Ergod. Th. & Dynam. Sys. (2007), **27**, 849–867
doi:10.1017/S0143385706000964
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Printed in the United Kingdom

Graph representatives for surface maps

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(*Received* 20 *December* 2004 *and accepted in revised form* 4 *October* 2006)

Abstract. Given a compact surface F with non-empty boundary and a homotopy class of self-maps on F, we consider the problem of finding a representative which carries the minimal number of fixed points possible for the class. When F is either the pants surface or the once punctured Möbius band this problem is solved by a graph map, where the graph is homotopy equivalent to F. A partial result is presented for the remaining two surfaces whose fundamental group is the free group on two generators.

0. *Introduction*

The modern study of the dynamics of homeomorphisms on surfaces came to prominence as a result of the work of Thurston [**23**], where he defines the notion of a pseudo-Anosov homeomorphism (see also the treatment given in [**9**]). By their definition, these homeomorphisms have a rigid geometrical structure coming from the hyperbolic geometry on the universal covering space of the surface. On the other hand, pseudo-Anosov homeomorphisms also have a very concrete and computable combinatorial aspect. Using the notion of folding introduced by Stallings [**22**], Bestvina and Handel [**6**] showed that the dynamical information for a pseudo-Anosov is carried by an action on a (one-dimensional (1D)) train-track. Similar results were independently obtained by Los [**20**], and by Franks and Misiurewicz [**11**], in the case that the surface is planar. A comprehensive treatment of these works and related dynamical properties of pseudo-Anosov homeomorphisms can be found in [**8**].

One feature of this study is that the train-track reduced the two-dimensional (2D) dynamical problem to a 1D problem by producing a graph representative. One can then use methods from 1D dynamics to obtain information about the original problem. For instance, information regarding the periodic orbit structure of the graph representative implies that any homeomorphism in the isotopy class of the pseudo-Anosov also has orbits with these prescribed periods. Recent results concerning periodic orbits for graph maps can be found in [**1, 7, 17, 19**]. There are numerous other papers dealing with the study of 1D dynamics. An excellent text which gives a survey of this area of research is [**3**].

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The focus of this paper is to study dynamics in two dimensions when a given map is no longer in the homotopy class of a homeomorphism. It is well known [**10**] that, except for the sphere and projective plane, such maps have degree zero. As a result, the homotopy class of the map contains a representative which has its image contained in a 1D complex in the surface.

The goal will be to find a reasonable representative map in a given homotopy class which somehow carries the minimal dynamical information for that class. In particular, we will be concerned with the fixed-point set of maps and ask for a concrete representative in the given homotopy class which carries the minimal number of fixed points possible. Such a map will be referred to as a *fixed-point minimal* map in this paper. This minimal number has been studied by this author in [**13**] and [**14**].

In a related direction, Llibre and Nunes [**18**] compute the minimal number of fixed points for maps on the figure-eight graph and compare these numbers to those obtained for the pants surface in [**13**]. Minimal models for 1D dynamical problems have also been considered in [**2**] and [**4**]. The topological problem presented in this paper has a natural algebraic version, which is the study of the fixed subgroup of a given group endomorphism. The case where the group is a finitely generated free group has been studied by a number of authors and has been applied to decision problems in geometric group theory (see, e.g., [**12**] and [**21**]). As the free groups are the fundamental groups of the surfaces with boundaries, it would be of interest to investigate the relationship between results obtained for the algebraic problem and the results presented in this paper.

In order to state the main result of this paper we first introduce some notation. Let F be a compact, connected surface with non-positive Euler Characteristic $\chi(F)$, and with non-empty boundary ∂F. The surface F admits handle structures consisting of one 0-handle, and $1-\chi(F)$ 1-handles. Let $\mathcal{H} = \{(H_i)\}$ denote a choice of such a handle structure, where each H_i, $1 \leq i \leq 1-\chi(F)$, denotes a single 1-handle. The single 0-handle is denoted by the letter D. Each H_i has an attaching region which consists of a pair of proper arcs in the surface. These two arcs are isotopic to each other (by a relative isotopy). Let A denote the union of all of the attaching arcs for the 1-handles. This notation drops the dependence on the choice of handle structure. Different handle structures give rise to different isotopy classes of curves in A.

The main result proved in this paper is given in the following theorem and its corollary which concerns the pants surface. The surface obtained by removing three open disks from the 2-sphere.

THEOREM 1. *Let P denote the pants surface and let $f : P \to P$ be a self-mapping. Then there exist a handle structure $\mathcal{H}$ and a fixed-point minimal map g, homotopic to f, such that $g^{-1}(A)$ consists of essential proper arcs each disjoint from $\bigcup H_i$.*

Given a self-mapping $f : X \to X$ let $MF[f]$ denote the least number of fixed points possible among all maps that are homotopic to f.

COROLLARY 1. *Given $f : P \to P$ there exist a graph G, an embedding $i : G \to P$ and a strong deformation retraction $r : P \to G$ such that $MF[f] = MF[r \circ f \circ i]$.*

The proof of Corollary 1 follows from Theorem 1 and the following proposition.

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PROPOSITION 1. *Let F be a compact surface having non-empty boundary. Suppose that there exist a handle structure $\mathcal{H}$ on F and a fixed-point minimal map $f : F \to F$ such that $f^{-1}(A)$ consists of essential proper arcs each disjoint from $\bigcup H_i$. Then there is a graph G, an embedding $i : G \to F$ and a strong deformation retraction $r : F \to G$ such that $MF[f] = MF[r \circ f \circ i]$.*

Proof. Clearly, for any such graph G we have that $MF[f] \leq MF[r \circ f \circ i]$. The reverse inequality will be obtained by constructing a graph G and a self-map of G which has exactly $MF[f]$ fixed points.

Let f be a map, and $\mathcal{H}$ a handle structure as given in the hypothesis. A *region* is a component of $F \setminus (f^{-1}(A) \cup A)$. Embed a graph G in the interior of F as follows: in each component of $f^{-1}(A) \cup A$ choose a single vertex located on the interior of this arc. Also, choose a vertex in the interior of each region that is bordered by at least three components of $f^{-1}(A) \cup A$. These we will call the exceptional vertices of G. In the pants surface there are either one or two such regions. In regions which do not contain an exceptional vertex, join the two vertices on its border by an edge which lies in the region. Finally, for each exceptional vertex, add one edge from that vertex to each of the vertices which lie on the border of the region containing the exceptional vertex. It is evident that there is a strong deformation retraction $r : F \to G$. Let $i : G \to F$ denote the inclusion map.

A self-map $f_0 : G \to G$ is obtained from f in the following manner. First, each point of $G \cap f^{-1}(A)$ is mapped to the obvious point in $G \cap A$. Now consider a component K of $G \setminus f^{-1}(A)$. First, consider the case when K does not contain an exceptional vertex. Then K is an arc and since G is homotopy equivalent to F, the image path is uniquely determined. Extend f_0 defined on the two endpoints by mapping one-to-one into the image path and so that at most one fixed point occurs. In the case when K contains an exceptional vertex and is mapped by f into a 1-handle we simply extend by mapping into this handle. By hypothesis, K must lie in D so no fixed points appear in such a K. Finally, the case when K contains exceptional vertices and is mapped by f into D. Set $f_0(v) = v$ for each exceptional vertex v and extend to the complimentary arcs in K with a one-to-one map onto the unique path in $G \cap D$ determined by the endpoints. This is done without having any fixed points on the interior of the path.

