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On a Tauberian theorem for absolute harmonic summability

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1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let the sequence $\{t_n^{(K)}\}$ be defined by

$$t_n^{(K)} = \frac{1}{P_n^{(K)}} \sum_{\mu=0}^{n} p_{n-\mu}^{(K)} S_\mu \quad (P_n^{(K)} = p_0^{(K)} + p_1^{(K)} + \ldots + p_n^{(K)}), \tag{1·1}$$

where $p_n^{(K)}$ is given by

$$\sum_{n=0}^{\infty} p_n^{(K)} x^n = \left(\sum_0^{\infty} \frac{x^n}{n+1}\right)^K \quad (|x| < 1). \tag{1·2}$$

The series Σa_n is defined to be summable $|N, K|$, if the series

$$\sum_n |t_n^{(K)} - t_{n-1}^{(K)}| \quad \text{is convergent.}$$

Mohanty(3) has given the following criterion for the absolute convergence of a Fourier series at a point, which is analogous to Hardy and Littlewood's(1) criterion for the convergence of a Fourier series.

THEOREM A. *If* (i) $\phi(t)\log(r/t)$ *is of bounded variation in* $(0, \pi)$ *and* (ii) $\{n^\delta A_n(x)\}$, $0 < \delta < 1$, *is a sequence of bounded variation, then* $\Sigma A_n(x)$ *is absolutely convergent.*

To prove this theorem Mohanty first proves that the condition (i) alone ensures the absolute Riesz summability of type $\exp(\omega^\delta)$, $0 < \delta < 1$, of the series $\Sigma A_n(x)$. The condition (ii) is then the appropriate Tauberian condition to ensure absolute convergence.

The object of this paper is to give an indirect proof of Theorem A, by first proving a Tauberian theorem for summability $|N, K|$ and then combining it with the following known theorem.

THEOREM B(6). *If* $\phi(t)\log(r/t)$ *is of bounded variation in* $(0, \pi)$ *then the Fourier series of* $f(t)$ *at* $t = x$, *is summable* $|N, K|$, $K \geqslant 2$.

We note that a similar indirect proof of Hardy and Littlewood's criterion for the convergence of Fourier series was given by Iyengar(2) and Rajagopal(4). We prove

THEOREM 1. *If the series* Σa_n *is summable* $|N, K|$ $(K \geqslant 1)$ *and if the sequence* $\{n^\delta a_n\}$ *is of bounded variation for some* δ $(0 < \delta < 1)$ *then the series* Σa_n *is absolutely convergent.*

2. Since we will be concerned with a fixed K, we will write

$$p_n = p_n^{(K)}, \quad P_n = P_n^{(K)} \quad \text{and} \quad t_n = t_n^{(K)}.$$

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Let c_n be defined by $$\left(1-\sum_1^\infty c_n x^n\right)\left(\sum_0^\infty p_n x^n\right) = 1, \tag{2·1}$$

from which it follows that

$$p_n = \sum_{\mu=1}^{n} c_\mu p_{n-\mu}, \quad P_n = 1+\sum_{\mu=1}^{n} c_\mu P_{n-\mu}. \tag{2·2}$$

In our case we have

$$c_n = O\left(\frac{1}{n\{\log(n+1)\}^{K+1}}\right), \quad \sum_1^\infty c_n = 1; \tag{2·3}$$

$$p_n = O\left(\frac{\log^{K-1} n}{n}\right), \quad P_n = O(\log^K n), \tag{2·4}$$

$$p_n > A_1\left(\frac{\log^{K-1} n}{n}\right) \quad (A_1 > 0); \quad P_n > A_2(\log^K n), \quad (A_2 > 0). \tag{2·4'}$$

Proofs of (2·3) and (2·4) are known ((2), (5)). The first inequality in (2·4'), which also yields the second inequality, is proved by induction on K, using

$$p_n^{(K+1)} = \frac{p_n^{(K)}}{1}+\frac{p_{n-1}^{(K)}}{2}+\ldots+\frac{p_0^{(K)}}{n+1}.$$

