]{\xi} n_{t-1} \cdot \frac{a}{2} - 2\sqrt[5]{\xi} n_{t-1} a > f(n_1, \ldots, n_r, 1, t),
\end{aligned} \tag{13}$$

a contradiction, where the last inequality holds since ξ is sufficiently small and $|\overline{P}_{i_1} \setminus \overline{V}_1| \geq \sqrt[10]{\xi} n_{t-1}$. The proof of Claim 4.7 is complete. □

Claim 4.8. *If $\overline{V}_i$ is integral in $\overline{P}_j$, then $|\overline{P}_j \setminus \overline{V}_i| \leq m + \sqrt[7]{\xi} n_{t-1}$.*

Proof. For convenience of notations, let $i = 1$. Suppose $|\overline{P}_j \setminus \overline{V}_1| \geq m + \sqrt[7]{\xi} n_{t-1}$. If $\overline{P}_{i_m}$ is integral, then $i_m \neq j$. Let $S = \bigcup_{i \neq i_m} \overline{P}_i \setminus \overline{V}_1$. Then $G^{k-1}[S]$ is $(t-2)$-partite by Claim 4.3. By (10), we have $|\overline{V}_1| \geq \sqrt[10]{\xi} n_{t-1}$. By Proposition 3.2, $G^{k-1}_{\overline{V}_1 \to S}$ is a K_t-free graph with more than

$$e(G^{k-1}) + |\overline{V}_1| \cdot \left(|\overline{P}_j \setminus \overline{V}_1| - m - |\overline{X}|\right) \geq e(G^{k-1}) + \sqrt{\xi} n_{t-1}^2 \overset{(5)}{>} f(n_1, \ldots, n_r, 1, t)$$

edges, a contradiction. Thus $\overline{P}_{i_m}$ must be partial. Without loss of generality, assume that $\overline{V}_2$ is partial in $\overline{P}_{i_m}$.

Suppose that $\overline{V}_2$ is partial in $\overline{P}_j$. Note that it is possible that $j = i_m$. Then we may assume that $\overline{V}_2$ is partial in $\overline{P}_j$ and $\overline{P}_{i'_m}$, where $j \neq i'_m$. By Claim 4.6, we have $|I(\overline{P}_{i'_m})| \leq m + \sqrt[5]{\xi} n_{t-1}$. Let $S = \bigcup_{i \neq i'_m} \overline{P}_i \setminus \overline{V}_1$. By Claim 4.3, $G^{k-1}[S]$ is $(t-2)$-partite. By Proposition 3.2, $H = G^{k-1}_{\overline{V}_1 \to S}$ is a K_t-free graph. By definition,

$$\begin{aligned}
e(H) \geq & e(G^{k-1}) + \left(|\overline{P}_j \setminus \overline{V}_1| - |\overline{P}_{i'_m}| - |\overline{X}|\right) |\overline{V}_1| \\
\geq & e(G^{k-1}) + \left(|\overline{P}_j \setminus \overline{V}_1| - |I(\overline{P}_{i'_m})| - |X_{2,i'_m}| - |\overline{X}|\right) |\overline{V}_1| \\
\geq & e(G^{k-1}) + \left(\sqrt[6]{\xi} n_{t-1} - |X_{2,i'_m}|\right) |\overline{V}_1|,
\end{aligned}$$

the last inequality holds by the assumption $|\overline{P}_j \setminus \overline{V}_1| \geq m + \sqrt[7]{\xi} n_{t-1}$ and $|\overline{X}| \leq \sqrt{\xi} n_{t-1}$.

Let $M_1 = \bigcup_{i \neq j, i'_m} \overline{P}_i \setminus \overline{V}_2$ and $M = \left(\overline{P}_{i'_m} \setminus \overline{V}_2\right) \cup \overline{V}_1 \cup M_1$. By Claim 4.3, $H[M]$ is $(t-2)$-partite. By Proposition 3.2, $F = H_{X_{2,i'_m} \to M}$ is a K_t-free graph. Observe that the degree of every vertex of X_{2,i'_m} in H is less than $|\overline{X}| + |M_1| + |\overline{P}_j \setminus (\overline{V}_1 \cup \overline{V}_2)|$. By the 1-partialness of $\mathcal{P}^{k-1}_{\overline{X}}$ and Claim 4.6, we have $|\overline{P}_j \setminus \overline{V}_2| = |I(\overline{P}_j)| \leq m + \sqrt[5]{\xi} n_{t-1}$. By $\overline{V}_1$ is integral in $\overline{P}_j$, we have $|\overline{P}_j \setminus (\overline{V}_1 \cup \overline{V}_2)| \leq m + \sqrt[5]{\xi} n_{t-1} - |\overline{V}_1|$. Thus the degree of every vertex of X_{2,i'_m} in H is less than $|M_1| + m - |\overline{V}_1| +$

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$\sqrt[5]{\xi}n_{t-1}+\sqrt{\xi}n_{t-1}$. Thus

$$\begin{aligned}e(F)\geq& e(H)+\left(|M|-|M_1|-m+|\overline{V}_1|-\sqrt[6]{\xi}n_{t-1}\right)|X_{2,i'_m}|\\ \geq& e(H)+\left(|\left(\overline{P}_{i'_m}\setminus\overline{V}_2\right)\cup\overline{V}_1|-m+|\overline{V}_1|-\sqrt[6]{\xi}n_{t-1}\right)|X_{2,i'_m}|\\ \geq& e(H)+\left(2|\overline{V}_1|-\sqrt[6]{\xi}n_{t-1}\right)|X_{2,i'_m}|\\ \overset{(10)}{\geq}& e(G^{k-1})+\sqrt[5]{\xi}n_{t-1}^2\overset{(5)}{>}f(n_1,\ldots,n_r,1,t),\end{aligned}$$

a contradiction. Thus $\overline{V}_2$ is not partial in $\overline{P}_j$.

Let a be a vertex in $\overline{V}_2$ with maximum number of neighbours in G^{k-1}. Let $a\in\overline{P}_{\ell_1}$ and $\overline{V}_2$ be partial in $\overline{P}_{\ell_2}$ with $i_m\in\{\ell_1,\ell_2\}$. By Proposition 3.3, $H=G^{k-1}_{\overline{V}_2\to N_{G^{k-1}}(a)}$ keeps K_t-freeness with $e(H)\geq e(G^{k-1})$. From Claim 4.6, we have $|I(\overline{P}_{\ell_2})|\leq m+\sqrt[5]{\xi}n_{t-1}$. Let $S'=(\bigcup_{i=1}^{t-1}\overline{P}_i)\setminus(V_1\cup I(\overline{P}_{\ell_2}))$. Note that $H[S']$ is $(t-2)$-partite by Claim 4.3. By Proposition 3.3, $H^1=H_{\overline{V}_1\to S'}$ is K_t-free with more than

$$\begin{aligned}&e(G^{k-1})+\left(|I(\overline{P}_j)|-|I(\overline{P}_{\ell_2})|-|\overline{X}|\right)\cdot|\overline{V}_1|\\ \geq& e(G^{k-1})+\left(\sqrt[7]{\xi}-\sqrt[5]{\xi}-4tr\xi\right)n_t\cdot|\overline{V}_1|\\ \overset{(5)}{>}& f(n_1,\ldots,n_r,1,t)\end{aligned}$$

edges, a contradiction. We finish the proof of this claim. □

Next we show the correctness of item (a).