As a consequence of the construction of f_0 the only vertices fixed by f_0 are exceptional ones. Moreover, the fixed-point index for f defined on a component of $D \setminus f^{-1}(A)$ is the same as that of f_0 on the component restricted to G. Since f is fixed-point minimal each such component has one fixed point if the index is non-zero, and none otherwise. Hence, ignoring the exceptional vertices of index zero we have a one-to-one correspondence between the fixed points of f and those of f_0. To complete the proof deform f_0 slightly near a given exceptional vertex with index zero so as to remove this inessential fixed point. □

The majority of this paper, §§1–4, is devoted to the proof of Theorem 1. Section 1 gives some basic lemmas for simplifying fixed-point minimal maps leading to a notion of a reduced map. In §2 we study the structure of these reduced maps near the boundary of F. Section 3 gives a refinement of the map reduction given in §1, and §4 contains the proof of Theorem 1. In §5 we apply the methods used in §§2–4 to some other surfaces.

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In particular, in Theorem 2 the same result is obtained for the punctured Möbius band, while a partial result is given in Theorem 3 for the punctured torus and the punctured Klein bottle. This partial result is applied in [**16**] to answer a question raised by Berrick *et al* [**5**] regarding homotopy idempotents on surfaces.

We conclude this introduction with a comment relating the results of this work with those in two other papers. In [**13**] an algorithm is given for computing the value of the minimal number of fixed points among maps homotopic to a given self-map on the pants surface. This is recovered in Corollary 1 as follows. Fix a map $f : P \to P$. As we may assume that there are no valence one vertices for our graph, there are exactly three distinct possible choices for G in the corollary. Since the mapping class group of P is finite there is a finite number of ways to embed G in P. Choosing a retraction $r : P \to G$ then determines the graph map $g = r \circ f \circ i$. The computation of $MF[g]$ is straightforward and so we obtain $MF[f]$ by taking the minimum over all choices of G and embeddings. In [**15**] bounds are given for the fixed-point indices of a fixed-point minimal map on a surface with non-empty boundary. These bounds are entirely consistent with the fixed-point index bounds for a fixed-point minimal graph map defined on a graph which is homotopy equivalent to the given surface.

1. *Map reduction for minimal maps*

In this section we develop methods which will allow us to recognize when a given map is fixed-point minimal. Variations of methods used in previous work by the author [**13**] will be used to simplify a given fixed-point minimal map. Such simplification is always relative to a given handle structure on the surface. So consider a given handle structure on a surface F, and, as defined in the previous section, A denotes the union of the attaching arcs for the 1-handles.

The most basic simplification comes from the following lemma [**13**, Lemma 3.2]. The conclusion of this lemma will be assumed throughout this paper.

LEMMA 1. *Given a map f there exists a map g homotopic to f such that:* $\#\mathrm{Fix}(g) \leq \#\mathrm{Fix}(f)$*; g has no fixed points on A; $g^{-1}(A)$ is a 1D proper submanifold of F transverse to A; for each open set $\mathcal{U}$ meeting $g^{-1}(A)$, $g(\mathcal{U})$ meets at least two components of $F \setminus A$, and each inessential simple closed curve or proper arc in $g^{-1}(A)$ intersects A.*

A *region* is a component of $F \setminus (A \cup f^{-1}(A))$. The region X is *critical* if X and $f(X)$ lie in the same component of $F \setminus A$. A point p is a *crossing point* if p and $f(p)$ lie on the same component of A.

We let $i(X)$ denote the topological fixed point index of the region X. In this setting, the computation of the fixed-point index depends on two things: (1) the curves from $f^{-1}(A)$ in the frontier of X, and (2) the crossing points in the frontier of X and the way in which f maps these points. This is illustrated in the following example. This example will also be used to illustrate some other ideas given later in the paper.

Example 1. Let F denote the surface obtained by removing an open disk from the Möbius band. Fix a handle structure as indicated by Figure 1. The components of A are indexed as in the figure, with A_1, A_2 the horizontal attaching for H_1 and A_3, A_4 the vertical attaching for H_2. We consider a class of self-maps where $f : F \to F$ is in the class if and only

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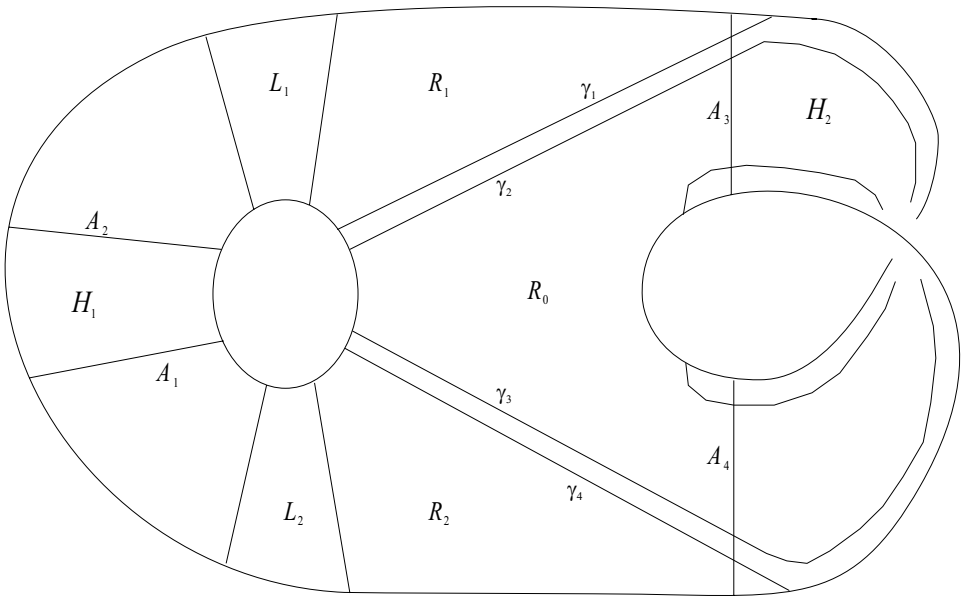


FIGURE 1. The family of maps used in Example 1.

if $f^{-1}(A)$ consists of proper arcs $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as indicated in Figure 1, together with collections of proper arcs which are contained in L_1 and L_2. Furthermore, we assume that the conclusions of Lemma 1 hold. Fix a map f in this class. If τ is a component of $f^{-1}(A)$ let $\mu(\tau)$ denote the subscript index of the component of A which contains $f(\tau)$. For $i = 1, 2$ let α_i denote the component of $f^{-1}(A)$ contained in L_i which borders R_i, if one exists.

As an illustration of the value of the fixed point index, consider the region R_1 and suppose that this region is critical. Then the value of $i(R_1)$ depends on the values of $\mu(\alpha_1)$ and $\mu(\gamma_1)$:

- if $\mu(\alpha_1) = 2$, $\mu(\gamma_1) = 1$ or 4, then $i(R_1) = -1$;
- if $\mu(\alpha_1) = 2$, $\mu(\gamma_1) = 2$, then $i(R_1) = 0$;
- if $\mu(\alpha_1) \neq 2$, $\mu(\gamma_1) = 1$ or 4, then $i(R_1) = 0$;
- if $\mu(\alpha_1) \neq 2$, $\mu(\gamma_1) = 2$, then $i(R_1) = +1$.