Making use of (1·1) and (2·1), we may write

$$S_n = P_n t_n-(c_1 P_{n-1} t_{n-1}+c_2 P_{n-2} t_{n-2}+\ldots+c_n P_0 t_0). \tag{2·5}$$

Furthermore, A will denote an absolute positive constant. We will use the following notations:

$$\alpha_0 = 1 = -c_0, \quad \alpha_1 = -1-c_1, \quad \alpha_n = c_{n-1}-c_n \quad (n > 1);$$

$$-d_n = c_0+c_1+\ldots+c_n \quad (n \geqslant 0),$$

$$\nu = [\tfrac{1}{2}n], \quad m = [n^{\frac{1}{3}\delta}];$$

$$L_n = \frac{1}{n\{\log(n+1)\}^2}.$$

3. We shall need the following relations

$$\left.\begin{aligned} &(a)\,\Delta p_n = O\left(\frac{\log^{K-1} n}{n^2}\right), \quad (b)\,\Delta(np_n) = O(\log^{K-2} n/n), \\ &(c)\,\Delta^2(np_n) = O(\log^{K-2} n/n^2) \quad (K \geqslant 2); \end{aligned}\right\} \tag{3·1}$$

$$\left|\sum_{\mu=0}^{n}(n-\mu+1)p_{n-\mu}\alpha_\mu\right| = O\left(\frac{1}{n\{\log(n+1)\}^2}\right) \quad (K \geqslant 1) \tag{3·2}$$

Relations (3·1 b) and (3·1 c) may be proved by differentiating (1·2) and multiplying respectively by $(x-1)$ and $(x-1)^2$ and then comparing the coefficient of x^n. (3·1 a) follows from (3·1 b).

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To prove (3·2) we use Abel's transformation twice. We have for $K \geqslant 2$

$$\begin{aligned}
\left|\sum_{\mu=0}^{n}(n-\mu+1)p_{n-\mu}\alpha_\mu\right| &= \left|\sum_{\mu=0}^{n}(n-\mu)p_{n-\mu}\alpha_\mu+\sum_{\mu=0}^{n}\alpha_\mu p_{n-\mu}\right| \\
&= \left|\sum_{\mu=0}^{n-1}-c_\mu\Delta\{(n-\mu)p_{n-\mu}\}-\sum_{\mu=0}^{n-1}c_\mu\Delta p_{n-\mu}-c_n\right| \\
&= \left|\left[\sum_{\mu=0}^{\nu+1}+\sum_{\nu+2}^{n-1}\right][-c_\mu\Delta\{(n-\mu)p_{n-\mu}\}]-\left[\sum_{\mu=0}^{\nu}+\sum_{\nu+1}^{n-1}\right][c_\mu\Delta p_{n-\mu}]-c_n\right| \\
&= \left|\sum_{\mu=0}^{\nu}d_\mu\Delta^2\{(n-\mu)p_{n-\mu}\}+d_{\nu+1}\Delta\{(n-\nu-1)p_{n-\nu-1}\}\right. \\
&\quad \left.+\sum_{\mu=\nu+2}^{n-1}-c_\mu\Delta\{(n-\mu)p_{n-\mu}\}-\sum_{\mu=0}^{\nu}c_\mu\Delta p_{n-\mu}-\sum_{\mu=\nu+1}^{n-1}c_\mu\Delta p_{n-\mu}-c_n\right| \\
&= O\left(\frac{1}{n\{\log(n+1)\}^2}\right).
\end{aligned}$$

For $K = 1$, we have

$$\left|\sum_{\mu=0}^{n}(n-\mu+1)p_{n-\mu}\alpha_\mu\right| = \left|\sum_{\mu=0}^{n}\alpha_\mu\right| = |-c_n| = O\left(\frac{1}{n\{\log(n+1)\}^2}\right).$$

4. We shall require the following lemmas:

LEMMA 1. *If the series, Σa_n is summable $|N, K|$, then*

$$\sum_n \frac{|\sigma_{n-1}|}{n+1} = \sum_n \frac{1}{n+1}\left|\sum_{i=0}^{n-1}\frac{p_i(n-i)a_{n-i}}{P_{n-1}}\right| < \infty. \tag{4·1}$$

