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Sluice-gate problems with gravity

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Abstract. In this paper, a free-boundary problem of the steady plane flow of an ideal fluid through a slot is solved. The fluid flows under the effect of gravity between rigid boundaries, and then out of a slot as a jet which becomes horizontal at infinity downstream. A constructive proof of the existence of solutions to a non-linear integral equation is given for a parameter range $0 < \epsilon < \epsilon^* \simeq 1{\cdot}02$ (where ϵ, the inverse Froude number, measures the effect of gravity). Approximations to the solution are then found for $\epsilon \to 0$.

1. *Introduction.* The steady plane flow of an ideal fluid and the geometry of Fig. 1 (*a*) are taken as a model of a sluice-gate problem. There is one parameter in the problem, the inverse Froude number ϵ, and flows are found when ϵ satisfies $0 < \epsilon < \epsilon^* \simeq 1{\cdot}02$. In fact, a more realistic problem such as that of Fig. 1 (*b*) can be solved merely with a complication of notation, and this is proved in section 5.

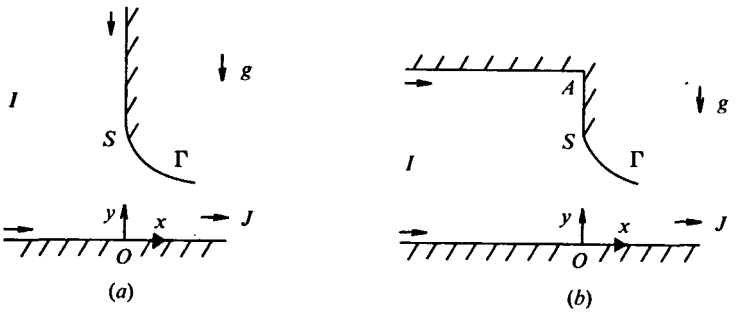


Fig. 1

A recent paper of Keady and Norbury (2) describes the formulation, in terms of a non-linear integral equation, of free-boundary problems similar to these. The idea is to use the complex potential $w = \phi + i\psi$ as the independent variable, so that the flow domain becomes a known strip (Fig. 2), and to solve for $\Omega(w) = \ln(u - iv)$, where u and v are the x and y components of the (non-dimensionalized) fluid velocity in Fig. 1 (*a*). This choice of independent variable means that one can state a mixed boundary-value problem (P) for $\Theta(w) = \operatorname{Im} \Omega(w)$, which is harmonic in the strip. (We do this in section 2.) Hence one finds a (non-linear) integral equation for $\Theta(\phi) \equiv \Theta(\phi + i)$, $\phi \geqslant 0$ (where $\Theta(\phi)$ is the acute angle that the free streamline Γ makes with the x-axis).

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We prove in section 3 that solving the integral equation solves (P). We denote the known solution to (P) when $\epsilon = 0$ by Θ_H (the Helmholtz solution), and find that we can solve an integral equation for $\theta(\phi) = \Theta(\phi) - \Theta_H(\phi)$ by the contraction mapping principle.

For the problems in (2), the velocity of the fluid in the jet tends to infinity far downstream (as $\phi \to \infty$). Here the velocity in the jet approaches a constant far downstream. This fact means that we have to modify the method used in (2), since we now expect $\theta(\phi) = O(\exp(-c(\epsilon)\phi))$ as $\phi \to \infty$ (instead of $\theta(\phi) = O(\phi^{-1})$ as $\phi \to \infty$). Thus we need to introduce a more restrictive convex set of functions with this exponential decay in order to find our solution, and this makes the proof of the contraction mapping theorem more difficult. Hence we establish existence and uniqueness of the solution for $0 < \epsilon < \epsilon^*$, where ϵ^* is restricted to approximately 1·02 because of technical difficulties.

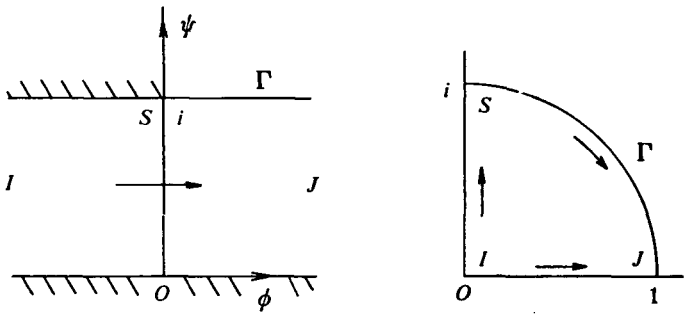


Fig. 2. The w and t planes.

The method of solution is constructive, and we produce in section 3 an iteration $\theta_n = \mathscr{A}_\epsilon \theta_{n-1}$, $n \geqslant 1$, $\theta_0 \equiv 0$, which converges to the solution. This sequence is used in section 4 to derive approximations to the solution, asymptotic as $\epsilon \to 0$. We then use these results to compute the contraction ratio, the ratio of the height of the jet at infinity downstream to the height of the slot, the quantity of most practical concern, as

$$C_c = 0{\cdot}6110 - 0{\cdot}1060\epsilon + 0{\cdot}0329\epsilon^2 + \ldots,$$

for the model problem of Fig. 1 (*a*) (the ϵ^2 coefficient is derived in (1)).

In section 5 we indicate how to extend these results to more general sluice-gate problems.

2. *Formulation of the model problem.* The fluid is assumed to be incompressible and inviscid, the flow to be plane, steady and irrotational. The geometry of the flow is as shown in Fig. 1 (*a*) with infinite depth upstream. Gravity g acts in the $-y$ direction, where the (x, y) axes are chosen to coincide with the solid boundaries, the origin O on the horizontal solid boundary. The (constant) volume flux from the slot per unit time is denoted by Q, the speed at the separation point $S = (0, d)$ by q_s, and the free streamline extending from S to downstream infinity J by Γ. Q and q_s are assumed known. We solve for d and the height of the jet at J, and hence obtain the contraction ratio of the jet.

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We non-dimensionalize velocities with respect to q_s, lengths with respect to Q/q_s, and introduce the following non-dimensional variables: the complex co-ordinate $z = x+iy$, complex potential $w = \phi+i\psi$, complex velocity $dw/dz = u-iv$. Further we introduce q, T, Θ by

$$\exp\{T+i\Theta\} = q\exp\{i\Theta\} = dw/dz = u-iv.$$

$T+i\Theta$ is an analytic function of $\phi+i\psi$ in the flow domain. The flow involves one parameter $\epsilon = 2gQ/q_s^3$. The problem is to find $\Omega = T+i\Theta$ analytic and bounded in the strip $\{\phi+i\psi \mid -\infty < \phi < \infty, 0 < \psi < 1\}$, with boundary conditions:

$$\left.\begin{aligned}
&\text{(i)}\quad \Theta = 0 \quad \text{on} \quad \psi = 0 \quad (-\infty < \phi < \infty),\\
&\text{(ii)}\quad \Theta = \tfrac{1}{2}\pi \quad \text{on} \quad \psi = 1 \quad (-\infty < \phi < 0),\\
&\text{(iii)}\quad \frac{\partial T}{\partial \phi} = \frac{\partial \Theta}{\partial \psi} = \frac{\epsilon}{2}\frac{\sin\Theta(\phi,1)}{1+\dfrac{3\epsilon}{2}\displaystyle\int_0^{\phi}\sin\Theta(\bar{\phi},1)\,d\bar{\phi}} \quad (\psi = 1,\ \ 0 < \phi < \infty),\\
&(S)\quad T\to 0, \quad |d\Omega/dw| = O(|w-i|^{-\frac{1}{2}}) \quad \text{as} \quad w\to i,\\
&(I)\quad T\to -\infty, \quad \Theta\to\pi\psi/2 \quad \text{as} \quad \phi\to-\infty \quad (0<\psi<1),\\
&(J)\quad \Theta\to 0, \quad T\to \text{const (independent of } \psi) \quad \text{as} \quad \phi\to\infty \quad (0<\psi<1).
\end{aligned}\right\} \qquad (P)$$

Here (i) and (ii) state that the flow is tangential to the rigid boundaries, while (iii) follows from differentiating Bernoulli's equation ($q^2+\epsilon y$ = constant) with respect to arc-length along Γ. At S we require a singularity of the same kind as the Helmholtz solution, the solution of (P) with $\epsilon = 0$. The flow at upstream infinity I is a simple source flow, while the flow at downstream infinity J is uniform and horizontal. T is uniquely determined in terms of Θ by the Cauchy–Riemann equations and by specifying $T = 0$ at S.

The partial differential equation problem for $\Theta(\phi,\psi)$ has mixed boundary conditions, the non-linearity appearing in the boundary condition on Γ. The strip is mapped conformally onto the upper-right quadrant of the unit circle in the $t = \rho\exp(i\sigma)$ plane by

$$w(t) = -i+\frac{2}{\pi}\ln\left\{\frac{2it}{1-t^2}\right\}, \qquad (2{\cdot}1)$$

where the logarithm is real when its argument is real and positive (see Fig. 2). Helmholtz (1867) noted that the exact solution of (P) when $\epsilon = 0$ is

$$T+i\Theta = T_H+i\Theta_H \equiv \ln t. \qquad (2{\cdot}2)$$

We form the difference $\tau+i\theta = (T-T_H)+i(\Theta-\Theta_H)$ between the exact solution for $\epsilon > 0$ and the Helmholtz solution, and restate boundary conditions (P) (i) to (P) (iii) as

$$\begin{aligned}
&\theta = 0 \quad \text{on} \quad \psi = 0, \quad -\infty < \phi < \infty, \quad \text{and on} \quad \psi = 1, \quad \phi < 0,\\
&\frac{\partial\tau}{\partial\phi} = \frac{\partial\theta}{\partial\psi} = \frac{\epsilon}{2}\frac{\sin(\sigma+\theta)}{1+\dfrac{3\epsilon}{2}\displaystyle\int_0^{\phi}\sin(\bar{\sigma}+\bar{\theta})\,d\bar{\phi}} \quad \text{on} \quad \psi = 1, \quad \phi > 0,
\end{aligned} \qquad (2{\cdot}3)$$

where $\bar{\theta} = \theta(\bar{\phi},1)$, etc.

