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Pointwise dimension and ergodic decompositions

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Abstract. We study the Hausdorff dimension and the pointwise dimension of measures that are not necessarily ergodic. In particular, for conformal expanding maps and hyperbolic diffeomorphisms on surfaces we establish explicit formulas for the pointwise dimension of an arbitrary invariant measure in terms of the local entropy and of the Lyapunov exponents. These formulas are obtained with a direct approach, pertaining to the fundamentals of ergodic theory, which in particular does not require Pesin theory. This allows us to show, for those systems, that the Hausdorff dimension of a (non-ergodic) invariant measure is equal to the essential supremum of the Hausdorff dimensions of the measures in an ergodic decomposition. We also establish corresponding results for non-uniformly hyperbolic diffeomorphisms on surfaces, now using Pesin theory. To the best of our knowledge, the formula for the Hausdorff dimension of an invariant measure along an ergodic decomposition is new even in the case of uniformly hyperbolic dynamics.

1. *Introduction*

In the theory of dynamical systems we are often interested in studying the complexity of a given dynamics. This is particularly relevant in the study of the chaotic behavior associated to the strange attractors that frequently occur in natural phenomena. The complexity of the evolution of a dynamical system in its phase space and of the associated invariant sets can be measured from many points of view. These include the topological, ergodic, and dimensional points of view. To each of these approaches correspond important invariants, often with deep relations between them. For example, when the dynamics preserves a given measure we can obtain important information about the dynamics from the Kolmogorov–Sinai entropy and from the Hausdorff dimension of the measure.

On the other hand it is often the case that a given *global* invariant has behind it or in a certain sense can even be built with the help of a *local* quantity. For example, in the case of

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the Kolmogorov–Sinai entropy and of the Hausdorff dimension, which are quantities of global nature, these can be built (in a rigorous mathematical sense) respectively with the local entropy and with the pointwise dimension. In the case of the entropy this goes back to the classical Shannon–McMillan–Breiman theorem: the Kolmogorov–Sinai entropy is obtained by integrating the local entropy. In this paper we are mostly interested in the second invariant: the Hausdorff dimension of an invariant measure.

We first introduce some notation. Let $\dim_{\mathrm{H}} Z$ be the Hausdorff dimension of the set Z. Given a measure μ on $\Lambda \subset \mathbb{R}^m$ the *Hausdorff dimension* of μ is defined by

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} Z : Z \subset \Lambda \text{ and } \mu(\Lambda \setminus Z) = 0\}. \tag{1}$$

This quantity can indeed be defined in terms of a local quantity. Namely

$$\dim_{\mathrm{H}} \mu = \operatorname{ess\,sup}\left\{\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} : x \in \Lambda\right\}, \tag{2}$$

where $B(x,r)$ denotes the ball of radius r centered at x, with the essential supremum taken with respect to μ (see Proposition 3). In particular, if there exists a number d such that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d \tag{3}$$

for μ-almost every $x \in \Lambda$ then $\dim_{\mathrm{H}} \mu = d$. This criterion was established by Young in [**10**]. The limit in (3), when it exists, is called the *pointwise dimension* of μ at x.

Now, let μ be a compactly supported finite measure invariant under a $C^{1+\varepsilon}$ diffeomorphism f. It follows from the work of Ledrappier and Young in [**8**] and the work of Barreira *et al* in [**2**] that if the measure μ is hyperbolic then the pointwise dimension exists almost everywhere. In the two-dimensional case this statement was established in [**10**] by Young, who also showed that if μ is an *ergodic* invariant measure of saddle type then

$$\dim_{\mathrm{H}} \mu = h_\mu(f)\left(\frac{1}{\lambda_u(\mu)} - \frac{1}{\lambda_s(\mu)}\right), \tag{4}$$

where $h_\mu(f)$ is the Kolmogorov–Sinai entropy of f with respect to μ, and $\lambda_u(\mu)$ and $\lambda_s(\mu)$ are respectively the μ-averages of the positive and negative values of the Lyapunov exponent (note that since μ is ergodic these values are constant μ-almost everywhere). We recall that a finite measure μ is ergodic if all f-invariant sets A (i.e. such that $f^{-1}A = A$) have either zero of full μ-measure.

Our main objective is to discuss the pointwise dimension and the Hausdorff dimension of measures *that are not necessarily ergodic*. We emphasize that when μ is not ergodic in general (4) does not hold (again on a two-dimensional manifold). Examples can be readily obtained from the fact that given invariant probability measures μ_1, μ_2 and constants c_1, $c_2 > 0$ with $c_1 + c_2 = 1$, the measure $\mu = c_1\mu_1 + c_2\mu_2$ satisfies

$$\dim_{\mathrm{H}} \mu = \max\{\dim_{\mathrm{H}} \mu_1, \dim_{\mathrm{H}} \mu_2\} \tag{5}$$

and

$$h_\mu(f) = c_1 h_{\mu_1}(f) + c_2 h_{\mu_2}(f). \tag{6}$$

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We now present our results in the case of diffeomorphisms on surfaces. Let $f: M \to M$ be a C^1 surface diffeomorphism with a compact f-invariant locally maximal hyperbolic set $\Lambda \subset M$. For each $x \in M$ and $v \in T_x M$ we consider the *Lyapunov exponent*

$$\lambda(x, v) = \limsup_{n \to +\infty} \frac{1}{n} \log \|d_x f^n v\|.$$

For each $x \in \Lambda$ the function $\lambda(x, \cdot)$ takes exactly two real values $\lambda_u(x) > 0$ and $\lambda_s(x) < 0$, the positive and negative values of the Lyapunov exponent at x (see [**1**] for details). Now, let μ be an f-invariant probability measure on Λ. By the work of Brin and Katok in [**4**], for μ-almost every $x \in \Lambda$ there exists the limit

$$h_\mu(x) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu\left(\bigcap_{k=0}^{n-1} f^{-k} B(f^k(x), \varepsilon)\right).$$

The number $h_\mu(x)$ is called the *local entropy* of μ at the point x.

Our first result considers measures that are not necessarily ergodic and establishes an explicit formula for the pointwise dimension at a given point x in terms of the positive and negative values $\lambda_u(x)$ and $\lambda_s(x)$ of the Lyapunov exponent, and of the local entropy $h_\mu(x)$.

THEOREM 1. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ, and let μ be an f-invariant probability measure on Λ. Then for μ-almost every $x \in \Lambda$ we have*

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right). \tag{7}$$

We refer to the main text for details and generalizations. We emphasize that for non-ergodic measures in general the Hausdorff dimension $\dim_{\mathrm{H}} \mu$ (not to mention the pointwise dimension in a set of positive measure) is not given by the right-hand side of (4). Nevertheless, Theorem 1 indicates that at the local level we do have an identity that imitates the one in (4) for an arbitrary measure. Namely, the new identity in (7) can be formally obtained by replacing each global quantity in (4) by the corresponding local quantity.

We note that Theorem 1 can also be obtained from the work of Ledrappier and Young [**8**] and Barreira *et al* [**2**], in the more general context of non-uniformly hyperbolic dynamics. However, while the approach in these papers requires the machinery of Pesin theory, we provide a direct and arguably short proof, clearly pertaining to the fundamentals of ergodic theory, although only in the case of uniformly hyperbolic dynamics.

