LIVŠIC REGULARITY THEOREMS FOR TWISTED COCYCLE EQUATIONS OVER HYPERBOLIC SYSTEMS

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Abstract

Let ϕ be a hyperbolic map. Cocycle equations of the form $f = u\phi \cdot g \cdot \alpha u^{-1}$ are considered, with f, g, u taking values in a compact connected Lie group G, α being an automorphism of G and f, g being Hölder continuous. When the eigenvalues of the derivative of α have modulus 1, it is proved that any measurable solution u has a Hölder continuous version. This condition on α is optimal. When f, g are C^k then u may be taken to be $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.

1. *Introduction*

Let ϕ be a diffeomorphism of a compact Riemannian manifold M, and let $\Lambda \subset M$ be a closed ϕ-invariant subset. Let G be a compact connected Lie group and let α be an automorphism of G. For $f: \Lambda \longrightarrow G$ define a skew product $\phi_f: \Lambda \times G \longrightarrow \Lambda \times G$ by $\phi_f(x, y) = (\phi(x), f(x)\,\alpha(y))$. We call f a *twisted cocycle* or α-*cocycle*.

We say that two twisted cocycles f, g are α-*cohomologous* if the map $U: (x, y) = (x, u(x)\,y)$, $u: \Lambda \longrightarrow G$, conjugates ϕ_f and ϕ_g: $\phi_f U = U\phi_g$. This is equivalent to u satisfying the *twisted cocycle equation*

$$f(x) = u(\phi x) \cdot g(x) \cdot \alpha u(x)^{-1} \tag{1}$$

(we often use a dot '$\cdot$' to denote multiplication in G for clarity). It is usually important to specify the class of function to which u belongs. Indeed, the main aim of this note is to give conditions on ϕ and α for when the existence of a measurable solution to (1) (holding almost everywhere with respect to a suitable measure) implies the existence of a more regular solution. We call such a result a *regularity theorem*.

Throughout, we shall assume that ϕ is hyperbolic. By 'hyperbolic' we essentially mean that ϕ can be represented as an Axiom A diffeomorphism restricted to its non-wandering set (we give a precise definition in §2); examples include (transitive) Anosov diffeomorphisms and shifts of finite type.

The classical case is when α is the identity (in which case we drop the prefix α-). In this case, the twisted cocycle equation (1) becomes the *cocycle equation*

$$f = u\phi \cdot g \cdot u^{-1} \text{ a.e.}. \tag{2}$$

Suppose that f and g are Hölder continuous and we have a measurable solution u to equation (2). Is u necessarily equal a.e. (almost everywhere) to a continuous function? This was first considered by Livšic [**9**], where it is proved to be the case for abelian Lie groups, under the assumptions that ϕ is C^2 Anosov and that the measure is smooth. For compact Lie group valued cocycles satisfying the cocycle equation (2), Parry and Pollicott proved the following theorem [**13**].

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THEOREM 1.1 [**13**]. *Let ϕ be a hyperbolic diffeomorphism and G a compact Lie group. Suppose that $f, g: \Lambda \longrightarrow G$ are Hölder continuous and satisfy $f = u\phi \cdot g \cdot u^{-1}$ almost everywhere (where the measure is the equilibrium state corresponding to a Hölder continuous function) for some measurable $u: \Lambda \longrightarrow G$. Then $u = u'$ almost everywhere for a Hölder continuous u' and $f = u'\phi \cdot g \cdot u'^{-1}$ everywhere.*

(Schmidt [**15**] has proved a similar theorem under the assumption that f and g both satisfy a 'bounded distortion' condition.)

Now suppose that Λ is a manifold (that is, ϕ is Anosov), f, g are C^r ($r = 1, 2, \ldots, \infty, \omega$) and u is a continuous solution to the untwisted cocycle equation (2). Does u possess some degree of differentiability? This problem has a long history (see [**12**]) culminating in the following theorem.

THEOREM 1.2 [**12**]. *Suppose that G is a compact Lie group and $f, g: \Lambda \longrightarrow G$ are C^r. Then a continuous solution u to $f = u\phi \cdot g \cdot u^{-1}$ is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.*

(With some extra hypotheses, Niţică and Török [**12**] also considered some classes of non-compact Lie groups.)

Let us now consider twisted cocycles. Let G be a torus, and let α be an ergodic automorphism of G. A theorem due to Lind [**8**] then says that any two α-cocycles are measurably α-cohomologous. Theorem 1.3 is more precise.

THEOREM 1.3 [**8**]. *Let ϕ be a hyperbolic diffeomorphism, G a torus and α an ergodic automorphism of G. Suppose that $f, g: \Lambda \longrightarrow G$ are Hölder continuous. Then there exists a measurable $u: \Lambda \longrightarrow G$ such that $f = u\phi \cdot g \cdot \alpha u^{-1}$.*

(In fact, Lind proved considerably more: ϕ need only be an (aperiodic) invertible measure preserving transformation (not even ergodic), G need only be compact and abelian and f, g need only be measurable.) An observation due to Lind shows that when α is a hyperbolic toral automorphism, the measurable solution guaranteed by Theorem 1.3 need not be continuous. This suggests the problem that we consider in this paper: finding the conditions on α that ensure that the regularity theorem holds for the twisted cocycle equation (1).

We shall mostly be interested in the case when α is a unipotent automorphism of a compact connected Lie group. We say that α is *unipotent* if all the eigenvalues of the derivative $d\alpha$ on the Lie algebra of G have modulus 1. This is equivalent to the eigenvalues being roots of unity. (Caution: This is not the usual meaning of the term 'unipotent', which is usually only defined for abelian groups. However, it is a generalisation appropriate for Lie groups. See the remark preceding Theorem 3.1 in §3.)

Our main result is the following theorem.

THEOREM A. *Let ϕ be a hyperbolic map. Let α be a unipotent automorphism of a compact connected Lie group G. Let f, g be Hölder continuous. Suppose that there exists a measurable solution $u: \Lambda \longrightarrow G$ to $f = u\phi \cdot g \cdot \alpha u^{-1}$ almost everywhere (where the measure is the equilibrium state corresponding to a Hölder continuous function). Then there exists a Hölder continuous u such that $u' = u$ almost everywhere and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.*

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The condition on α is optimal in the following sense. Let α be a non-unipotent automorphism of G. Then, after replacing α and G by locally isomorphic copies, there exists a hyperbolic ϕ, Hölder f, g and a measurable u such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u does not have a continuous version.

Now suppose that ϕ is a transitive Anosov diffeomorphism and f, g are C^r. Suppose that u is a continuous solution to $f = u\phi \cdot g \cdot \alpha u^{-1}$. Using techniques used by Niţică and Török in the untwisted case we prove Theorem B.