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It follows from (2·1) and (2·2) that $\Theta_H \equiv \sigma$, and that, as ϕ goes from 0 to ∞ on $\psi = 1$, σ goes from $\pi/2$ to 0 on $\rho = 1$ with $\sin\sigma = \exp(-\pi\phi/2)$.

We write

$$\frac{\partial\theta}{\partial\rho} = f(\sigma) = \frac{2}{\pi}\cot\sigma\frac{\partial\theta}{\partial\psi}(\sigma), \quad \text{on} \quad \rho = 1, \quad 0 < \sigma < \pi/2, \tag{2·4}$$

and convert the mixed boundary-value problem for θ in the quadrant into a Neumann problem in the unit disc D, with boundary ∂D, in the t-plane by defining

$$f(\sigma) = -f(\pi-\sigma) = f(\pi+\sigma) = -f(2\pi-\sigma) \quad (0 < \sigma < \pi/2). \tag{2·5}$$

Then f is continuous on $\partial D\backslash\{\pm 1\}$ if we define $f(\pm\pi/2) = 0$. (Note that f is defined in terms of the unknown solution θ so that, while equations (2·3) and (2·4) imply that $f(\sigma)\to 0$ as $\sigma\to\pm\pi/2$, they do not imply that the limit of $f(\sigma)$ as $\sigma\to 0$ or π exists.)

We now derive an integral representation for functions defined on $\bar{D}\backslash\{\pm 1\}$ that lie in a certain function-class $\mathscr{F}$ (where $\mathscr{F}$ is defined below). We seek a solution $T+i\Theta$ of (P) such that $\theta = \Theta - \sigma$ has this integral representation, and produce the integral equation (2·13) for the boundary values of θ. In section 3 we will solve this integral equation, obtain θ and show that θ indeed has the integral representation. Then we will show that $T+i\Theta$ is a solution of (P).

Define $\mathscr{F}$ as the set of functions v which are harmonic in D, are continuously differentiable on $\bar{D}\backslash\{\pm 1\}$, and satisfy the following conditions:

$$\left.\begin{aligned}
&\text{(i)} \quad v \text{ is odd about the real and imaginary axes,}\\
&\text{(ii)} \quad \delta \sup_{|t-1|=\delta} |v| \to 0 \quad \text{as} \quad \delta\to 0,\\
&\text{(iii)} \quad \delta^2 \sup_{|t-1|=\delta} |\nabla v| \to 0 \quad \text{as} \quad \delta\to 0,\\
&\text{(iv)} \quad \text{on} \quad \rho = 1, \quad \sigma\,\partial v/\partial\rho \text{ is bounded independently of } \sigma \text{ on} \quad 0 < \sigma \leqslant \pi/2.
\end{aligned}\right\} \tag{2·6}$$

THEOREM 2·1. *If $v\in\mathscr{F}$ is such that $\partial v/\partial\rho = g(\sigma)$ on $\partial D\backslash\{\pm 1\}$, then v has the integral representation*

$$v(\hat{t}) = \int_0^{2\pi} G(e^{i\sigma},\hat{t})\,g(\sigma)\,d\sigma, \quad \hat{t}\in\bar{D}\backslash\{\pm 1\},$$

where G is the Green's function defined by equation (2·7) and the integral is to be interpreted in the sense of the Cauchy principal value at $\sigma = 0$ and π.

Proof. Suppose $\hat{t}\in D$. Let D_1, D_2 be the intersections of D with discs of radii δ centred at $+1$, -1 respectively, where δ is chosen sufficiently small so that $0, \hat{t}\notin D_1\cup D_2$. Let $C_i = \partial D_i\cap D$, $C_i^* = \partial D_i\backslash C_i (= \partial D_i\cap\partial D)$ $(i = 1, 2)$. The Green's function for the Neumann problem of a function harmonic in the unit disc in the t-plane is, with $\hat{t} = \hat{\rho}\exp(i\hat{\sigma})$,

$$G(t,\hat{t}) = (-\tfrac{1}{4}\pi)\ln\{\rho^{-2}(\rho^2+\hat{\rho}^2-2\rho\hat{\rho}\cos(\sigma-\hat{\sigma}))\,(\rho^2+\hat{\rho}^{-2}-2\rho\hat{\rho}^{-1}\cos(\sigma-\hat{\sigma}))\}. \tag{2·7}$$

Suppose $v\in\mathscr{F}$ is such that $\partial v/\partial\rho = g(\sigma)$ on $\partial D\backslash\{\pm 1\}$. We apply Green's second identity to v and G in $\bar{D}\backslash\{D_1\cup D_2\}$ and obtain, as is usual in potential theory (see (3)),

$$v(\hat{t}) = \int_{\partial(D\backslash\{D_1\cup D_2\})} (G\,\partial v/\partial n - v\,\partial G/\partial n)\,ds, \tag{2·8}$$

where n is the outward-drawn unit normal and s is arc-length along $\partial(D\backslash\{D_1\cup D_2\})$.

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We claim that

$$\int_{C_i} (v\,\partial G/\partial n - G\,\partial v/\partial n)\,ds \to 0 \quad \text{as} \quad \delta \to 0, \quad i = 1, 2,$$

and need only prove the result for $i = 1$ (the argument being identical for $i = 2$). G is continuously differentiable in $\bar{D}\backslash\{0, \hat{t}\}$ and so G and ∇G are continuous and bounded in any closed subset of $\bar{D}\backslash\{0, \hat{t}\}$. If we write $t = 1 - \delta \exp(-i\beta)$, $|\beta| \leqslant \gamma < \pi/2$, on C_1, we note that, since $v \in \mathscr{F}$ satisfies (2·6)(ii),

$$\left| \int_{C_1} v\,\partial G/\partial n\,ds \right| \leqslant (\text{const})\,\delta \int_{-\gamma}^{\gamma} |v|\,d\beta \to 0 \quad \text{as} \quad \delta \to 0.$$

As v is odd about $\operatorname{Im} t = 0$ and $G(1, \hat{t})$ is independent of t,

$$\int_{C_1} G\,\partial v/\partial n\,ds = \delta \int_{-\gamma}^{\gamma} \{G(t, \hat{t}) - G(1, \hat{t})\}\,\partial v/\partial n\,d\beta.$$

But, for δ sufficiently small,

$$|G(t, \hat{t}) - G(1, \hat{t})| \leqslant 2|t-1|\,|(\partial G/\partial t)_{t=1}| \leqslant (\text{const})\,\delta,$$

and therefore, since $v \in \mathscr{F}$ satisfies (2·6) (iii),

$$\int_{C_1} G\,\partial v/\partial n\,ds \to 0 \quad \text{as} \quad \delta \to 0.$$

It remains to show that $\int_{\partial D} G\,\partial v/\partial n\,ds$ is a well-defined integral in the sense of the Cauchy principal value; that is, to show that

$$\int_{C_i^*} G\,\partial v/\partial n\,ds \to 0 \quad \text{as} \quad \delta \to 0, \quad i = 1, 2. \tag{2·9}$$

Again we need consider only the case when $i = 1$. Let $\lambda = \arg(1 - \delta \exp(-i\gamma))$. Then we have

$$\begin{aligned}
\left| \int_{C_1^*} G\,\partial v/\partial n\,ds \right| &= \left| \int_{-\lambda}^{\lambda} G(e^{i\sigma}, \hat{t})\,g(\sigma)\,d\sigma \right|, \\
&= \left| \int_0^{\lambda} \{G(e^{i\sigma}, \hat{t}) - G(e^{-i\sigma}, \hat{t})\}\,g(\sigma)\,d\sigma \right|, \\
&\leqslant \int_0^{\lambda} |G(e^{i\sigma}, \hat{t}) - G(e^{-i\sigma}, \hat{t})|\,|g(\sigma)|\,d\sigma, \\
&\leqslant (\text{const}) \int_0^{\lambda} |\sigma g(\sigma)|\,d\sigma \quad \text{(by the mean value theorem).}
\end{aligned}$$

Since $|\sigma g(\sigma)|$ is bounded independently of σ on $(0, \lambda]$ for $0 < \lambda \leqslant \pi/2$ by (2·6) (iv), and since $\lambda \to 0$ as $\delta \to 0$, we have established (2·9).

Suppose $\hat{t} \in \partial D\backslash\{\pm 1\}$ is fixed. Then there exists a $\delta^* > 0$ such that, for $0 < \delta < \delta^*$, $\hat{t} \in \bar{D}\backslash\{\bar{D}_1 \cup \bar{D}_2\}$. For $0 < \delta < \delta^*$ equation (2·8) holds and the proof proceeds as before. Hence, for fixed $\hat{t} \in \bar{D}\backslash\{\pm 1\}$, we may now let $\delta \to 0$ in equation (2·8) and prove the theorem.

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THEOREM 2·2. *If $g(\sigma)$ is continuous on $(0,\pi)$, if $\sigma g(\sigma)$ is bounded independently of σ on $(0,\pi/2)$, if $g(\sigma)$ and $g(\pi/2+\sigma)$ are odd functions of σ and if $g(2\pi+\sigma)=g(\sigma)$, then*

$$u(\hat{t}) \equiv \int_0^{2\pi} G(e^{i\sigma},\hat{t})\,g(\sigma)\,d\sigma, \quad \hat{t}\in\bar{D}\backslash\{\pm 1\},$$

is harmonic in D and $\partial u/\partial\hat{\rho}\to g(\hat{\sigma})$ as $\hat{t}\to\exp(i\hat{\sigma})$ when $0<\hat{\sigma}<\pi$.