Theorem 1 provides crucial information of local nature that in its turn can be used to describe how the Hausdorff dimension $\dim_{\mathrm{H}} \mu$ of an invariant measure on a hyperbolic set Λ behaves under an ergodic decomposition. We recall that each ergodic decomposition of the measure μ can be identified with a probability measure τ on the metrizable space of f-invariant probability measures on Λ such that the subspace $\mathcal{M}_{\mathrm{E}}$ of ergodic measures on Λ has full τ-measure (see §2).

THEOREM 2. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ, and let μ be an f-invariant probability measure on Λ.*

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For any ergodic decomposition τ of μ we have

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\}, \tag{8}$$

with the essential supremum taken with respect to τ.

This is a particular case of our results which also include a generalization to non-uniformly hyperbolic dynamics. We emphasize that without the presence of hyperbolicity the identity in (8) may not hold, and in fact it is possible to give very simple examples where it fails (we refer to §2 for examples and for a related discussion).

In the case of the entropy it is well known that (see for example [**5**])

$$h_\mu(f) = \int_{\mathcal{M}_{\mathrm{E}}} h_\nu(f)\, d\tau(\nu). \tag{9}$$

The formulas (8) and (9) are generalizations respectively of (5) and (6) for an arbitrary number (possibly uncountable) of ergodic invariant probability measures.

Our approach to these problems can be described as pertaining to the fundamentals of ergodic theory, although with several technical complications. A rough description of the proof of Theorem 2 is the following.

(1) We first reduce our problem to f-invariant sets Γ where the Lyapunov exponents and the local entropy change little, that is,

$$|\lambda_u(y) - \lambda_u(x)| < \varepsilon, \quad |\lambda_s(y) - \lambda_s(x)| < \varepsilon, \quad |h_\mu(y) - h_\mu(x)| < \varepsilon$$

for each $y \in \Gamma$ and some fixed $x \in \Lambda$. In view of Theorem 1 the pointwise dimension also changes little almost everywhere in Γ and thus we are able to use (2) in order to obtain sharp estimates for the Hausdorff dimension of the invariant measure induced by μ on Γ in terms of the pointwise dimension at the given point x.

(2) Furthermore, we can use the Birkhoff ergodic theorem and the Shannon–McMillan–Breiman theorem to relate local quantities (respectively the Lyapunov exponents and the local entropy) and global quantities (respectively the averages of the Lyapunov exponents and of the Kolmogorov–Sinai entropy) up to some small error terms on each set Γ. In particular, this allows us to obtain sharp lower and upper bounds for the Hausdorff dimension of μ in terms of each *ergodic* invariant measure supported on Γ.

(3) Finally, since in the case of ergodic invariant measures we can use (4) to compute their Hausdorff dimension, the outcome of this approach is that the pointwise dimension at the given point x, and thus at almost every point of Γ, can be arbitrarily approximated by the Hausdorff dimension of *ergodic* measures (and in fact by the Hausdorff dimension of ergodic measures on a positive measure set with respect to some ergodic decomposition).

We now briefly describe the contents of the paper. In §2 we discuss basic relations between the pointwise dimension and the Hausdorff dimension of an invariant measure. We also discuss what happens in the context of Pesin theory. In the remaining sections we consider several classes of dynamical systems, and in particular we establish appropriate versions of Theorems 1 and 2. We first formulate our results in the case of repellers (in §3) since the approach is a bit simpler, instead of the need to deal simultaneously with the stable

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and unstable directions as in the case of hyperbolic sets. In §4 we establish the statements in Theorems 1 and 2 in the more general context of conformal diffeomorphisms on manifolds of arbitrary dimension. This allows us to reduce the study of invariant measures of maximal dimension to that of *ergodic* invariant measures of maximal dimension. Section 5 is dedicated to the study of non-uniformly hyperbolic diffeomorphisms. Even though the results are more general than those in §4, the main point is that the approaches in the two sections are different. While in §4 the approach is direct and pertains to the fundamentals of ergodic theory, in §5 we use several non-trivial results from the theory of non-uniformly hyperbolic dynamics.

2. *Pointwise dimension and dimension of measures*

Let μ be a Borel measure on a metric space $\Lambda \subset \mathbb{R}^m$. The *lower* and *upper pointwise dimensions* of μ at the point $x \in \Lambda$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We first recall two basic properties relating these quantities with the Hausdorff dimension of subsets of $\mathbb{R}^m$ (see, for example, [**9**, Theorems 7.1 and 7.2]):

(i) if $\underline{d}_\mu(x) \geq \alpha$ for μ-almost every $x \in \Lambda$ then $\dim_{\mathrm{H}} \mu \geq \alpha$;

(ii) if $\underline{d}_\mu(x) \leq \alpha$ for every $x \in Z \subset \Lambda$ then $\dim_{\mathrm{H}} Z \leq \alpha$.

Here $\dim_{\mathrm{H}} Z$ and $\dim_{\mathrm{H}} \mu$ denote respectively the Hausdorff dimensions of the set Z and of the measure μ (see equation (1) for the definition). By Whitney's embedding theorem, the above statements also hold on subsets of smooth manifolds.

The following statement should be considered well known, although it is difficult to find a straightforward presentation in the literature. For completeness we include a very simple argument.

PROPOSITION 3. *If μ is a finite Borel measure on a set $\Lambda \subset \mathbb{R}^m$ then* $\dim_{\mathrm{H}} \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in \Lambda\}$.

Proof. Let

$$\alpha = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in \Lambda\} \quad \text{and} \quad Z = \{x \in \Lambda : \underline{d}_\mu(x) \leq \alpha\}.$$

Then $\mu(Z) = 1$ and $\dim_{\mathrm{H}} \mu \leq \dim_{\mathrm{H}} Z \leq \alpha$ (using property (ii)). Now, for $\varepsilon > 0$, let

$$Z_\varepsilon = \{x \in \Lambda : \underline{d}_\mu(x) \geq \alpha - \varepsilon\}.$$

By the definition of essential supremum we have $\mu(Z_\varepsilon) > 0$ for every $\varepsilon > 0$. Hence, $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}}(\mu | Z_\varepsilon) \geq \alpha - \varepsilon$ (using property (i)), and the arbitrariness of ε implies that $\dim_{\mathrm{H}} \mu \geq \alpha$. □

We also want to discuss how the Hausdorff dimension of an invariant measure relates with its ergodic decompositions. Let $f : \Lambda \to \Lambda$ be a Borel measurable transformation. We denote by $\mathcal{M}$ the set of f-invariant probability measures on Λ, and by $\mathcal{M}_{\mathrm{E}} \subset \mathcal{M}$ the subset of ergodic measures. Recall that a measure μ is ergodic if for every f-invariant set $A \subset \Lambda$ (i.e. such that $f^{-1}A = A$) we have $\mu(A) = 0$ or $\mu(\Lambda \setminus A) = 0$.

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We say that a probability Borel measure τ on $\mathcal{M}$ (with the weak$*$ topology) is an *ergodic decomposition* of a measure $\mu \in \mathcal{M}$ if $\tau(\mathcal{M}_{\mathrm{E}}) = 1$ and

$$\int_\Lambda \varphi \, d\mu = \int_{\mathcal{M}} \left(\int_\Lambda \varphi \, d\nu \right) d\tau(\nu)$$

for every μ-integrable measurable function $\varphi \colon \Lambda \to \mathbb{R}$.