The above calculations assume that a curve α_1 exists and covers all possible cases except where R_1 has a crossing point on its border. If L_1 contains no curves in $f^{-1}(A)$, then $i(R_1)$ is $+1$ when $\mu(\gamma_1) = 2$ and 0 when either 1 or 4. Finally, when a crossing point occurs, $\mu(\gamma_1) = 3$. The value of $i(R_1)$ depends now on $\mu(\alpha_1)$ and the direction on A_3 which the crossing point is mapped: if downward, the result is the same as above with $\mu(\gamma_1) = 1$ or 4, if upward, with $\mu(\gamma_1) = 2$. Note that we always obtain $-1 \leq i(R_1) \leq 1$. To obtain other values for the index we would need that γ_1 meets A_3 in more than a single crossing point.

For the main result in this paper we will be able to avoid such index computations. In fact the only setting that will require knowing the value of the fixed-point index is

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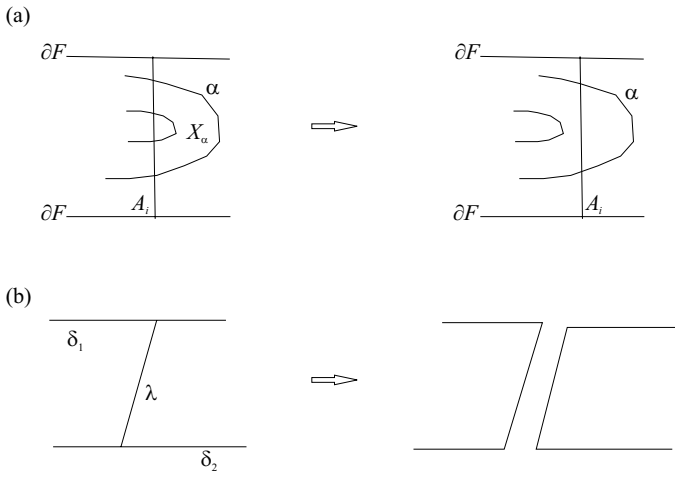


FIGURE 2. The natural homotopy from f to g of Lemma 2: (a) disk reduction, (b) arc reduction.

presented in Lemma 5 at the end of this section. Our main concern is that the value of $MF[f]$ is thought of as a minimization problem where we vary $f^{-1}(A)$ and count the number of critical regions with non-zero index. Lemma 1 provides a first step in the search for an optimal collection of curves $f^{-1}(A)$. The following two lemmas give further restrictions on the map f and the curves in $f^{-1}(A)$ needed for this minimization problem. Lemma 2 is part of the main technical lemma [**13**, Lemma 3.5] used in that paper. It is rephrased using the language and notation of this paper.

Let α be a subarc of $f^{-1}(A)$ with one endpoint x lying on some A_i, the other endpoint y on $\partial F \cup A_i$ and such that $\alpha \cap (A \cup \partial F) = \{x, y\}$. There is a unique disk D_α such that ∂D_α consists of α together with a subarc J_α of $\partial F \cup A_i$. Let X_α denote the closure of the component of $F \setminus (f^{-1}(A) \cup A)$ that contains α and lies in D_α. We assume that X_α is a disk.

LEMMA 2. *Let f be a map with an arc $\alpha \subset f^{-1}(A)$ as above. Suppose that either $f(\alpha) \not\subset A_i$, or X_α is a critical region, or $X_\alpha \neq D_\alpha$. Then there exist a non-empty subset S_α of $J_\alpha \cap f^{-1}(A)$ and a map g homotopic to f such that g satisfies the conclusion of Lemma* 1 *and $g^{-1}(A) \cap A = (f^{-1}(A) \cap A) \setminus S_\alpha$.*

Remarks. (1) Only the case $y \in A_i$ is proved in [**13**]. The other case, when $y \in \partial F$ can be proved using the same argument. (2) The natural homotopy from f to g in the lemma is illustrated in Figure 2(a). It shows the case when $X_\alpha \neq D_\alpha$ where the homotopy removes components of $f^{-1}(A)$ inside D_α from A_i. In the two other cases the homotopy removes α from A_i without adding fixed points.

One of the steps used in the proof of Lemma 2 is a construction similar to the result in the following lemma. It is presented here as a separate lemma as the reduction provided by this lemma will be useful in our results later on. By a *segment* we mean the closure of a component of $f^{-1}(A) \setminus A$.

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LEMMA 3. *Suppose that δ_1 is a segment and that δ_2 is either a segment or the closure of a component of $\partial F \setminus A$. We assume that $\delta_1 \cap \delta_2 = \emptyset$ and that if δ_2 is a segment, then $f(\delta_1)$ and $f(\delta_2)$ are contained in the same component of A. Suppose that there is an arc λ in $F \setminus A$ such that its interior is disjoint from $f^{-1}(A) \cup \partial F$, and it has one endpoint on each δ_i. Let U be a closed disk in $F \setminus A$ containing λ so that $\partial U \setminus \left(\bigcup \delta_i \right)$ has four components. Among the four, let σ denote the union of the two components parallel to λ. If the interior of λ is not in a critical region, then there is a map g which agrees with f outside a neighbourhood of U, has the same fixed point set as f, and $g^{-1}(A) = (f^{-1}(A) \setminus U) \cup \sigma$.*

Proof. This is straightforward. The hypothesis of the lemma implies that $f(U)$ does not intersect U so that the fixed-point set is not affected by the natural homotopy which corresponds to cutting along λ. Figure 2(b) gives an illustration of the lemma and proof. □

Remark. Lemma 3 has a boundary version where the arc λ is contained in $\partial F \setminus A$, and we allow for the possibility that it lies in a critical region.

Let λ be an arc as in Lemma 3 (or the remark above), and let R denote the region that contains its interior. Furthermore, suppose that for any curve η in $f^{-1}(A) \cup \partial F$ lying in the closure of R, $\eta \cup \lambda$ does not bound a disk in R. If λ can be transported in R by an isotopy that keeps endpoints in $f^{-1}(A) \cup \partial F$ to an arc which is contained in either A or ∂F, then λ will be referred to as a *reduction arc* for the map f, and the process from f to g given in Lemma 3 an *arc reduction*. Similarly, reducing by the previous Lemma 2 is called a *disk reduction*, due to the role of the disk D_α.

We finish this section with an example and an elementary application of these reductions. Again, this information is based on [**13**, Lemma 3.6].

Example 2. A map as given in Example 1 has no disk reductions exactly when $\mu(\gamma_1) = 3$, $\mu(\gamma_2) = 4$, $\mu(\gamma_3) = 3$, $\mu(\gamma_4) = 4$ and each region R_i is mapped into D. Hence each R_i is a critical region. On the other hand, if $\mu(\gamma_i) = \mu(\gamma_{i+1})$, $i = 1, 2, 3$, then there is a reduction arc near either A_3, A_4 or ∂F.