THEOREM B. *Let ϕ be a C^k (transitive) Anosov diffeomorphism ($1 \leqslant k \leqslant \infty$ or $k = \omega$) on a compact manifold M. Let α be a unipotent automorphism of a compact connected Lie group G. Let $f, g : M \longrightarrow G$ be C^r. Suppose that $f = u\phi \cdot g \cdot \alpha u^{-1}$ almost everywhere (with respect to a Hölder equilibrium state) for a measurable u. Then there exists a continuous u' such that $u = u'$ almost everywhere and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere. Moreover, u' is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0,1)$. (When $r = 1, \infty, \omega$ we define $r - 1 + \varepsilon = 1, \infty, \omega$ respectively, for any $\varepsilon \in (0,1)$.)*

This paper is organised as follows. In §2, we briefly review some definitions. In §3, we consider the case of a torus and prove the regularity theorem for a unipotent toral automorphism. In §4, we consider automorphisms of compact semi-simple Lie groups. In §5, we combine the results of §§3, 4 together to obtain Theorem A. We prove Theorem B in §6. In §7 we show that Theorem 1.1 can fail for a class of non-compact Lie groups.

2. *Preliminaries*

2.1. *Notation*

The following notation will be useful in the sequel. For $f : \Lambda \longrightarrow G$ define

$$f^n(x) = f(\phi^{n-1}x) \cdot \alpha f(\phi^{n-2}x) \dots \alpha^{n-1} f(x)$$
$$f^{-n}(x) = \alpha^{-1} f(\phi^{-n}x)^{-1} \cdot \alpha^{-2} f(\phi^{-n+1}x)^{-1} \dots \alpha^{-n} f(\phi^{-1})^{-1}$$

for $n > 0$ and $f(x, 0) = e$, the group identity, so that $\phi_f^n(x, y) = (\phi^n x, f^n(x)\, \alpha^n(y))$. Clearly,

$$f^{n+m}(x) = f^n(\phi^m x) \cdot \alpha^n f^m(x). \tag{3}$$

2.2. *Hyperbolic dynamics*

Let M be a smooth compact Riemannian manifold, and let ϕ be a C^1 diffeomorphism of M. We call a compact ϕ-invariant subset $\Lambda \subset M$ a *basic set* and ϕ restricted to Λ a *hyperbolic diffeomorphism* if the following conditions hold:

(1) The tangent bundle restricted to Λ can be written as a continuous Whitney sum $E^s \oplus E^u$ of $d\phi$-invariant sub-bundles for which there exist constants $C > 0$, $\lambda \in (0,1)$ such that

$$\|d\phi^n|_{E^s}\| \leqslant C\lambda^n, \qquad n \geqslant 0,$$
$$\|d\phi^{-n}|_{E^u}\| \leqslant C\lambda^n, \qquad n \geqslant 0.$$

(2) The periodic orbits in Λ are dense in Λ and Λ is not a single orbit.
(3) There is a dense orbit in Λ.
(4) There is an open neighbourhood $U \supset \Lambda$ such that $\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^n U$.

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Thus, in defining a basic set, we are merely abstracting the properties of the basic sets appearing in Smale's spectral decomposition of the non-wandering set of an Axiom A diffeomorphism. In particular (see [7] for example), by replacing ϕ by ϕ^n, for some $n > 0$, there is no loss in assuming that $\phi|_\Lambda$ is topologically mixing.

Give Λ the metric d induced from the Riemannian metric on M. For a metric space (Y, ρ) and $\theta \in (0,1)$, let $C^\theta(Y)$ denote the space of Hölder continuous functions $\Lambda \longrightarrow Y$ of exponent θ, that is, those $f: \Lambda \longrightarrow Y$ that satisfy $\rho(f(x), f(y)) \leqslant Cd(x,y)^\theta$. We denote the least possible such C by $|f|_\theta$. It is easy to see that if $\theta_1 \leqslant \theta_2$ then $C^{\theta_1} \supset C^{\theta_2}$.

Let $f \in C^\theta(\mathbb{R})$. As μ ranges over all ϕ-invariant probability measures, the quantity $h_\mu(\phi) + \int f d\mu$ (h_μ denotes entropy) is maximised by a unique ϕ-invariant measure [2]. We call this the *equilibrium state* of f, and denote it by m_f or m. Equilibrium states corresponding to Hölder continuous functions enjoy many nice properties. In particular, they assign positive measure to non-empty open sets and have no atoms.

A standard, and very useful, result allows us to view a hyperbolic diffeomorphism as a quotient of a shift of finite type in a particularly nice way. Before describing this, let us first discuss shifts of finite type.

Recall that a shift of finite type is defined as follows. Let A be an aperiodic $k \times k$ matrix with entries from $\{0, 1\}$. Let

$$\Sigma = \{(x_j)_{j=-\infty}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}} \mid A_{x_j x_{j+1}} = 1\}.$$

Then Σ is a totally disconnected zero-dimensional compact metrisable space. Define the shift $\sigma: \Sigma \longrightarrow \Sigma$ by $(\sigma x)_j = x_{j+1}$ and note that $\sigma(\Sigma) = \Sigma$. We call σ a *shift of finite type*.

For each $\theta \in (0,1)$ we obtain a metric on Σ by defining

$$d^\theta((x_j), (y_j)) = \theta^{n(x,y)},$$

where $n(x,y) = \sup\{|n| \mid x_j = y_j, |j| \leqslant n\}$. For a metric space (Y, ρ) we define $F_\theta(Y)$ to be the space of all Lipschitz continuous functions with respect to ρ and d^θ:

$$F_\theta(Y) = \{f: \Sigma \longrightarrow Y \mid \rho(f(x), f(y)) \leqslant Cd^\theta(x,y) \text{ for some } C > 0\}.$$

Write $|f|_\theta$ for the least possible C appearing above. Often, we shall abuse this definition and speak of functions in $F_\theta(Y)$ as being Hölder continuous of exponent θ. In particular, observe that if $\theta_1 \leqslant \theta_2$ then $F_{\theta_1} \subset F_{\theta_2}$.

Again, if $f \in F_\theta(\mathbb{R})$ then we can define the equilibrium state m_f of f.

We first prove Theorem A in the case of a shift of finite type. To deduce Theorem A for a hyperbolic diffeomorphism we use the fact that a hyperbolic diffeomorphism is a quotient of a shift of finite type in a nice way via Markov partitions. We give a quick review of the parts of this construction that will be needed in §5.

For $x \in \Lambda$ and small $\varepsilon > 0$, define

$$W_\varepsilon^s(x) = \{y \in M \mid d(\phi^n x, \phi^n y) \leqslant \varepsilon,\ n \geqslant 0\},$$
$$W_\varepsilon^u(x) = \{y \in M \mid d(\phi^{-n} x, \phi^{-n} y) \leqslant \varepsilon,\ n \geqslant 0\}.$$

We call these the local stable and unstable manifolds through x, respectively. They are tangent to E^s and E^u, respectively. If $z \in W_\varepsilon^s(x)$ then we have the estimate $d(\phi^n x, \phi^n z) \leqslant C\lambda^n d(x,z)$ for $n \geqslant 0$; if $z \in W_\varepsilon^u(y)$ then $d(\phi^{-n} y, \delta^{-n} z) \leqslant C\lambda^n d(y,z)$ for $n \geqslant 0$. If $x, y \in \Lambda$ are sufficiently close then $\langle x, y \rangle = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ is a single point and this point lies in Λ.

