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Collisions at infinity in hyperbolic manifolds

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Abstract

For a complete, finite volume real hyperbolic n-manifold M, we investigate the map between homology of the cusps of M and the homology of M. Our main result provides a proof of a result required in a recent paper of Frigerio, Lafont and Sisto.

1. *Introduction*

Let M be a cusped finite volume hyperbolic n-orbifold. Recall that the thick part of M is the quotient $M_0 = X_M/\pi_1(M)$, where X_M is the complement in $\mathbf{H}^n$ of a maximal $\pi_1(M)$-invariant collection of horoballs (see for instance [**11**]). It is known that M and M_0 are homotopy equivalent and M_0 is a compact orbifold with boundary components $E_1, \ldots, E_r$. Each E_j is called a *cusp cross-section* of M. Since horoballs in $\mathbf{H}^n$ inherit a natural Euclidean metric, each cusp cross-section is naturally a flat $(n-1)$-orbifold. Changing the choice of horoballs preserves the flat structure up to similarity.

The aim of this paper is to provide a proof of a result required in Frigerio, Lafont, and Sisto [**3**] for their construction in every $n \geqslant 4$ of infinitely many n-dimensional graph manifolds that do not support a locally CAT(0) metric. Specifically, the following is our principal result.

THEOREM 1·1. *For every $n \geqslant 3$ and $n > k \geqslant 2$, there exist infinitely many commensurability classes of orientable non-compact finite volume hyperbolic n-manifolds M containing a properly embedded totally geodesic hyperbolic k-submanifold N with the following properties. Let $\mathcal{E} = \{E_1, \ldots, E_r\}$ be the cusp cross-sections of M and $\mathcal{F} = \{F_1, \ldots, F_s\}$ the cusp cross-sections of N. Then*:

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(1) *M and N both have at least two ends, and all cusp cross-sections are flat tori*;
(2) *the inclusion $N \to M$ induces an injection of $H_{k-1}(\mathcal{F}; \mathbf{Q})$ into $H_{k-1}(\mathcal{E}; \mathbf{Q})$*;
(3) *the induced homomorphism from $H_{k-1}(\mathcal{F}; \mathbf{Q})$ to $H_{k-1}(M; \mathbf{Q})$ is not an injection.*

That every finite volume hyperbolic n-orbifold has a finite covering for which (1) holds is a folklore result for which we give a complete proof in Section 3 (see Theorem 3·1). A proof for $k = 2$, the case needed in [**3**], is given in [**3**, chapter **12**] assuming that (1) is known. However, our proof yields the more general result stated above. The main difficulty in proving Theorem 1·1 is (2), which will follow from a separation result that requires some terminology.

Consider a totally geodesic hyperbolic k-manifold N immersed in M. Assume that N is also noncompact with cusp cross-sections $F_1, \ldots, F_s$. Though they are not freely homotopic in N, it is possible that two distinct ends of N become freely homotopic inside M. When this occurs, we say that the two ends of N *collide at infinity* inside M. For certain N, it is a well-known consequence of separability properties of $\pi_1(N)$ in $\pi_1(M)$ (e.g., see [**1**] and [**6**]) that one can find a finite covering M' of M into which N embeds. However, N may still have collisions at infinity in M' and such collisions can lead to the continued failure of (2). Removing these collisions is the content of the next result; see §2 for the definition of virtual retractions.

THEOREM 1·2. *Suppose M is a cusped finite volume hyperbolic n-manifold and N is an immersed totally geodesic cusped hyperbolic k-manifold. If $\pi_1(M)$ virtually retracts onto $\pi_1(N)$, then there exists a finite covering M' of M such that N embeds in M' and has no collisions at infinity.*

That there are infinitely many commensurability classes of manifolds for which Theorem 1·2 applies is discussed in the remark at the end of Section 2. In particular, any noncompact arithmetic hyperbolic n-manifold has the required property, and these manifolds determine infinitely many commensurability classes in every dimension $n > 2$. Given that there are infinitely many commensurability classes to which Theorem 1·2 applies, we now assume Theorem 1·2 and (1) and prove (2) and (3) of Theorem 1·1.

Proof of Theorem 1·1 (2) *and* (3). Let M and N satisfy the conditions of Theorem 1·2 and assume that M satisfies (1). We replace M with the covering M' satisfying the conclusions of Theorem 1·2 and let $\mathcal{E}' = \{E'_j\}$ be the set of cusp ends of M'. Note that M' also satisfies (1). Now, each $(k-1)$-torus F_j is realized as an embedded homologically essential submanifold of some $(n-1)$-torus $E'_{i(j)}$. In particular, (3) is immediate as the homology class $\sum[F_j]$ bounds the class $[N_0] = [N \cap M'_0]$ inside M'. Since F_j is not freely homotopic to F_k in M' for any $j \neq k$, it follows that they cannot be freely homotopic in $\mathcal{E}'$ for any $j \neq k$. Hence the induced map on $(k-1)$-homology is an injection, which gives (2). This completes the proof.