LEMMA 4. *Suppose that $f : F \to F$ satisfies the conclusion of Lemma 1 and also has no reduction disks or arcs. Then $f^{-1}(A)$ does not contain any inessential simple closed curves or any inessential proper arcs. In addition, adjacent points of $f^{-1}(A)$ on any component of ∂F map to distinct components of A.*

Proof. Inessential means that there is a disk E embedded in F such that ∂E is either in $f^{-1}(A)$ or consists of a proper arc in $f^{-1}(A)$ together with an arc in ∂F. Suppose that $f^{-1}(A)$ has an inessential simple closed curve or proper arc and consider an innermost such one. Then the interior of E is disjoint from $f^{-1}(A)$. As E is a disk either it meets at most one component of A or it contains a reduction arc. Hence, we can assume that E meets exactly one component of A. However, this forces a reduction disk and so no such E can exist.

The additional part follows directly from the remark after Lemma 3. □

LEMMA 5. *Suppose that $f : F \to F$ satisfies the conclusion of Lemma 1 and also has no reduction disks or arcs. Let R be a critical region for f with $R \cap f^{-1}(A)$ consisting of the segments $\alpha_1, \ldots, \alpha_\ell$ on the boundary of R. Let X denote the closure of the component of*

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$F \setminus A$ for which $R \subset X$. Furthermore, suppose that for each i, $f\left(\bigcup \alpha_i\right) \cap \alpha_i$ is at most one point. Set $x_i = 0$ if $f(\alpha_i)$ and $R \setminus \alpha_i$ lie in the same component of $X \setminus \alpha_i$, otherwise set $x_i = 1$. Then

$$i(R) = 1 - \sum_{i=1}^{\ell} x_i.$$

Proof. Consider a tree T in X which has one vertex in the interior of R and one vertex on each component of $A \cap X$, with these boundary vertices chosen to coincide with the images of the α_i whenever possible. Also, we will assume that each edge of T meets $\cup\alpha_i$ in at most one point so that $T \cap R$ is a sub-tree.

By a homotopy relative to $\bigcup \alpha_i$ we deform f restricted to R to a map τ which has its image in T. Since each component of $R \setminus \bigcup \alpha_i$ is contained in $A \cup \partial F$, this can be done so that no fixed points occur on the boundary of R during the homotopy. Thus, $i(R)$ is the same as the total fixed-point index of the graph map $\tau : (T \cap R) \to T$. Consider a valence one vertex v of $T \cap R$, which is a point on α_j for some j. By the definition of T, $\tau(v)$ and v lie on the same edge of T exactly when $x_j = 1$. The result now follows from the definition of the fixed-point index. □

Remark. The assumption in Lemma 5 that $f\left(\bigcup \alpha_i\right) \cap \alpha_i$ be at most one point is quite restrictive. However, for the instances in this paper where this will be used the condition will be easy to verify.

2. *Finding reductions near the boundary*

The purpose of this section is to study some of the structure of the preimage of A for maps that admit no arc or disk reductions. Consequently, we assume that all maps have no arc or disk reductions.

An *end-arc* is a maximal subarc of $f^{-1}(A)$ which has one endpoint in ∂F, the other either in A or ∂F, and which only meets one component of A. A consequence of disk reduction is that end-arcs have a simple description as seen in the following lemma, which first requires some notation. Let η be an end-arc meeting A_i and choose a point $q_\eta \in \eta \cap \partial F$. Let p_η denote the point of $\partial F \cap A_i$ which is contained in the unique disk bounded by the segment of η containing q_η, a subarc of A_i and a subarc of ∂F. Orient η starting from q_η and A_i starting from p_η.

Figure 3 illustrates the two types of end-arcs that are possible as a consequence of the following lemma.

LEMMA 6. *Let η be an end-arc for the map f which meets A_i. Then $f(\eta) \subset A_i$ and the points of $\eta \cap A_i$ follow the same order along η as they do along A_i.*

Proof. That $f(\eta) \subset A_i$ is immediate from disk reduction. Following the oriented arc η, the first such violation of the ordering in the conclusion of the lemma would create a nested pair of segments with endpoints on $A_i \cup \partial F$. However, this fits the hypothesis for a disk reduction. So no such violation can occur. □

LEMMA 7. *Given $f : F \to F$ suppose that p is a point of $f^{-1}(A) \cap A$ which is nearest the boundary. Then p is contained in an end-arc.*

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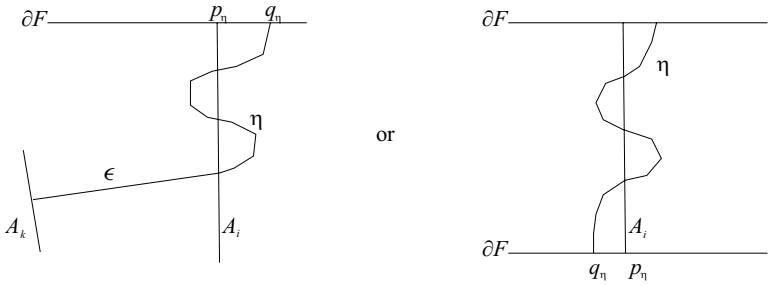


FIGURE 3. The two types of end-arcs.

Proof. Suppose p is not contained in an end-arc and let γ be a subarc of A whose interior misses $f^{-1}(A)$ and has as endpoints p and a point in ∂F. Consider two arcs, parallel to (and near) γ, one on each side of γ. As their interiors do not meet $f^{-1}(A)$ the one that is not in a critical region is a candidate for a reduction arc. In fact, because p does not extend to a segment meeting ∂F, it follows from the definition that this is a reduction arc. □

LEMMA 8. *Given a map $f : F \to F$ each* 1*-handle satisfies either* (1) *each of its two attaching arcs meets at most one end-arc, or* (2) *both attaching arcs meet exactly two end-arcs.*

Proof. Fix a 1-handle H. Since each attaching arc for H meets at most two end-arcs the proof is completed by assuming that one of its attaching arcs, denoted by J, meets at most one end-arc and showing the same for its dual, denoted by J'. First, observe from Lemma 7 and disk reduction that at most one component of $f^{-1}(A)$ meets J and if there is such a component it must be an end-arc. Consequently, any segment in H which has one endpoint on J' has its other endpoint either on ∂F or also on J'. We need to show that any such segment is contained in a single end-arc meeting J'.

If τ is such a segment disk reduction implies that $f(\tau) \subset J'$. Also, the disk bordered by τ and a subarc of J' is not a critical region. Disk reduction also ensures that no nesting occurs among such segments. Now, if one has two adjacent segments as above consider the subarc J_0 of J' which joins them. By Lemma 1 we see that the region in D with J_0 on its border is not critical. Thus, ∂J_0 must be the endpoints for a segment in D, for otherwise we see a reduction arc in D near J_0. The result now follows. □

Let β be a subarc of $f^{-1}(A)$ with an endpoint e contained in A. The *extension* of β at e is the segment which intersects β at e, but is not contained in β. An end-arc which is not a component of $f^{-1}(A)$ has a unique extension. This is illustrated by the segment labelled ϵ in Figure 3. In general an extension lies either in D or some 1-handle. The next lemma deals with the nature of extensions near the boundary of F. For this we use the notion of a dual of an end-arc. If η is such an end-arc, and meets the 1-handle H, we define its *dual* to be the end-arc $\eta^\perp$ such that p_η and $p_{\eta^\perp}$ lie on the same component of $H \cap \partial F$. This is well defined as a result of Lemma 8.