Claim 4.9. *$\mathcal{P}^{k-1}$ is an (X_{k-1},ϵ_{k-1})-stable partition of G^{k-1}.*

Proof. We only need to find the required X_{k-1}. For $i\in[r],j\in[t-1]$, let

$$Z_3^{i,j}=\left\{v\in X_{i,j}\colon d_{G^{k-1}}(v)\leq\sum_{\ell\neq j}|\overline{P}_\ell\setminus\overline{V}_i|-\sqrt{\epsilon_k}n_{t-1}\right\}.$$

Let $S=\bigcup_{\ell\neq j}\overline{P}_\ell\setminus\overline{V}_i$. By Claim 4.3 and Proposition 3.2, $G^{k-1}_{Z_3^{i,j}\to S}$ is K_t-free. By (5),

$$\delta n_{t-1}^2\geq f(n_1,\ldots,n_r,1,t)-e(G^{k-1})\geq e(G^{k-1}_{Z_3^{i,j}\to S})-e(G^{k-1})\geq|Z_3^{i,j}|\cdot\sqrt{\epsilon_k}n_{t-1},$$

implying $|Z_3^{i,j}|\leq\sqrt{\epsilon_k}n_{t-1}$. Let

$$Z_3=\bigcup_{i\in[r],j\in[t-1]}Z_3^{i,j}.$$

Thus $|Z_3|\leq rt\sqrt{\epsilon_k}n_{k-1}$. By Claims 4.4-4.8, $\mathcal{P}^{k-1}_{\overline{X}}$ is $\sqrt[8]{\xi}n_{t-1}$-stable to $\mathcal{V}_{\overline{X}}$. Thus $\mathcal{P}^{k-1}_{\overline{X}\cup Z_3}$ is $2\sqrt[8]{\xi}n_{t-1}$-stable to $\mathcal{V}_{\overline{X}\cup Z_3}$.

By Lemma 2.6, we can remove at most $4tr\sqrt[7]{\xi}n_{t-1}$ vertices, say Z_4, to ensure that $\mathcal{P}^{k-1}_{\overline{X}\cup Z_3\cup Z_4}$ is stable to $\mathcal{V}_{\overline{X}\cup Z_3\cup Z_4}$. Let

$$X_{k-1}=\overline{X}\cup Z_3\cup Z_4.$$

Thus $|X_{k-1}|\leq\sqrt[8]{\xi}n_{t-1}\leq\epsilon_{k-1}n_{t-1}$. By the choice of X_{k-1}, we know that $\mathcal{P}^{k-1}_{X_{k-1}}$ is stable to $\mathcal{V}_{X_{k-1}}$. Since $Z_3\subseteq X_{k-1}$, $G^{k-1}-X_{k-1}$ is $\sqrt{\epsilon_k}n_{t-1}$-close to $K_{n_1,\ldots,n_r}[\mathcal{P}^{k-1}]-X_{k-1}$. Thus $\mathcal{P}^{k-1}$ is an (X_{k-1},ϵ_{k-1})-stable partition of G. □

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By the operations of symmetrisation, every $V_{i\le k-1}\setminus X_{k-1}$ keeps integral in $\mathcal{P}^{k-1}_{X_{k-1}}$, and we prove the item (b).

Thus recursively, we obtain an (X,ϵ)-stable partition $\mathcal{P}$ of G. By $\mathcal{P}_X$ is stable to $\mathcal{V}_X$, it follows from Proposition 2.4 that the size of integral part of each class of $\mathcal{P}_X$ is at least $(1-\epsilon)n_{t-1}$. We finish the proof of Theorem 4.1.

Now we start to prove Theorem 1.15.

Proof of Theorem 1.15. Note that each graph with chromatic number t is a subgraph of a t-partite graph. We only need to prove that the theorem holds for K_t^s, where s is an integer depending on F.

Let G be a K_t^s-free r-partite graph with parts $\mathcal{V}=(V_1,\dots,V_r)$. Let $0<\delta\ll\delta_1\ll\epsilon<1$, where δ and δ_1 will be determined later by ϵ, Lemma 2.1 and Theorem 4.1. Let $n\ge 1/\delta$. Suppose that

$$e(G)\ge f(n_1,\dots,n_r,1,t)-\delta n_{t-1}^2. \tag{14}$$

We shall show that G has an (X,ϵ)-stable $(t-1)$-partition $\mathcal{P}$ after removing ϵn_{t-1}^2 edges. Let

$$\tau=\tau(n_1,\dots,n_r,t,tr).$$

Case 1. $\tau=0$. Since $\tau=0$, by the definition of τ, we have $n_{t-1}\ge(1/tr)^{4t-1}n_1\ge(1/tr)^{4t}n$. For $\alpha=(1/tr)^{8t}\delta_1$, by Lemma 2.1, G contains a K_t-free graph $\widetilde{G}$ with $e(\widetilde{G})\ge e(G)-(1/tr)^{8t}\delta_1 n^2\ge e(G)-\delta_1 n_{t-1}^2$ edges. Hence by (19), the resulting graph $\widetilde{G}$ is K_t-free with $e(\widetilde{G})\ge f(n_1,\dots,n_t,1,t)-2\delta_1 n_{t-1}^2$. Therefore, by Theorem 4.1, $\widetilde{G}$ has an (X,ϵ)-stable $(t-1)$-partition $\mathcal{P}$ and each class of $\mathcal{P}$ has size larger than $(1-\epsilon)n_{t-1}$. Thus we finish the proof of the theorem in this case.