PROPOSITION 4. *Let $f \colon \Lambda \to \Lambda$ be a Borel measurable transformation preserving a probability measure μ on $\Lambda \subset \mathbb{R}^m$. For any ergodic decomposition τ of μ we have*

$$\dim_{\mathrm{H}} \mu \geq \operatorname{ess\,sup}\{\dim_{\mathrm{H}} \nu : \nu \in \mathcal{M}_{\mathrm{E}}\}, \tag{10}$$

with the essential supremum taken with respect to τ.

Proof. If $\mu(\Lambda \setminus Z) = 0$ then $\nu(\Lambda \setminus Z) = 0$ for τ-almost every $\nu \in \mathcal{M}$. Hence, $\dim_{\mathrm{H}} Z \geq \dim_{\mathrm{H}} \nu$ for τ-almost every $\nu \in \mathcal{M}$, and thus

$$\dim_{\mathrm{H}} Z \geq \operatorname{ess\,sup}\{\dim_{\mathrm{H}} \nu : \nu \in \mathcal{M}_{\mathrm{E}}\}.$$

Taking the infimum over all the sets Z with $\mu(\Lambda \setminus Z) = 0$ we obtain the desired result. □

We remark that the inequality in (10) may be strict. A simple example is given by a rational rotation of the circle: in this case each measure supported on a periodic orbit has zero Hausdorff dimension and thus when μ is the Lebesgue measure we have a strict inequality in (10). Another example was mentioned to one of us by Manfred Denker: it is a translation on the 2-torus given by the product of an irrational rotation and the identity; in this case the ergodic components are the 'horizontal' circles and when μ is the Lebesgue measure we have a strict inequality in (10).

It should be noted that when the number of ergodic components is finite or even infinite countable it is straightforward to verify that (10) is an identity, i.e.

$$\dim_{\mathrm{H}} \mu = \operatorname{ess\,sup}\{\dim_{\mathrm{H}} \nu : \nu \in \mathcal{M}_{\mathrm{E}}\}. \tag{11}$$

More precisely, assume that one can write $\Lambda = \bigcup_{n \in \mathbb{N} \cup \{0\}} \Lambda_n$ with pairwise disjoint f-invariant sets Λ_n for $n \in \mathbb{N} \cup \{0\}$ such that $f|\Lambda_n$ is ergodic with respect to μ for each $n \in \mathbb{N}$ and $\mu(\Lambda_0) = 0$ (that is, up to a zero measure set the number of ergodic components is countable). Then it is simple to show that

$$\dim_{\mathrm{H}} \mu = \sup\{\dim_{\mathrm{H}}(\mu|\Lambda_n) : n \in \mathbb{N}\}.$$

In particular, it follows from work of Pesin that (11) holds in the context of smooth ergodic theory. Namely, let μ be a hyperbolic finite Borel measure invariant under a $C^{1+\varepsilon}$ diffeomorphism of a compact manifold (see §5 for the definition of hyperbolic measure). Pesin showed that if μ is absolutely continuous with respect to the volume, then up to a zero measure set the number of ergodic components is countable, and thus (11) holds. See, for example, [**1**] for details (see also §5).

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3. *Expanding maps*

3.1. *Basic notions.* Let $f: M \to M$ be a differentiable map of a smooth manifold M. Consider a compact f-invariant set $J \subset M$. We say that f is *expanding* on J, and that J is a *repeller* of f if there exist constants $c > 0$ and $\beta > 1$ such that $\|d_x f^n v\| \geq c\beta^n \|v\|$ for all $x \in J$, $v \in T_x M$, and $n \in \mathbb{N}$. Furthermore, we say that f is *conformal* on J if $d_x f$ is a multiple of an isometry for every $x \in J$. From now on we assume that J is a repeller of f and that f is conformal on J.

Let μ be an f-invariant probability measure on J. By Birkhoff's ergodic theorem, for μ-almost every $x \in J$ there exists the limit

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log\|d_x f^n\| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)), \tag{12}$$

where $\phi(y) = \log\|d_y f\|$, and $x \mapsto \lambda(x)$ is f-invariant μ-almost everywhere.

Moreover, by work of Brin and Katok in [**4**], for μ-almost every $x \in J$ there exists the limit

$$h_\mu(x) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} -\frac{1}{n} \log \mu(B(x, n, \varepsilon)), \tag{13}$$

where

$$B(x, n, \varepsilon) = \{y \in M : d(f^k(y), f^k(x)) < \varepsilon \text{ for } k = 0, \ldots, n-1\}, \tag{14}$$

and d denotes the distance on M. The number $h_\mu(x)$ is called the *local entropy* of μ at x. Furthermore, the function $x \mapsto h_\mu(x)$ is μ-integrable, f-invariant μ-almost everywhere, and the Kolmogorov–Sinai entropy $h_\mu(f)$ of f with respect to μ can be computed by

$$h_\mu(f) = \int_J h_\mu(x)\, d\mu(x). \tag{15}$$

For the purpose of the proofs we require the relation of the local entropy with the corresponding version in terms of partitions instead of the sets $B(x, n, \varepsilon)$ in (14). Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a partition of the repeller J. Given integers $i_0, \ldots, i_n \in \{1, \ldots, k\}$ we define the *rectangle*

$$R_{i_0 \cdots i_n} = \{x \in J : f^j(x) \in R_{i_j} \text{ for } j = 0, \ldots, n\}. \tag{16}$$

By the Shannon–McMillan–Breiman theorem, for μ-almost every $x \in J$ we have

$$h_\mu(x) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(R_n(x)), \tag{17}$$

where $R_n(x) = R_{i_0 \cdots i_n}$ is any rectangle such that $x \in R_n(x)$. For definiteness, we always assume that for each x a particular choice of rectangles $R_n(x)$ is made from the beginning for all $n \in \mathbb{N}$. We denote by X a fixed full μ-measure f-invariant set of points $x \in J$ such that:

(1) the number $\lambda(x)$ in (12) is well defined;

(2) the number $h_\mu(x)$ in (13) is well defined and satisfies (17).

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3.2. *Formula for the pointwise dimension.* We are now ready to establish a local formula for the pointwise dimension of invariant measures that are not necessarily ergodic.

THEOREM 5. *Let J be a repeller of a $C^{1+\varepsilon}$ map f such that f is conformal on J, and let μ be an f-invariant probability measure on J. Then, for μ-almost every $x \in J$ we have*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = \frac{h_\mu(x)}{\lambda(x)}.$$

Proof. Fix $\varepsilon > 0$ such that $\varepsilon < \min \phi/2$. We define 'Pesin sets' Q_ℓ related to the functions h_μ and λ. For each x in the full measure set X there exists $p(x) \in \mathbb{N}$ such that if $n \geq p(x)$, then

$$\lambda(x) - \varepsilon < \frac{1}{n} \log \|d_x f^n\| < \lambda(x) + \varepsilon, \tag{18}$$

$$-h_\mu(x) - \varepsilon < \frac{1}{n} \log \mu(R_n(x)) < -h_\mu(x) + \varepsilon. \tag{19}$$

For each $\ell \in \mathbb{N}$ we define the set $Q_\ell = \{x \in X : p(x) \leq \ell\}$. Note that $\bigcup_{\ell \in \mathbb{N}} Q_\ell = X$. For each $x \in X$ there exists $r(x) > 0$ such that for every $r \in (0, r(x))$ we can choose $n = n(x, r) \geq p(x)$ such that

$$\|d_x f^n\|^{-1} \geq r \quad \text{and} \quad \|d_x f^{n+1}\|^{-1} < r. \tag{20}$$

We write $R(x, r) = R_{n(x,r)}(x)$ for each $x \in X$ and $r \in (0, r(x))$.