LEMMA 9. *Let $f : F \to F$ be given and let η be an end-arc which is not a component of $f^{-1}(A)$. Then exactly one of η, $\eta^\perp$ extends into D.*

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Proof. Let H denote the 1-handle meeting η. Clearly, both cannot extend into H. So we assume that η and $\eta^{\perp}$ both extend into D and will derive a contradiction.

First, we assume the case where $f^{-1}(A)$ contains a segment in H which meets both attaching arcs for H. In this case there is a unique region R in H which meets η and $\eta^{\perp}$. This is also bordered by a segment τ which meets both attaching arcs for H. Then disk reduction implies that R must be a critical region, and so τ is mapped to one of the attaching arcs for H. Hence there is a reduction arc in D near one of the attaching arcs.

Now suppose that $f^{-1}(A)$ does not contain any segment in H which meets both attaching arcs for H. In this case all of the four end-arcs meeting H extend into D and disk reduction ensures that there are no other segments in H. As in the proof of Lemma 8 we find a reduction arc near each attaching arc. □

3. *A generalized method of reduction*

In this section we present a method for reducing the cardinality of $f^{-1}(A) \cup A$ which generalizes the notion of a disk reduction.

A *null-arc* for f is a subarc ζ of $f^{-1}(A)$ which meets exactly one component of A, has its endpoints in $A \cup \partial F$ and if ζ meets A in an odd number of points, then $\zeta \cap \partial F \neq \emptyset$. Null-arcs are a generalization of the notion of an end-arc and as such they satisfy the conclusion of Lemma 6 once suitable orientations are given. The primary feature of a null-arc is that it can be pulled off of A by an isotopy of F which has support in some neighbourhood of the component of A that contains the null-arc. This feature sets up the following reduction process.

Let $\mathcal{C}$ be a pairwise disjoint collection of null-arcs for f. Define a new map homotopic to f obtained as follows. First, change $f^{-1}(A)$ by an isotopy which pulls each null-arc in $\mathcal{C}$ off of the one component of A that it meets (but no other component). Now choose a map $f_{\mathcal{C}} : F \to F$ that has this new collection of curves as its preimage of A.

In this process the set of critical regions for f transforms into a new collection of critical regions for $f_{\mathcal{C}}$. The process is called an *essential reduction* if choices can be made so that the new map $f_{\mathcal{C}}$ has no more critical regions with non-zero index than does f.

Some observations of this process are as follows.

(1) As an example consider a disk reduction where the disk D_{α} is a critical region. It follows from Lemmas 1 and 4 that there are two additional critical regions which have an endpoint of α on their boundary. By pulling the null-arc α off of A these three critical regions for f merge into one. The fixed point index of this resulting region is just the sum of the indices of the three critical regions for f. Since there is no increase in the number which have non-zero index we see that this disk reduction is one type of essential reduction. In general, not all disk reductions are obtained by this reduction process. The proof of [**13**, Lemma 3.5] makes use of arc reduction as well.

(2) Consider D_{α} as above except that now it is not a critical region. Then there are exactly two critical regions which meet α and there are two corresponding critical regions for $f_{\mathcal{C}}$. Consequently, if f has no disk reductions, then to each critical region for f there is a unique one for $f_{\mathcal{C}}$ corresponding to it.

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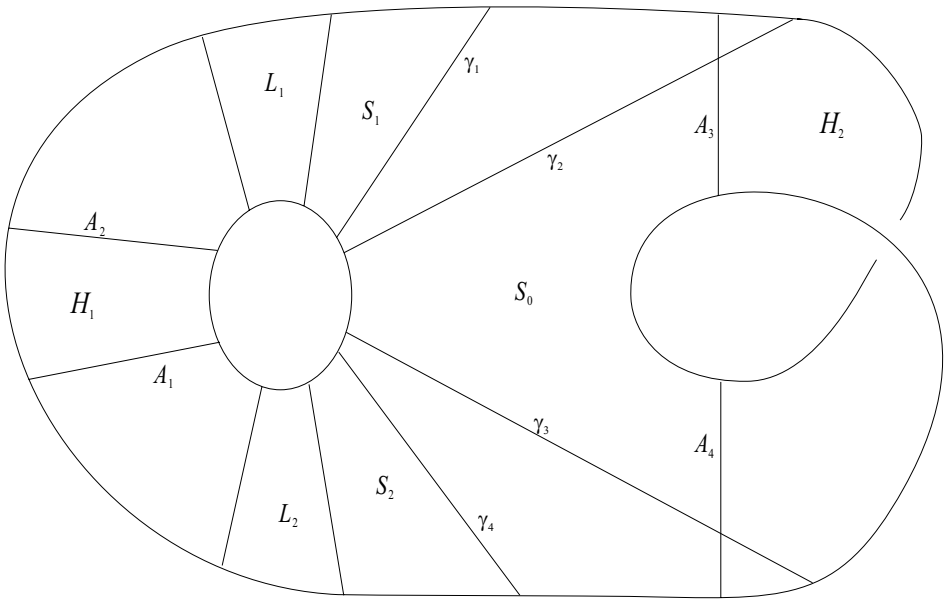


FIGURE 4. The deformed map given in Example 3.

(3) Even if f has no arc or disk reductions the map $f_{\mathcal{C}}$ may have some. This depends on $\mathcal{C}$.

Example 3. We return to the previous examples for an illustration of this reduction process. We assume that no disk or arc reductions occur as in Example 2. Apply the reduction process described above to the collection consisting of the four end-arcs (see Figure 1). The resulting map is illustrated in Figure 4. Moreover, if a reduction was applied to any one of the four end-arcs there would be a reduction arc for the resulting map. After arc reduction the remaining three end-arcs are removed, resulting in the same map.

Such reductions may or may not be essential. For example, if f is such that $\mu(\alpha_1) \neq 2$, then apply Lemma 5 with $\ell = 2, x_1 = 0, x_2 = 1$ to obtain $i(S_1) = 0$. So removing the end-arc contained in γ_1 from A_3 is an essential reduction. Similarly, we obtain an essential reduction when $\mu(\alpha_2) \neq 1$ or if either α_1 or α_2 does not exist. On the other hand, if f is such that $\mu(\alpha_1) = 2$ and $\mu(\alpha_2) = 1$, we see from Lemma 5 (here $\ell = 2, x_1 = 1, x_2 = 1$) that $i(S_1) = i(S_2) = -1$. However, f can be defined so that $i(R_1) = i(R_2) = 0$ and the index for each of the two critical regions in H_2 is also zero. Thus, no collection from the four end-arcs results in an essential reduction in this case.

We say that the map f *admits no reductions* if it satisfies the conclusion of Lemma 1 and it is not possible to find a reduction of any of the following types: arc reduction, disk reduction or essential reduction.