Proof. The function $u(\hat{t})$ is harmonic in D since $G(e^{i\sigma},\hat{t})$ is harmonic when $\hat{t}\in D$.

For $\hat{\sigma}\in(0,\pi)$ and $\hat{\rho}<1$, we have

$$\frac{\partial u}{\partial\hat{\rho}} = \frac{1}{2\pi\hat{\rho}}\int_0^{2\pi}\frac{(1-\hat{\rho}^2)g(\sigma)}{1+\hat{\rho}^2-2\hat{\rho}\cos(\sigma-\hat{\sigma})}\,d\sigma \equiv \frac{1}{2\pi\hat{\rho}}\int_0^{2\pi} P(\hat{\rho},\hat{\sigma},\sigma)\,g(\sigma)\,d\sigma,$$

and hence, using a property of the Poisson kernel $P(\hat{\rho},\hat{\sigma},\sigma)$ we see that

$$\left|\frac{\partial u}{\partial\hat{\rho}}-\frac{g(\hat{\sigma})}{\hat{\rho}}\right| = \frac{1}{2\pi\hat{\rho}}\left|\int_0^{2\pi} P(\hat{\rho},\hat{\sigma},\sigma)\{g(\sigma)-g(\hat{\sigma})\}\,d\sigma\right|.$$

For fixed $\hat{\sigma}\in(0,\pi)$, we claim that this integral tends to zero as $\hat{\rho}\to 1$. We separate the range of integration into $[-\delta,\delta]$, $[\delta,\pi-\delta]$, $[\pi-\delta,\pi+\delta]$ and $[\pi+\delta,2\pi-\delta]$, where δ is chosen sufficiently small so that $0<\delta<\hat{\sigma}<\pi-\delta$. We have

$$\begin{aligned}\left|\int_{-\delta}^{\delta} P(\hat{\rho},\hat{\sigma},\sigma)\{g(\sigma)-g(\hat{\sigma})\}\,d\sigma\right| &= \left|\int_0^{\delta} g(\sigma)\{P(\hat{\rho},\hat{\sigma},\sigma)-P(\hat{\rho},\hat{\sigma},-\sigma)\}\,d\sigma\right.\\ &\quad \left.-g(\hat{\sigma})\int_0^{\delta}\{P(\hat{\rho},\hat{\sigma},\sigma)+P(\hat{\rho},\hat{\sigma},-\sigma)\}\,d\sigma\right|,\\ &\leqslant 2\sup_{-\delta\leqslant\sigma\leqslant\delta}|\partial P/\partial\sigma|\int_0^{\delta}|\sigma g(\sigma)|\,d\sigma+(1-\hat{\rho}^2)|g(\hat{\sigma})|\\ &\qquad\times|\hat{\sigma}-\delta|^{-2}k_1\delta\\ &\leqslant \delta(1-\hat{\rho}^2)\{k_2|\hat{\sigma}-\delta|^{-3}+|g(\hat{\sigma})|\,k_1|\hat{\sigma}-\delta|^{-2}\},\\ &\to 0 \quad \text{as} \quad \hat{\rho}\to 1,\end{aligned}$$

for fixed δ and $\hat{\sigma}$ (where k_1 and k_2 are positive constants).

An identical argument holds for the integral over $[\pi-\delta,\pi+\delta]$.

The two integrals that remain tend to zero as $\hat{\rho}\to 1$ since g is continuous over the corresponding ranges of integration. We refer to standard proofs of the Poisson integral formula for details. This completes the proof of the theorem.

We now suppose that $\theta(\hat{t})$ has the integral representation given by Theorem 2·1. Let $\hat{t}\to\exp(i\hat{\sigma})$, $0<\hat{\sigma}\leqslant\pi/2$. Then, using Theorem 2·1, we obtain for $0<\hat{\sigma}\leqslant\pi/2$

$$\begin{aligned}\theta(e^{i\hat{\sigma}}) &= \int_0^{2\pi} G(e^{i\sigma},e^{i\hat{\sigma}})f(\sigma)\,d\sigma,\\ &= \int_0^{\pi/2} K(\sigma,\hat{\sigma})f(\sigma)\,d\sigma, \qquad (2\cdot10)\end{aligned}$$

where

$$\begin{aligned}K(\sigma,\hat{\sigma}) &= G(e^{i\sigma},e^{i\hat{\sigma}})+G(-e^{i\sigma},e^{i\hat{\sigma}})-G(-e^{-i\sigma},e^{i\hat{\sigma}})-G(e^{-i\sigma},e^{i\hat{\sigma}}),\\ &= \frac{1}{\pi}\ln\left|\frac{\sin(\sigma+\hat{\sigma})}{\sin(\sigma-\hat{\sigma})}\right|,\\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\sigma\sin 2n\hat{\sigma}}{n}, \qquad (2\cdot11)\end{aligned}$$

and

$$K(\sigma,\hat{\sigma})\geqslant 0 \quad \text{for} \quad 0\leqslant\sigma,\quad \hat{\sigma}\leqslant\pi/2.$$

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We adopt the conventions $\theta(\hat{\phi}) \equiv \theta(\exp(i\hat{\sigma}))$ and $K(\phi, \hat{\phi}) \equiv K(\sigma, \hat{\sigma})$, $0 \leqslant \phi, \hat{\phi} < \infty$ $(0 < \sigma, \hat{\sigma} \leqslant \pi/2)$, where $\phi + i$, $\hat{\phi} + i$ correspond respectively to $\exp(i\sigma)$, $\exp(i\hat{\sigma})$ under the transformation (2·1), and write equation (2·10) in the following form, where $0 \leqslant \hat{\phi} < \infty$:

$$\theta(\hat{\phi}) = \int_0^\infty K(\phi, \hat{\phi})\, \partial\theta/\partial\psi|_{\psi=1}\, d\phi, \tag{2·12}$$

$$= \frac{\epsilon}{2} \int_0^\infty K(\phi, \hat{\phi})\, \frac{\sin(\sigma + \theta)}{1 + \frac{3}{2}\epsilon \int_0^\phi \sin(\overline{\sigma} + \overline{\theta})\, d\overline{\phi}}\, d\phi \equiv \mathscr{A}_\epsilon \theta(\hat{\phi}). \tag{2·13}$$

In section 3, we wish to use the integral equalities (2·10) and (2·12) for certain known harmonic functions $\tau_j + i\theta_j$ in D, and we note that we may do so provided (2·6) (ii) to (iv) are satisfied at $t = 1$.

We now solve the integral equation (2·13) for $\theta(\hat{\phi})$ when $0 < \epsilon < \epsilon^*$ and then construct from $\theta(\hat{\phi})$ a solution of the original problem (P).

3. *Solution of the integral equation.* We seek a solution of equation (2·13) such that $\theta(\phi) \geqslant 0$ for $0 \leqslant \phi < \infty$.

The linear space of continuous bounded functions on $[0, \infty)$ is a Banach space under the supremum norm, and is denoted henceforth by $C[0, \infty)$. Define, for any $\alpha \in (0, 1)$, the closed convex subset $X_\alpha \subset C[0, \infty)$ by

$$X_\alpha = \{\theta \in C[0, \infty) \,|\, 0 \leqslant \theta(\phi) \leqslant [\sigma(\phi)]^\alpha, \quad 0 \leqslant \phi < \infty\}. \tag{3·1}$$

In this section, we use Banach's contraction mapping theorem to show that equation (2·13) has a unique solution in X_α, for any $\alpha \in (0, 1)$, if $0 < \epsilon < \epsilon^*(\alpha)$, where $\epsilon^*(\alpha)$ is to be defined.

THEOREM 3·1. *The function $K(\phi, \hat{\phi})$ satisfies the following equations for*

$$0 \leqslant \hat{\phi} < \infty \quad (0 < \hat{\sigma} \leqslant \pi/2):$$

$$\int_0^\infty K(\phi, \hat{\phi})\, d\phi = \frac{2}{\pi}\left(\frac{\pi}{2} - \hat{\sigma}\right),$$

$$\int_0^\infty K(\phi, \hat{\phi}) \sin\sigma\, d\phi = \frac{4}{\pi^2} \sin\hat{\sigma} \ln\{\cot\tfrac{1}{2}\hat{\sigma}\} = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin 2n\hat{\sigma}}{4n^2 - 1}.$$

Proof. For the first equation, consider $\tau_0 + i\theta_0 = \ln\{(1 - t^2)/2\}$ in

$$0 \leqslant |t| \leqslant 1, \quad 0 \leqslant \arg t \leqslant \pi/2,$$

where the logarithm is real when its argument is real and positive. Then $\theta_0 = 0$ on $\arg t = 0$, $0 \leqslant |t| < 1$, and on $\arg t = \pi/2$, $0 \leqslant |t| \leqslant 1$, and θ_0 satisfies (2·6) (ii) to (iv) since θ_0 is bounded and $|\partial\theta_0/\partial n| = O(|t - 1|^{-1})$ as $t \to 1$. The integral relation (2·10) therefore applies. If $t = \rho\exp(i\sigma)$, we have, for $0 < \sigma \leqslant \pi/2$,

$$\theta_0(\exp(i\sigma)) = \arg((1 - t^2)/2)|_{t=\exp(i\sigma)} = \sigma - \pi/2,$$

$$\tau_0(\exp(i\sigma)) = \ln|(1 - t^2)/2||_{t=\exp(i\sigma)} = \ln\sin\sigma,$$

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and (2·10) implies that, for $0 < \hat{\sigma} \leqslant \pi/2$,

$$\begin{aligned}\hat{\sigma} - \pi/2 &= \int_0^{\pi/2} K(\sigma, \hat{\sigma})\, \partial\theta_0/\partial\rho \, d\sigma, \\ &= -\int_0^{\pi/2} K(\sigma, \hat{\sigma})\, \partial\tau_0/\partial\sigma \, d\sigma, \\ &= -(\pi/2)\int_0^{\infty} K(\phi, \hat{\phi})\, d\phi. \end{aligned} \tag{3·2}$$

The result follows by multiplying through by $-2/\pi$.