Proof. Using (1·1) and (2·5), we have

$$\begin{aligned}
t_n - t_{n-1} &= \sum_{i=0}^{n}\left(\frac{p_{n-i}}{P_n}-\frac{p_{n-i-1}}{P_{n-1}}\right)S_i \\
&= \sum_{i=0}^{n}\left(\frac{P_{n-i}}{P_n}-\frac{P_{n-i-1}}{P_{n-1}}\right)a_i \\
&= \frac{1}{P_nP_{n-1}}\left\{\sum_{i=0}^{n-1}(P_np_i-p_nP_i)a_{n-i}\right\} \\
&= \frac{1}{(n+1)}\sum_{i=0}^{n-1}\frac{(n-i)p_i}{P_{n-1}}a_{n-i}+\frac{1}{P_nP_{n-1}}\sum_{i=0}^{n-1}\left\{P_np_i\left(\frac{1+i}{1+n}\right)-p_nP_i\right\}a_{n-i} \\
&= \frac{\sigma_{n-1}}{n+1}+\frac{1}{(n+1)P_{n-1}}\sum_{i=0}^{n}p_i(1+i)a_{n-i}-\frac{p_nt_n}{P_{n-1}} \\
&= \frac{\sigma_{n-1}}{n+1}+\rho_n, \quad \text{say,}
\end{aligned} \tag{4·2}$$

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where

$$|\rho_n| = \left|\frac{1}{(n+1)P_{n-1}}\sum_{i=0}^{n} p_i(1+i)\,a_{n-i} - \frac{p_n t_n}{P_{n-1}}\right|$$

$$= \left|\frac{1}{(n+1)P_{n-1}}\sum_{i=0}^{n}(n-i+1)\,p_{n-i}a_i - \frac{p_n t_n}{P_{n-1}}\right|$$

$$= \left|\frac{1}{(n+1)P_{n-1}}\left\{\sum_{i=0}^{n}(n-i+1)\,p_{n-i}\left(\sum_{\mu=0}^{i}P_\mu t_\mu \alpha_{i-\mu}\right) - \frac{p_n t_n}{P_{n-1}}\right|$$

$$= \left|\frac{1}{(n+1)P_{n-1}}\left[\sum_{\mu=0}^{n}P_\mu(t_\mu - t_n)\left\{\sum_{i=\mu}^{n}p_{n-i}(n-i+1)\,\alpha_{i-\mu}\right\}\right]\right|$$

$$= \left|\frac{1}{(n+1)P_{n-1}}\left[\sum_{\mu=0}^{n}P_{n-\mu}(t_{n-\mu}-t_n)\left(\sum_{i=0}^{\mu}p_{\mu-i}(\mu-i+1)\,\alpha_i\right)\right]\right|$$

$$< \frac{A}{(n+1)P_{n-1}}\left[\left(\sum_{\mu=1}^{\nu+1}+\sum_{\nu+2}^{n}\right)L_\mu P_{n-\mu}|t_{n-\mu}-t_n|\right]$$

$$= \rho_n(1)+\rho_n(2), \quad \text{say.} \tag{4·3}$$

Since the sequence $\{t_n\}$ is of bounded variation, we obtain

$$\sum_n |\rho_n(1)| = A\sum_n \frac{1}{(n+1)P_{n-1}}\left\{\sum_{\mu=1}^{\nu+1}L_\mu P_{n-\mu}|t_n - t_{n-\mu}|\right\}$$

$$< A\sum_n \frac{1}{(n+1)}\left\{\sum_{\mu=1}^{\nu+1}L_\mu\left(\sum_{i=n-\mu}^{n-1}|\Delta t_i|\right)\right\}$$

$$< A\sum_n \frac{1}{(n+1)}\left\{\sum_{i=n-\nu-1}^{n-1}|\Delta t_i|\left(\sum_{\mu=\nu+1}^{n-i}L_\mu\right)\right\}$$