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The following describes the properties of Markov partitions that we shall need in the sequel.

PROPOSITION 2.1. *Let ϕ be a hyperbolic diffeomorphism. Then there exists a shift of finite type σ and a semi-conjugacy $\pi:\Sigma \longrightarrow \Lambda$ (so that $\pi\sigma = \phi\pi$) such that the following hold*:

(1) *The semi-conjugacy π is one-to-one on a residual subset of Λ and on a set of full measure for any equilibrium state. (Call a point $x \in \Lambda$ 'good' if it has a unique pre-image under π.)*

(2) *There is a dense set of good periodic points for ϕ.*

(3) *If $x, y \in \Lambda$ are good periodic points such that $\langle x, y\rangle$ is defined, then $\langle x, y\rangle$ is good.*

(4) *The semi-conjugacy π is Lipschitz continuous with respect to d^λ. In particular, if $f:\Lambda \longrightarrow G \in C^\theta(G)$ then $f\pi:\Sigma \longrightarrow G \in F_{\lambda^\theta}(G)$.*

Statements 1 and 4 are well-known (see [**2**], for example). Statements 2 and 3 are easy consequences of well-known properties of Markov partitions; see [**13**].

3. *Toral automorphisms*

Let $G = \mathbb{R}^k/\mathbb{Z}^k$ be the k-torus for which we use additive notation. Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism.

We first consider the case when α is unipotent. In our terminology, this means that all the eigenvalues of α have modulus 1. It is well-known that this forces all the eigenvalues to be roots of unity. Hence, for some $p \geqslant 1$, α^p has 1 as its only eigenvalue. For definiteness, we fix p to be the least such $p \geqslant 1$ with this property. It follows that, for some $n \geqslant 1$,

$$(\alpha^p - \mathrm{id})^n\, G = \{e\}; \tag{4}$$

here, id is the identity map on G and e is the group identity. We call the least positive integer n for which equation (4) holds the *unipotency order* of α^p.

REMARK. Observe that (4) is equivalent to $\bigcap_{n=0}^{\infty} (\alpha^p - \mathrm{id})^n\, G = \{e\}$. This is the usual definition of what it means for α^p to be a unipotent automorphism of the abelian group G. Thus our definition of unipotent can be viewed as a generalisation of the usual definition to the case of a non-abelian Lie group. Note also that we say α is unipotent, whereas one would usually say that some power of α is unipotent.

We prove the following theorem.

THEOREM 3.1. *Let σ be a shift of finite type equipped with a Hölder equilibrium state. Let α be a unipotent toral automorphism. If $f \in F_\theta(G)$ satisfies $f = u\sigma - \alpha u$ almost everywhere for some measurable $u:\Lambda \longrightarrow G$, then there exists $u' \in F_\theta(G)$ such that $u' = u$ almost everywhere and $f = u'\sigma - \alpha u'$ everywhere.*

Proof. Define $f^p = \sum_{j=0}^{p-1} \alpha^j f \sigma^{p-1-j}$ so that $f^p = u\sigma^p - \alpha^p u$ almost everywhere. Note that σ^p is a shift of finite type. Hence there is no loss in assuming that all the eigenvalues of α are equal to 1.

We induct on the unipotency order n of α. When $n = 1$, α is the identity and the result follows from Theorem 1.1.

Let α have unipotency order $n > 1$. Let $H = (\alpha - \mathrm{id})\, G$. Then H is a closed subtorus of G (in particular, H cannot be dense in G, as $\alpha - \mathrm{id}$ has zero as its only

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eigenvalue; hence all the eigenvectors can be taken to be rational). Clearly H is α-invariant. Moreover, the induced automorphism α_H on G/H is the identity. (To see this, observe that $\alpha g+H=g+\alpha g-g+H=g+H$.) It is also clear that the H has unipotency order $n-1$.

Let π_H denote the quotient map $G\longrightarrow G/H$. Then, as $\alpha_H=\mathrm{id}$, we have $\pi_H f=(\pi_H u)\sigma-\pi_H u$ almost everywhere. By Theorem 1.1, there exists a Hölder v'': $\Sigma\longrightarrow G/H$ such that $\pi_H u=v''$ almost everywhere and $\pi_H f=v''\sigma-v''$ everywhere.

As Σ is totally disconnected, choose a Hölder lift of v'', that is, a function $v':\Sigma\longrightarrow G$ such that $\pi_H v'=v''$. Then $u-v'\in H$ almost everywhere. Moreover,

$$f-v'\sigma+\alpha v'=(u-v')\sigma-\alpha(u-v') \text{ a.e.}.$$

As $\pi_H(f-v'\sigma+\alpha v')=0$, this is an H-valued twisted cocycle equation. As α restricted to H has unipotency order $n-1$, the inductive hypothesis ensures we can write $u-v'=v$ almost everywhere for some Hölder v. Then $u'=v+v'$ is the required Hölder solution to $f=u'\sigma-\alpha u'$ and $u=u'$ almost everywhere. □

4. *Semi-simple Lie groups*

We first prove the regularity theorem for an inner automorphism of a compact Lie group in the context of a shift of finite type. This follows from a simple modification to the proof of Theorem 1.1 given in [**13**]. Throughout, we use the notation $U(Y)$ to denote the group of unitary operators on a Euclidean space Y. The key fact is as follows.

PROPOSITION 4.1. *Let $M(d)$ denote the space of $d\times d$ matrices over $\mathbb{C}$ with the Euclidean structure induced from $\mathbb{C}^{d^2}$. Let $\Psi:\Lambda\longrightarrow U(M(d))\in F_\theta(U(M(d)))$ and let $u:\Lambda\longrightarrow M(d)$ be measurable. Let σ be a shift of finite type equipped with a Hölder equilibrium state. Suppose that $u(\sigma x)=\Psi(x)\,u(x)$ almost everywhere. Then there exists $u'\in F_\theta(M(d))$ such that $u'=u$ almost everywhere and $u'(\sigma x)=\Psi(x)\,u'(x)$ everywhere.*

REMARK. Parry and Pollicott only concluded that $u'\in F_{\theta^{1/2}}(M(d))$. To see that $u'\in F_\theta(M(d))$, it is sufficient to show that $u'\in F_\theta(M(d))$ along local stable and unstable 'manifolds' [**7**, Proposition 19.1.1] (the local stable 'manifold' through $x\in\Sigma$ is $\{y\in\Sigma\,|\,x_j=y_j \text{ for } j\geqslant 0\}$). Let $\Psi^n(x)=\Psi(\sigma^{n-1}x)\ldots\Psi(x)\in U(M(d))$. Then

$$\Psi^n(y)\,(u'(x)-u'(y))=u'(\sigma^n x)-u'(\sigma^n y)-(\Psi^n(x)-\Psi^n(y))\,u'(x).$$

By ergodicity and the fact that Ψ is unitary, we may assume that $\|u'\|=1$ so that

$$\|u'(x)-u'(y)\|\leqslant\|u'(\sigma^n x)-u'(\sigma^n y)\|+\|\Psi^n(x)-\Psi^n(y)\|. \tag{5}$$

It is easy to see that when y belongs to the same local stable 'manifold' as x, we have $\|\Psi^n(x)-\Psi^n(y)\|\leqslant\sum_{j=0}^{n-1}\|\Psi\|_\theta\theta^n d^\theta(x,y)\leqslant\text{Const.}\,d^\theta(x,y)$ for some constant independent of n. Letting $n\to\infty$ in (5), we see that $\|u'(x)-u'(y)\|\leqslant\text{Const.}\,d^\theta(x,y)$.