The first part of the second equation follows by noting that

$$\int_0^{\infty} K(\phi, \hat{\phi}) \sin\sigma \, d\phi = (2/\pi)\int_0^{\pi/2} K(\sigma, \hat{\sigma}) \cos\sigma \, d\sigma, \tag{3·3}$$

and integrating by parts. If we write $K(\sigma, \hat{\sigma})$ in its series form (2·11) and integrate term by term we obtain the latter part of the second equation.

THEOREM 3·2. *If $\theta \in X_\alpha$ then $\theta(\phi) = O(\exp(-\pi\alpha\phi/2))$ as $\phi \to \infty$ and*

$$\int_0^{\infty} K(\phi, \hat{\phi})\, \theta(\phi)\, d\phi \leqslant (2/\pi)^{1-\alpha} \alpha^{-1} \tan(\pi\alpha/2)\, \hat{\sigma}^{\alpha} \quad (0 \leqslant \hat{\phi} < \infty). \tag{3·4}$$

Proof. For $0 \leqslant \phi < \infty$, and for $\theta \in X_\alpha$, we have

$$\theta(\phi) \leqslant (\sigma(\phi))^{\alpha} \leqslant (\pi/2)^{\alpha} (\sin\sigma)^{\alpha} = (\pi/2)^{\alpha} \exp(-\pi\alpha\phi/2),$$

which proves the first part of the theorem.

For the second part consider $\tau_1 + i\theta_1 = ((1-t^2)/2)^{\alpha}$ in $0 \leqslant |t| \leqslant 1$, $0 \leqslant \arg t \leqslant \pi/2$, where $\theta_1 = 0$ when $1 - t^2$ is real and positive. Then $\theta_1 = 0$ on $\arg t = 0$, $0 \leqslant |t| \leqslant 1$, and on $\arg t = \pi/2$, $0 \leqslant |t| \leqslant 1$, and θ_1 satisfies (2·6) (ii) to (iv) since θ_1 is bounded and $|\partial\theta_1/\partial n| = O(|t-1|^{\alpha-1})$ as $t \to 1$. Hence we may apply the integral relations (2·10) and (2·12). On $t = \exp(i\sigma)$, $0 \leqslant \sigma \leqslant \pi/2$, we have

$$\begin{aligned}\tau_1(\exp(i\sigma)) &= (\sin\sigma)^{\alpha} \cos[\alpha(\sigma - \pi/2)], \\ \theta_1(\exp(i\sigma)) &= (\sin\sigma)^{\alpha} \sin[\alpha(\sigma - \pi/2)]. \end{aligned}$$

The integral relations then imply that, for $0 < \hat{\sigma} \leqslant \pi/2$, we have

$$\begin{aligned}(\sin\hat{\sigma})^{\alpha} \sin[\alpha(\hat{\sigma} - \pi/2)] &= \int_0^{\pi/2} K(\sigma, \hat{\sigma})\, (-\partial\tau_1/\partial\sigma)\, d\sigma, \\ &= (-\pi\alpha/2)\int_0^{\infty} K(\phi, \hat{\phi})\, (\sin\sigma)^{\alpha+1} \left\{ \frac{\cos[\alpha(\sigma - \pi/2)]}{\sin\sigma} \right. \\ &\qquad \left. - \frac{\sin[\alpha(\sigma - \pi/2)]}{\cos\sigma} \right\} d\phi, \\ &= (-\pi\alpha/2)\int_0^{\infty} K(\phi, \hat{\phi})\, (\sin\sigma)^{\alpha} \frac{\cos[\sigma + \alpha(\sigma - \pi/2)]}{\cos\sigma} \, d\phi. \end{aligned} \tag{3·5}$$

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For $0 < \alpha < 1$, $\cos[\sigma+\alpha(\sigma-\pi/2)]/\cos\sigma$ has a positive minimum in the interval $0 \leqslant \sigma \leqslant \pi/2$ of $\cos(\alpha\pi/2)$. Hence, on multiplying (3·5) by -1 (since both sides are negative), we obtain, for $\theta \in \chi_\alpha$, the following:

$$\begin{aligned}(\sin\hat{\sigma})^\alpha \sin[\alpha(\pi/2-\hat{\sigma})] &\geqslant \frac{\pi}{2}\alpha\cos(\pi\alpha/2)\int_0^\infty K(\phi,\hat{\phi})(\sin\sigma)^\alpha d\phi,\\ &\geqslant (2/\pi)^{\alpha-1}\alpha\cos(\pi\alpha/2)\int_0^\infty K(\phi,\hat{\phi})\sigma^\alpha d\phi,\\ &\geqslant (2/\pi)^{\alpha-1}\alpha\cos(\pi\alpha/2)\int_0^\infty K(\phi,\hat{\phi})\theta(\phi)\,d\phi. \qquad (3\cdot6)\end{aligned}$$

But $\sin[\alpha(\pi/2-\hat{\sigma})](\sin\hat{\sigma})^\alpha \leqslant \sin(\pi\alpha/2)\hat{\sigma}^\alpha$. If we multiply (3·6) by $(2/\pi)^{1-\alpha}(\alpha\cos(\pi\alpha/2))^{-1}$ we have proved the theorem.

THEOREM 3·3. *$\mathscr{A}_\epsilon: X_\alpha \to X_\alpha$ if $0 < \epsilon \leqslant c_1(\alpha)$, where c_1 is defined below.*

Proof. Suppose $\theta \in X_\alpha$. We have to show that $\mathscr{A}_\epsilon\theta \in X_\alpha$, where

$$\mathscr{A}_\epsilon\theta(\hat{\phi}) = \frac{\epsilon}{2}\int_0^\infty K(\phi,\hat{\phi})\,\frac{\sin(\sigma+\theta)}{1+\dfrac{3\epsilon}{2}\displaystyle\int_0^\phi \sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}}\,d\phi \quad (0 \leqslant \hat{\phi} < \infty). \qquad (3\cdot7)$$

We note that $K(\phi,\hat{\phi}) \geqslant 0$ and $0 \leqslant \sigma(\phi)+\theta(\phi) \leqslant \pi$ $(0 \leqslant \phi < \infty)$, imply that the integrand in (3·7) is non-negative, and hence that $\mathscr{A}_\epsilon\theta$ is a non-negative function of $\hat{\phi}$.

$\mathscr{A}_\epsilon$ maps X_α into continuous functions. (For a proof, refer to the appendix in (2).)

$\mathscr{A}_\epsilon\theta$ is bounded since for $0 \leqslant \phi < \infty$ we have

$$0 \leqslant \mathscr{A}_\epsilon\theta(\hat{\phi}) \leqslant (\epsilon/2)\int_0^\infty K(\phi,\hat{\phi})\,d\phi \leqslant \epsilon/2. \qquad (3\cdot8)$$

This implies that $\mathscr{A}_\epsilon\theta \in C[0,\infty)$. To prove that $\mathscr{A}_\epsilon\theta \in X_\alpha$, note that

$$\begin{aligned}0 \leqslant \mathscr{A}_\epsilon\theta(\hat{\phi}) &\leqslant (\epsilon/2)\int_0^\infty K(\phi,\hat{\phi})\sin(\sigma+\theta)\,d\phi,\\ &\leqslant (\epsilon/2)\int_0^\infty K(\phi,\hat{\phi})\sin\sigma\,d\phi + (\epsilon/2)\int_0^\infty K(\phi,\hat{\phi})\theta(\phi)\,d\phi.\end{aligned}$$

Hence, for $0 < \hat{\sigma} \leqslant \pi/2$, using Theorems 3·1 and 3·2 and the inequality

$$\sin\hat{\sigma}\ln(\cot\tfrac{1}{2}\hat{\sigma}) \leqslant \hat{\sigma}\ln(2/\hat{\sigma}),$$

we have

$$\begin{aligned}\mathscr{A}_\epsilon\theta(\hat{\phi}) &\leqslant (2\epsilon/\pi^2)\hat{\sigma}\ln(2/\hat{\sigma}) + (\epsilon/2\alpha)(2/\pi)^{1-\alpha}\tan(\pi\alpha/2)\hat{\sigma}^\alpha,\\ &\leqslant \left\{\frac{2^{2-\alpha}}{\pi^2 e(1-\alpha)} + \frac{1}{2\alpha}\left(\frac{2}{\pi}\right)^{1-\alpha}\tan(\pi\alpha/2)\right\}\epsilon\hat{\sigma}^\alpha \equiv \epsilon\hat{\sigma}^\alpha/c_1(\alpha),\end{aligned}$$

(where e is the base of the natural logarithms), upon computing the maximum of $\hat{\sigma}^{1-\alpha}\ln(2/\hat{\sigma})$ on $[0,\pi/2]$. Thus, for $0 \leqslant \phi < \infty$, we have

$$\mathscr{A}_\epsilon\theta(\hat{\phi}) \leqslant \hat{\sigma}^\alpha \quad \text{if} \quad \epsilon \leqslant c_1(\alpha). \qquad (3\cdot9)$$

Lengthy elementary calculations show that $dc_1(\alpha)/d\alpha < 0$ on $0 < \alpha < 1$, that $c_1 \to (4/\pi^2 e + \frac{1}{2})^{-1} \simeq 1{\cdot}54$ as $\alpha \to 0$ and that $c_1 \to 0$ as $\alpha \to 1$.