Case 2. $\tau\ge 1$. Let $U_L=V_1\cup\ldots\cup V_\tau$ and $U_S=V(K)-V_L$. Clearly, we have $\tau\le t-2$ (recall the definition of τ). Hence,

$$f(n_1,\dots,n_r,1,t)\ge e(K_{n_1,\dots,n_\tau})-|U_S|^2\ge e(K_{n_1,\dots,n_r})-(rn_{\tau+1})^2. \tag{15}$$

Let Z_i^1 be the vertices of large V_i with degree less than $n-|V_i|-2rn_{\tau+1}$. Thus $e(G)\le e(K_{n_1,\dots,n_r})-2rn_{\tau+1}|Z_i^1|$. Hence by (19) and (20), we have $2rn_{\tau+1}|Z_i^1|\le 2(rn_{\tau+1})^2$ and thus $|Z_i^1|\le rn_{\tau+1}\le r(1/tr)^4 n_\tau\le(1/tr)^3 n_\tau$. Note that every vertex of V_i except Z_i^1 is adjacent to all but at most $2rn_{\tau+1}\le 2rn_\tau/(tr)^4$ vertices in other classes of $K_{n_1,\dots,n_\tau}$. Therefore, we can easily find a copy of K_τ^s in the subgraph of G induced by $V_1,\dots,V_\tau$. In fact, for any si vertices from $(V_1\setminus Z_1^1)\cup\ldots\cup(V_i\setminus Z_i^1)$, their common neighbours in $V_{i+1}\setminus Z_{i+1}^1$ is at least s. Thus we can find a copy of K_τ^s from $V_1\setminus Z_1^1,\dots,V_\tau\setminus Z_\tau^1$ recursively.

Now we will prove that $G[U_S]$ can be made $K_{t-\tau}^s$-free by removing at most $\delta_1 n_{\tau+1}^2$ edges. Suppose $G[U_S]$ contains a copy of $K_{t-\tau}^s$. Then there is at least one vertex of $K_{t-\tau}^s$ having neighbours less than $|U_L|-2trn_{\tau+1}$ in U_L, as otherwise all vertices of $K_{t-\tau}^s$ will have more than $n_i-2(tr)^2n_{\tau+1}$ common neighbours in each set of $V_1,\dots,V_\tau$. Thus we can extend $K_{t-\tau}^s$ to K_t^s with some s vertices from each V_i as in the end of last paragraph, a contradiction to that G is K_t^s-free. Therefore, at least one vertex v_1 of $K_{t-\tau}^s$ has degree at most $|U_L|-2trn_{\tau+1}$ in U_L, implying $d_G(v_1)\le|U_L|-trn_{\tau+1}$.

Let $H^1=G_{\{v_1\}\to U_L}$. Thus $e(H^1)\ge e(G)+|U_L|-d_G(v_1)\ge e(G)+trn_{\tau+1}$. Since G is K_t^s-free and v_1 is not adjacent to any copy of K_{t-1} in H^1, H^1 is still K_t^s-free.

If there is a copy of $K_{t-\tau}^s$ in $H^1[U_S]$, then let $H^2=H^1_{\{v_2\}\to U_L}$, where v_2 is a vertex in $K_{t-\tau}^s$ with $d_{H_1}(v_2)\le|U_L|-trn_{\tau+1}$. We repeat this process recursively and obtain graph H^ℓ such that $H^\ell[U_S]$ is $K_{t-\tau}^s$-free. By Lemma 2.1, $H^\ell[U_S]$ contains a $K_{t-\tau}$-free subgraph H_S with $e(H_S)\ge e(H^\ell[U_S])-(1/tr)^{8t}\delta_1 n_{\tau+1}^2$. Thus by Theorem 1.1, we have

$$e(H^\ell[U_S])\le f(n_{\tau+1},\dots,n_r,1,t-\tau)+(1/tr)^{8t}\delta_1 n_{\tau+1}^2. \tag{16}$$

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It follows from Proposition 3.4 and (19) that

$$\begin{aligned} e(H^\ell[U_S]) \geq & e(H^\ell) - \sum\nolimits_{1\leq i<j\leq \tau} n_i n_j - \sum\nolimits_{i=1}^{\tau} n_i \sum\nolimits_{i=\tau+1}^{r} n_i \\ \geq & e(G) + \ell t r n_{\tau+1} - \sum\nolimits_{1\leq i<j\leq \tau} n_i n_j - \sum\nolimits_{i=1}^{\tau} n_i \sum\nolimits_{i=\tau+1}^{r} n_i \qquad (17) \\ \geq & f(n_1,\ldots,n_r,1,t) - \delta n_{t-1}^2 + \ell t r n_{\tau+1} \\ & - \sum\nolimits_{1\leq i<j\leq \tau} n_i n_j - \sum\nolimits_{i=1}^{\tau} n_i \sum\nolimits_{i=\tau+1}^{r} n_i \\ \geq & f(n_{\tau+1},\ldots,n_r,1,t-\tau) + \ell t r n_{\tau+1} - \delta n_{t-1}^2. \qquad (17) \end{aligned}$$

Therefore, by (21) and (23) we have $\ell t r n_{\tau+1} \leq (1/tr)^{8t}\delta_1 n_{\tau+1}^2 + \delta n_{t-1}^2 \leq (1/tr)^{8t} tr\delta_1 n_{\tau+1}^2$. By the definition of τ, we have $n_{\tau+1} \leq (tr)^{t-1} n_{t-1}$. Thus we only need to remove in total

$$\ell|U_S| \leq \ell \cdot r n_{\tau+1} \leq (1/tr)^{8t} tr\delta_1 n_{\tau+1}^2 = \frac{\delta_1 n_{\tau+1}^2}{(tr)^{8t-1}} \leq \delta_1 n_{t-1}^2$$

edges to make $G[U_S]$ $K_{t-\tau}^s$-free.

We denote the new graph by $\widetilde{G}$ after removing these $\delta_1 n_{t-1}^2$ edges in G. Thus

$$\begin{aligned} e(\widetilde{G}[U_S]) \geq & \, e(G[U_S]) - \delta_1 n_{t-1}^2 \\ \geq & \, e(G) - \sum\nolimits_{1\leq i<j\leq \tau} n_i n_j - \sum\nolimits_{i=1}^{\tau} n_i \sum\nolimits_{i=\tau+1}^{r} n_i - \delta_1 n_{t-1}^2 \\ \geq & \, f(n_1,\ldots,n_r,1,t) - \sum\nolimits_{1\leq i<j\leq \tau} n_i n_j - \sum\nolimits_{i=1}^{\tau} n_i \sum\nolimits_{i=\tau+1}^{r} n_i - 2\delta_1 n_{t-1}^2 \\ \geq & \, f(n_{\tau+1},\ldots,n_r,1,t-\tau) - 2\delta_1 n_{t-1}^2. \end{aligned}$$

Hence by Theorem 4.1 there exists an $(X_S,\epsilon/2)$-stable $(t-\tau-1)$-partition $\mathcal{P}$ of $\widetilde{G}[U_S]$ and each integral class of $\mathcal{P}_{X_S}$ has size larger than $(1-\epsilon/2)n_{t-1}$.

