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Non-uniform (hyper/multi)coherence spaces

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In (hyper)coherence semantics, proofs/terms are cliques in (hyper)graphs. Intuitively, the vertices represent the results of computations and the edge relation witnesses the ability to carry out the computation assembled into a single piece of data or a single (strongly) stable function, at arrow types.
In (hyper)coherence semantics, the argument of a (strongly) stable functional is always a (strongly) stable function. As a consequence, compared to the relational semantics where there is no edge relation, some vertices are missing. Recovering these vertices is essential if we are to reconstruct proofs/terms from their interpretations. It will also be useful for comparing with other semantics, such as game semantics.
Bucciarelli and Ehrhard (2001) introduced a *non-uniform coherence space semantics*, where no vertex is missing. By constructing the co-free exponential, we get a new version of this semantics, together with non-uniform versions of hypercoherences and multicoherences. This provides a new semantics in which an edge is a finite multiset. Thanks to the co-free construction, these non-uniform semantics are deterministic in the sense that the intersection of a clique and an anti-clique contains at most one vertex, which is a result of interaction, and they then extensionally collapse onto the corresponding uniform semantics.

1. Introduction

Notation

In this paper, when we refer to a multiset we will always mean a *finite* multiset. We use the notation [ ] for multisets and, as usual, the notation { } for sets. The pairwise union of multisets is denoted by a $+$ sign and, following this notation, the generalised union is denoted by a $\sum$ sign. The neutral element for this operation, the empty multiset, is denoted by []. If $k \in \mathbb{N}$, we use $k[a]$ to denote the multiset $\sum_1^k [a]$. If $[a_i \mid i \in I]$ is a multiset, its support is the set $\{a_i \mid i \in I\}$. If m is a multiset, we use $\mathrm{supp}(m)$ to denote its support. The cardinality $\sharp[a_i \mid i \in I]$ of a multiset $[a_i \mid i \in I]$ is the cardinality $\sharp I$ of the set I. The disjoint sum operation on sets is defined by setting $A + B = \{1\} \times A \cup \{0\} \times B$. Instead of the usual notation $g \circ f$ for the categorical composition of f and g, we prefer here to use the notation $f \fatsemi g$ (the reading order is driven by the arrows in diagrams).

1.1. *Strong stability and hypercoherences*

Strong stability was introduced in Bucciarelli and Ehrhard (1994) in order to give a purely 'extensional' definition of sequentiality at all types, that is, a description of sequential

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computations that does not involve the atomic description of each step of interaction between an agent (function, term) and its environment (argument or, more generally, context), as game semantics do. The results obtained in Ehrhard (1999) (and proved again, using different methods, in Longley (2002), van Oosten (1997) and Melliès (2005)) showed that strong stability does indeed correspond to sequentiality at all types. Ehrhard showed that the strongly stable semantics is the extensional collapse of the sequential algorithm semantics designed in the late 1970's by Berry and Curien (Berry and Curien 1982). Unlike the continuous or stable interpretations of PCF, the sequential algorithm interpretation (which is now better understood as a deterministic game semantics) is very 'operational' in nature: Cartwright, Curien and Felleisen showed in (Cartwright *et al.* 1994) that sequential algorithms are fully abstract (and fully complete) for the extension of PCF by a *catch and throw* mechanism.

The comparison between the strongly stable semantics with more operational interpretations was made possible by Ehrhard's discovery of *hypercoherences* (Ehrhard 1993). Moreover, the introduction of these objects simplified the presentation of the strongly stable semantics and provided a strongly stable interpretation of (second-order) linear logic. A hypercoherence is very similar to a coherence space (Girard 1987) and consists of a set, the *web*, together with a coherence relation on this web. However, in a hypercoherence, the coherence relation is not a binary relation, but a set of finite subsets of the web containing all singletons (these sets are said to be coherent). An 'element' of a hypercoherence X is then a *clique* of X, that is, a subset of the web of X that has the property that all its finite and non-empty subsets are coherent.

Hypercoherences are a semantics of linear logic, so they provide an interpretation of intuitionistic implication, which has the shape $X \Rightarrow Y = (!X) \multimap Y$ where '$\multimap$' is a linear implication and ! is a so-called 'exponential'. The basic operational intuition behind this decomposition is as follows: a linear map represents a program that uses its argument exactly once, and an element of $!X$ is obtained essentially by taking an element of X and making it available as many times as required.