Let G be a compact Lie group. Then for some d, G is a closed subgroup of $U(d)$, the $d\times d$ unitary matrices. Let $N(G)$ denote the normaliser of G in $\mathrm{Gl}(d)$, the general linear group. Then any $H\in N(G)$ determines an automorphism α of G by conjugation: $\alpha(A)=HAH^{-1}$.

THEOREM 4.2. *Let ϕ be a shift of finite type equipped with a Hölder equilibrium state. Let G be a compact Lie group and α an automorphism of G arising from an*

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element of the normaliser. Let $f, g \in F_\theta(G)$, and suppose that $f = u\sigma \cdot g \cdot \alpha u^{-1}$ almost everywhere for some measurable u. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ almost everywhere and $f = u'\sigma \cdot g \cdot \alpha u'^{-1}$ everywhere.

Proof. We modify the corresponding argument given in [**13**] for the case when α is the identity.

Equip $M(d)$ with the inner product $\langle A, A' \rangle = \operatorname{trace} AA'^*$ (* denotes the Hermitian conjugate). Define $\Psi : \Lambda \longrightarrow U(M(d))$ by $\Psi(x)\, A = f(x)\, HAH^{-1} g(x)^{-1}$. Clearly, Ψ is linear. To see that Ψ is unitary, observe that

$$\begin{aligned} \langle \Psi(x)\, A, \Psi(x)\, A' \rangle &= \operatorname{trace} f(x)\, HAH^{-1} g(x)^{-1}\, g(x)\, (H^{-1})^*\, A'^* H^* f(x)^{-1} \\ &= \operatorname{trace} AA'^* \\ &= \langle A, A' \rangle. \end{aligned}$$

Finally, if $f = u\sigma \cdot g \cdot \alpha u^{-1}$ almost everywhere, then $u(\sigma x) = f(x)\, Hu(x)\, H^{-1} g(x)^{-1}$ a.e. $= \Psi(x)\, u(x)$ a.e. and Proposition 4.1 applies. □

COROLLARY 4.3. *The regularity theorem holds for an inner automorphism of a compact Lie group.*

We now consider automorphisms of compact connected semi-simple Lie groups. Our first observation is that *all* automorphisms of such groups are unipotent (in the sense defined above). This fact is surely known, but we have been unable to locate a reference.

PROPOSITION 4.4. *Let α be an automorphism of a compact connected semi-simple Lie group G. Then the eigenvalues of $d\alpha$ are roots of unity.*

Proof. It is well-known (see [**16**], for example) that the inner automorphisms have finite index in the automorphism group of a compact semi-simple Lie group. Choose n such that α^n is inner. It is clear that the eigenvalues of $d\alpha^n$ have modulus 1; this follows from the existence of a bi-invariant metric. We prove that the eigenvalues are roots of unity. A standard result allows us to assume that, for some d, G is a closed subgroup of $O(d)$, the $d \times d$ orthogonal matrices (previously we had embedded G in a unitary group $U(d)$). Let $\mathfrak{g}$ be the Lie algebra of G, and note that it is a sub-algebra of $\mathfrak{o}(d) = \{X \in M(d) \mid X^T + X = 0\}$.

Suppose that $\alpha(A) = HAH^{-1}$ for some $H \in G$. Then for $X \in \mathfrak{g}$ we have $d\alpha(X) = HXH^{-1}$. Suppose that μ is an eigenvalue of $d\alpha$ with eigenvector $X \neq 0$. Then $\mu X = d\alpha(X) = HXH^{-1}$. Thus, when viewed as matrices, μX and X are similar. Therefore they have the same eigenvalues. We need the following easily-proved sub-lemma.

SUB-LEMMA. *If $X \in \mathfrak{o}(d)$ has zero as its only eigenvalue, then $X = 0$.*

Hence multiplying the spectrum of X by μ merely permutes the eigenvalues of X. As this is a finite set containing a non-zero entry, some power of μ is equal to 1. Hence μ is a root of unity. □

We prove the regularity theorem for automorphisms of compact connected semi-simple Lie groups.

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THEOREM 4.5. *Let σ be a shift of finite type equipped with a Hölder equilibrium state. Let G be a compact connected semi-simple Lie group and α an automorphism of G. Let $f, g \in F_\theta(G)$ and suppose that $f = u\phi \cdot g \cdot \alpha u^{-1}$ almost everywhere for some measurable $u: \Lambda \longrightarrow G$. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ almost everywhere and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.*

Proof. Choose $n > 0$ such that α^n is inner. Then $f^n = u\phi^n \cdot g^n \cdot \alpha^n u^{-1}$ almost everywhere. As σ^n is a shift of finite type, Corollary 4.3 applies. □

5. *Unipotent automorphisms of compact connected Lie groups*

We now prove Theorem A. Our strategy is to prove it first for a shift of finite type, using the structure theory for compact connected Lie groups. For a hyperbolic diffeomorphism, we use Markov partitions to push down solutions from a symbolic cover.

Let G be a compact connected Lie group. Let $\hat{A}$ denote the identity component of the centre of G; hence $\hat{A}$ is a torus. Let $\hat{S}$ denote the product of the simple normal subgroups of G (of which there are only finitely many); hence $\hat{S}$ is semi-simple. Then a standard result allows us to write

$$G = \frac{\hat{A} \times \hat{S}}{D}$$

where D is a discrete normal subgroup contained in the centre of $\hat{A} \times \hat{S}$. Let $\pi_D: \hat{A} \times \hat{S} \longrightarrow G$ denote the natural projection. Let α be an automorphism of G. Then α lifts to an automorphism $\hat{\alpha}$ of $\hat{A} \times \hat{S}$ such that $\pi_D \hat{\alpha} = \alpha \pi_D$. Moreover, $\hat{A}$ and $\hat{S}$ are $\hat{\alpha}$-invariant.

Let $A = \pi_D \hat{A}$, $S = \pi_D \hat{S}$. Then A is a torus and S is semi-simple. The properties of A and S that are important for our work are listed below.