We first establish an upper bound for the pointwise dimension. The conformality of f on J guarantees that there exists $\kappa > 0$ (independent of x and r) such that $B(x, \kappa r) \supset R(x, r)$ for each $x \in X$ and $r \in (0, r(x))$, and hence

$$\mu(B(x, \kappa r)) \geq \mu(R(x, r)) \geq \exp[(-h_\mu(x) - \varepsilon)n].$$

Applying (18) and (20) we obtain

$$\mu(B(x, \kappa r)) \geq \exp\left[(h_\mu(x) + \varepsilon)\frac{\log r}{\lambda(x) - \varepsilon}\right],$$

and thus

$$\overline{d}_\mu(x) \leq \frac{h_\mu(x) + \varepsilon}{\lambda(x) - \varepsilon}.$$

The arbitrariness of ε implies that $\overline{d}_\mu(x) \leq h_\mu(x)/\lambda(x)$ for every $x \in X$ and hence for μ-almost every $x \in J$.

We now establish a lower bound for the pointwise dimension. Given $x \in X$ we define

$$\Gamma(x) = \{y \in X : |h_\mu(y) - h_\mu(x)| < \varepsilon \text{ and } |\lambda(y) - \lambda(x)| < \varepsilon\}. \tag{21}$$

The sets $\Gamma(x)$ form a cover of X and we can choose points $y_i \in X$ for $i = 1, 2, \ldots$ such that $\Gamma_i = \Gamma(y_i)$ satisfies $\mu(\Gamma_i) > 0$ for each i, and $\bigcup_{i \in \mathbb{N}} \Gamma_i$ has full μ-measure.

Fix $i, \ell \in \mathbb{N}$. We now proceed in a similar way to that in [**9**, §22] to construct an appropriate cover of $\Gamma_i \cap Q_\ell$ by sets of the form $R(x, r) \cap \Gamma_i \cap Q_\ell$. For each $x \in \Gamma_i \cap Q_\ell$ and $r > 0$, we denote by $R'(x, r)$ the largest rectangle containing x (among those in (16)) with the property that $R'(x, r) = R(y, r)$ for some $y \in R'(x, r) \cap \Gamma_i \cap Q_\ell$ and that

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$R(z,r) \subset R'(x,r)$ for any $z \in R'(x,r) \cap \Gamma_i \cap Q_\ell$. Two sets $R'(x,r)$ and $R'(y,r)$ either coincide or intersect at most along their boundaries.

The Borel density lemma (see, for example, [**6**, Theorem 2.9.11]) tells us that for μ-almost every $x \in \Gamma_i \cap Q_\ell$ we have

$$\lim_{r\to 0} \frac{\mu(B(x,r) \cap \Gamma_i \cap Q_\ell)}{\mu(B(x,r))} = 1.$$

This implies that for μ-almost every $x \in \Gamma_i \cap Q_\ell$ there exists a constant $r(x) > 0$ such that for each $r \in (0, r(x))$ we have

$$\mu(B(x,r)) \le 2\mu(B(x,r) \cap \Gamma_i \cap Q_\ell). \tag{22}$$

By the conformality of f on J, there exist a constant $K > 0$ (independent of x and r) and points $x_1, \ldots, x_k \in \Gamma_i \cap Q_\ell$ with $k \le K$ such that

$$B(x,r) \cap \Gamma_i \cap Q_\ell \subset \bigcup_{j=1}^{k} R'(x_j, r).$$

Applying (19) and (22) we obtain

$$\mu(B(x,r)) \le 2\mu(B(x,r) \cap \Gamma_i \cap Q_\ell) \le 2\sum_{j=1}^{k} \mu(R'(x_j,r))$$
$$\le 2\sum_{j=1}^{k} \exp[(-h_\mu(x_j) + \varepsilon) n(x_j, r)].$$

By the definition of Γ_i and (20) we conclude that

$$\mu(B(x,r)) \le 2\sum_{j=1}^{k} \exp\left[(h_\mu(y_i) - 2\varepsilon)\frac{\log r + \max\phi}{\lambda(x_j) + \varepsilon}\right]$$
$$\le 2K \exp\left[(h_\mu(x) - 3\varepsilon)\frac{\log r + \max\phi}{\lambda(x) + 2\varepsilon}\right],$$

and thus

$$\underline{d}_\mu(x) \ge \frac{h_\mu(x) - 3\varepsilon}{\lambda(x) + 2\varepsilon}.$$

The arbitrariness of ε implies that

$$\underline{d}_\mu(x) \ge h_\mu(x)/\lambda(x) \tag{23}$$

for μ-almost every $x \in \Gamma_i \cap Q_\ell$. Letting $\ell \to \infty$ we conclude that (23) holds for μ-almost every $x \in \Gamma_i$. Since $\bigcup_{\ell\in\mathbb{N}} \Gamma_i$ has full μ-measure, (23) holds for μ-almost every $x \in J$. □

Combining Theorem 5 with Proposition 3 and using the μ-almost everywhere f-invariance of the functions h_μ and λ, we obtain the following formulas for the Hausdorff dimension of an invariant measure (that is not necessarily ergodic).

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COROLLARY 6. *If f is a $C^{1+\varepsilon}$ map with a repeller J on which f is conformal, and μ is an f-invariant probability measure on J, then*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{\frac{h_\mu(x)}{\lambda(x)} : x \in J\right\}.$$

If, in addition, μ is ergodic then

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = \dim_{\mathrm{H}}\mu = \frac{h_\mu(f)}{\int_J \log\|df\|\,d\mu} \tag{24}$$

for μ-almost every $x \in J$.

The identity (24) was established by Pesin in [**9**].

3.3. *Hausdorff dimension of measures and ergodic decompositions.* The following statement provides a formula for the Hausdorff dimension of a measure in terms of an ergodic decomposition. We recall that $\mathcal{M}_{\mathrm{E}}$ is the set of ergodic f-invariant probability measures on J.

THEOREM 7. *Let J be a repeller of a $C^{1+\varepsilon}$ map f such that f is conformal on J, and let μ be an f-invariant probability measure on J. For any ergodic decomposition τ of μ we have*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\},$$

with the essential supremum taken with respect to τ.