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THEOREM 3·4. *There exists a* $\beta \in (0, 1)$ *such that for each* $\eta, \theta \in X_\alpha$ *we have*

$$||\mathscr{A}_\epsilon \eta - \mathscr{A}_\epsilon \theta|| \leqslant \beta ||\eta - \theta|| \text{ if } \quad 0 < \epsilon < c_3(\alpha),$$

where c_3 *is defined below.*

Proof. For any $\eta, \theta \in X_\alpha$, we have

$$|\mathscr{A}_\epsilon \eta - \mathscr{A}_\epsilon \theta| = \frac{\epsilon}{2}\left| \int_0^\infty K(\phi, \hat{\phi}) \left\{ \frac{\sin(\sigma+\eta)}{1+\frac{3}{2}\epsilon \int_0^\phi \sin(\bar{\sigma}+\bar{\eta}) d\bar{\phi}} - \frac{\sin(\sigma+\theta)}{1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}} \right\} d\phi \right|,$$

$$\leqslant \frac{\epsilon}{2} \int_0^\infty K(\phi, \hat{\phi}) |J(\phi)|\, d\phi,$$

where, for $0 \leqslant \phi < \infty$,

$$J(\phi) = \frac{\sin(\sigma+\eta) - \sin(\sigma+\theta)}{1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\eta})\, d\bar{\phi}} + \sin(\sigma+\theta) \left\{ \frac{1}{1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\eta})\, d\bar{\phi}} - \frac{1}{1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}} \right\},$$

$$= \frac{2\cos[\sigma+(\theta+\eta)/2]\sin[(\eta-\theta)/2]}{1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\eta})\, d\bar{\phi}} + \frac{3\epsilon}{2} \frac{\sin(\sigma+\theta)\int_0^\phi 2\cos[\bar{\sigma}+(\bar{\theta}+\bar{\eta})/2]\sin[(\bar{\theta}-\bar{\eta})/2]\, d\bar{\phi}}{\left(1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}\right)\left(1+\frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\eta})\, d\bar{\phi}\right)},$$

and hence, for $0 \leqslant \phi < \infty$,

$$|J(\phi)| \leqslant 2|\sin[(\eta-\theta)/2]| + (3\epsilon/2)\sin(\sigma+\theta)\int_0^\phi |\bar{\eta}-\bar{\theta}|\, d\bar{\phi},$$

$$\leqslant |\eta-\theta| + (3\epsilon/2)\sin(\sigma+\theta)\,\phi\, ||\eta-\theta||,$$

$$\leqslant ||\eta-\theta||\{1+(3\epsilon/2)\,\phi\sin(\sigma+\theta)\} \leqslant ||\eta-\theta||\{1+\epsilon c_2(\alpha)\}, \tag{3·10}$$

where

$$c_2(\alpha) = \sup_{0<\sigma\leqslant\pi/2}\left[\frac{3}{\pi}\ln\left(\frac{1}{\sin\sigma}\right)\sin(\sigma+\sigma^\alpha)\right].$$

Hence we have, for $0 \leqslant \phi < \infty$,

$$|\mathscr{A}_\epsilon \eta - \mathscr{A}_\epsilon \theta| \leqslant \frac{\epsilon}{2}||\eta-\theta||\{1+\epsilon c_2(\alpha)\},$$

and therefore

$$||\mathscr{A}_\epsilon \eta - \mathscr{A}_\epsilon \theta|| \leqslant \frac{\epsilon}{2}||\eta-\theta||\{1+\epsilon c_2(\alpha)\}.$$

We conclude that there exists a β satisfying the requirements of the theorem if

$$0 < \frac{\epsilon}{2}(1+\epsilon c_2) = \beta < 1,$$

that is,

$$0 < \epsilon < \left[\frac{1}{4c_2^2} + \frac{2}{c_2}\right]^{\frac{1}{2}} - \frac{1}{2c_2} \equiv c_3(\alpha). \tag{3·11}$$

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Lengthy calculations show that $dc_2(\alpha)/d\alpha < 0$ for $0 < \alpha < 1$, and hence $dc_3(\alpha)/d\alpha > 0$ for $0 < \alpha < 1$, that $c_3 \to 0$ as $\alpha \to 0$ and that $c_3 \to$ constant $\simeq 1{\cdot}14$ as $\alpha \to 1$.

THEOREM 3·5. *There exists a unique solution of* $\theta = \mathscr{A}_\epsilon \theta$ *in* X_α, *for any* $\alpha \in (0, 1)$, *if* $0 < \epsilon < \epsilon^*(\alpha) \equiv \min(c_1, c_3)$, *where* c_1 *and* c_3 *are defined by equations* (3·9) *and* (3·11) *respectively.*

Proof. The proof is an immediate consequence of Theorems 3·3 and 3·4 using Banach's contraction mapping theorem.

COROLLARY *For* $0 < \epsilon < \epsilon^*(\alpha)$, *the sequence* $\{\theta_n\}_{n=0}^\infty \subset X_\alpha$, *where* $\theta_0 \equiv 0$, $\theta_n = \mathscr{A}_\epsilon \theta_{n-1}$, $n \geqslant 1$, *converges to the unique solution of* $\theta = \mathscr{A}_\epsilon \theta$ *in* X_α.

Remarks. It follows from the monotonicity of both $c_1(\alpha)$ and $c_3(\alpha)$ that $\epsilon^*(\alpha) \to 0$ as $\alpha \to 0$ and as $\alpha \to 1$, and that $\sup\{\epsilon^*(\alpha), 0 < \alpha < 1\}$ is achieved at a unique $\alpha^* \in (0, 1)$. Calculations show that $\alpha^* \simeq 0{\cdot}483$ and that $\epsilon^*(0{\cdot}483) = 1{\cdot}0168$. The sequence $\{\theta_n\}_{n=0}^\infty$ converges uniformly on $[0, \infty)$ to θ, the unique solution of equation (2·13) in $X_{0{\cdot}483}$, if $0 < \epsilon < 1{\cdot}0168$. Further, we note that if $0 < \epsilon < \min(c_2^{-1}, 1{\cdot}0168) = 1{\cdot}0168$ then $|\theta_n - \theta_{n-1}| < \frac{1}{2}\epsilon^n$, and that if $0 < \epsilon < 1$ then

$$|\theta - \theta_n| < \tfrac{1}{2}\epsilon^{n+1}(1-\epsilon)^{-1} \quad (0 \leqslant \phi < \infty, n \geqslant 1).$$

We now construct a solution of (P). For $0 < \epsilon < \epsilon^*(\alpha)$ let $\theta(\exp(i\sigma))$, $0 < \sigma \leqslant \pi/2$, denote the unique solution of equation (2·13) in X_α. Define

$$\theta(\exp(i(\pi-\sigma))) = -\theta(\exp(i(\pi+\sigma))) = \theta(\exp(i(2\pi-\sigma))) = -\theta(\exp(i\sigma))$$
$$(0 < \sigma \leqslant \pi/2),$$

and $\theta(\pm 1) = 0$. Then θ is continuous on ∂D. (Note that $\theta(\exp(i\sigma)) \to 0$ as $\sigma \to 0$.) Using the Poisson integral formula, we construct the unique function $\theta(t)$, $t \in D$, harmonic in D and continuous on $\bar{D}$ which is equal to $\theta(\exp(i\sigma))$ on ∂D. We note that $|\theta|$ is bounded on $\bar{D}$ by $\epsilon/2$.

We construct the conjugate function $\tau(t)$ such that $\tau = 0$ at $t = \pm i$ and

$$\partial\tau/\partial\sigma = -\partial\theta/\partial\rho = -f(\sigma) \quad \text{on} \quad \partial D \backslash \{\pm 1\}.$$

(The required information about $\partial\theta/\partial\rho$ on $\partial D\backslash\{\pm 1\}$ is provided in Lemmas 3·6–8 which follow.) Thus τ is defined on $\partial D\backslash\{\pm 1\}$ by

$$\tau(\exp(i\sigma)) = \int_\sigma^{\pi/2} f(x)\,dx \quad (0 < \sigma \leqslant \pi/2),$$

$$\tau(\exp(i\sigma)) = \tau(\exp(i(\pi-\sigma))) = \tau(\exp(i(\pi+\sigma))) = \tau(\exp(i(2\pi-\sigma))) \quad (0 < \sigma \leqslant \pi/2),$$

and is continuous on $\partial D\backslash\{\pm 1\}$. Note that $\tau(1)$, the limit of $\tau(\exp(i\sigma))$ as $\sigma \to 0$, exists since $f(x) = O(x^{\alpha-1})$ as $x \to 0$. Hence we have τ continuous on ∂D. We construct $\tau(t)$, $t \in D$, using the Poisson integral formula (as we constructed $\theta(t)$). Then our unique τ is continuous and bounded on $\bar{D}$. Since $0 \leqslant \tau(t) \leqslant \tau(1)$ on ∂D, it follows from the Maximum Principle that $0 < \tau(t) < \tau(1)$ throughout D. Hence we have constructed $\tau + i\theta$. It remains to show that $T + i\Theta = (T_H + \tau) + i(\Theta_H + \theta)$ solves (P). This is a consequence of the lemmas and theorem that follow.