LEMMA 5.1. (1) *A and S are α-invariant normal subgroups of G.*
(2) *$A \cap S$ is a discrete α-invariant normal subgroup of G.*
(3) *$G/A \cap S \cong A/A \cap S \times S/A \cap S$.*
(4) *$A/A \cap S$ is a torus and $S/A \cap S$ is semi-simple.*

By Lemma 5.1.2, there is a well-defined quotient automorphism on $G/A \cap S$. We continue to denote this automorphism by α. By Lemma 5.1.1, both $A/A \cap S$ and $S/A \cap S$ are α-invariant.

REMARK. As both D and $A \cap S$ are discrete, the groups $\hat{A} \times \hat{S}$, G and $G/A \cap S$ are all locally isomorphic. Hence, the eigenvalues of $d\alpha$ are the same, irrespective of whether α is viewed as an automorphism of $\hat{A} \times \hat{S}$, G or $G/A \cap S$.

Proof of Theorem A (*for a shift of finite type*). Let $\sigma: \Sigma \longrightarrow \Sigma$ be a shift of finite type. We assume that

$$f = u\sigma \cdot g \cdot \alpha u^{-1} \text{ a.e.} \tag{6}$$

where $f, g \in F_\theta(G)$ and $u: \Sigma \longrightarrow G$ is measurable. Apply the quotient map $\pi_{A \cap S}; G \longrightarrow G/A \cap S$ to equation (6) to obtain

$$\pi_{A \cap S} f = (\pi_{A \cap S} u)\, \sigma \cdot \pi_{A \cap S} g \cdot \alpha(\pi_{A \cap S} u)^{-1} \text{ a.e.}.$$

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By Lemma 5.1.3, this is an equation holding in the direct product of a torus and a semi-simple Lie group. Observe that α is the direct product of a unipotent toral automorphism and an automorphism of a semi-simple Lie group. By Theorems 3.1 and 4.5, there exists $v \in F_\theta(G/A \cap S)$ such that

$$\pi_{A\cap S} f = v\sigma \cdot \pi_{A\cap S} g \cdot \alpha v^{-1} \text{ everywhere}$$
$$v = \pi_{A\cap S} u \text{ a.e.}.$$

As Σ is totally disconnected, we can find $w \in F_\theta(G)$ such that $\pi_{A\cap S} w = v$. By construction, $\pi_{A\cap S} w = \pi_{A\cap S} u$ almost everywhere. Hence, there exists a measurable $p : \Sigma \longrightarrow A \cap S$ such that $p \cdot w = u$ almost everywhere.

We claim that p has a continuous version. Observe that as $A \cap S$ is a discrete (and hence finite) α-invariant subgroup of G, some power of α restricted to $A \cap S$ is the identity. Suppose that $\alpha^n|_{A\cap S} = \mathrm{id}$. As $f^n = u\sigma^n \cdot g^n \cdot \alpha^n u^{-1}$ almost everywhere, we have

$$f^n = p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot \alpha^n p^{-1} \text{ a.e.}$$
$$= p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot p^{-1} \text{ a.e.}.$$

As f^n, $w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \in F_\theta(G)$ and σ^n is a shift of finite type, we can use Theorem 1.1 to find $p' \in F_\theta(A \cap S)$ such that $p' = p$ almost everywhere and the above equation holds everywhere for p'.

Define $u = p' \cdot w$. Then $u' \in F_\theta(G)$, $u' = u$ almost everywhere, and $f = u'\sigma \cdot g \cdot \alpha u'^{-1}$ everywhere. □

We now want to prove Theorem A for hyperbolic diffeomorphisms. We will need the fact that α^n expands distances at an arbitrarily small exponential rate. The following proposition is well-known.

PROPOSITION 5.2. *Let ρ be a Riemannian metric on G. Let $\delta > 0$. Then there exists $C_\delta > 0$ such that, for all $n \in \mathbb{Z}$ and for all $g, h \in G$, $\rho(\alpha^n g, \alpha^n h) \leqslant C_\delta (1+\delta)^{|n|} \rho(g, h)$.*

Proof (sketch). Let $\|\cdot\|_e$ denote the norm on the tangent space at $e \in G$, identified with the Lie algebra of G. It follows from the spectral radius formula that $\|d\alpha^n X\|_e \leqslant C^+_\delta (1+\delta)^n \|X\|_e$ for $n \geqslant 0$, for some constant $C^+_\delta > 0$. From the definition of the Riemannian metric, it is easy to check that $\rho(\alpha^n g, \alpha^n h) \leqslant C^+_\delta (1+\delta)^n \rho(g, h)$ for $n > 0$. A similar argument yields the same for $n < 0$. □

Before proving Theorem A we need some preliminary estimates.

Let $f, g \in C^\theta(G)$. Recall that λ is the exponential rate of contraction and expansion of ϕ on stable and unstable manifolds. Choose $\delta > 0$ such that

$$(1+\delta)\lambda^\theta < 1; \tag{7}$$

this is possible as both $\lambda, \theta \in (0, 1)$.

For $z \in W^s_\varepsilon(x)$ and $n \geqslant 0$, define $F^n(z, x) = \alpha^{-n}(f^n(z)^{-1} f^n(x))$. Similarly, define g^n and G^n.

Let ρ be a bi-invariant metric on G.

LEMMA 5.3. *Let $z \in W^s_\varepsilon(x)$. Then*

(1) *$F(z, x) = \lim_{n\to\infty} F^n(z, x)$ exists;*

(2) *$\rho(F(z, x), e) \leqslant \mathrm{Const.}\, d(x, z)^\theta$, where the constant is independent of x, z.*

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Proof. We prove part (1). Using the identity $f^{n+1}(x) = f(\phi^n x)\cdot \alpha f^n(x)$, Proposition 5.2 and the bi-invariance of the metric ρ on G, it is a simple matter to estimate

$$\rho(F^{n+1}(z,x), F^n(z,x)) \leqslant \text{Const.}\,((1+\delta)\,\lambda^\theta)^n\, d(x,z)^\theta$$

where the constant is independent of x, z and n. As $(1+\delta)\,\lambda^\theta < 1$, this estimate tends to zero exponentially fast as $n \to \infty$, and the result follows.

We prove part (2). Note that

$$\rho(F^n(z,x), e) \leqslant \sum_{j=0}^{n-1} \rho(F^{j+1}(z,x), F^j(z,x)) \leqslant \text{Const.}\,((1+\delta)\,\lambda^\theta)^n\, d(x,z)^\theta$$

by the above. Letting $n \to \infty$ and observing that by inequality (7) the sum is finite, the result follows. □

Proof of Theorem A (*for a hyperbolic diffeomorphism*). Suppose that $f, g \in C^\theta(G)$ are as above and u is a measurable function such that $f = u\phi\cdot g\cdot \alpha u^{-1}$ almost everywhere. Let $\pi : \Sigma \longrightarrow \Lambda$ be a semi-conjugacy as described in Proposition 2.1. Let $\hat{f} = f\pi$, $\hat{g} = g\pi$ and $\hat{u} = u\pi$. Then $\hat{f}, \hat{g} \in F_\lambda^\theta(G)$ and $\hat{u}$ is measurable. Moreover, $\hat{f} = \hat{u}\sigma\cdot\hat{g}\cdot\alpha\hat{u}^{-1}$ almost everywhere (with respect to a Hölder equilibrium state for σ). By Theorem A in the case of a shift of finite type, we can find a Hölder continuous $\hat{u}' \in F_\lambda^\theta(G)$ such that $\hat{u}' = \hat{u}$ almost everywhere and $\hat{f} = \hat{u}'\sigma\cdot\hat{g}\cdot\alpha\hat{u}'^{-1}$ everywhere. We show that $\hat{u}' = u'\pi$ for some $u' : \Lambda \longrightarrow G \in C^\theta(G)$.