Proof. By Proposition 4 we have

$$\dim_{\mathrm{H}}\mu \geq \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\}.$$

We now establish the opposite inequality. By Corollary 6 we have

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{\frac{h_\mu(x)}{\lambda(x)} : x \in X\right\}. \tag{25}$$

Fix $\varepsilon > 0$ such that $\varepsilon < \min\phi/2$. As in the proof of Theorem 5, for each x in the full μ-measure set X we consider the set $\Gamma(x)$ defined by (21). Again we choose points $y_i \in X$ for $i = 1, 2, \ldots$ such that the f-invariant set $\Gamma_i = \Gamma(y_i)$ satisfies $\mu(\Gamma_i) > 0$ for each i, and $\bigcup_{i\in\mathbb{N}}\Gamma_i$ has full μ-measure. Fix i and consider the normalized restriction μ_i of μ to Γ_i. It follows from (15) and (21) that

$$h_{\mu_i}(f|\Gamma_i) = \frac{1}{\mu(\Gamma_i)}\int_{\Gamma_i} h_\mu(x)\,d\mu(x) \geq h_\mu(y_i) - \varepsilon. \tag{26}$$

Let $\mathcal{M}_i = \{\nu \in \mathcal{M} : \nu(J \setminus \Gamma_i) = 0\}$, where $\mathcal{M}$ denotes the set of f-invariant probability measures on J. Since Γ_i is f-invariant, there is a one-to-one correspondence between the ergodic f-invariant probability measures on Γ_i and the measures in $\mathcal{M}_i \cap \mathcal{M}_{\mathrm{E}}$. It is thus straightforward to verify that $\tau(\mathcal{M}_i \cap \mathcal{M}_{\mathrm{E}}) > 0$ and that the normalization τ_i of $\tau|\mathcal{M}_i$ provides an ergodic decomposition of μ_i. Since

$$h_{\mu_i}(f|\Gamma_i) = \int_{\mathcal{M}_i} h_\nu(f)\,d\tau_i(\nu)$$

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(see for example [**5**]), there exists a set $A_i \subset \mathcal{M}_i \cap \mathcal{M}_{\mathrm{E}}$ of positive τ_i-measure and thus also with positive τ-measure such that $h_\nu(f) > h_{\mu_i}(f|\Gamma_i) - \varepsilon$ for every $\nu \in A_i$. Using (26), for each $\nu \in A_i$ and $x \in \Gamma_i$ we have

$$h_\nu(f) > h_{\mu_i}(f|\Gamma_i) - \varepsilon \geq h_\mu(y_i) - 2\varepsilon > h_\mu(x) - 3\varepsilon$$

and

$$\lambda(\nu) = \int_{\Gamma_i} \phi \, d\nu \leq \lambda(y_i) + \varepsilon \leq \lambda(x) + 2\varepsilon.$$

Therefore, for all $\nu \in A_i$ we have

$$\frac{h_\mu(x)}{\lambda(x)} \leq \frac{h_\nu(f) + 3\varepsilon}{\lambda(\nu) - 2\varepsilon} \leq \dim_{\mathrm{H}} \nu + C(\varepsilon),$$

where $\varepsilon \mapsto C(\varepsilon)$ is a function (independent of i and ν) that tends to zero as $\varepsilon \to 0$. Since $\tau(A_i) > 0$, it follows from (25) and Corollary 6 that

$$\dim_{\mathrm{H}} \mu \leq \operatorname{ess\,sup}\{\dim_{\mathrm{H}} \nu : \nu \in \mathcal{M}_{\mathrm{E}}\} + C(\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain the desired result. □

All the statements in §3 can also be established when the conformality is replaced by the following more general situation: assume that there exist a continuous function $\psi : J \to \mathbb{R}^+$ and a constant $\kappa \geq 1$ such that for every $x \in J$ we have

$$\kappa^{-1}\|v\| \exp \sum_{k=0}^{n-1} \psi(f^k(x)) \leq \|d_x f^n v\| \leq \kappa \|v\| \exp \sum_{k=0}^{n-1} \psi(f^k(x))$$

for every $v \in T_x M$ and $n \in \mathbb{N}$.

4. *Hyperbolic diffeomorphisms*

We now consider hyperbolic diffeomorphisms and derive formulas for the pointwise dimension and the Hausdorff dimension of an invariant measure that is not necessarily ergodic. These formulas are versions of those in the case of expanding maps (see §3). Our approach is similar to that in §3 although it is now necessary to deal simultaneously with the stable and unstable directions.

4.1. *Basic notions.* Let M be a smooth manifold and $f : M \to M$ a diffeomorphism. Let also $\Lambda \subset M$ be a compact f-invariant locally maximal hyperbolic set for f. This means that there exist a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$, and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

(1) $d_x f(E^s_x) = E^s_{f(x)}$ and $d_x f(E^u_x) = E^u_{f(x)}$;
(2) $\|d_x f^n v\| \leq c\lambda^n \|v\|$ whenever $v \in E^s_x$ and $n > 0$;
(3) $\|d_x f^{-n} v\| \leq c\lambda^n \|v\|$ whenever $v \in E^u_x$ and $n > 0$.

Furthermore, there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$. We assume that f is *conformal* on Λ, i.e. that the maps

$$d_x f|E^s_x : E^s_x \to E^s_{f(x)} \quad \text{and} \quad d_x f|E^u_x : E^u_x \to E^u_{f(x)}$$

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are multiples of isometries for each $x \in \Lambda$. For example, this is the case when $\dim E^s_x = \dim E^u_x = 1$ for every $x \in \Lambda$. A situation more general than conformality is considered at the end of §4.2.

Now, let μ be an f-invariant probability measure on Λ. Since f is conformal on Λ, it follows from Birkhoff's ergodic theorem that for μ-almost every $x \in \Lambda$ there exist the limits

$$\lambda_s(x) = \lim_{n\to\infty} \frac{1}{n} \log \|d_x f^n | E^s_x\| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi_s(f^k(x)), \tag{27}$$

and

$$\lambda_u(x) = \lim_{n\to\infty} \frac{1}{n} \log \|d_x f^n | E^u_x\| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi_u(f^k(x)), \tag{28}$$

where

$$\phi_s(y) = \log \|d_y f | E^s_y\| \quad \text{and} \quad \phi_u(y) = \log \|d_y f | E^u_y\|.$$

The numbers $\lambda_s(x)$ and $\lambda_u(x)$ are, respectively, the negative and positive values of the Lyapunov exponent at x.

We define again the local entropy of μ at x by (13). As in the case of expanding maps, the Kolmogorov–Sinai entropy $h_\mu(f)$ of f with respect to μ satisfies (15).

Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a Markov partition of Λ (see, for example, [**9**] for the definition). Given integers $i_{-m}, \ldots, i_n \in \{1, \ldots, k\}$ we define the *rectangle*

$$R_{i_{-m}\cdots i_n} = \{x \in \Lambda : f^j(x) \in R_{i_j} \text{ for } j = -m, \ldots, n\}.$$

The Shannon–McMillan–Breiman theorem shows that for μ-almost every $x \in \Lambda$ we have

$$h_\mu(x) = \lim_{n,m\to\infty} -\frac{1}{n+m} \log \mu(R_{n,m}(x)), \tag{29}$$

for any choice of rectangles $R_{n,m}(x) = R_{i_{-m}\cdots i_n}$ such that $x \in R_{n,m}(x)$. For definiteness, we always assume that for each x a particular choice of rectangles $R_{n,m}(x)$ is made from the beginning for all $n, m \in \mathbb{N}$. Let X be a fixed full μ-measure f-invariant set of points $x \in \Lambda$ such that:

(1) the numbers $\lambda_s(x)$ and $\lambda_u(x)$ in (27)–(28) are well defined;

(2) the number $h_\mu(x)$ in (13) is well defined and satisfies (29).