PROPOSITION 2. *Given any $f : F \to F$, there is a map g homotopic to f such that g is fixed point minimal and admits no reductions.*

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Proof. Both disk reductions and essential reductions have the effect of reducing the cardinality of $f^{-1}(A) \cap A$. So there is a fixed point minimal map h in the homotopy class of f with $h^{-1}(A) \cap A$ of smallest cardinality. Now h may admit arc reductions, but there are finitely many so apply arc reduction to each to obtain a map h'. Repeat the above to the map h'. Noting that since $f^{-1}(A) \cap A$ is finite this process must stabilize. That is, either h admits no arc reductions or h' admits no disk or essential reductions. This gives the desired map g. □

Outside of a disk reduction or a specific example it would be difficult in general to recognize when an essential reduction might occur. In fact, in item (2), above, it is most likely that f has fewer critical regions of non-zero index than $f_{\mathcal{C}}$, simply because the presence of the points on $\partial\alpha$ allow for more flexibility for the values of the fixed-point indices. There is one setting where we will be able to possibly recognize such a reduction. This was illustrated in Example 3 and will be the only type used in this paper. This essential reduction is along the boundary of F and proceeds as follows: fix a boundary component of F and consider the collection of end-arcs which meet this component. As each end-arc is a null-arc we can apply the reduction process to any subcollection of these end-arcs.

A simple setting where an essential reduction of this type occurs is provided by the following lemma.

LEMMA 10. *Let H be a 1-handle in F such that $H \cup D$ contains a component of ∂F. Suppose $f^{-1}(A)$ contains a segment in D that has one endpoint in an attaching arc for H and the other endpoint either in the dual attaching arc for H or on a component of $\partial F \cap D$ which meets H. Assuming the segment is not a null-arc, then f admits an essential reduction.*

Proof. Let C denote the unique boundary component contained in $H \cup D$, and let ϵ denote the segment among those which satisfy the hypothesis of the lemma that is nearest to C.

Suppose that ϵ has both endpoints in attaching arcs. It then follows that ϵ is an extension of an end-arc η_1 with its other endpoint on C. Let R_1 denote the region with ϵ and a subarc of $C \cap \partial F$ on its border. Then R_1 meets another end-arc η_2 which meets C and is such that both end-arcs meet a unique region R_2 in H. Consider the reduction involving η_1, η_2 creating the corresponding regions S_1, S_2. This setting and reduction are illustrated in Figure 5. Applying Lemma 5 to the region S_1 we have that $\ell = 1$ and Lemma 6 implies that $x_1 = 1$. Thus $i(S_1) = 0$, and as there are only two regions involved, the reduction is essential.

Now suppose that all such segments have an endpoint on ∂F. The existence of ϵ ensures that there two end-arcs at the attaching arc which ϵ meets, and by Lemma 8 there are two end-arcs meeting the dual attaching arc for H. Because there are no segments in D joining the two attaching arcs, we see that no other curves in $f^{-1}(A)$ meet this dual attaching arc. Hence, a reduction arc occurs near this attaching arc. □

4. *The pants surface*

The purpose of this section is to give a proof of Theorem 1. To do so we first give two lemmas which allow for changing the handle structure. Recall that since there is only

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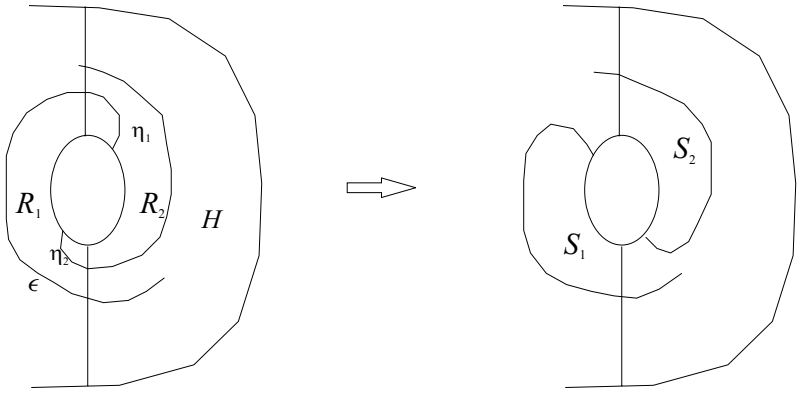


FIGURE 5. The essential reduction in Lemma 10.

one 0-handle in a handle structure, no attaching arc separates F into two components. These two lemmas give conditions on when a non-separating arc may be an attaching arc for a 1-handle in some handle structure.ϵ

LEMMA 11. *Let $\{(H_i)\}$ be a handle structure for the surface F. Let $f : F \to F$ be a map (which satisfies the conclusion of Lemma 1). Suppose that γ is a component of $f^{-1}(A)$ which is a proper arc in F which does not separate and is such that $f(\gamma)$ is a single point. Let U be an open neighbourhood of γ in F with $U \cap f^{-1}(A) = \gamma$. Then there is a handle structure $\{(K_i)\}$ for F such that $K_1 \subset U$ and $f^{-1}\big(\bigcup K_i\big) \cap K_1 = \emptyset$.*

Proof. Set $\gamma = \gamma_1$. As $f(\gamma)$ is a single point it follows from Lemma 1 that γ does not intersect $f(\gamma)$. Let $k = 1 - \chi(F)$, which is the number of 1-handles. Since γ does not separate we can choose $k - 1$ pairwise disjoint proper arcs $\gamma_2, \ldots, \gamma_k$ in $F \setminus \gamma$ so that $F \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$ is connected and simply-connected. Moreover, we choose so that each γ_i does not contain the point $f(\gamma)$. It follows that $f^{-1}\big(\bigcup \gamma_i\big)$ is disjoint from γ. For if $p \in f^{-1}\big(\bigcup \gamma_i\big) \cap \gamma$, then $f(p) \in f\big(f^{-1}\big(\bigcup \gamma_i\big)\big) \cap f(\gamma)$, and hence $f(p) \in \big(\bigcup \gamma_i\big) \cap f(\gamma)$. However, $\big(\bigcup \gamma_i\big) \cap f(\gamma) = \bigcup(\gamma_i \cap f(\gamma)) = \emptyset$. Now choose the desired handle structure by taking a suitable small regular neighbourhood of the arcs γ_i. □

This lemma has the following relative version.

LEMMA 12. *Let γ be as in Lemma 11 and suppose that it does not meet any of the 1-handles $H_2, \ldots, H_\ell$. If $F \setminus (H_2 \cup \cdots \cup H_\ell \cup \gamma)$ is connected, then we can choose $\{(K_i)\}$ so that $K_i \subset H_i$ for each $2 \le i \le \ell$.*

Proof. Following the proof of Lemma 11 the additional hypothesis allows us to choose γ_i inside of H_i for each i from 2 up to ℓ. □

Let P denote the pants surface and consider a handle structure $\{(H_1, H_2)\}$. One feature of the pants surface is that for each $i = 1, 2$ there is a unique boundary component C_i which only meets the handle H_i. Let C_0 denote the remaining boundary component of P. Another feature of the pants is that any essential proper arc which joins two distinct boundary components is non-separating in P.