Recall from §2 that a point $x \in \Lambda$ is called *good* if it has a unique pre-image under π. Let $x, y \in \Lambda$ be good periodic points. Suppose that $\pi\hat{x} = x$, $\pi\hat{y} = y$. Define u' on the orbit of x by $u'(\phi^n x) = \hat{u}'(\sigma^n x)$, and similarly define u' on the orbit of y. Thus we have defined u' on a dense subset of Λ. Suppose that x and y are sufficiently close so that $z = \langle x, y\rangle$ is defined. Then z is good and we may define u' on the orbit of z as above. When u' is defined, clearly we have $f = u'\phi\cdot g\cdot\alpha u'^{-1}$. We claim that u' extends to a Hölder continuous function defined on Λ. It then follows that the cocycle equation holds everywhere for u'. Moreover, $u' = u$ almost everywhere as π is one-to-one on a set of full measure. Thus Theorem A is proved.

It remains to check that u' does indeed extend to Λ. We claim that the following holds:

$$F(z,x) = u'(z)\, G(z,x)\, u'(x)^{-1}. \tag{8}$$

To see this, note that

$$\begin{aligned} F^n(z,x) &= \alpha^{-n}(f^n(z)^{-1} f^n(x)) \\ &= u'(z)\,\alpha^{-n}(g^n(z)^{-1}\, u'(\phi^n z)^{-1}\, u'(\phi^n x)\, g^n(x))\, u'(x)^{-1}. \end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned} &\rho(G^n(z,x), \alpha^{-n}(g^n(z)^{-1}\, u'(\phi^n z)^{-1}\, u'(\phi^n x)\, g^n(x))) \\ &\qquad = \rho(\alpha^{-n}(g^n(z)^{-1} g^n(x)), \alpha^{-n}(g^n(z)^{-1}\, u'(\phi^n z)^{-1}\, u'(\phi^n x)\, g^n(x))) \\ &\qquad \leqslant C_\delta(1+\delta)^n\, \rho(u'(\phi^n z), u'(\phi^n x)) \\ &\qquad = C_\delta(1+\delta)^n\, \rho(\hat{u}'(\sigma^n \tilde{z}), \hat{u}'(\sigma^n \tilde{x})) \\ &\qquad \leqslant C_\delta(1+\delta)^n\, |\hat{u}|_{\lambda^\theta}\, \lambda^{\theta n}, \end{aligned}$$

and this tends to zero as $n \to \infty$ by the assumption on δ. Letting $n \to \infty$ in equation (9) yields equation (8).

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Define

$$F'(z,y) = \lim_{n\to\infty} \alpha^n(f^{-n}(z)^{-1} f^{-n}(y));$$

this makes sense as $z \in W^u_\varepsilon(y)$. Then all of the above estimates on F and G continue to hold for F' and G'. In particular,

$$F'(z,y) = u'(z)\, G'(z,y)\, u'(y)^{-1}. \tag{10}$$

Combining equation (10) with equation (8) gives

$$u'(y) = F'(z,y)^{-1} F(z,x)\, u'(x)\, G(z,x)^{-1} G'(z,y).$$

By Lemma 5.3.2 (and its analogues for F' and G') and the bi-invariance of ρ, we have

$$\begin{aligned}\rho(u'(x),u'(y)) &\leqslant \rho(F(z,x),e)+\rho(G(z,x),e)+\rho(F'(y,z),e)+\rho(G'(y,z),e)\\ &\leqslant \text{Const.}\,(d(x,z)^\theta + d(y,z)^\theta)\end{aligned}$$

for some constant independent of x, y, z. By [**7**, Proposition 19.1.1], u' is Hölder continuous of exponent θ on the dense set of 'good' periodic orbits. Hence u' extends (uniquely) to a function $u' \in C^\theta(G)$. This completes the proof of Theorem A. □

We give examples to show that the unipotency hypothesis is optimal in the following sense. Let α be a non-unipotent automorphism of G. Then (up to a local isomorphism in G) there exists a hyperbolic ϕ, Hölder f, g and a measurable u such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u does not have a continuous version.

By Proposition 4.4, we may assume that G is a torus. Moreover, by taking a subtorus, we may assume that α is either hyperbolic or quasi-hyperbolic.

In the hyperbolic case, we use the following example due to Lind [**8**]. Take $\phi = \alpha$, so that ϕ is an Anosov toral automorphism. Take $f(x) = x$. By Theorem 1.3, $f = u\phi \cdot \alpha u^{-1}$ almost everywhere for some measurable u. Hence the maps ϕ_f and $\phi \times \alpha$ are metrically isomorphic. Moreover, ϕ_f and $\phi \times \alpha$ can be represented by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ \mathrm{id} & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

respectively (here, 'id' denotes the $k \times k$ identity matrix). If u had a continuous version then ϕ_f and $\phi \times \alpha$ would be topologically conjugate, and hence, by a result of Adler and Palais [**1**], algebraically conjugate. However, this is not the case as ϕ_f and $\phi \times \alpha$ have (generalised) eigenspaces of different dimensions.

We now consider the quasi-hyperbolic case, when α has an eigenvalue of modulus 1. Such an α is still ergodic, so Theorem 1.3 still applies. We give an example of an f that cannot be written as $f = u\phi \cdot \alpha u^{-1}$ for a continuous u.

We use multiplicative notation. Let K denote the unit circle in $\mathbb{C}$. Let ϕ be a hyperbolic automorphism of the m-torus $\Lambda = K^m$. Let α be a quasi-hyperbolic automorphism of K^n. Recall that a K^n-valued function is null-homotopic if it lifts to $\mathbb{R}^n$. The key observation is as follows.

SUB-LEMMA. *Suppose that $f = u\phi \cdot \alpha u^{-1}$ where $f, u : K^m \longrightarrow K^n$ are continuous and ϕ, α have disjoint spectra. If f is null-homotopic then u is null-homotopic.*

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Suppose that f is null-homotopic and that $f = u\phi \cdot \alpha u^{-1}$ for some continuous u. As u is null-homotopic, we can lift this equation to $\mathbb{R}^n$ to obtain (after possibly subtracting an integer)

$$F = U\phi - \alpha U \tag{11}$$

where F is a lift of f, U is a lift of u and α is viewed as a linear map on $\mathbb{R}^n$. Conversely, if $F, U : \Lambda \longrightarrow \mathbb{R}^n$ are continuous and satisfy equation (11), then we can obtain a continuous solution $u = e^{2\pi i U}$ to $\exp(2\pi i F) = u\phi \cdot \alpha u^{-1}$. Therefore, it is sufficient to choose a Hölder $F : \Lambda \longrightarrow \mathbb{R}^n$ such that equation (11) does not have a continuous solution U.