Let Z_i^2 be the vertices of large V_i with degree less than $n-|V_i|-2\sqrt{\delta}n_{t-1}$. Recall that $\widetilde{G}[U_S]$ is $K_{t-\tau}^s$-free. By Theorem 1.1, we have $e(\widetilde{G}[U_S]) \leq f(n_{\tau+1},\ldots,n_r,1,t-\tau)$. From (19) and Proposition 3.4, we have

$$\begin{aligned} e(\widetilde{G}) - e(\widetilde{G}[U_S]) \geq & \, f(n_1,\ldots,n_t,1,t) - \delta n_{t-1}^2 - f(n_{\tau+1},\ldots,n_r,1,t-\tau) \\ \geq & \sum_{1\leq i<j\leq \tau} n_i n_j - \sum_{i=1}^{\tau} n_i \sum_{i=\tau+1}^{r} n_i - \delta n_{t-1}^2. \end{aligned}$$

By definition of Z_i^2, we have

$$e(\widetilde{G}) - e(\widetilde{G}[U_S]) \leq \sum_{1\leq i<j\leq \tau} n_i n_j - \sum_{i=1}^{\tau} n_i \sum_{i=\tau+1}^{r} n_i - |Z_i^2| \cdot 2\sqrt{\delta} n_{t-1}.$$

Thus we have $|Z_i^2| \leq 2\sqrt{\delta}n_{t-1}$. Let $X = X_S \cup Z_1^2 \cup \ldots \cup Z_\tau^2$. Therefore, $\mathcal{P}$ is an (X,ϵ)-stable $(t-1)$-partition of G after removing ϵn_{t-1}^2 edges with $|X| \leq (2r\sqrt{\delta}+\epsilon/2)n_{t-1} \leq \epsilon n_{t-1}$ and the size of each integral class of $\mathcal{P}$ is larger than $(1-\epsilon)n_{t-1}$. We finish the proof of Theorem 1.15.

Proof of Theorem 1.17. Let F be a graph with chromatic number $t \geq 3$ and $G \subseteq K_{n_1,\ldots,n_r}$ be an extremal graph for F. Let $\epsilon \gg \delta_3 \gg \delta_2 \gg \delta_1 > 0$ and $n_{t-1} \geq 1/\delta_1$.

By Theorem 1.1, there is an extremal graph for K_t which is $(t-1)$-partite, implying $e(G) \geq f(n_1,\ldots,n_r,1,t)$ (since every $(t-1)$-partite graph is F-free). Since $\delta_2 \gg \delta_1$, by Theorem 1.15, after removing at most $\delta_2 n_{t-1}^2$ edges, the resulting graph $\widetilde{G}$ has an (X,δ_2)-stable $(t-1)$-partition $\mathcal{P} = (P_1,\ldots,P_{t-1})$. Moreover, $|I(P_i \setminus X)| \geq (1-\delta_2)n_{t-1}$ for any $i \in [t-1]$. Thus we finish the proof of Theorem 1.17(a). Moreover, we may edit $\mathcal{P}$ such that $X_i \subseteq P_i$ and each vertex of X has the smallest number of neighbours in its own class of $\mathcal{P}$.

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By Proposition 2.3 there exists an $\eta \in [\delta_2, \sqrt[10^{(t-1)r}]{\delta_2}]$ such that every set of $(\mathcal{P} \wedge \mathcal{V})_X$ with size either less than ηn_{t-1} or more than $\sqrt[10]{\eta} n_{t-1}$. Let Z' be the union of sets in $(\mathcal{P} \wedge \mathcal{V})_X$ with size less than ηn_{t-1} and $Z = Z' \cup X$. Thus $|Z| \le tr\eta n_{t-1} + \delta_2 n_{t-1} \le 2tr\eta n_{t-1}$ and $\mathcal{P}_Z$ is $\sqrt{\eta} n_{t-1}$-stable to $\mathcal{V}_Z$. Let

$$\widetilde{V}_i = V_i \setminus Z,\ \widetilde{P}_i = P_i \setminus Z,\ X_{i,j} = \widetilde{V}_i \cap \widetilde{P}_j,\ \widetilde{\mathcal{P}} = (\widetilde{P}_1, \ldots, \widetilde{P}_{t-1}) \text{ and } \widetilde{\mathcal{V}} = (\widetilde{V}_1, \ldots, \widetilde{V}_r).$$

Let $v \in V_k$ and P_{i_k} be a set with $\sum_{i \ne i_k} |\widetilde{P}_i \setminus \widetilde{V}_k| = \max\{\sum_{i \ne \ell} |\widetilde{P}_i \setminus \widetilde{V}_k| : 1 \le \ell \le t-1\}$. Note that it is possible that $\widetilde{V}_k = \emptyset$. For any integral $\widetilde{V}_s$ in $\widetilde{P}_{i_k}$, we can choose $B \subseteq \widetilde{V}_s$ with size $|F| \le \sqrt[10]{\eta} n_{t-1}$. Since $\mathcal{P}$ is an (X, δ_2)-stable $(t-1)$-partition of G and $X \subseteq Z$, we have $|N_{\widetilde{G}}[B]| \ge \sum_{i \ne i_k} |\widetilde{P}_i \setminus \widetilde{V}_k| - \delta_2 |F| n_{t-1}$. By Proposition 3.3, $G_{\{v\} \to N_{\widetilde{G}}[B]}$ is also F-free. Since G is an extremal graph, we have

$$d_G(v) \ge |N_{\widetilde{G}}[B]| \ge \sum\nolimits_{i \ne i_k} |\widetilde{P}_i \setminus \widetilde{V}_k| - \delta_2 |F| n_{t-1},$$

as otherwise $G_{\{v\} \to N_{\widetilde{G}}[B]}$ is F-free with more edges, a contradiction. Thus, if $v \in V_j$, then we have

$$d_G(v) \ge \max\left\{\sum\nolimits_{i \ne \ell} |P_i \setminus V_j| : 1 \le \ell \le t-1\right\} - \epsilon n_{t-1}, \tag{18}$$

where $\epsilon = \sqrt[5]{\eta}$. We finish the proof of Theorem 1.17(b).

For any $v \in V_k \cap Z$, let $\widehat{\mathcal{P}} = (\widetilde{P}_1, \ldots, \widetilde{P}_{i_k} \cup \{v\}, \ldots, \widetilde{P}_{t-1})$ and $\widehat{\mathcal{V}} = (\widetilde{V}_1, \ldots, \widetilde{V}_{j_k} \cup \{v\} \ldots, \widetilde{V}_{t-1})$, where $\widetilde{V}_{j_k}$ is integral in $\widetilde{P}_{i_k}$. Let $G^* = G[\widehat{\mathcal{P}}]$. Since $\mathcal{P}_X$ is stable to $\mathcal{V}_X$, $\widehat{\mathcal{P}}$ is $\sqrt{\eta} n_{t-1}$-stable to $\widehat{\mathcal{V}}$. Moreover, the size of each part of $\widehat{\mathcal{P}} \wedge \widehat{\mathcal{V}}$ is at least $\sqrt[10]{\eta} n_{t-1}$. Note that $\sqrt{\eta} \ll \sqrt[5]{\eta}$. By (24), we can apply Proposition 2.5 to G^* to get that there exists an integer $m(v) \in [t-1]$ such that v is adjacent to $\sqrt[10]{\eta} n_{t-1}/4$ vertices from the integral part of each $\widetilde{P}_{i \ne m(v)}$. Thus we can partition Z into $Z_1, \ldots, Z_{t-1}$ such that $v \in Z_i$ is adjacent to $\sqrt[10]{\eta} n_{t-1}/4$ vertices from the integral part of each $\widetilde{P}_{j \ne i}$. Moreover, by $\mathcal{P} = (P_1, \ldots, P_{t-1})$ is an (X, δ_2)-stable $(t-1)$-partition and $\delta_2 \ll \sqrt[10]{\eta}$, we can edit $P_i = \widetilde{P}_i \cup Z_i$ for $i \in [t-1]$.