The purely relational semantics is arguably the simplest semantics of linear logic. In this semantics, formulae are sets and proofs are relations. The constructions of the relational semantics underlie both the coherence space semantics and the hypercoherence semantics. Barreiro and Ehrhard traced back the introduction of the relational semantics to an unpublished remark by van de Wiele about the co-freeness of the exponentials in coherence semantics.

The hypercoherence semantics is said to be static, as opposed to games semantics, which involve a direct representation of the dynamics of computation. In game semantics, time is explicit: such semantics interpret terms by focusing on the history of an atomic interaction between a player (the program implemented by the term) and an opponent (the environment). For instance, an interaction inside a function type $A \to B$ is an interleaving of an interaction querying a piece of A data and an interaction producing a piece of B data.

There is no such reference to time in hypercoherences. For instance, the web of a linear function space is the Cartesian product of the webs.

However, the strong relation between hypercoherences and sequentiality means that the semantics carries an *implicit* representation of time.

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Melliès (2005) investigated the game theoretic counterpart of this implicit representation of time by introducing sequential games in which the coherence relation can be expressed in game terms.

In a complementary direction, we used an unfolding of hypercoherences introduced in Ehrhard (2000) to uncover the game structures of hypercoherences (Boudes 2004). For the reverse direction, the idea was to project the usual games directly onto hypercoherences by mapping the interaction history onto the results of the interaction. But some interaction histories were not mapped anywhere in the hypercoherence semantics. Indeed, not only do the representations of interactions differ between games and hypercoherences, but they do not even agree on what are the possible interactions between terms. However, one can circumvent the problem by projecting onto the relational semantics rather than hypercoherences (Boudes 2005). In the relational semantics, the representation of an interaction is the same as in hypercoherences, but there are fewer (or no) assumptions in the relational semantics about the possibilities of interactions.

1.2. *Uniformity*

The relational semantics (almost) consists of that part of the hypercoherence semantics dealing with webs, except that in hypercoherences the web of the exponentials depends on the coherence relations. To be precise, the web of $!A$ in the relational semantics is the set of multisets of elements of the web of A, while in hypercoherences, the web of $!A$ contains only multisets whose supports are a clique.

The dependence of webs on coherence is called the uniformity of the exponentials. This terminology, which is mainly used by Ehrhard and Girard, comes from the fact that in such semantics the context of an agent behaves uniformly, that is, as if this context is produced by a single agent. The hypercoherence interpretation of a term omits points relative to its relational interpretation, so the hypercoherence semantics loses information about some parts of the term. The same holds for the coherence semantics.

As a (very standard) example, consider the relational semantics of the simply typed term

$$\lambda b^{\mathbf{bool}}.\,\mathtt{if}\ b\ \mathtt{then}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{t})\ \mathtt{else}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{f}\ \mathtt{else}\ \mathtt{f}). \tag{1}$$

(where `t` stands for true and `f` for false). The relational semantics is the relation

$$\{([\mathtt{t},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{t}],\mathtt{f}),([\mathtt{f},\mathtt{f}],\mathtt{f})\}$$

but its hypercoherence semantics is just $\{([\mathtt{t},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{f}],\mathtt{f})\}$.

The hypercoherence semantics of the term trusts its environment and makes the assumption that the boolean b has a single fixed value during the time of the computation. Of course, this is reasonable from an interactive point of view since the environment complies with coherence conditions in the same way as programs do. But if we are to reconstruct terms from their semantics, some information is missing. Our example is very simple, but it is easy to imagine terms for which the part missed by the uniform interpretation contains some large sub-routines and not just constants.

Intuitively, one can think of uniformity as a technique for removing a particular kind of dead code, as in the example above. However, it is worth noting that in general,

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and especially at functional types, it is not easy to match what is lost by uniformity in coherence spaces and hypercoherences to well-identified pieces of syntax (sub-terms or sub-proofs). First, these semantics are not fully complete, so (a part of) a semantical agent does not always correspond to (a part of) a syntactical agent. Second, the uniformity restriction not only takes into account the accessibility of branches of code but also, in a more subtle manner, reachability constraints – in particular, on the copying discipline of pieces of data (see Example 2).

Non-uniform static semantics will interpret terms in exactly the same way as the relational semantics does. This will allow us to *combine* semantics in order to take advantage of their different features. For instance, we can define a semantics where proofs will be cliques in:

— the non-uniform coherence space semantics;
— the non-uniform hypercoherence semantics; and
— the non-uniform bipartite semantics we present in this paper.

The uniformity/non-uniformity issue in static semantics can be related to games where some uniformity conditions were originally designed for the exponential type: interactions in $!A$ are *deterministic* (in the sense of games) interleavings of interactions of A – see Abramsky *et al.* (1994). Recent work in the game semantics area is more permissive: conditions like this (games determinism) on the semantics of types are postponed to conditions on the semantics of terms.