Let $\mu = e^{2\pi i\theta}$ be an eigenvalue of α of unit modulus. By diagonalising equation (11) and considering the eigenspace of μ, we obtain a solution to

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} U_1\phi \\ U_2\phi \end{pmatrix} - R(\theta)\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{12}$$

where $F_i, U_i : \Lambda \longrightarrow \mathbb{R}$ and $R(\theta)$ denotes the rotation of $\mathbb{R}^2$ by θ. Alternatively, equation (12) can be viewed as an equation in $\mathbb{C}$:

$$\hat{F} = \hat{U}\phi - \mu\hat{U}. \tag{13}$$

Let $\eta \in \hat{\Lambda}$ be an arbitrary non-zero character of Λ. Define $\hat{F} = \eta : \Lambda \longrightarrow \mathbb{C}$, and observe that we can recover a Hölder continuous (indeed analytic) function $F : \Lambda \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ by taking real and imaginary parts. Write $\hat{U} = \sum_{\gamma \in \hat{\Lambda}} u_\gamma \gamma$ where $u_\gamma \in \mathbb{C}$ and $\sum_{\gamma \in \hat{\Lambda}} |u_\gamma|^2 < \infty$. Then equation (13) becomes

$$\eta = \sum_{\gamma \in \hat{\Lambda}} u_{\gamma\phi^{-1}}\gamma - \mu \sum_{\gamma \in \hat{\Lambda}} \gamma$$

so that

$$u_{\gamma\phi^{-1}} = \begin{cases} \mu u_\gamma & \gamma \neq \eta, \\ 1 + \mu u_\gamma & \gamma = \eta. \end{cases} \tag{14}$$

Hence $u_{\eta\phi^{-p}} = \mu^{p-1}(1 + \mu u_\eta)$ for $p > 0$ so that $|u_{\eta\phi^{-p}}| = |1 + \mu u_\eta|$. As ϕ is hyperbolic, and in particular ergodic, all the $\eta\phi^{-p}$ are distinct. Hence $u_\eta = -\mu^{-1}$. However, it also follows from equation (14) that for $p \geqslant 0$, $u_\eta = \mu^p u_{\eta\phi^p}$, whence $|u_{\eta\phi^p}| = |u_\eta|$ for $p \geqslant 0$. Hence $u_\eta = 0$, a contradiction.

6. *Higher regularity results*

In this section we prove Theorem B. Let ϕ be a C^r (transitive) Anosov diffeomorphism M. Let $f, g : M \longrightarrow G$ be C^r and such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ almost everywhere (with respect to a Hölder equilibrium state) where α is unipotent. By Theorem A, we may assume that $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere and u is Lipschitz.

We will show that u possesses some global degree of differentiability by showing that it is sufficiently differentiable along the stable and unstable manifolds of ϕ. That this is sufficient follows from the following result.

PROPOSITION 6.1 [**6**, **10**]. *Let ϕ be a C^r Anosov diffeomorphism on M ($1 \leqslant r \leqslant \infty$ or $r = \omega$). Let $f : M \longrightarrow U \subset \mathbb{R}^n$ be uniformly C^r on stable and unstable manifolds. Then f is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.*

This result was first proved for an Anosov diffeomorphism in the smooth case in [**11**]. Another proof of the smooth case, as well as a C^r version, can be found in [**5**].

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The techniques of [**5**] were used in [**10**] to deduce the analytic case. The proof of the C^r case given in [**6**] has the advantage of being completely elementary: it only uses the facts that the stable and unstable manifolds are uniformly transverse, continuously varying C^r sub-manifolds.

In [**12**] the analogue of Theorem B is proved in the case where $\alpha = \mathrm{id}$ (indeed, they prove regularity results for cocycles taking values in a diffeomorphism group). This was done by viewing the associated skew product as a partially hyperbolic system. Recall that a diffeomorphism $\Phi: M \longrightarrow M$ is *partially hyperbolic* if the tangent bundle TM has a continuous splitting $TM = E^s \oplus E^c \oplus E^u$ into $d\Phi$-invariant sub-bundles for which there exist constants $C > 0$, $0 < \lambda < 1 \leqslant \mu$ such that $\lambda\mu < 1$ and

$$\|d\Phi^n|_{E^s}\| \leqslant C_1 \lambda^n,$$
$$\|d\Phi^{-n}|_{E^u}\| \leqslant C_2 \lambda^n,$$
$$C_3 \mu^{-n} \leqslant \|d\Phi^n|_{E^c}\| \leqslant C_4 \mu^n,$$

for all $n > 0$ and some constants $C_1, \ldots, C_4 > 0$. Hence $d\Phi$ expands E^u more sharply than it expands E^c, and contracts E^s more sharply than it contracts E^c.

The following proposition lists the properties of partially hyperbolic diffeomorphisms that we shall need. The case when $1 \leqslant r \leqslant \infty$ follows from [**4**, Theorem 5.5] and the analytic case can be found in [**12**].

PROPOSITION 6.2. *Let Φ be a C^r partially hyperbolic diffeomorphism.*

(1) *The distributions E^s, E^u are integrable. We call the resulting manifolds the* stable *and* unstable manifolds, *respectively. They are C^r submanifolds and depend continuously on x in the C^r topology on embeddings.*

(2) *The stable manifold $W^s(x)$ through x is characterised as follows. We have $y \in W^s(x)$ if and only if for some ν with $\lambda < \nu < \mu^{-1}$ we have*

$$\frac{d(\Phi^n x, \Phi^n y)}{\nu^n} \to 0$$

as $n \to \infty$. A similar criterion holds for unstable manifolds.

As in the introduction, we define for $f: M \longrightarrow G$ the diffeomorphism $\phi_f: M \times G \longrightarrow M \times G: (x, y) \longmapsto (\phi x, f(x)\,\alpha(y))$. By using [**3**, Theorem 2.2] and the unipotency assumption on α we have the following proposition.

PROPOSITION 6.3. *Let $f: M \longrightarrow G$ be C^r. Then ϕ_f is a C^r partially hyperbolic diffeomorphism and we can take $\mu = 1 + \delta$ in the definition of partial hyperbolicity, for any small $\delta > 0$. Moreover, if $W_f^s(x_0, y_0)$ denotes the stable manifold of ϕ_f through $(x_0, y_0) \in M \times G$ then $\pi W_f^s(x_0, y_0) = W^s(x_0)$, the stable manifold of ϕ through x. Hence we can write*

$$W_f^s(x_0, y_0) = \{(x, \gamma^f_{(x_0, y_0)}(x)) \mid x \in W^s(x_0)\}$$

where $\gamma^f_{(x_0, y_0)}: M \longrightarrow G$ is a C^r function whose first r derivatives depend continuously on (x_0, y_0).