Let Y_i be the vertices of P_i with more than $\sqrt[4]{\eta} n_{t-1}$ neighbours in their own part in G, and let $Y = \bigcup_{i=1}^{t-1} Y_i$. We will show that Y satisfies (c) and (d). For $v \in Y_i$, let $Q_{j \ne i}(v)$ be the neighbours of v in the integral part of $\widetilde{P}_j$ and Q_i be the neighbours of v in $\widetilde{P}_i$. From the last paragraph, we see that $|Q_i(v)| \ge \sqrt[3]{\eta} n_{t-1}$ for each $i \in [t-1]$. Since $\mathcal{P}$ is an (X, δ_2)-stable partition of $\widetilde{G}$, we can choose $1/\delta_3$ vertices from Q_i and recursively pick up $1/\delta_3$ vertices from $Q_j(v)$ for each $j \ne i$ such that they induce a copy of K_{t-1}^{1/δ_3} (similar as the proof in the first paragraph of Case 2 in Theorem 1.15). Thus Theorem 1.17(d) holds.

Let $\mathcal{Q}(v) = (Q_1(v), \ldots, Q_{t-1}(v))$. For a set of vertices R, let

$$\bigcap\nolimits_{v \in R} \mathcal{Q}(v) = \left(\bigcap\nolimits_{v \in R} Q_1(v), \ldots, \bigcap\nolimits_{v \in R} Q_{t-1}(v)\right).$$

Let $\tau = \tau(n_1, \ldots, n_r, t, r)$, $\alpha = \sqrt[4]{\eta}/r^{4t}$ and

$$c_{r,\epsilon,F} = tN(\alpha, \ldots, \alpha; 1/2; |F|),$$

where the N function is defined by Lemma 2.2.

Case 1. If $\tau = 0$, then $n_{t-1} \ge (1/r)^{4t} n$. Recall that for $i \in [t-1]$, every $v \in Y$ satisfies $|Q_i(v)| \ge \sqrt[4]{\eta} n_{t-1}$. Thus $|Q_i(v)| \ge \sqrt[4]{\eta} n_{t-1} \ge \sqrt[4]{\eta}(1/r)^{4t} n \ge \sqrt[4]{\eta}/r^{4t} \cdot |P_i| = \alpha |P_i|$. If $|Y_1| \ge N(\alpha, \ldots, \alpha; 1/2; |F|)$, then by Lemma 2.2 there exist $|F|$ vertices, say C_F, of Y_1 such that every part of $\bigcap_{v \in C_F} \mathcal{Q}(v)$ has size larger than $(\sqrt[4]{\eta}/2r^{4t})^{|F|} n_{t-1}$. By the definition of $\bigcap_{v \in C_F} \mathcal{Q}(v)$, at least $(t-2)$ parts of $\bigcap_{v \in C_F} \mathcal{Q}(v)$ belong to the integral parts of $\mathcal{P}_Z$. As in the previous proofs, we can choose $|F|$ vertices from $\bigcap_{v \in C_F} Q_1(v)$ and pick up to $|F|$ from $\bigcap_{v \in C_F} Q_i(v)$ for $i = 2, \ldots, t-1$ recursively such that they induce a copy of $K_{t-1}^{|F|}$. Together with C_F, there is a copy of $K_t^{|F|}$, implying $F \subset G$, a contradiction. Hence $\sum_{i=1}^{t-1} |Y_i| < tN(\alpha, \ldots, \alpha; 1/2; |F|) \le c_{r,\epsilon,F}$.

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Case 2. Suppose $\tau \geq 1$. Then $1 \leq \tau \leq t-2$. We will show that large sets $V_1, \ldots, V_\tau$ are parts of $\mathcal{P}$. We first show $\widetilde{V}_1, \ldots, \widetilde{V}_\tau$ are parts of $\mathcal{P}_Z$. Let $j \in [\tau]$ be the minimum integer such that $\widetilde{V}_j$ is not a part of $\mathcal{P}_Z$. Without loss of generality, let $\widetilde{V}_j \cap \widetilde{P}_1 \neq \emptyset$. If $\widetilde{V}_j$ is partial in $\widetilde{P}_1$, then since $\mathcal{P}_Z$ is $\sqrt{\eta}n_{t-1}$-stable to $\mathcal{V}_Z$ we have $|I(\widetilde{P}_i)| = |\widetilde{P}_i \setminus \widetilde{V}_j| \geq |\widetilde{V}_j| - \sqrt{\eta}n_{t-1}$ and each part of $\mathcal{P}_Z$ has size larger than $|\widetilde{V}_j| - 2\sqrt{\eta}n_{t-1}$. If $\widetilde{V}_j$ is integral in $\widetilde{P}_1$, then since $\mathcal{P}_Z$ is $\sqrt{\eta}n_{t-1}$-stable to $\mathcal{V}_Z$, each part of $\mathcal{P}_Z$ has size larger than $|\widetilde{P}_1 \setminus \widetilde{V}_{i_*}| - \sqrt{\eta}n_{t-1} \geq |\widetilde{V}_j| - 2\sqrt{\eta}n_{t-1}$, where $\widetilde{V}_{i_*} \cap \widetilde{P}_1 \neq \emptyset$. In either case, from the definition of τ, we have $|\bigcup_{i=1}^{t-1} \widetilde{P}_i| \geq |\bigcup_{i=1}^{j-1} \widetilde{V}_i| + (t-j)(|\widetilde{V}_j| - 2\sqrt{\eta}n_{t-1}) > n$, a contradiction. Thus we have $\widetilde{P}_1 = \widetilde{V}_1, \ldots, \widetilde{P}_\tau = \widetilde{V}_\tau$. By (24) and $n_\tau > (tr)^4 n_{\tau+1}$, we obtain $P_1 = V_1, \ldots, P_\tau = V_\tau$.