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LEMMA 3·6. *The function $f(\sigma)$, defined on $\partial D\backslash\{\pm 1\}$ by equations* (3·12) *and* (2·5) *in terms of θ, is continuous on $(0,\pi)$ and is such that $\sigma f(\sigma)$ is bounded independently of σ on $(0,\pi/2]$.*

Proof. With equations (2·3) and (2·4) in mind we define

$$f(\sigma) = (\epsilon/\pi)\cot\sigma \frac{\sin(\sigma+\theta)}{1+\frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}} \quad (0<\sigma<\pi/2), \tag{3·12}$$

$$= (\epsilon/\pi)\cot\sigma\, p(\sigma) \quad (0<\sigma<\pi/2),$$

where $p(\sigma)$ is continuous and non-negative on $0<\sigma\leqslant\pi/2$ and tends to zero as $\sigma\to 0$. We define $f(\pi/2)=0$ and the lemma follows.

LEMMA 3·7. *The function*

$$u(\hat{t}) \equiv \int_0^{2\pi} G(e^{i\sigma},\hat{t})f(\sigma)\,d\sigma, \quad \hat{t}\in\overline{D}\backslash\{\pm 1\},$$

where f is defined in Lemma 3·6, *is a bounded harmonic function in D and is such that $\partial u/\partial\hat{\rho}\to f(\hat{\sigma})$ as $\hat{t}\to\exp(i\hat{\sigma})$ for $0<\hat{\sigma}<\pi$.*

Proof. The fact that u is harmonic and the condition on its derivative follow from Theorem 2·2. To prove the boundedness, note that

$$u(\hat{t}) = \int_0^{\pi/2} K^*(\hat{\rho},\hat{\sigma},\sigma)f(\sigma)\,d\sigma, \quad \hat{t}\in\overline{D}\backslash\{\pm 1\},$$

where

$$K^*(\hat{\rho},\hat{\sigma},\sigma) = \frac{1}{2\pi}\ln\left\{\frac{[(1-\hat{\rho})^2+4\hat{\rho}\cos^2\tfrac{1}{2}(\sigma+\hat{\sigma})]\,[(1-\hat{\rho})^2+4\hat{\rho}\sin^2\tfrac{1}{2}(\sigma+\hat{\sigma})]}{[(1-\hat{\rho})^2+4\hat{\rho}\cos^2\tfrac{1}{2}(\sigma-\hat{\sigma})]\,[(1-\hat{\rho})^2+4\hat{\rho}\sin^2\tfrac{1}{2}(\sigma-\hat{\sigma})]}\right\},$$

$$= \frac{1}{2\pi}\ln\left\{1+\frac{4\sin 2\sigma\sin 2\hat{\sigma}}{4\sin^2(\sigma-\hat{\sigma})+\hat{\rho}^{-2}(1-\hat{\rho}^2)^2}\right\} \quad (\hat{\rho}\neq 0).$$

Now, for $0\leqslant\sigma,\hat{\sigma}\leqslant\pi/2$, $0\leqslant\hat{\rho}\leqslant 1$, we have

$$0\leqslant K^*(\hat{\rho},\hat{\sigma},\sigma)\leqslant K(\sigma,\hat{\sigma}), \quad K^*(1,\hat{\sigma},\sigma)=K(\sigma,\hat{\sigma}),$$

and so it follows that, for $0\leqslant\hat{\rho}\leqslant 1$, $0<\hat{\sigma}\leqslant\pi/2$,

$$\begin{aligned} 0\leqslant u(\hat{t}) &\leqslant \int_0^{\pi/2} K(\sigma,\hat{\sigma})f(\sigma)\,d\sigma,\\ &\leqslant (\epsilon/\pi)\int_0^{\pi/2} K(\sigma,\hat{\sigma})\cot\sigma\,d\sigma,\\ &= (\epsilon/2)\int_0^{\infty} K(\phi,\hat{\phi})\,d\phi,\\ &\leqslant \epsilon/2. \end{aligned}$$

The boundedness of u in $\overline{D}\backslash\{\pm 1\}$ follows from the above and its oddness properties.

LEMMA 3·8. *For $\hat{t}\in\overline{D}\backslash\{\pm 1\}$, the functions $u(t)$ and $\theta(t)$ are identical.*

Proof. Consider the difference $u(t)-\theta(t)$. Then $u-\theta$ is harmonic in D and bounded on $\overline{D}\backslash\{\pm 1\}$. Moreover $u-\theta=0$ on $\partial D\backslash\{\pm 1\}$. The Phragmén–Lindelöf Principle (see Protter and Weinberger (4), p. 100) asserts that $u-\theta=0$ in D.

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THEOREM 3·9. *The function* $T+i\Theta = (T_H+\tau)+i(\Theta_H+\theta)$ *is a solution of* (P).

Proof. The oddness of θ about $\mathrm{Re}\,t = 0$ and $\mathrm{Im}\,t = 0$ implies that $\theta = 0$ on $\psi = 0$, $-\infty < \phi < \infty$, and on $\psi = 1$, $0 < \phi < \infty$. Hence Θ satisfies (P) (i) and (ii). Condition (P) (iii) follows since $\partial\theta/\partial\hat{\rho} \equiv \partial u/\partial\hat{\rho} \to f(\hat{\sigma})$ as $\hat{t} \to \exp(i\hat{\sigma})$, $0 < \hat{\sigma} < \pi/2$, by Lemmas 3·7 and 3·8. Also $\theta \to 0$ and $\tau \to$ constant as $t \to 0$ and as $t \to i$. Thus (P), (I) and (J) follow if we note that $T_H, \Theta_H \to 0$ as $\phi \to \infty$ $(0 \leqslant \psi \leqslant 1)$ and that $T_H \to -\infty$, $\Theta_H \to \pi\psi/2$ as $\phi \to -\infty$ $(0 \leqslant \psi \leqslant 1)$. We have normalized τ to be equal to 0 at $t = i$. Therefore $T \to 0$ as $\phi + i\psi \to i$. If we note that $|dt/dw| = O(|w-i|^{-\frac{1}{2}})$ as $w \to i$ and that $|d\Omega/dt|$ is bounded near $t = i$, then (P), (S) follows. This completes the proof of the theorem.

LEMMA 3·10. *The function* Θ *satisfies* $0 < \Theta \leqslant \pi/2$ *on* Γ, *the equality occurring only at* S.

Proof. By definition $\Theta = \sigma + \theta$, so we must show that $0 \leqslant \theta \leqslant \pi/2 - \sigma$ on Γ. But, for $0 \leqslant \hat{\phi} < \infty$, $\hat{\psi} = 1$, we have

$$\begin{aligned} 0 \leqslant \theta(\hat{\phi}, 1) = \mathscr{A}_\epsilon \theta(\hat{\phi}, 1) &\leqslant (\epsilon/2)\int_0^\infty K(\phi, \hat{\phi})\, d\phi, \\ &= \frac{\epsilon}{\pi}\left(\frac{\pi}{2} - \hat{\sigma}\right) \quad \text{by Theorem 3·1}, \\ &\leqslant \pi/2 - \hat{\sigma} \quad \text{since} \quad 0 < \epsilon < \epsilon^* < \pi. \end{aligned}$$

We note that equality occurs only when $\hat{\sigma} = \pi/2$ $(\hat{\phi} = 0)$. This proves the lemma.

Remarks. We note that the Maximum Principle implies that $0 < \Theta < \pi/2$ in the interior of the flow domain. The fact that $\partial x/\partial\phi = e^{-T}\cos\Theta$, $\partial y/\partial\phi = -e^{-T}\sin\Theta$ implies that x and $-y$ are strictly monotonic increasing functions of arc-length along Γ (and any streamline $0 < \psi < 1$).

It is reasonable to ask whether the local uniqueness result for the solution of the integral equation (2·13) carries over into a uniqueness result for the solution of (P). We have immediately that, for $0 < \epsilon < \epsilon^*(\alpha)$, any solution of (P) for which $\Theta(t) \in \mathscr{F}$ (hence the boundary values of $\Theta - \sigma$ satisfy our integral equation), and

$$\sigma \leqslant \Theta(e^{i\sigma}) \leqslant \sigma + \sigma^\alpha \quad \text{on} \quad \Gamma$$

(so that $\Theta - \sigma \in X_\alpha$ on Γ) must be identical to the solution we have constructed. Rather than refine Theorem 2·1 (and conditions (2·6)), in order to improve this local uniqueness result, we prove in (1) a global uniqueness result that includes all the solutions in X_α of our integral equation. In (1) we show that given any ϵ, $0 < \epsilon < 2$, there exists at most one solution of (P) which is such that $0 \leqslant \Theta \leqslant \pi/2$.

4. *The slot height and contraction ratio.* We now obtain the asymptotic expansions of the slot height and contraction ratio to first order in ϵ. The difference between the quantities and their first-order approximations is proved to be $O(\epsilon^{\frac{3}{2}})$.