Proof of Theorem B. Define $U: M \times G \longrightarrow M \times G: (x, y) \longmapsto (x, u(x)\,y)$. Then clearly U is Lipschitz and $U\phi_g = \phi_f U$.

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We claim that $UW_g^s(x_0, y_0) \subset W_f^s(U(x_0, y_0))$. To see this, let $(x, y) \in W_g^s(x_0, y_0)$. Then if $(1+\delta)\lambda < \nu < 1$ we have

$$\begin{aligned}\frac{d(\phi_f^n U(x,y), \phi_f^n U(x_0,y_0))}{\nu^n} &= \frac{d(U\phi_g^n(x,y), U\phi_g^n(x_0,y_0))}{\nu^n} \\ &\leqslant |U|_{\mathrm{Lip}} \frac{d(\phi_g^n(x,y), \phi_g^n(x_0,y_0))}{\nu^n} \\ &\to 0\end{aligned}$$

as $n \to \infty$. By Proposition 6.2.2, $U(x, y) \in W_f^s(U(x_0, y_0))$.

Take $(x, y) \in W_g^s(x_0, y_0)$. Then

$$(x, y) = (x, \gamma^g_{(x_0, y_0)}(x)). \tag{15}$$

Moreover, as $U(x, y) \in W_f^s(U(x_0, y_0))$, we can write

$$(x, u(x)\, y) = (x, \gamma^f_{U(x_0, y_0)}(x)). \tag{16}$$

Combining equations (15) and (16) we see that

$$u(x) = \gamma^f_{U(x_0, y_0)}(x)\, \gamma^g_{(x_0, y_0)}(x)^{-1}$$

and the right-hand side is uniformly C^r along stable manifolds. As is remarked above, this suffices to prove the theorem. □

REMARKS. (1) The assumption that the Anosov diffeomorphism is transitive is only used to deduce that measurable solutions are Lipschitz.

(2) Let α be an automorphism of G such that the spectrum of $d\alpha$ is contained in the annulus $\{z \in \mathbb{C} \mid \lambda < |z| < \lambda^{-1}\}$. Let f, g be C^r functions and suppose that $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere for some $u \in C^\theta(G)$, $\theta > \log \mu / \log \lambda^{-1}$. Then the above proof shows that u is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.

7. *A counter-example*

Consider the coboundary equation $f = u\phi \cdot u^{-1}$ almost everywhere, where u is measurable. In [**14**] it is proved that u must be Hölder when G is a connected Lie group (provided that f satisfies a partial hyperbolicity assumption on the semi-simple part of G). For the equation $f = u\phi \cdot g \cdot u^{-1}$ almost everywhere taking values in a nilpotent Lie group, u must again be Hölder [**14**]. We give an example to show that this is false for soluble Lie groups.

Choose a hyperbolic or quasi-hyperbolic toral automorphism $\alpha \in \mathrm{Aut}(\mathbb{R}^k/\mathbb{Z}^k)$. Using the examples at the end of §5, choose a hyperbolic diffeomorphism ϕ and analytic functions $f, g : \Lambda \to \mathbb{R}^k/\mathbb{Z}^k$ for which there exists a measurable solution u: $\Lambda \longrightarrow \mathbb{R}^k/\mathbb{Z}^k$ to $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u has no continuous version. We may take $g = 0$.

Let G denote the semi-direct product $\mathbb{Z} \ltimes \mathbb{R}^k/\mathbb{Z}^k$ with group structure given by $(n, g) + (m, h) = (n+m, g + \alpha^n h)$. Then G is soluble.

Define $\bar{f} = (1, f)$, $\bar{g} = (1, 0)$, $\bar{u} = (1, u) : \Lambda \longrightarrow G$. Then $\bar{f}, \bar{g}$ are analytic and $\bar{u}$ is measurable. Moreover, one can easily check that $\bar{f} = \bar{u}\phi \cdot \bar{g} \cdot \bar{u}^{-1}$ almost everywhere and by the assumption on u, $\bar{u}$ has no continuous version.

REMARKS. (1) Suppose that G_1 and G_2 are Lie groups (not necessarily compact) such that G_1 acts smoothly on G_2 by automorphisms. Suppose that the Livšic

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regularity theorem (Theorem 1.1) holds for the coboundary equation taking values in G_1 and G_2. Then one can show that the regularity theorem for the coboundary equation continues to hold for the semi-direct product $G_1 \ltimes G_2$.

(2) In [**12**], an example is given (based on one due to de la Llave) of an $Gl(d, \mathbb{R})$-valued cocycle equation $f = u\phi \cdot g \cdot u^{-1}$ where f, g are smooth, and u is continuous but u is not C^r.

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References

1. R. L. Adler and R. Palais, 'Homeomorphic conjugacy of automorphisms on the torus', *Proc. Amer. Math. Soc*. 16 (1965) 1222–1225.
2. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470 (Springer, Berlin, 1975).
3. M. I. Brin and Y. B. Pesin, 'Partially hyperbolic dynamical systems', *Izv. Math*. 8 (1974) 177–218.
4. M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics 583 (Springer, Berlin, 1977).
5. S. Hurder and A. Katok, 'Differentiability, rigidity and Godbillon–Vey classes for Anosov flows', *Publ. Math. I.H.E.S*. 72 (1990) 5–61.
6. J.-L. Journé, 'A regularity lemma for functions of several variables', *Rev. Mat. Iberoamericana* 4 (1988) 187–193.
7. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems* (Cambridge University Press, 1995).
8. D. A. Lind, 'Split skew products, a related functional equation, and specification', *Israel J. Math*. 30 (1978) 236–254.
9. A. N. Livšic, 'Cohomology of dynamical systems', *Izv. Math*. 6 (1972) 1278–1301.
10. R. de la Llave, 'Analytic regularity of solutions of Livšic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems', *Ergodic Theory Dynam. Systems* 17 (1997) 649–662.
11. R. de la Llave, J. M. Marco and E. Moriyon, 'Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomological equation', *Ann. of Math*. 123 (1986) 537–611.
12. V. Niţică and A. Török, 'Regularity of the coboundary for cohomologous cocycles', *Ergodic Theory Dynam. Systems* 18 (1998) 1187–1209.
13. W. Parry and M. Pollicott, 'The Livšic cocycle equation for compact Lie group extensions of hyperbolic systems', *J. London Math. Soc*. 56 (1997) 405–416.
14. M. Pollicott and C. P. Walkden, 'The measurable Livšic theorem for connected Lie groups', preprint, Manchester University, 1998.
15. K. Schmidt, 'Remarks on Livšic theory for nonabelian cocycles', *Ergodic Theory Dynam. Systems* 19 (1999) 703–721.
16. J. Tits, 'Liesche Gruppen und Algebren', lecture notes, Mathematisches Institut, Bonn, 1965.

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