2. *The proof of Theorem* 1·2

The proof of Theorem 1·2 is an easy consequence of the *virtual retract property* of [**7**] (see also [**2**]) which has found significant applications in low-dimensional topology and geometric group theory of late (see [**2**], [**7**] and the references therein).

Definition 1. Let G be a group and $H < G$ be a subgroup. Then G *virtually retracts* onto H if there exists a finite index subgroup $G' < G$ with $H < G'$ and a homomorphism

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$\rho\colon G' \to H$ such that $\rho|_H = \mathrm{id}_H$. In addition we say that G' retracts onto H, and ρ is called the retraction homomorphism.

With this definition we note the following lemma.

LEMMA 2·1. *Let G be a group and $H < G$ a subgroup such that G retracts onto H. Then two subsets S_1, S_2 of H are conjugate in G if and only if they are conjugate in H.*

Proof. One direction is trivial. Suppose that there exists $g \in G$ such that $S_1 = gS_2g^{-1}$. Then

$$S_1 = \rho(S_1) = \rho(gS_2g^{-1}) = \rho(g)S_2\rho(g)^{-1},$$

so S_1 and S_2 are conjugate in H.

Proof of Theorem 1·2. Let $M = \mathbf{H}^n/\Gamma$ be a cusped finite volume hyperbolic n-manifold, $N = \mathbf{H}^k/\Lambda$ be a noncompact finite volume totally geodesic hyperbolic k-manifold immersed in M such that that Γ virtually retracts onto Λ. Let $F_1, \ldots, F_r$ be the cusp cross-sections of N and $\Delta_1, \ldots, \Delta_r < \Lambda$ representatives for the associated Λ-conjugacy classes of peripheral subgroups, i.e., $\Delta_j = \pi_1(F_j)$.

Two ends F_{j_1} and F_{j_2} of N collide at infinity in M if and only if any two representatives Δ_{j_1} and Δ_{j_2} for the associated Λ-conjugacy classes of peripheral subgroups are conjugate in Γ but not in Λ. Let Γ_N denote the finite index subgroup of Γ that retracts onto Λ, and $\rho\colon \Gamma_N \to \Lambda$ the retracting homomorphism. By Lemma 2·1, Δ_{j_1} and Δ_{j_2} are not conjugate in Γ_N for any $j_1 \neq j_2$. Thus N has no collisions at infinity inside $M' = \mathbf{H}^n/\Gamma_N$.

Moreover, since Λ is a retract of Γ_N, it follows that Λ is separable in Γ_N (see [**4**, lemma 9·2]). Now a well-known result of Scott [**12**] shows that we can pass to a further covering M'' of M' such that the immersion of N into M' lifts to an embedding in M''. This proves the theorem.

Remark.

(i) Examples where the virtual retract property holds are abundant. From [**2**], if $M = \mathbf{H}^n/\Gamma$ is any non-compact finite volume hyperbolic n-manifold, which is arithmetic or arises from the construction of Gromov–Piatetskii-Shapiro, then Γ has the required virtual retract property. Briefly, the arithmetic case follows from [**2**, theorem 1·4] and the discussion at the very end of [**2**, section **9**], and for the examples from the Gromov–Piatetskii-Shapiro construction it follows from [**2**, theorem 9·1] and the same discussion at the very end of [**2**, section **9**].

(ii) We have in fact shown something stronger, namely that two essential loops in a cusp cross-section F_j of N are homotopic inside M' if and only if they are freely homotopic in N. Therefore, the kernel of the induced map from $H_*(\mathcal{F}; \mathbf{Q})$ to $H_*(M'; \mathbf{Q})$ is precisely equal to the kernel of the homomorphism from $H_*(\mathcal{F}; \mathbf{Q})$ to $H_*(N; \mathbf{Q})$.

(iii) Lemma 2·1 also implies that N cannot have positive-dimensional essential self-intersections inside M'. In particular, if $n < 2k$, then N automatically embeds in M'.

3. *Covers with torus ends*

The following will complete the proof of Theorem 1·1.

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THEOREM 3·1. *Let M be a complete finite volume cusped hyperbolic n-manifold. Then M has a finite covering M' such that M' has at least two ends and each cusp cross-section is a flat $(n-1)$-torus.*

Proof of Theorem 3·1 Let $M = \mathbf{H}^n/\Gamma$ be a cusped hyperbolic n-manifold of finite volume. Let $\Delta_1, \ldots, \Delta_{r_j}$ be representatives for the conjugacy classes of peripheral subgroups of Γ. For each Δ_j the Bieberbach Theorem [**11**, section 7·4] gives a short exact sequence

$$1 \longrightarrow \mathbf{Z}^{n-1} \longrightarrow \Delta_j \longrightarrow \Theta_j \longrightarrow 1$$

where Θ_j is finite. Then E_j is a flat $(n-1)$-torus if and only if Θ_j is the trivial group. Note in the case when M is a surface, the statement is trivial and thus we will assume $n > 2$.