Since each vertex in every $P_{i \leq \tau}$ has no neighbour in its own part, we only need to consider the sets $Y_{i>\tau}$'s. Note that $n_{t-1} \geq (1/r)^{4t} n_{\tau+1}$ (by the definition of τ). Thus $|P_i| \leq r n_{\tau+1} \leq r^{5t} n_{t-1}$ for $i > \tau$. For each vertex $a \in Y_i$ with $i \geq \tau + 1$, each part of $\mathcal{Q}(a)$ in $P_{i>\tau}$ has size larger than $\sqrt[4]{\eta}n_{t-1} \geq \alpha |P_i|$. If $|Y_i| \geq N(\alpha_1, \ldots, \alpha_{t-\tau-1}; 1/2; |F|)$, as the proof in Case 1, there is a copy of $K_{t-\tau}^{|F|}$ in $P_{i>\tau}$'s. Note that each vertex of $K_{t-\tau}^{|F|}$ has at least $|V_i| - \delta_2 n_{t-1}$ neighbours in each V_i for each $i \leq \tau$. As in the previous proofs, we can extend this copy of $K_{t-\tau}^{|F|}$ to a copy of $K_t^{|F|}$, implying $F \subset G$, a contradiction. Thus we have $|Y_i| < N(\alpha_1, \ldots, \alpha_{t-\tau-1}; 1/2; |F|)$, implying $|Y| \leq c_{r,\epsilon,F}$.

For vertex $v \in V_k \setminus Y$ with $k \in [r]$, since the size of the neighbours of a in its own part is less than $\sqrt[4]{\eta}n_{t-1}$, by (24) we have $|N_{K[\mathcal{P}_Y]}(v) \triangle N_{G-Y}(v)| \leq \epsilon n_{t-1}$. Thus Theorem 1.17(c) holds and we finish the proof the strong stability theorem.

5. Applications

Proofs of Theorems 1.2 and 1.4. Given $r \geq t \geq 3$ and $k \geq 2$, let $n_1 \geq \ldots \geq n_r$ and n_{t-1} be sufficiently large. Let $G \subseteq K_{n_1,\ldots,n_r}$ be an extremal graph for kK_t. We will show that $e(G) = g(n_1, \ldots, n_r, k, t)$. Moreover, the extremal graphs are obtained from $(t-1)$-partite graphs by joining all possible edges incident with $k-1$ fixed vertices.

Since the graphs are obtained from $(t-1)$-partite graphs by joining all possible edges incident with $k-1$ fixed vertices are kK_t-free, we have

$$e(G) \geq g(n_1, \ldots, n_r, k, t) \geq f(n_1, \ldots, n_r, 1, t), \tag{19}$$

where the second inequality follows by the definition of $g(n_1, \ldots, n_r, 1, t)$. Let $\epsilon > 0$ be a small constant. By Theorem 1.17 there exist constants $\delta_1 \ll \delta_2 \ll \delta_3 \ll \epsilon$ and $c_{r,\epsilon,F}$ depending on and r, ϵ and F such that if $n_{t-1} \geq 1/\delta_1$, then

(a) there exists an (X, δ_2)-stable $(t-1)$-partition $\mathcal{P}$,
(b) $d_G(v) \geq \max\{\sum_{i \neq \ell} |P_i \setminus V_k| : 1 \leq \ell \leq t-1\} - \epsilon n_{t-1}$ for each $v \in V_k$,
(c) there exists a set Y with $|Y| \leq c_{r,\epsilon,F}$ such that every vertex v of $G - Y$ satisfies $|N_G(v) \triangle N_{K_{n_1,\ldots,n_r}[\mathcal{P}_Y]}(v)| \leq \epsilon n_{t-1}$ and
(d) every vertex of Y is adjacent to at lest $1/\delta_3$ vertices of each class of $\mathcal{P}$ such that they induce a complete $(t-1)$-partite graph.

For an edge xy inside $P_s \setminus Y$, we say xy *good* if x and y are adjacent to each vertex of a copy of K_{t-2}^{kt} consisting of edges between different P_i's and say xy *bad* otherwise (see Figure 4). Let E be the set of good edges.

Claim 1. $|E| \leq 2k\epsilon n_{t-1}$.

Proof. Clearly, there is no matching on k edges in E, as otherwise we can easily find a copy of kK_t. Since each vertex is adjacent to at most ϵn_{t-1} vertices in their own class of $\mathcal{P}$, we have $|E| \leq 2k\epsilon n_{t-1}$. □

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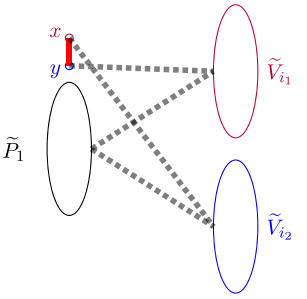


Figure 4. A bad edge xy with $x \in V_{i_1}, y \in V_{i_2}$.

Claim 2. *If $Y = \emptyset$, then $e(G) - |E| \le g(n_1, \dots, n_r, 1, t)$. Moreover, if the equality holds, then $G - E$ is a $(t-1)$-partite graph.*

Proof. Let $\widetilde{V}_i = V_i \setminus X$, $\widetilde{P}_j = P_j \setminus X$, $X_{i,j} = \widetilde{V}_i \cap \widetilde{P}_j$ and $X_i = X \cap P_i$. Let m be the minimum size of the integral part of all classes of $\mathcal{P}_X$. Let xy be a bad edge in P_s with $x \in V_{i_1}$ and $y \in V_{i_2}$ (it is possible that $x \in X$ or $y \in X$). We will show that there exists a unique $\ell = \ell(i_1, i_2) \neq s$ such that

$$|X_{i_1,\ell}| = m + \Theta(\epsilon n_{t-1}), |X_{i_2,\ell}| = m + \Theta(\epsilon n_{t-1}) \text{ and } P_\ell \setminus (V_{i_1} \cup V_{i_2}) = \emptyset. \tag{20}$$

First, we will show that there exists an $\ell \neq s$ such that $m - 6\epsilon n_{t-1} \le |X_{i_1,\ell}|$. Suppose $|X_{i_1,j}| < m - 6\epsilon n_{t-1}$ for each $j \in [t-1]$ distinct from s. Since $Y = \emptyset$, by (c) x is adjacent to at least $|\widetilde{P}_j \setminus \widetilde{V}_{i_1}| - \epsilon n_{t-1}$ vertices of $\widetilde{P}_j$ for any $j \neq s$. Hence, by (a) x is adjacent to at least $m - |X_{i_1,j}| - \epsilon n_{t-1} \ge 5\epsilon n_{t-1}$ vertices of each integral part of $\widetilde{P}_j$ for any $j \neq s$. We divide the proof into the following cases.

Case 2.1.