$$< A\sum_i |\Delta t_i|\left(\sum_{n=1+i}^{2i+3}\frac{1}{(n+1)}\right)$$

$$< A. \tag{4·4}$$

Again, since the sequence $\{t_n\}$ is convergent, we have

$$\sum_n |\rho_n(2)| = A\sum_n \frac{1}{(n+1)P_{n-1}}\sum_{\mu=\nu+2}^{n}L_\mu|t_n - t_{n-\mu}|\,P_{n-\mu}$$

$$< A\sum_n \frac{1}{(n+1)}\frac{L_{\nu+2}+\ldots+L_n}{1}$$

$$< A\sum_n \frac{1}{(n+1)\{\log(n+1)\}^2}$$

$$< A. \tag{4·5}$$

Combining (4·2) through (4·5), we find that

$$\sum_n \frac{|\sigma_{n-1}|}{n+1} \leqslant \sum_n |\Delta t_{n-1}| + \sum_n |\rho_n| < A,$$

in view of the fact that the sequence $\{t_n\}$ is of bounded variation.

This completes the proof of the lemma.

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LEMMA 2. *If the condition* (4·1) *holds, then*

$$\sum_n \frac{1}{(n+1)P_{n-1+m}}\left|\sum_{i=0}^{n-1} p_{i+m}(n-i)a_{n-i}\right| < \infty, \tag{4·6}$$

where $m = [n^{\frac{1}{2}\delta}]$.

Proof. Proceeding as in the case of (2·5) and using (2·2), we obtain

$$na_n = P_{n-1}\sigma_{n-1} - (c_1 P_{n-2}\sigma_{n-2} + c_2 P_{n-3}\sigma_{n-3} + \ldots + c_{n-1}P_0\sigma_0).$$

Now inserting the expression for ia_i $(i = 0, 1, \ldots, n)$, we obtain

$$\sum_{i=0}^{n} p_{m+i}(n-i)a_{n-i} = \sum_{i=0}^{n-1} A_{n,i}P_i\sigma_i,$$

where
$$A_{n,i} = c_{n-i}p_{m-1} + c_{n-i+1}p_{m-2} + \ldots + c_{n-i+m-1}p_0 \quad (i \leqslant n-1)$$
$$< A\frac{P_{m-1}}{(n-i)\log^{K+1}(n-i+1)}. \tag{4·7}$$

Now

$$\sum_n \frac{1}{(n+1)P_{n+m-1}}\left|\sum_{i=0}^{n}(n-i)a_{n-i}p_{m+i}\right| = \sum_n \frac{1}{(n+1)P_{n+m-1}}\left|\sum_{i=0}^{n-1} A_{n,i}P_i\sigma_i\right|$$
$$\leqslant \sum_i \frac{|\sigma_i|}{1+i}\left\{\sum_{n=1+i}^{\infty}\frac{(1+i)A_{n,i}P_i}{(n+1)P_{n+m-1}}\right\}. \tag{4·8}$$

Using (2·3) and (2·4), we get

$$\sum_{n=1+i}^{i+\lambda}\frac{(1+i)A_{n,i}P_i}{(n+1)P_{n+m-1}} \leqslant \sum_{n=i+1}^{i+\lambda}|A_{n,i}|$$
$$= \sum_{n=i+1}^{i+\lambda}\left|\sum_{\mu=0}^{m-1} p_\mu c_{n-i+m-1-\mu}\right|$$
$$= \sum_{n=i+1}^{i+\lambda}\left|\sum_{\mu=0}^{[\frac{1}{2}m]} p_\mu c_{n-i+m-1-\mu} + \sum_{\mu=[\frac{1}{2}m]+1}^{m-1} p_\mu c_{n-i+m-1-\mu}\right|$$
$$< A\sum_{n=i+1}^{i+\lambda} P_{[\frac{1}{2}m]}c_{n-i+[\frac{1}{2}m]} + A\sum_{n=i+1}^{i+\lambda} p_{[\frac{1}{2}m]}$$
$$< A\frac{\lambda}{i^{\frac{1}{2}\delta}}\frac{\log^K i}{\log^{K+1} i} + A\frac{\lambda}{i^{\frac{1}{2}\delta}}\log^{K-1} i.$$
$$< A, \tag{4·9}$$

where $\lambda = [(1+i)^{\frac{1}{2}\delta}]$. On the other hand using (4·7), we have

$$\sum_{n=i+\lambda+1}^{\infty}\frac{(i+1)A_{n,i}P_i}{(n+1)P_{n+m-1}} < A\sum_{n=i+\lambda+1}^{\infty}\frac{P_i}{(n-i)\log^{K+1}(n-i+1)} < A. \tag{4·10}$$

Collecting (4·8), (4·9) and (4·10) and using (4·1), we find that (4·6) is proved.