1.3. *Previous (hyper)coherence semantics*

Providing a coherence space or hypercoherence semantics with non-uniform exponentials is not a trivial job. The main difficulty lies in defining the interpretation of the exponentials. One has to design a semantics where, for instance, one point of the web is incoherent with itself. This must be the case for the point $[\mathtt{t},\mathtt{f}]$ since the valid term above maps it to an incoherent piece of data $\{\mathtt{t},\mathtt{f}\}$. The situation where two different points are coherent and incoherent at the same time may also arise (see item 4 below). In coherence spaces this will mean that the semantics does not enjoy *determinism* – we will come back to this later.

Bucciarelli and Ehrhard have designed a general tool for producing non-uniform semantics (Bucciarelli and Ehrhard 2001). As observed in Girard (1996), in order to get closer to full completeness for linear logic, the coherence space semantics can be enriched by indexing each clique on a monoid. In brief, by doing this, and thanks to a clever handling of *indexes* (locations), Bucciarelli and Ehrhard showed that when this monoid comes with a *phase space* structure of a certain sort (more precisely, a symmetric phase space that is a truth-value semantics of an *indexed* linear logic), this leads to a denotational semantics of linear logic – for details see Bucciarelli and Ehrhard (2000; 2001). This leaves us with a potentially infinite number of denotational semantics of linear logic. Bruasse-Bac studied many of them in her Ph.D. thesis (Bruasse-Bac 2001), and amongst them there is one rejecting the *Mix* rule. A quite simple phase space produces an earlier version of non-uniform coherence semantics. According to a suggestion of Ehrhard and Bucciarelli,

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by generalising the construction to all arities, one obtains non-uniformity for something resembling non-uniform hypercoherence semantics. However:

(1) Each of these non-uniform semantics relates badly to the usual versions.
(2) Neither the coherence nor the hypercoherence non-uniform semantics is deterministic.
(3) As we have already observed, the non-uniform hypercoherence semantics lacks one important feature of the usual hypercoherence semantics in that not all the finite cliques of type $\mathbf{bool}^n \to \mathbf{bool}$ are *sub-definable* (that is, included in the interpretation of a term).
(4) The non-uniform coherence semantics is also a little bit puzzling. For instance, in $!\mathbf{bool}$, for any $p, q \in \mathbb{N}$ such that $p + q > 1$, the points $p[\mathtt{t}]$ and $q[\mathtt{t}]$ are both coherent and incoherent at the same time (while, once $p \neq q$, they are strictly coherent in the usual coherence spaces). So, one can find in the former non-uniform semantics a semantical agent mixing the term (1), which queries its argument twice, and the term $\lambda b.\mathtt{t}$, which does not use its argument. (This is not the case in the former hypercoherence semantics).

1.4. *The contribution of this paper*

The present work is an extended version of Boudes (2003) – some parts were also presented in my Ph.D. thesis.

Starting from the existing non-uniform (hyper)coherence semantics and mainly focusing on hypercoherences, we observed that:

(1) Unlike what happens in the usual hypercoherence semantics, there is at least one clique f of type $\mathbf{bool}^n \to \mathbf{bool}$ that is not included in the interpretation of any term. The clique f is a variant of one that was originally designed by Berry to reveal the same failure in coherence spaces[†]. It is important to note that f does not contain points related to non-uniformity, such as $[\mathtt{t}, \mathtt{f}]$. Hence the set f can be presented to the usual (multiset-based) hypercoherence semantics, which successfully refutes it.
(2) Many other definitions of the interpretation of exponentials are possible.

Amongst all the possibilities for the interpretation of exponentials, we found the *co-free* exponential (which can be thought of as an *infinite* tensor product), which led to a more satisfactory setting for both coherence spaces and hypercoherences. We also introduced a new coherence-like semantics, *viz*. multicoherences.

Summarising the results:

(1) The co-free exponential is maximal in a sense we make precise in Corollary 1, but basically means that any clique of type $\mathbf{bool}^n \to \mathbf{bool}$ would also be a clique with other variants of the exponentials. The bad news is that there still exists a clique of type $\mathbf{bool}^n \to \mathbf{bool}$ that is not included in the interpretation of any (sequential) term. But, in hypercoherences (and multicoherences), such cliques necessarily contain points arising

[†] Berry's example is often called *Gustave's function* after a private joke about the large number of French scientists whose first name is Gérard, amongst whom is Berry.