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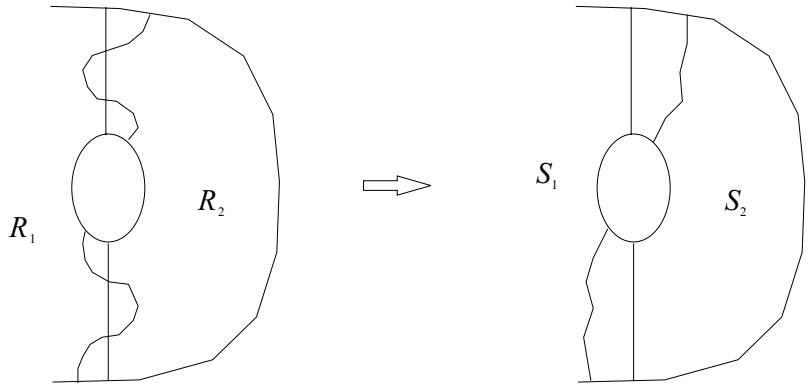


FIGURE 6. The essential reduction in Proposition 3.

PROPOSITION 3. *Let (H_1, H_2) be a handle structure for P. Suppose $f : P \to P$ admits no reductions and is fixed point minimal. Furthermore, suppose that $f^{-1}(H_1 \cup H_2) \cap H_1 = \emptyset$. Then there exists H_2' such that $\{H_1, H_2'\}$ gives a handle structure for P and $f^{-1}(H_1 \cup H_2') \subset D$, where D denotes the 0-handle.*

Proof. First, consider the case when $f^{-1}(A)$ has a component which meets C_2 and is disjoint from A. Let γ be such a component and observe that since γ meets C_2 but not H_1 that it is non-separating and is not isotopic to an attaching arc for H_1. Without adding additional fixed points we can deform f so that $f(\gamma)$ becomes a single point. So the result follows from Lemma 12.

In the case $C_2 \cap f^{-1}(A) = \emptyset$ the conclusion of the proposition is already satisfied, so set $H_2' = H_2$.

Finally, when neither of the above occur it follows from Lemma 7 that each of the attaching arcs for H_2 is met by an end-arc. We show that in this case there must be an essential reduction.

According to Lemma 8 either H_2 meets four end-arcs or two end-arcs. In the case of two end-arcs, each one has both endpoints in ∂P. In this setting there are exactly two critical regions which meet C_2: R_1 in D and R_2 in H_2, and there are choices as to which component of $P \setminus A$ the end-arcs are moved into under a reduction resulting in the corresponding regions S_1 and S_2. There are two choices leading to $i(S_2) = 0$ as indicated in Figure 6. If H_2 meets four end-arcs, then by Lemma 9 of the two meeting C_2 one extends into H_2 and the other into D. Again, there are two critical regions which meet C_2 and one obtains a reduction just as in the proof of Lemma 10.

Hence, we have an essential reduction in either case. □

The following proposition will now complete the proof of Theorem 1.

PROPOSITION 4. *Given any $f : P \to P$ there is a map g homotopic to f and a handle structure such that g is fixed point minimal, admits no reductions and the preimage of the 1-handles is disjoint from one of the two 1-handles.*

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Proof. Let (H_1, H_2) be a handle structure for P, with boundary components C_0, C_1, C_2 as above. By Proposition 2 there is a fixed-point minimal map which admits no reductions. Without loss of generality we denote by f such a map. As in the proof of Proposition 3 we first use Lemma 8 to divide the proof into two cases.

For the case where one of the 1-handles meets at most two end-arcs choose one of these handles and relabel if necessary so that it is H_1. Now follow the proof of Proposition 3 to obtain the desired 1-handle.

We now assume that there are four end-arcs meeting each of the two 1-handles. The proof is completed by showing that there must be an essential reduction.

By Lemma 9 two of the end-arcs meeting C_0 extend into D, one from each H_i. First, consider the one that meets H_1 and let τ denote the extending segment in D. The set $\partial D \setminus (A \cap \partial F)$ has eight components. Label the closure of these components as follows: let J_1 be the component of A which meets the end-arc, J_2 the component which is in C_0 and meets J_1, finally $J_3, \ldots, J_8$ are chosen cyclically around ∂D. One endpoint of τ lies on J_1. Let q denote the other. We consider the various possible locations for q.

The component J_2 is ruled out by Lemma 4, and J_8 is the previous case of one end-arc meeting a 1-handle. The cases of q being in either J_6 or J_7 is taken care of by Lemma 10. If q lies on J_4, then the two end-arcs meeting the attaching arc J_3 must extend into the 1-handle. Moreover, no other curves in $f^{-1}(A)$ can intersect J_3 so there is a reduction arc near J_3. Now consider the case that q is on J_5, which is an attaching arc for the other 1-handle. Continue extending this curve from q through the 1-handle and back into D. Owing to the position of the arc formed by the end-arc and τ and disk reduction we see that this curve will never enter the handle H_1. It can only spiral around H_2. As a consequence, the final segment of this curve satisfies the hypothesis for a disk reduction.

As a result, the argument is complete if either of the two end-arcs which meet C_0 and extend into D extend to any of the corresponding J_k, where $k \neq 3$. So we now assume that each extends to J_3. However, this implies that the end-arc meeting C_0 at J_3 extends into the corresponding 1-handle. Hence, the only curves in $f^{-1}(A)$ meeting C_0 are the four end-arcs, which in turn determine four critical regions which meet C_0. After a reduction which removes these four end-arcs, each of the corresponding critical regions has zero index. This implies that the reduction is essential and so this last case is not possible. □

5. *Some considerations for other surfaces with two 1-handles*

One can naturally ask as to what extent the structure of the pants surface is used to obtain the conclusion of Theorem 1. We note that for the surfaces with exactly one 1-handle, the annulus and the Möbius band, this result is trivial. A simple test case would be to consider the other surfaces which have exactly two 1-handles. In fact, this is the reason for the choice of the punctured Möbius band surface used in Examples 1–3.

The maps defined in Example 3 illustrate a situation where a map has no essential reductions and, since each preimage curve in L_i is isotopic to an attaching arc for H_1, there is no way to apply Lemma 12 to obtain the desired result. Of course, one needs to be sure that these examples do have the minimal number of fixed points. In general this is quite difficult to check. We point out that this occurs when each L_i contains exactly two curves and on the level of π_1; $a \mapsto baab, b \mapsto 1$, where a is the loop traversing H_1

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oriented so that it meets A_1 first and b traverses H_2 meeting A_3 first. Here there is exactly one essential fixed point (in R_0).

On the other hand, if we are a little more careful about reducing we see that the difficultly given in these examples can be removed. We now present a rough outline of the process. First, change the map f corresponding to adding one 2-sided simple closed curve to $f^{-1}(A)$ which loops twice around H_2. This curve is mapped to either A_3 or A_4, and the new map, which we still denote by f, can be defined so that no new fixed points are introduced. Reduce by an arc reduction which coalesces the simple closed curve with one of the proper arcs in $f^{-1}(A)$ which meets H_2. Further reduction (there are some choices as to the essential reductions used) leads to a map which has a preimage curve in D that is non-separating and is not isotopic to an attaching arc for H_1 or H_2. Now apply Lemma 12 to this curve to obtain a handle structure with the map satisfying the conclusion of Theorem 1.