5. *Proof of Theorem* 1. Let us write $G_n = n^\delta a_n$. Our hypothesis, that the sequence $\{n^\delta a_n\}$ is of bounded variation, implied that

$$\sum_n |\Delta G_n| < \infty. \tag{5·1}$$

Since
$$\sum_{i=0}^{n+m} p_{n+m-i}ia_i = \sum_{i=0}^{n} p_{n+m-i}ia_i + \sum_{i=n+1}^{n+m}(ia_i - na_n)p_{n+m-i} + na_n P_{m-1},$$

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we have

$$\sum_n \frac{n|a_n|P_{m-1}}{(n+1)P_{n+m-1}} \leqslant \sum_n \frac{|\sigma_{n+m-1}|}{(n+1)} + \sum_n \frac{1}{(n+1)}\left|\sum_{i=0}^{n} \frac{ia_i p_{n+m-i}}{P_{n+m-1}}\right| + \sum_n \frac{1}{(n+1)}\left|\sum_{i=n+1}^{n+m} \frac{(ia_i - na_n)p_{n+m-i}}{P_{n+m-1}}\right|. \quad (5{\cdot}2)$$

In view of Lemmas 1, 2 and (2·4′) the proof of our theorem will be complete, if we show that under the hypothesis of our theorem

$$\begin{aligned}\sum_n \frac{1}{(n+1)P_{n+m-1}}&\left|\sum_{i=n+1}^{n+m}(ia_i - na_n)p_{n+m-i}\right| \\ &\leqslant \sum_n \frac{1}{(n+1)P_{n+m-1}}\left|\sum_{i=n+1}^{n+m} p_{n+m-1}G_n(i^{1-\delta}-n^{1-\delta})\right| \\ &\qquad + \sum_n \frac{1}{(n+1)P_{n+m-1}}\left|\sum_{i=n+1}^{n+m} p_{n+m-i}\,i^{1-\delta}(G_i - G_n)\right| \\ &= S_1(n) + S_2(n) \\ &< \infty. \end{aligned} \quad (5{\cdot}3)$$

Since the sequence $\{G_n\}$ is bounded, we have

$$\begin{aligned} S_1(n) &< A\sum_n \frac{1}{(n+1)P_{n+m-1}}\left\{\sum_{i=n+1}^{n+m} p_{n+m-i}(i^{1-\delta}-n^{1-\delta})\right\} \\ &< A\sum_n \left\{\frac{(n+m)^{1-\delta}-n^{1-\delta}}{(n+1)P_{n+m-1}}\right\}P_{m-1} \\ &< A\sum_n \left\{\frac{(n+m)^{1-\delta}-n^{1-\delta}}{n+1}\right\} \\ &< A\sum_n \frac{m}{(n+1)^{1+\delta}} < A\sum_n \frac{1}{(n+1)^{1+\frac{1}{2}\delta}} \\ &< A, \end{aligned} \quad (5{\cdot}4)$$

while, using (5·1), we obtain

$$\begin{aligned} S_2(n) &= \sum_n \frac{1}{(n+1)P_{n+m-1}}\left|\sum_{i=n+1}^{n+m} i^{1-\delta}(G_i - G_n)p_{m+n-i}\right| \\ &< A\sum_n \frac{1}{(n+1)^{\delta}}\frac{P_{m-1}}{P_{n+m-1}}\sum_{i=n}^{n+m-1}|\Delta G_i| \\ &< A\sum_i |\Delta G_i| \\ &< A. \end{aligned} \quad (5{\cdot}5)$$

Collecting (5·4) and (5·5) we see that the relation (5·3) holds. This completes the proof of our theorem.

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