On Γ we have

$$\frac{\partial\tau}{\partial\phi} = \frac{\epsilon}{2}\frac{\sin(\sigma+\theta)}{1 + \frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}} \quad (0 < \phi < \infty),$$

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and so, using $\tau = 0$ at $\phi = 0$, we obtain

$$\tau(\hat{\phi}, 1) = \frac{\epsilon}{2}\int_0^{\hat{\phi}} \frac{\sin(\sigma+\theta)}{1+\frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}}\,d\phi \quad (0 \leqslant \hat{\phi} < \infty).$$

If we denote the limit as $\phi \to \infty$ of $\tau(\phi, 1)$ by τ_∞, and use the subscript ∞ with other variables to denote a similar limit, we have (since the flow becomes uniform in $0 \leqslant \psi \leqslant 1$ as $\phi \to \infty$),

$$\tau_\infty = \frac{\epsilon}{2}\int_0^{\infty} \frac{\sin(\sigma+\theta)}{1+\frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}}\,d\phi, \tag{4·1}$$

and

$$\exp(\tau_\infty) y_\infty = q_\infty y_\infty = 1. \tag{4·2}$$

From Bernoulli's equation ($q^2 + \epsilon y = 1 + \epsilon y_s$) on Γ, where y_s is the non-dimensionalized slot height, it follows that

$$q_\infty^2 + \epsilon y_\infty = 1 + \epsilon y_s. \tag{4·3}$$

Hence the contraction ratio $C_c = y_\infty / y_s$ and y_s are given by

$$y_s = (q_\infty^3 + \epsilon - q_\infty)/\epsilon q_\infty, \tag{4·4}$$

$$C_c = \epsilon/(q_\infty^3 + \epsilon - q_\infty). \tag{4·5}$$

THEOREM 4·1. *The function $\theta(\hat{\phi})$ satisfies*

$$\theta(\hat{\phi}) = (2\epsilon/\pi^2)\sin\hat{\sigma}\ln\{\cot\tfrac{1}{2}\hat{\sigma}\} + O(\epsilon^2) \quad (0 \leqslant \hat{\phi} < \infty).$$

Proof. Since $\theta \in X_\alpha$ we have, for $0 \leqslant \phi < \infty$,

$$\begin{aligned}
&\left|\theta(\hat{\phi}) - \frac{\epsilon}{2}\int_0^{\infty} K(\phi,\hat{\phi})\sin\sigma\,d\phi\right| \\
&\qquad = \frac{\epsilon}{2}\left|\int_0^{\infty} K(\phi,\hat{\phi})\left\{\frac{\sin(\sigma+\theta) - \sin\sigma\left[1+\frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}\right]}{1+\frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}}\right\}d\phi\right|, \\
&\qquad \leqslant \frac{\epsilon}{2}\int_0^{\infty} K(\phi,\hat{\phi})\left\{|\sin(\sigma+\theta) - \sin\sigma| + \frac{3\epsilon}{2}\int_0^{\phi}\sin(\overline{\sigma}+\overline{\theta})\,d\overline{\phi}\sin\sigma\right\}d\phi, \\
&\qquad \leqslant \frac{\epsilon}{2}\int_0^{\infty} K(\phi,\hat{\phi})\left\{\theta(\phi) + \frac{3\epsilon}{2}\phi\exp(-\pi\phi/2)\right\}d\phi, \\
&\qquad \leqslant \left(\frac{1}{4} + \frac{3}{2\pi e}\right)\epsilon^2,
\end{aligned}$$

using Theorem 3·1 and (3·8). If we multiply the second equation given in the statement of Theorem 3·1 by $\epsilon/2$, we have proved the theorem.

Henceforth we write $(2\epsilon/\pi^2)\sin\hat{\sigma}\ln\{\cot\tfrac{1}{2}\hat{\sigma}\} = \theta^{(1)}(\hat{\phi})$ $(0 \leqslant \hat{\phi} < \infty)$.

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The sequence $\{\theta_n\}_{n=0}^{\infty}$ defined in the Corollary to Theorem 3·5 yields (after successive substitutions into $\mathscr{A}_\epsilon$ starting from $\theta_0 = 0$), a series in integer powers of ϵ. The first term, the term in ϵ, is $\theta^{(1)}$. Theorem 4·1 and the remark following it indicate that no single asymptotic series of Poincaré form describes Θ uniformly on $[0, \infty)$. The method of matched asymptotic expansions would lead to the construction of inner and outer asymptotic expansions, the outer asymptotic under the limit $\epsilon \to 0$, $\hat{\phi}$ fixed, $0 \leqslant \hat{\phi} \leqslant \epsilon^{-1}$, the inner asymptotic under the limit $\epsilon \to 0$, $\epsilon\hat{\phi}$ fixed, $\epsilon^{-1} \leqslant \hat{\phi} < \infty$. Then the sum of the first two terms of the outer expansion is $\sigma + \theta^{(1)}$. In this paper we call the region of uniformity of the asymptotic expansion for Θ derived in Theorem 4·1 and the following remarks the outer region, that is $0 \leqslant \hat{\phi} \leqslant \epsilon^{-1}$, and that expansion the outer expansion. That the inner expansion is irrelevant to the determination of y_s and C_c by the method described in this paper is a consequence of the following theorem.

THEOREM 4·2. *The quantity* τ_∞ *satisfies*

$$\tau_\infty = \frac{\epsilon}{2}\int_0^\infty \sin\sigma\, d\phi + \frac{\epsilon}{2}\int_0^\infty \cos\sigma\theta^{(1)} d\phi - \tfrac{3}{4}\epsilon^2 \int_0^\infty \sin\sigma \left\{\int_0^\phi \sin\bar{\sigma}\, d\bar{\phi}\right\} d\phi + O(\epsilon^{\frac{5}{2}}).$$

Proof. Using (4·1) and the fact that $\theta \in X_\alpha$, we have

$$\left|\frac{\epsilon}{2}\int_{\epsilon^{-\frac{1}{2}}}^\infty \frac{\sin(\sigma+\theta)}{1+\dfrac{3\epsilon}{2}\displaystyle\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}}\, d\phi\right| \leqslant \frac{\epsilon}{2}\int_{\epsilon^{-\frac{1}{2}}}^\infty (\sigma+\sigma^\alpha)\, d\phi,$$

$$= O(\exp[-\tfrac{1}{2}\pi\alpha\epsilon^{-\frac{1}{2}}]).$$

Therefore

$$\tau_\infty = \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \sin\sigma\, d\phi + \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \cos\sigma \sin\theta^{(1)} d\phi - \tfrac{3}{4}\epsilon^2 \int_0^{\epsilon^{-\frac{1}{2}}} \sin\sigma \left\{\int_0^\phi \sin\bar{\sigma}\, d\bar{\phi}\right\} d\phi$$

$$+ \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \left\{\frac{\sin(\sigma+\theta)}{1+\dfrac{3\epsilon}{2}\displaystyle\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}} - \sin\sigma - \cos\sigma\sin\theta^{(1)} + \frac{3\epsilon}{2}\sin\sigma\int_0^\phi \sin\bar{\sigma}\, d\bar{\phi}\right\} d\phi$$

$$+ \text{exponentially small terms.}$$

We show that the last integral (which we denote by I) in the above expression is $O(\epsilon^{\frac{5}{2}})$. Then

$$|I| \leqslant \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \left\{|\sin(\sigma+\theta) - \sin\sigma - \sin\theta^{(1)}\cos\sigma| + \tfrac{9}{4}\epsilon^2 \sin\sigma \int_0^\phi \sin\bar{\sigma}\, d\bar{\phi} \int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi} + \frac{3\epsilon}{2}\int_0^\phi |\sin\bar{\sigma} - \sin(\bar{\sigma}+\bar{\theta})|\, d\bar{\phi} + \frac{3\epsilon}{2}\cos\sigma\sin\theta^{(1)}\int_0^\phi \sin(\bar{\sigma}+\bar{\theta})\, d\bar{\phi}\right\} d\phi.$$

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From Theorem 4·1 we know that $\theta(\phi) = \theta^{(1)}(\phi) + R(\phi, \epsilon)$, where $|R| = O(\epsilon^2)$ for $0 \leqslant \phi < \infty$. Using this and $0 \leqslant \theta(\phi) \leqslant \epsilon/2$, we may show that

$$\frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \left\{\frac{3\epsilon}{2}\cos\sigma\sin\theta^{(1)}\int_0^{\phi}\sin(\bar{\sigma}+\bar{\theta})\,d\bar{\phi}\right\}d\phi = O(\epsilon^{\frac{5}{2}}),$$

$$\frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} |\sin(\sigma+\theta)-\sin\sigma-\sin\theta^{(1)}\cos\sigma|\,d\phi = O(\epsilon^{\frac{5}{2}}),$$

$$\frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \left\{\tfrac{9}{4}\epsilon^2\sin\sigma\int_0^{\phi}\sin\bar{\sigma}\,d\bar{\phi}\int_0^{\phi}\sin(\bar{\sigma}+\bar{\theta})\,d\bar{\phi}\right\}d\phi = O(\epsilon^3),$$

$$\frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}} \left\{\tfrac{3}{2}\epsilon\sin\sigma\int_0^{\phi}|\sin(\bar{\sigma}+\bar{\theta})-\sin\bar{\sigma}|\,d\bar{\phi}\right\}d\phi = O(\epsilon^3),$$

and conclude that $|I| = O(\epsilon^{\frac{5}{2}})$. Hence

$$\tau_\infty = \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}}\sin\sigma\,d\phi + \frac{\epsilon}{2}\int_0^{\epsilon^{-\frac{1}{2}}}\cos\sigma\sin\theta^{(1)}d\phi - \tfrac{3}{4}\epsilon^2\int_0^{\epsilon^{-\frac{1}{2}}}\sin\sigma\int_0^{\phi}\sin\bar{\sigma}\,d\bar{\phi}\,d\phi + O(\epsilon^{\frac{5}{2}}). \quad (4\cdot6)$$

We may replace $\sin\theta^{(1)}$ by $\theta^{(1)}$ in (4·6) to the required order and, further, replace $\epsilon^{-\frac{1}{2}}$, the upper limit of integration in the outer integrals, by ∞ and make only exponentially small errors (since the integrands are all exponentially decaying functions of ϕ). This completes the proof of the theorem.