Taking advantage of this specialized reduction leads to the following theorem.

THEOREM 2. *Let M denote the once punctured Möbius band and let $f : M \to M$ be a self-mapping. Then there exist a handle structure $\mathcal{H}$ and a fixed-point minimal map g, homotopic to f, such that $g^{-1}(A)$ consists of proper arcs each disjoint from $\bigcup H_i$.*

Proof. The surface M has two boundary components. We let C_1 denote the one that does not meet the handle H_2, and let C_0 denote the other. The proof of Proposition 4 in the case that there are four end-arcs adapts quite well to M. Follow this proof up to the last paragraph and then include the two other end-arcs meeting C_0 (at H_2) to reduce to the case of an essential reduction involving the six end-arcs and six critical regions meeting C_0. The details are left to the reader.

The proof of Proposition 3 works fine in the case that $f^{-1}(A) \cap H_2 = \emptyset$, but when this condition does not hold a problem occurs when H_2 meets four end-arcs. Here we do not have the boundary component C_2 to set up an essential reduction. However, with no reductions and assuming Lemmas 11 and 12 cannot be applied, this leaves only the situation as described in Example 3. Here we use the reduction as outlined above to complete the proof of the proposition. □

This result points out that the notion of essential reduction defined in this paper is not enough to obtain the conclusion of Theorem 1 for arbitrary surfaces. In particular, the topological structure of the pants surface, which has three boundary components, as opposed to the two boundary components for the punctured Möbius band made the difference in detecting essential reductions.

At this time the validity for such a result for maps on the once punctured torus (or Klein bottle) is not known. For most maps the proof given for the pants surface can be adapted. However, owing to the topology of this surface it is possible to produce examples where one fails to detect essential reductions. The remainder of this section is devoted to a partial result which uses the methods developed in this paper to produce a representative map which helps to compute the value of $MF[\cdot]$ for a family of maps defined on the once punctured torus or Klein bottle. This partial result is used in [**16**] to answer a question raised by the authors in [**5**].

First we provide a lemma which replaces Lemma 10.

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LEMMA 13. *Let F be a surface with one boundary component and Euler characteristic $\chi(F) = -1$. Fix a handle structure for F and suppose that $f : F \to F$ is a map which has a critical region that is bordered by three segments, none of that is a null-arc and two map to the same component of A. If f admits no reductions, then $f(\partial F)$ is essential.*

Proof. Suppose that f admits no reductions. The existence of such a region ensures that $f^{-1}(A)$ has exactly eight end-arcs, four extending into D. Given an end-arc extending into D recall the notation $J_1, \ldots, J_8$ used for the components of $\partial D \setminus A$ in the proof of Proposition 4. With the four end-arcs extending into D take those that do not extend to J_3 and by a small isotopy with support in a neighbourhood of each end-arc replace each by a segment which lies in D. The corresponding map will be denoted by g.

Let Z denote the set of segments of $g^{-1}(A)$ meeting ∂F which are not end-arcs. By construction, each member of Z has one endpoint in A. Two members of Z are said to be *parallel* if their endpoints are in the same two components of $D \setminus (A \cap \partial F)$. The segments given in the hypothesis of the lemma together with the segments in Z restrict the structure of the complimentary regions in D. Since no reductions occur it follows that there are at most two parallel classes among members of Z together with segments from the hypothesis which meet the boundary. As each proper arc is either an end-arc or in Z, if we assume that $f(\partial F) = 1$, then there must exist adjacent members of Z which map to the same component of A. These segments are parallel and as a result determine a reduction arc for g along the arc of A joining the endpoints of the two segments. However, since g was obtained by deforming only some end-arcs, this reduction arc is also one for f. This is a contradiction, so $f(\partial F) \neq 1$. □

Fix a handle structure for F with 1-handles H_1, H_2 and let e_1, e_2 be based loops in F so that each e_i meets H_i in a single arc and $e_1 \cap e_2$ is the base point for the fundamental group. If F is the punctured Klein bottle we assume that e_1 is a two-sided loop and e_2 is one-sided. Consider a family of maps which on fundamental group are of the form $e_1 \mapsto 1, e_2 \mapsto W$, where W is any word in the free group generated by e_1, e_2.

THEOREM 3. *For any $g : F \to F$ which is as above, there is a representative map f which has exactly $MF[g]$ fixed points and $f^{-1}(A)$ consists of the following elements:*

(1) *proper arcs in D isotopic to an attaching arc for H_2;*
(2) *simple closed curves disjoint from H_2 and meeting H_1 in a single arc; and*
(3) *proper arcs which are isotopic to those in* (1) *but intersect H_1.*

Moreover, in (3) *there are either zero or four arcs in total, and pulling these arcs into D produces a map which has at most four additional fixed points.*

Proof. Let C be a simple closed curve in F which is disjoint from H_2, meets H_1 in an arc. Among all such curves we also assume that C has minimal self-intersection with $f^{-1}(A)$ with all intersections in D. Since $f(C)$ is null-homotopic if $C \cap f^{-1}(A) \neq \emptyset$, then there is a region in D which is bordered by two segments which map to the same component of A. Assuming no reductions this region satisfies the hypothesis in Lemma 13. We now have a contradiction, as $f(\partial F)$ is inessential. Hence $f^{-1}(A) \subset F \setminus N$, where N is some closed neighbourhood of C. However, the closure of $F \setminus N$ is the pants surface so it follows that all proper arcs are isotopic to those in (1) and all simple closed curves are isotopic to those in (2).

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We now show that there are no end-arcs meeting H_2. First, note that if no end-arcs meet H_1, then the existence of C implies that no curve meets H_1 and the only curves that can meet H_2 are end-arcs which are proper arcs in F. As needed, remove as in Figure 6 to obtain a map which satisfies the conclusion of the theorem. If exactly two end-arcs meet H_2 remove in the same manner. So we now assume that there are eight end-arcs. Those that meet H_2 and extend into D must extend to J_3. Otherwise, a continual extension of this curve creates an infinite spiral around C. A similar observation holds for those meeting H_1 (but using a different curve). Hence the only curves meeting ∂F are end-arcs. However, this implies that $f(\partial F)$ is homotopic to ∂F. This is a contradiction to the assumption of end-arcs meeting H_2.

Since there are no end-arcs meeting H_2, $f^{-1}(A) \cap H_2 = \emptyset$. This takes care of (1). Note that any simple closed curve which maps to H_2 satisfies (2). Given a curve that maps to H_1 and is adjacent to one mapping to H_2 we assume (2) holds for this curve and is mapped to a point on A so that no fixed points occur in the annular region between the two curves. This preserves minimality for fixed points, so we proceed inductively to obtain (2) for all curves. For the special case that all of the simple closed curves map to H_1 a direct construction satisfying (2) produces no fixed points in the annular region bounded by these curves.

If (2) is non-empty, for a reduced map we must have four end-arcs meeting H_1. The extensions of these to proper arcs are the curves in (3). If one proceeds to pull all four end-arcs off of A we obtain a map with all curves in (1) and (2) at the expense of possibly adding at most one fixed point in each critical region in D bordered by an end-arc. □

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https://doi.org/10.1017/S0143385706000964 Published online by Cambridge University Press