We shall restrict ourselves to the set X whenever appropriate.

4.2. *Formula for the pointwise dimension.* We now present the main result of this section: a local version of Young's formula in (4) for measures which are not necessarily ergodic. More precisely, we show that the pointwise dimension at a given point can be expressed in terms of the local entropy and of the positive and negative values of the Lyapunov exponent at that point.

THEOREM 8. *Let f be a $C^{1+\varepsilon}$ diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ on which f is conformal, and let μ be an f-invariant probability measure on Λ. Then for μ-almost every $x \in \Lambda$ we have*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right). \tag{30}$$

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Proof. The proof is an elaboration of the proof of Theorem 5. We only describe the arguments that require modifications. Fix $\varepsilon > 0$ such that

$$\varepsilon < \tfrac{1}{2}\min\{\min\phi_u, -\max\phi_s\}. \tag{31}$$

For each x in the full μ-measure set X (as defined in §4.1) there exists $p(x) \in \mathbb{N}$ such that if $n, m \geq p(x)$ then

$$\lambda_s(x) - \varepsilon < \frac{1}{n}\log\|d_x f^n|E^s_x\| < \lambda_s(x) + \varepsilon, \tag{32}$$

$$\lambda_u(x) - \varepsilon < \frac{1}{n}\log\|d_x f^n|E^u_x\| < \lambda_u(x) + \varepsilon, \tag{33}$$

$$-h_\mu(x) - \varepsilon < \frac{1}{n+m}\log\mu(R_{n,m}(x)) < -h_\mu(x) + \varepsilon. \tag{34}$$

Now, fix $\ell \in \mathbb{N}$ and consider the set $Q_\ell = \{x \in X : p(x) \leq \ell\}$. Clearly $\bigcup_{\ell\in\mathbb{N}} Q_\ell = X$. For each $x \in X$ there exists $r(x) > 0$ such that for every $r \in (0, r(x))$ one can choose integers $n = n(x,r) \geq p(x)$ and $m = m(x,r) \geq p(x)$ such that

$$\|d_x f^m|E^s_x\| \geq r \quad \text{and} \quad \|d_x f^{m+1}|E^s_x\| < r, \tag{35}$$

$$\|d_x f^n|E^u_x\|^{-1} \geq r \quad \text{and} \quad \|d_x f^{n+1}|E^u_x\|^{-1} < r. \tag{36}$$

We use the notation $R(x,r) = R_{n(x,r),m(x,r)}(x)$ (we recall that the rectangles $R_{n,m}(x)$ are fixed from the beginning). Combining (32) with (35), and (33) with (36) (also using (31)) we obtain

$$m(\lambda_s(x) - \varepsilon) < \log r - \min\phi_s \quad \text{and} \quad \log r < m(\lambda_s(x) + \varepsilon), \tag{37}$$

$$-\log r - \max\phi_u < n(\lambda_u(x) + \varepsilon) \quad \text{and} \quad n(\lambda_u(x) - \varepsilon) < -\log r. \tag{38}$$

We first establish an upper bound for the pointwise dimension. It follows from the conformality of f on Λ that there exists $\kappa > 0$ (independent of x and r) such that $B(x, \kappa r) \supset R(x,r)$ and hence

$$\mu(B(x,\kappa r)) \geq \mu(R(x,r)) \geq \exp[(-h_\mu(x) - \varepsilon)(n+m)].$$

Therefore

$$\mu(B(x,\kappa r)) \geq \exp\left[(h_\mu(x) + \varepsilon)\left(\frac{\log r}{\lambda_u(x) - \varepsilon} - \frac{\log r}{\lambda_s(x) + \varepsilon}\right)\right].$$

Taking logarithms and letting $r \to 0$ we obtain

$$\overline{d}_\mu(x) \leq (h_\mu(x) + \varepsilon)\left(\frac{1}{\lambda_u(x) - \varepsilon} - \frac{1}{\lambda_s(x) + \varepsilon}\right).$$

The arbitrariness of ε implies that

$$\overline{d}_\mu(x) \leq h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right)$$

for every $x \in X$ and hence for μ-almost every $x \in \Lambda$.

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We now establish a lower bound for the pointwise dimension. We continue to consider a fixed $\varepsilon > 0$ satisfying (31). Given x in the full μ-measure set X we define

$$\Gamma(x) = \{y \in X : |\lambda_s(y) - \lambda_s(x)| < \varepsilon, |\lambda_u(y) - \lambda_u(x)| < \varepsilon, \text{ and } |h_\mu(y) - h_\mu(x)| < \varepsilon\}.$$

Note that each set $\Gamma(x)$ is f-invariant. The sets $\Gamma(x)$ cover X and we can choose points $y_i \in X$ for $i = 1, 2, \ldots$ such that $\Gamma_i = \Gamma(y_i)$ satisfies $\mu(\Gamma_i) > 0$ for each i, and $\bigcup_{i\in\mathbb{N}} \Gamma_i$ has full μ-measure.

We can now proceed in a similar way to that in the proof of Theorem 5 to construct an appropriate cover of $\Gamma_i \cap Q_\ell$ by sets $R'(x, r)$ of the form $R(x, r) \cap \Gamma_i \cap Q_\ell$. Also as in the proof of Theorem 5 it follows from the Borel density lemma that for μ-almost every $x \in \Gamma_i \cap Q_\ell$ there exists a constant $r(x) > 0$ such that for each $r \in (0, r(x))$ we have

$$\mu(B(x, r)) \leq 2\mu(B(x, r) \cap \Gamma_i \cap Q_\ell).$$

By the conformality and the uniform transversality of the stable and unstable manifolds, we conclude that there exist a constant $K > 0$ (independent of x and r) and points $x_1, \ldots, x_k \in \Gamma_i \cap Q_\ell$ with $k \leq K$ such that

$$B(x, r) \cap \Gamma_i \cap Q_\ell \subset \bigcup_{j=1}^{k} R'(x_j, r).$$

We obtain

$$\mu(B(x, r)) \leq 2\mu(B(x, r) \cap \Gamma_i \cap Q_\ell) \leq 2\sum_{j=1}^{k} \mu(R'(x_j, r))$$
$$\leq 2\sum_{j=1}^{k} \exp[(-h_\mu(x_j) + \varepsilon)(n(x_j, r) + m(x_j, r))],$$

using (34). By (37)–(38) and the definition of Γ_i we conclude that

$$\mu(B(x, r)) \leq 2\sum_{j=1}^{k} \exp\left[(h_\mu(y_i) - 2\varepsilon)\left(\frac{\log r + \max \phi_u}{\lambda_u(x_j) + \varepsilon} - \frac{\log r - \min \phi_s}{\lambda_s(x_j) - \varepsilon}\right)\right]$$
$$\leq 2K \exp\left[(h_\mu(x) - 3\varepsilon)\left(\frac{\log r + \max \phi_u}{\lambda_u(x) + 2\varepsilon} - \frac{\log r - \min \phi_s}{\lambda_s(x) - 2\varepsilon}\right)\right].$$

Taking logarithms and letting $r \to 0$ yields

$$\underline{d}_\mu(x) \geq (h_\mu(x) - 3\varepsilon)\left(\frac{1}{\lambda_u(x) + 2\varepsilon} - \frac{1}{\lambda_s(x) - 2\varepsilon}\right).$$

The arbitrariness of ε implies that

$$\underline{d}_\mu(x) \geq h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) \tag{39}$$

for μ-almost every $x \in \Gamma_i \cap Q_\ell$. Letting $\ell \to \infty$ we conclude that (39) holds for μ-almost every $x \in \Gamma_i$. Since $\bigcup_{i\in\mathbb{N}} \Gamma_i$ has full μ-measure the inequality (39) holds for μ-almost every $x \in \Lambda$. □

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We note that the coincidence of the lower and upper pointwise dimensions is known in much greater generality. Namely, for any hyperbolic measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism (see §5), it was shown by Barreira *et al* in [**2**] that $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for μ-almost every x.