If we substitute the series representation of $\theta^{(1)}$ into the expression for τ_∞ given by Theorem 4·2, we obtain

$$\tau_\infty = \frac{\epsilon}{\pi} + \left\{\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{I_n}{4n^2-1} - \frac{3}{2\pi^2}\right\}\epsilon^2 + \ldots,$$

where

$$I_n = 2\sum_{r=1}^{n}\frac{(-1)^{r+1}}{2r-1} + \frac{2(-1)^n}{4n^2-1},$$

and therefore

$$q_\infty = 1 + \frac{\epsilon}{\pi} + \left\{\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{I_n}{4n^2-1} - \frac{1}{\pi^2}\right\}\epsilon^2 + \ldots.$$

Substitution of the above into the expressions (4·4) and (4·5) for y_s and C_c yields

$$y_s = 1{\cdot}6366 - 0{\cdot}2370\epsilon + \ldots, \quad (4\cdot7)$$

$$C_c = 0{\cdot}6110 - 0{\cdot}1060\epsilon + \ldots. \quad (4\cdot8)$$

The coefficients of ϵ^2 could be obtained after adaption of Theorem 4·2, and this is done in (1).

5. *Other sluice-gate problems.* In this section we generalize the method for the model problem to physically more realistic flow geometries, such as that of Fig. 1 (*b*). To be more precise, we consider a family of geometries in which the depth y (for $x < 0$) is bounded. We consider a finite number of straight line-segments extending from upstream infinity I to S, the separation point. In the w plane these line-segments form the semi-infinite line $\psi = 1$, $-\infty < \phi < 0$.

We proceed as follows. We find, by use of an appropriate conformal mapping (such as (2·1)), the Helmholtz ($\epsilon = 0$) solution for $\Omega(w) = T_H + i\Theta_H$ and formulate an

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integral equation for the deviation θ_b of the slope of the free-streamline from Θ_H as in section 2. If $\eta(\phi) = \Theta_H(\phi, 1)$, $\phi \geqslant 0$, then the integral equation takes the form

$$\theta_b(\hat{\phi}) = \mathscr{B}_\epsilon \theta_b(\hat{\phi}) \equiv \frac{\epsilon}{2}\int_0^\infty K(\phi, \hat{\phi}) \frac{\sin(\eta + \theta_b)}{1 + \frac{3\epsilon}{2}\int_0^\phi \sin(\bar{\eta} + \bar{\theta}_b)\, d\bar{\phi}}\, d\phi, \tag{5·1}$$

for $0 \leqslant \hat{\phi} < \infty$, where $K(\phi, \hat{\phi})$ is defined in section 2.

THEOREM 5·1. *If* $0 \leqslant \eta(\phi) \leqslant \sigma(\phi)$, $0 \leqslant \phi < \infty$, *then there exists a unique solution of equation* (5·1) *in* X_α, *provided* $\epsilon < \epsilon^*(\alpha)$, *for any* $\alpha \in (0, 1)$.

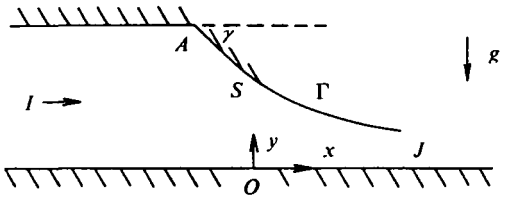


Fig. 3. The variable-angle sluice gate.

THEOREM 5·2. *If* $\epsilon < \epsilon^*(\alpha)$, *the unique solution of equation* (5·1) *in* X_α *satisfies*

$$\theta_b(\hat{\phi}) = (\epsilon/2)\int_0^\infty K(\phi, \hat{\phi}) \sin\eta\, d\phi + O(\epsilon^2) \quad (0 \leqslant \hat{\phi} < \infty).$$

Proofs. We refer to the proofs of Theorems 3·3, 3·4 and 4·1 and see that the inequalities $0 \leqslant \eta \leqslant \sigma$ and $0 \leqslant \sin\eta \leqslant \sin\sigma$ enable the proofs of those Theorems to go through with $\mathscr{A}_\epsilon$ replaced by $\mathscr{B}_\epsilon$.

We write

$$\theta_b^{(1)} = \frac{\epsilon}{2}\int_0^\infty K(\phi, \hat{\phi}) \sin\eta\, d\phi, \quad 0 \leqslant \hat{\phi} < \infty.$$

THEOREM 5·3. *The quantity* τ_∞ *satisfies*

$$\tau_\infty = \frac{\epsilon}{2}\int_0^\infty \sin\eta\, d\phi + \frac{\epsilon}{2}\int_0^\infty \cos\eta\, \theta_b^{(1)}\, d\phi - \tfrac{3}{4}\epsilon^2 \int_0^\infty \sin\eta \left\{\int_0^\phi \sin\bar{\eta}\, d\bar{\phi}\right\} d\phi + O(\epsilon^{\frac{5}{2}}).$$

Proof. The proof follows from that of Theorem 4·2 using $0 \leqslant \eta \leqslant \sigma$.

Hence we may deduce the contraction ratio and the slot height to order ϵ as in section 4.

We consider two specific examples, the flows of Fig. 1 (*b*) and 3. The flow of Fig. 1 (*b*) corresponds to that of Fig. 3 with $\gamma = \pi/2$. We restrict γ to lie in $(0, \pi/2]$.

EXAMPLE 1. $\gamma = \pi/2$. Let $\phi_A = \phi(x, y)$ at A. The function

$$w(t) = \frac{1}{\pi}\ln\left\{\frac{-4a^2(t^2 - a^2)(t^2 - a^{-2})}{(1 - t^2)^2(1 + a^2)^2}\right\},$$

where the logarithm is real when its argument is real and positive, maps the strip conformally onto $0 \leqslant |t| \leqslant 1$, $0 \leqslant \arg t \leqslant \pi/2$, with

$$\phi_A = (2/\pi)\ln b, \quad b = 2a/(1 + a^2) \quad (0 < a < 1).$$

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The Helmholtz solution $T_H + i\Theta_H = \ln t$ gives

$$\sin\eta(\phi) = (1-b^2)^{\frac{1}{2}}(e^{\pi\phi}-b^2)^{-\frac{1}{2}} \quad (0 \leqslant \phi < \infty),$$

and we note that $(1-b^2)^{\frac{1}{2}}\sin\sigma \leqslant \sin\eta \leqslant \sin\sigma$, since

$$\begin{aligned}\sin\eta &= e^{-\pi\phi/2}\{(1-b^2)/(1-b^2e^{-\pi\phi})\}^{\frac{1}{2}},\\ &= \sin\sigma\{(1-b^2)/(1-b^2\sin^2\sigma)\}^{\frac{1}{2}}.\end{aligned}$$

This implies that $0 \leqslant \eta(\phi) \leqslant \sigma(\phi)$ and hence that Theorems 5·1–5·3 apply. Then $\eta(\phi)\to\sigma(\phi)$ as $b\to 0$ $(a\to 0)$, and so $\theta_b(\phi)\to\theta(\phi)$ as $b\to 0$, where θ_b is the solution of (5·1) in X_α and θ the solution of (2·13) in X_α, if $0 < \epsilon < \epsilon^*(\alpha)$.

The upstream height is denoted by H (see Fig. 3) and the speed of the fluid far upstream by U. The condition of unit flux requires that $UH = 1$. For $\gamma = \pi/2$ we have the following theorem.

THEOREM 5·4. *If $H = \epsilon^{-1} + O(1)$, then the contraction ratio C_c and slot height y_s are given to order ϵ by the results* (4·7) *and* (4·8) *for the model problem.*

Proof. We have that $UH = 1$, and so $U = \epsilon + O(\epsilon^2)$. Also

$$q = \exp(T) = |t|\exp(\tau)\to a\exp(\tau)|_{t=a} = U \quad \text{as} \quad t\to a.$$

Therefore $a = U\{1+O(\epsilon)\} = \epsilon + O(\epsilon^2)$ and $b = 2\epsilon + O(\epsilon^2)$. This implies that

$$|\sin\eta - \sin\sigma| = O(\epsilon^2\sin\sigma), \quad |\theta_b^{(1)} - \theta^{(1)}| = O(\epsilon^2\theta^{(1)})$$

uniformly on $[0,\infty)$, and consequently that the difference between τ_∞ given by Theorem 5·3 and τ_∞ given by Theorem 4·2 is of higher order than ϵ^2. This completes the proof of the theorem, using (4·4) and (4·5) (the general expressions for y_s and C_c).

Remark. Under the conditions of Theorem 5·4, Bernoulli's theorem evaluated on $\psi = 1$ between I and A (the stagnation point) shows the pressure difference between them to be of order ϵ^2. In this problem the pressure is bounded below whereas in the model problem $p\to-\infty$ as $\phi\to-\infty$. The conditions of Theorem 5·4 are satisfied if we ask that the pressure on $\psi = 1$ tend to its constant value on Γ as $\phi\to-\infty$. (This is the usual model of the situation where $\psi = 1, \phi < \phi_A$ is a free surface.)

Example 2. $0 < \gamma \leqslant \pi/2$ (see Fig. 3). The function

$$w(t) = \frac{1}{\pi}\ln\left\{\frac{-4a^2(t^{\pi/\gamma}-a^2)(t^{\pi/\gamma}-a^{-2})}{(1-t^{\pi/\gamma})^2(1+a^2)^2}\right\},$$

where the logarithm is real when its argument is real and positive, maps the strip in the w plane conformally onto $0 \leqslant |t| \leqslant 1$, $0 \leqslant \arg t \leqslant \gamma$. The Helmholtz solution, $T_H + i\Theta_H = \ln t$, is such that

$$\sin(\pi\eta/[2\gamma]) = (1-b^2)^{\frac{1}{2}}(e^{\pi\phi}-b^2)^{-\frac{1}{2}} \quad (0 \leqslant \phi < \infty).$$

This implies that $(1-b^2)^{\frac{1}{2}}\sin\sigma \leqslant \sin(\pi\eta/[2\gamma]) \leqslant \sin\sigma$ and hence that $\eta \leqslant (2\gamma/\pi)\sigma \leqslant \sigma$. Theorems 5·1 to 5·3 apply.

More complicated geometries may be handled in a similar fashion.

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