$V_{i_2} \subseteq X$ or $\widetilde{V}_{i_2}$ is integral in $\widetilde{P}_s$. By (c), y is adjacent to all but at most ϵn_{t-1} vertices of $\widetilde{P}_{j\neq s} \setminus V_{i_2}$ and hence x and y are adjacent to a copy of K^{kt}_{t-2} consisting of edges between different $P_{j\neq s}$'s, a contradiction to that xy is a bad edge.

Case 2.2.

$\widetilde{V}_{i_2}$ is partial in $\mathcal{P}_X$. By (c), y is adjacent to all but at most ϵn_{t-1} vertices of each integral part of $\widetilde{P}_{j\neq s}$. Again, x and y are adjacent to a copy of K^{kt}_{t-2} consisting of edges between different integral parts of $P_{j\neq s}$'s, a contradiction to that xy is a bad edge.

Case 2.3.

$\widetilde{V}_{i_2}$ is integral in $\widetilde{P}_q$ with $q \neq s$. It follows from (b), (c) and $Y = \emptyset$ that $\sum_{i\neq q} |\widetilde{P}_i \setminus V_{i_2}| - \epsilon n_{t-1} \le d_G(y) \le \sum_{i\neq s} |\widetilde{P}_i \setminus V_{i_2}| + |X| + \epsilon n_{t-1}$. Thus we have $|\widetilde{P}_s| \le |\widetilde{P}_q \setminus \widetilde{V}_{i_2}| + 3\epsilon n_{t-1}$. Note that $|\widetilde{P}_s| \ge m$ and $|\widetilde{P}_q \setminus \widetilde{V}_{i_2}| \le m$ (by (a), $\mathcal{P}_X$ is stable to $\mathcal{V}_X$). Therefore, $m \le |\widetilde{P}_s| \le m + 3\epsilon n_{t-1}$ and $m - 3\epsilon n_{t-1} \le |\widetilde{P}_q \setminus \widetilde{V}_{i_2}| = |\widetilde{P}_q| - |X_{i_2,q}| \le m$. Recall that $|X_{i_1,q}| < m - 6\epsilon n_{t-1}$. Then $|\widetilde{P}_q| - |X_{i_2,q}| - |X_{i_1,q}| \ge 3\epsilon n_{t-1}$. By (c), we can easily see that x and y have at least kt common neighbours in $\widetilde{P}_q \setminus (X_{i_2,q} \cup X_{i_1,q})$. Hence x and y are adjacent to each vertex of a copy of K^{kt}_{t-2} consisting of edges between different $P_{j\neq s}$'s (at least $t-3$ parts of K^{kt}_{t-2} are from integral parts in $P_{j\neq s,q}$'s), a contradiction to that xy is a bad edge.

Therefore, there exist $\ell_1, \ell_2 \neq s$ such that

$$m - 6\epsilon n_{t-1} \le |X_{i_1,\ell_1}| \quad \textit{and} \quad m - 6\epsilon n_{t-1} \le |X_{i_2,\ell_2}|. \tag{21}$$

Moreover, for $\eta = 1, 2$ if there is an $\alpha \notin \{\ell_\eta, s\}$ such that $|X_{i_\eta,\alpha}| \ge m - 6\epsilon n_{t-1}$, then $\widetilde{V}_{i_\eta}$ is partial with size at least $2m - 12\epsilon n_{t-1}$, a contradiction to (a).

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Now we will show that $\ell_1 = \ell_2 = \ell$ and $|X_{i_1,\ell}| = m + \Theta(\epsilon n_{t-1})$ and $|X_{i_2,\ell}| = m + \Theta(\epsilon n_{t-1})$. Consider vertices $x \in V_{i_1}$ and $z \in X_{i_1,\ell_1}$. Applying (c), we have $d_G(x) = \sum_{i\neq s} |\widetilde{P}_i \setminus V_{i_1}| + \Theta(\epsilon n_{t-1})$ and $d_G(z) = \sum_{i\neq \ell_1} |\widetilde{P}_i \setminus V_{i_1}| + \Theta(\epsilon n_{t-1})$. Combining with (b) we have

$$\max\left\{\sum_{i\neq j} |\widetilde{P}_i \setminus V_{i_1}| : j \in [t-1]\right\} = \sum_{i\neq s} |\widetilde{P}_i \setminus V_{i_1}| + \Theta(\epsilon n_{t-1}) = \sum_{i\neq \ell_1} |\widetilde{P}_i \setminus V_{i_1}| + \Theta(\epsilon n_{t-1}),$$

implying

$$|\widetilde{P}_{\ell_1}| - |X_{i_1,\ell_1}| = |\widetilde{P}_s| - |X_{i_1,s}| + \Theta(\epsilon n_{t-1}). \tag{22}$$

Note that $|\widetilde{P}_s| \geq m$, $|\widetilde{P}_{\ell_1}| - |X_{i_1,\ell_1}| \leq m$ and $|X_{i_1,s}| \leq 6\epsilon n_{t-1}$ (by (a), $\mathcal{P}_X$ is stable to $\mathcal{V}_X$). Thus (28) implies $|\widetilde{P}_s| = m + \Theta(\epsilon n_{t-1})$ and $|\widetilde{P}_{\ell_1}| - |X_{i_1,\ell_1}| = m + \Theta(\epsilon n_{t-1})$. By symmetry, we have $|\widetilde{P}_{\ell_2}| - |X_{i_2,\ell_2}| = m + \Theta(\epsilon n_{t-1})$. If $\ell_1 = \ell_2 = \ell$, then combining with (27), we have $|X_{i_2,\ell}| = m + \Theta(\epsilon n_{t-1})$ and $|X_{i_2,\ell}| = m + \Theta(\epsilon n_{t-1})$ and we are done. Assume that $\ell_1 \neq \ell_2$. Then $|X_{i_1,i\neq \ell_1}| = \Theta(\epsilon n_{t-1})$ and $|X_{i_2,i\neq \ell_2}| = \Theta(\epsilon n_{t-1})$ for each $i \in [t-1]$ by $\mathcal{P}_X$ is stable to $\mathcal{V}_X$ and (27). Recall that the size of each partial sets $\widetilde{V}_i$ in $\mathcal{P}_X$ is at most m. It follows from (c) and $|\widetilde{P}_{\ell_\eta}| - |X_{i_\eta,\ell_\eta}| = m + \Theta(\epsilon n_{t-1})$ for $\eta = 1, 2$ that x and y are adjacent to each vertex of a copy of K^{kt}_{t-2}, a contradiction.