Combining Theorem 8 with Proposition 3 yields the following formula for the Hausdorff dimension of an invariant measure.

COROLLARY 9. *If f is a $C^{1+\varepsilon}$ diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ on which f is conformal, and μ is an f-invariant probability measure on Λ, then*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) : x \in \Lambda\right\}. \tag{40}$$

We emphasize that the essential supremum in (40) is to be understood in the sense that for any full μ-measure subset of Λ in which the quantities $h_\mu(x)$, $\lambda_s(x)$, and $\lambda_u(x)$ are well defined (and in particular for the set X defined in §4.1), the number $\dim_{\mathrm{H}}\mu$ coincides with the essential supremum over that set.

When M is a surface and μ is an *ergodic* measure, Theorem 8 (or Corollary 9) can be used to recover Young's formula in (4) (in the uniformly hyperbolic case; see also §5). Theorem 8 can also be used to recover the corresponding result for conformal diffeomorphisms on higher-dimensional manifolds (see [**9**, Theorem 24.2]). These are immediate consequences of the μ-almost everywhere f-invariance of the functions h_μ, λ_s, and λ_u in the right-hand sides of (30) and (40).

Theorem 8 and Corollary 9 have been obtained assuming that f is conformal on the hyperbolic set Λ. The same statements can also be established in the following more general situation: assume that there exist continuous functions $\psi_s : \Lambda \to \mathbb{R}^-$ and $\psi_u : \Lambda \to \mathbb{R}^+$ and a constant $\kappa \geq 1$ such that for every $x \in \Lambda$ we have

$$\kappa^{-1}\|v\| \exp \sum_{k=0}^{n-1} \psi_s(f^k(x)) \leq \|d_x f^n v\| \leq \kappa\|v\| \exp \sum_{k=0}^{n-1} \psi_s(f^k(x))$$

and

$$\kappa^{-1}\|w\| \exp \sum_{k=0}^{n-1} \psi_u(f^k(x)) \leq \|d_x f^n w\| \leq \kappa\|w\| \exp \sum_{k=0}^{n-1} \psi_u(f^k(x))$$

for every $v \in E^s_x$, $w \in E^u_x$, and $n \in \mathbb{N}$.

4.3. *Hausdorff dimension of measures and ergodic decompositions.* Using Theorem 8 we can now describe how the dimension of measures behaves under ergodic decompositions on hyperbolic sets.

THEOREM 10. *Let f be a $C^{1+\varepsilon}$ diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ on which f is conformal, and let μ be an f-invariant probability measure on Λ. For any ergodic decomposition τ of μ we have*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\},$$

with the essential supremum taken with respect to τ.

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Proof. The proof is a direct elaboration of the proof of Theorem 7 and thus we only give a brief sketch. By Corollary 9 we can write

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) : x \in X\right\}. \tag{41}$$

Fix $\varepsilon > 0$ satisfying (31). As in the proof of Theorem 8, we consider the sets $\Gamma_i = \Gamma(y_i)$. Fix i and consider the normalized restriction μ_i of μ to Γ_i. Proceeding as in the proof of Theorem 7 we show that there exists a set $A_i \subset \mathcal{M}_i \cap \mathcal{M}_{\mathrm{E}}$ of positive τ_i-measure (where τ_i is the normalization of $\tau|\mathcal{M}_i$) such that for each $\nu \in A_i$ and $x \in \Gamma_i$ we have

$$h_\nu(f) > h_{\mu_i}(f|\Gamma_i) - \varepsilon \geq h_\mu(y_i) - 2\varepsilon > h_\mu(x) - 3\varepsilon.$$

Furthermore

$$\lambda_s(\nu) = \int_{\Gamma_i} \phi_s \, d\nu \geq \lambda_s(y_i) - \varepsilon \geq \lambda_s(x) - 2\varepsilon,$$
$$\lambda_u(\nu) = \int_{\Gamma_i} \phi_u \, d\nu \leq \lambda_u(y_i) + \varepsilon \leq \lambda_u(x) + 2\varepsilon.$$

We conclude that

$$h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) \leq (h_\nu(f) + 3\varepsilon)\left(\frac{1}{\lambda_u(\nu) - 2\varepsilon} - \frac{1}{\lambda_s(\nu) + 2\varepsilon}\right)$$

for every $\nu \in A_i$. Since $\tau(A_i) > 0$, it follows from (41) that

$$\dim_{\mathrm{H}}\mu \leq \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\} + C(\varepsilon),$$

where $\varepsilon \mapsto C(\varepsilon)$ is a function (independent of i and ν) that tends to zero as $\varepsilon \to 0$. By Proposition 4, the arbitrariness of ε implies the desired result. □

The following is now an immediate application of Theorem 10.

COROLLARY 11. *If f is a $C^{1+\varepsilon}$ diffeomorphism with a compact f-invariant locally maximal hyperbolic set Λ on which f is conformal, then*

$$\sup\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}\} = \sup\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\}.$$

Corollary 11 allows one to reduce the study of invariant measures of maximal dimension to that of *ergodic* invariant measures of maximal dimension (see [**3**] for a detailed discussion). Recall that an invariant probability measure μ on Λ is said to be a measure of maximal dimension if $\dim_{\mathrm{H}}\mu = \sup\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}\}$.

We note that in Theorem 10 and Corollary 11 the hypothesis on the conformality of f on Λ can be replaced by the weaker assumption at the end of §4.2.

5. *Non-uniformly hyperbolic diffeomorphisms*

We can also establish related results in the more general case of non-uniformly hyperbolic dynamics. We stress that the emphasis in §4 is that the results are obtained with a direct approach that in particular does not require Pesin theory. In the present section the emphasis is different. Namely, we want to show to what extent the results in the

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former section have corresponding versions in the non-uniformly hyperbolic setting, now necessarily resorting to the appropriate results from smooth ergodic theory.