Let $\gamma_{j,1}, \ldots, \gamma_{j,r_j}$ be lifts of the distinct nontrivial elements of Θ_j to Δ_j. Since $n \geqslant 3$, it is a well-known consequence of Weil Local Rigidity (see [**10**, theorem 7·67]) that we can conjugate Γ in $\mathrm{PO}_0(n,1)$ inside $\mathrm{GL}_N(\mathbf{C})$ so that it has entries in some number field k. Since Γ is finitely generated, we can further assume that it has entries in some finitely generated subring $R \subset k$. Then $R/\mathfrak{p}$ is finite for every prime ideal $\mathfrak{p} \subset R$.

This determines a homomorphism from Γ to $\mathrm{GL}_N(R/\mathfrak{p})$. For every $\gamma_{j,k}$, the image of $\gamma_{j,k}$ in the finite group $\mathrm{GL}_N(R/\mathfrak{p})$ is nontrivial for almost every prime ideal $\mathfrak{p}$ of R. Indeed, any off-diagonal element is congruent to zero modulo $\mathfrak{p}$ for only finitely many $\mathfrak{p}$ and there are only finitely many $\mathfrak{p}$ so that a diagonal element is congruent to 1 modulo $\mathfrak{p}$. Since there are finitely many $\gamma_{j,k}$, this determines a finite list of prime ideals $\mathcal{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ such that $\gamma_{j,k}$ has nontrivial image in $\mathrm{GL}_N(R/\mathfrak{p})$ for any $\mathfrak{p} \notin \mathcal{P}$ and every j, k. If $\Gamma(\mathfrak{p})$ is the kernel of this homomorphism, then $\Gamma(\mathfrak{p})$ contains no conjugate of any of the $\gamma_{j,k}$.

The peripheral subgroups of $\Gamma(\mathfrak{p})$ are all of the form $\Gamma(\mathfrak{p}) \bigcap \gamma\Delta_j\gamma^{-1}$ for some $\gamma \in \Gamma$. Since no conjugate of any $\gamma_{j,k}$ is contained in $\Gamma(\mathfrak{p})$, we see that $\Gamma(\mathfrak{p}) \bigcap \gamma\Delta_j\gamma^{-1}$ is contained in the kernel of the above homomorphism $\gamma\Delta_j\gamma^{-1} \to \Theta_j$. It follows that every cusp cross-section of $\mathbf{H}^n/\Gamma(\mathfrak{p})$ is a flat torus. This proves the second part of the theorem.

To complete the proof of Theorem 3·1, it suffices to show that if M is a noncompact hyperbolic n-manifold with k ends, then M has a finite sheeted covering M' with strictly more than k ends. We recall the following elementary fact from covering space theory. Let $\rho\colon \Gamma \to Q$ be a homomorphism of Γ onto a finite group Q and Γ_ρ be the kernel of ρ. If Δ_j is a peripheral subgroup of Γ, then the number of ends of $\mathbf{H}^n/\Gamma_\rho$ covering the associated end of $\mathbf{H}^n/\Gamma$ equals the index $[Q : \rho(\Delta_j)]$ of $\rho(\Delta_j)$ in Q. Therefore, it suffices to find a finite quotient Q of Γ and a peripheral subgroup Δ_j of Γ that $\rho(\Delta_j)$ is a proper subgroup of Q.

In our setting, the proof is elementary. From above, we can pass to a finite sheeted covering of M, for which all the cusp cross-sections are tori, i.e., all peripheral subgroups are abelian. It follows that for $\rho|_{\Delta_j}$ to be onto, $\rho(\Gamma)$ must be abelian. However, it is well-known that the above reduction quotients $\Gamma/\Gamma(\mathfrak{p})$ are central extensions of non-abelian finite simple groups for all but finitely many prime ideals $\mathfrak{p}$ [**8**, Chapter 6]. The theorem follows.

Remark. Constructing examples with a small number of ends is much more difficult. For example, there are no known one-cusped hyperbolic n-orbifolds for $n > 11$. Furthermore, it is shown in [**13**] that for every d, there is a constant c_d such that d-cusped *arithmetic* hyperbolic n-orbifolds do not exist for $n > c_d$. For example, in the case $d = 1$, there are no 1-cusped arithmetic hyperbolic n-orbifolds for any $n \geqslant 30$. Very recently, [**5**] announced the first construction of a one-cusped hyperbolic n-manifold.

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