Finally, we show that $P_\ell \setminus (V_{i_1} \cup V_{i_2}) = \emptyset$. Suppose that there exists a vertex $z \in P_\ell \setminus (V_{i_1} \cup V_{i_2})$ and let $z \in V_{i_3}$. Thus by (*c*) and $Y = \emptyset$, we have $d_G(z) = \sum_{i\neq \ell} |P_i \setminus V_{i_3}| + \Theta(\epsilon n_{t-1}) \leq n - 2m + \Theta(\epsilon n_{t-1})$. Since $|\widetilde{P}_s| = m + \Theta(\epsilon n_{t-1})$, it follows from (b) that $d_G(z) \geq \sum_{i\neq s} |\widetilde{P}_i| + \Theta(\epsilon n_{t-1}) = n - m + \Theta(\epsilon n_{t-1})$, a contradiction. Thus $P_\ell \setminus (V_{i_1} \cup V_{i_2}) = \emptyset$, proving (26).

We call such $P_{\ell(i_1,i_2)}$ a *bad class* of the bad edge xy and call such (i_1, i_2) *bad pair*. We conclude that

$$\text{there is no } i \in [r] \text{ which appears in two bad pairs.} \tag{23}$$

Otherwise, we have bad pairs $(i_1, i_2), (i_2, i_3)$, and hence $\ell(i_1, i_2) \neq \ell(i_2, i_3)$ by (26). Therefore, $X_{i_2,\ell(i_1,i_2)}$ and $X_{i_2,\ell(i_2,i_3)}$ are partial classes with size $m + \Theta(\epsilon n_{t-1})$ in $\mathcal{P}_X$, implying $|\widetilde{V}_{i_2}| = 2m + \Theta(\epsilon n_{t-1})$ a contradiction to (a).

Let $G' = G - E$. Suppose that there is a bad edge in G'. Let K_α be any maximal clique in G'. By (29), K_α contains at most one bad edge in P_i for any $i \in [t-1]$, and by (26) if K_α contains β bad edges in different P_i's, then it cannot contain vertices in β corresponding bad classes of those β edges. Therefore, $\alpha \leq 2\beta + (t - 1 - \beta) - \beta \leq t - 1$, whence G' is K_t-free.

Suppose that there is a bad edge v_iv_j in P_1 with $v_i \in V_i$ and $v_j \in V_j$, and let $\ell = \ell(i, j)$. By (29), there is no bad edge between $P_1 \setminus (V_i \cup V_j)$ and $P_1 \cap (V_i \cup V_j)$. Moreover, there is no bad edge between V_i and V_j in P_ℓ, as otherwise by (26) there exists an $\alpha \neq \ell$ such that $|X_{i_1,\alpha}| = m + \Theta(\epsilon n_{t-1})$, a contradiction to (a) since $|X_{i_1,\ell}| = m + \Theta(\epsilon n_{t-1})$ and $\widetilde{V}_{i_1}$ is partial in $\mathcal{P}_X$.

If $|P_1 \setminus (V_i \cup V_j)| > |V_j \cap P_\ell|$, then let $H_1 = G'_{V_i\cap P_1 \to S}$ with $S = \bigcup_{\iota\neq\ell} P_\iota \setminus V_i$. By Proposition 3.3, H_1 is K_t-free with $e(G') = e(H_1) + |V_i \cap P_1|(|P_1 \setminus (V_i \cup V_j)| - |V_j \cap P_\ell|)$. Hence, by Theorem 1.1, $e(G') < e(H_1) \leq g(n_1, \ldots, n_r, 1, t)$. Thus we may assume that $|P_1 \setminus (V_i \cup V_j)| \leq \min\{|V_i \cap P_\ell|, |V_j \cap P_\ell|\}$.

Let H_2 be the graph obtained from G' by deleting all edges between $P_1 \setminus (V_i \cup V_j)$ and $V_i \cap P_\ell$, then adding all possible edges between $V_j \cap P_\ell$ and $V_i \cap P_\ell$ and all possible edges between $V_j \cap P_1$ and $P_1 \setminus (V_i \cap V_j)$. Let $A = (P_1 \cup P_\ell) \cap V_j$ and $B = (P_1 \cup P_\ell) \setminus V_j$. Then $H_2[A \cup B]$ is a bipartite graph since no bad edge between $P_1 \setminus (V_i \cup V_j)$ and $P_1 \cap (V_i \cup V_j)$. Similar as the proof for that G' is K_t-free, $H_2[V(G' \setminus (A \cup B)]$ is K_{t-2}-free, implying H_2 is K_t-free. Let $|V_i \cap P_\ell| = a$, $|V_j \cap P_\ell| = b$, $|P_1 \cap V_j| = c$ and $|P_1 \setminus (V_i \cup V_j)| = d$. Then $e(H_2) \geq e(G') + ab + cd - ad > e(G')$, where the last strict inequality holds by $b \geq d$ and $cd > 1$. By Theorem 1.1, $e(G') < e(H_2) \leq g(n_1, \ldots, n_r, 1, t)$, a contradiction. Therefore, there is no bad edge. Then $e(G') \leq g(n_1, \ldots, n_r, 1, t)$ and $G - E$ is a $(t-1)$-partite graph. We finish the proof of Claim 2. □

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We first prove Theorem 1.2. Clearly, (d) implies that Y is empty. Moreover, by Claim 2 and (25) we have $|E| = 0$ and $G[P_i]$ is an independent set for each $i \in [t-1]$, and hence the result follows by Theorem 1.1.

We prove Theorem 1.4 by induction on k. Suppose that the theorem holds for $k-1$. If there is a vertex $a \in Y$, i.e., Y is not empty, then by (d), a is adjacent to a copy of K_{t-1}^{kt}. Thus $G - \{a\}$ must be $(k-1)K_t$-free. Let $a \in V_\alpha$. By induction hypothesis, we have

$$e(G) \le e(G-\{a\}) + n - n_\alpha \le g(n_1, \dots, n_\alpha - 1, \dots, n_r, k-1, t) + n - n_\alpha \le g(n_1, \dots, n_r, k, t),$$

where the last inequality holds by the definition of $g(n_1, \dots, n_r, k, t)$. Moreover, the inequality holds only if $d_G(a) = n - n_\alpha$ (α depends on $n_1, \dots, n_r, k, t$) and $G - \{a\}$ is obtained from $K_{n_1,\dots,n_\alpha-1,\dots,n_r}[\mathcal{P}_\alpha]$ by joining all possible edges incident with $k-2$ fixed vertices, where $\mathcal{P}_\alpha$ is a $(t-1)$-partition for $K_{n_1,\dots,n_\alpha-1,\dots,n_r}$ (by induction hypothesis). Thus, the extremal graphs are obtained from $K_{n_1,\dots,n_r}[\mathcal{P}]$ by joining all possible edges incident with $k-1$ fixed vertices, where $\mathcal{P}$ is a $(t-1)$-partition for $K_{n_1,\dots,n_r}$.

Therefore, we may assume that Y is empty. Combining with Claims 1 and 2, we have $e(G) \le g(n_1, \dots, n_r, 1, t) + k\epsilon n_{t-1} < g(n_1, \dots, n_r, k, t)$, a contradiction, hence we finish the proof of Theorem 1.4.

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