We briefly review some notions from the theory of non-uniformly hyperbolic dynamical systems (see [**2, 1**] for details). Consider a $C^{1+\varepsilon}$ diffeomorphism $f: M \to M$. We recall that a finite measure μ on M is said to be *hyperbolic* if for each x in a full μ-measure set and each $v \in T_x M$ we have

$$\lambda(x, v) = \limsup_{n \to +\infty} \frac{1}{n} \log \|d_x f^n v\| \neq 0.$$

We assume from now on that μ is a compactly supported f-invariant hyperbolic probability measure on M. There exist stable and unstable local manifolds $V^s(x)$ and $V^u(x)$ at x, of size $r(x) > 0$ depending measurably on x. For each $r \in (0, r(x))$ we consider the balls $B^s(x, r) \subset V^s(x)$ and $B^u(x, r) \subset V^u(x)$ centered at x with respect to the induced distances on $V^s(x)$ and $V^u(x)$ respectively.

We also need the families of conditional measures μ_x^s and μ_x^u generated by certain measurable partitions constructed by Ledrappier and Young [**8**], based on the former work of Ledrappier and Strelcyn [**7**]. As shown by Rohklin, any measurable partition ξ of M has associated a family of conditional measures: for μ-almost every point $x \in M$ there exists a probability measure μ_x defined on the element $\xi(x)$ of ξ containing x. Furthermore, the conditional measures are characterized completely by the following property: if $\mathcal{B}_\xi$ is a σ-subalgebra of the Borel σ-algebra generated by the unions of elements of ξ then for each Borel set $A \subset M$, the function $x \mapsto \mu_x(A \cap \xi(x))$ is $\mathcal{B}_\xi$-measurable and

$$\mu(A) = \int_A \mu_x(A \cap \xi(x))\, d\mu.$$

It is established in [**8**] that there exist two measurable partitions ξ^s and ξ^u of M such that for μ-almost every point $x \in M$ we have:

(1) $\xi^s(x) \subset V^s(x)$ and $\xi^u(x) \subset V^u(x)$;

(2) $\xi^s(x) \supset V^s(x) \cap B(x, \gamma)$ and $\xi^u(x) \supset V^u(x) \cap B(x, \gamma)$ for some $\gamma = \gamma(x) > 0$.

We denote by μ_x^s and μ_x^u the conditional measures associated respectively to the partitions ξ^s and ξ^u.

In [**8**], Ledrappier and Young established the existence of the limits

$$d_\mu^s(x) = \lim_{r \to 0} \frac{\log \mu_x^s(B^s(x, r))}{\log r} \quad \text{and} \quad d_\mu^u(x) = \lim_{r \to 0} \frac{\log \mu_x^u(B^u(x, r))}{\log r}$$

for μ-almost every $x \in M$. We call these quantities respectively *stable* and *unstable pointwise dimensions* of μ at x. The following is a combination of results in [**8**] with results of Barreira *et al* in [**2**].

THEOREM 12. *Let f be a $C^{1+\varepsilon}$ diffeomorphism and μ a compactly supported f-invariant hyperbolic probability measure. For μ-almost every x we have*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu^s(x) + d_\mu^u(x).$$

More precisely, it is established in [**8**] that $\overline{d}_\mu \leq d_\mu^s + d_\mu^u$ μ-almost everywhere and it is proven in [**2**] that $\underline{d}_\mu \geq d_\mu^s + d_\mu^u$ for μ-almost every point. It should be noted that

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in [**8**] the authors consider a more general class of measures, for which some values of the Lyapunov exponent may be zero. On the other hand, Ledrappier and Young require the diffeomorphism f to be of class C^2. The only place in [**8**] where f is required to be of class C^2 concerns the Lipschitz regularity of the holonomies generated by the intermediate foliations. In the case of hyperbolic measures a new argument was given by Barreira *et al* in [**2**] establishing the Lipschitz regularity for $C^{1+\varepsilon}$ diffeomorphisms. This ensures that the inequality $\overline{d}_\mu \leq d^s_\mu + d^u_\mu$ of Ledrappier and Young holds even when f is only of class $C^{1+\varepsilon}$. See [**2**] for details.

Combining Theorem 12 with Proposition 3 yields the following formula for the Hausdorff dimension of a hyperbolic measure.

COROLLARY 13. *If $f\colon M \to M$ is a $C^{1+\varepsilon}$ diffeomorphism and μ is a compactly supported f-invariant hyperbolic probability measure then*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\{d^s_\mu(x) + d^u_\mu(x) : x \in M\}.$$

Under the hypotheses of Corollary 13, when M is a surface it follows from work of Ledrappier and Young in [**8**] that

$$d^s_\mu(x) = -\frac{h_\mu(x)}{\lambda_s(x)} \quad \text{and} \quad d^u_\mu(x) = \frac{h_\mu(x)}{\lambda_u(x)}$$

for μ-almost every $x \in M$, where $\lambda_s(x)$ and $\lambda_u(x)$ denote the negative and positive values of the Lyapunov exponent at x. Therefore

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) : x \in M\right\}. \tag{42}$$

This formula can now be used to establish the behavior under ergodic decompositions for hyperbolic measures. We say that a hyperbolic measure μ on a surface M is of *saddle type* if for μ-almost every $x \in M$ there exist one positive value and one negative value of the Lyapunov exponent.

COROLLARY 14. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism and μ a compactly supported f-invariant hyperbolic probability measure of saddle type. Then, for any ergodic decomposition τ of μ we have*

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\},$$

with the essential supremum taken with respect to τ.

Proof. The argument is a modification of the corresponding argument in the proof of Theorem 10. Let X be a full μ-measure f-invariant set of points for which $\lambda_s(x)$, $\lambda_u(x)$, and $h_\mu(x)$ are well defined. By (42) one can write

$$\dim_{\mathrm{H}}\mu = \operatorname{ess\,sup}\left\{h_\mu(x)\left(\frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)}\right) : x \in X\right\}.$$

Fix $\varepsilon > 0$. For each $x \in X$ we define

$$\Gamma(x) = \{y \in X : |\lambda_s(y) - \lambda_s(x)| < \varepsilon, |\lambda_u(y) - \lambda_u(x)| < \varepsilon, \text{ and } |h_\mu(y) - h_\mu(x)| < \varepsilon\}.$$

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Now, choose points $y_i \in X$ for $i = 1, 2, \ldots$ such that the f-invariant sets $\Gamma_i = \Gamma(y_i)$ satisfy $\mu(\Gamma_i) > 0$ for each i, and $\bigcup_{i\in\mathbb{N}} \Gamma_i$ has full μ-measure. We can now repeat arguments in the proof of Theorem 10 to show that

$$\dim_{\mathrm{H}}\mu \leq \operatorname{ess\,sup}\{\dim_{\mathrm{H}}\nu : \nu \in \mathcal{M}_{\mathrm{E}}\}.$$

This completes the proof. □

We emphasize that in Corollary 14 we are not assuming the measure μ to be absolutely continuous with respect to the volume, and thus the number of ergodic components may not be countable up to a zero measure set (this should be compared with the discussion in §2).

The statement in Corollary 14 easily extends to higher-dimensional manifolds provided that for μ-almost every x there exist numbers $\lambda_s(x)$ and $\lambda_u(x)$ such that all negative values of the Lyapunov exponent at x are equal to $\lambda_s(x)$ and that all positive values of the Lyapunov exponent at x are equal to $\lambda_u(x)$. This can be considered as a certain notion of conformality in the non-uniformly hyperbolic setting.

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