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from non-uniformity. So this phenomenon is now constrained to the non-uniform web and disappears when restricting to the uniform web.

(2) Our non-uniform semantics are *deterministic* in the sense that the intersection of a clique (say of type A) and an anti-clique (a clique of the dual type $A^{\perp}$) contains at most one point, as in the uniform semantics. We call the points at the intersection of a clique and an anti-clique the *results of interactions*. However, not all points can be results of interactions. For instance $[\mathtt{t}, \mathtt{f}] \in !\mathbf{bool}$ is not, since it is strictly incoherent with itself (and so does not belong to any clique of type $!\mathbf{bool}$).

(3) For coherence spaces, hypercoherences and multicoherences, we derive the usual uniform interpretation of types and terms by a straight set restriction of the web to the results of interactions (both at the level of types and at the level of proofs). More precisely, the restriction gives the multiset-based uniform interpretation (the web of $!A$ is made of multisets whose supports are cliques). Note that it was already known that the restriction of the relational interpretation of a proof to the web of the coherence space interpretation of a formula is the coherence space interpretation of the proof (Tortora de Falco 2000). The novelty introduced here is mainly that the uniform web is characterised by the coherence relation.

(4) In each case, the non-uniform and multiset-based uniform semantics are extensionally equivalent in that they have the same extensional collapse. It is already known that for the multiset-based semantics, this extensional collapse is the set-based uniform semantics (Barreiro and Ehrhard 1997; Melliès 2004).

(5) The existence of deterministic non-uniform semantics implies an unexpected property. Consider an extension of linear logic with new rules, typically, the daemon of Girard's Ludics (Girard 2001), for which the semantics is still valid and such that A and $A^{\perp}$ are both provable. Then a cut between a proof of A and a proof of $A^{\perp}$ induces an interaction that involves at most one point in the relational semantics. This unexpected result would have been hard to prove without introducing non-uniform (hyper/multi)coherence semantics. It may prove useful with other semantics based on the relational semantics. For instance, in Ehrhard's finiteness spaces (Ehrhard 2005), points of interaction are equipped with multiplicities. Since there is at most one point for each interaction, one can use its multiplicity to carry out some quantitative analysis of proof interactions.

(6) The multicoherence semantics arose as the general case in our approach to non-uniformity. In fact, we derive the non-uniform hypercoherence semantics from multicoherences. The difference between this and hypercoherences is that the coherence relation is made of multisets rather than sets. As for uniform hypercoherences, at function types (for example, first-order simple types), each finite clique in the uniform multicoherence semantics is sub-definable. But unlike hypercoherences, each clique of the multicoherence semantics is a clique in coherence spaces. At a functional type, there exist sets that are cliques in the hypercoherence semantics but are not cliques in the multicoherence semantics (even in the set-based uniform case). Since the set-based uniform multicoherence semantics is extensional, there are at least two extensionally different semantics of sequentiality of higher order: hypercoherences and multicoherences.

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Table 1. *A summary*

Semantics :	relational	coherence spaces	hypercoherences	multicoherences	bipartite
Coherence :	empty	pairs	sets	multisets	singletons
Category :	**Rel**	$\mathbf{NCoh}_{\{2\}}$	**NHc**	$\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$	**Bip**
Web of the exponentials (variants)					
multisets (co-free)	yes	new version	new	new	new
multisets uniform	no	yes	yes	new	new
sets uniform	no	yes	yes	new	new

Table 1 compares the principal variants of coherence-based semantics, including our new ones. Note that there are two axis where one can chose between sets and multisets: either for the coherence relation (the power of the coherence) or for the web of the exponentials. A third one, which we do not consider in this paper, is the shift from sets to multisets for cliques, as in finiteness spaces (Ehrhard 2005).

The (non-uniform) bipartite semantics of linear logic we present in Section 3 arises originally from a simple remark about the relational semantics: one can set polarities (positive/negative) on points of the web in such a manner that the orthogonal exchanges polarities and that every proof is interpreted by a set of positive points. This gives a kind of coherence spaces semantics where the coherence relation is of arity one. The semantics one obtains in this way is non-uniform. We discovered that this bipartite semantics also admits a uniform version where every proof of a *why not* formula is interpreted by the empty set. Apart from this radical lapse of memory of points in proof interpretations, the uniform bipartite semantics (like the non-uniform bipartite semantics) is equivalent to the relational semantics on simple types (proof and type interpretations are the same and these three semantics are extensionally equivalent).

In recent work, Michele Pagani showed that there is a syntactical counterpart, *visible acyclicity*, to non-uniform coherence (Pagani 2006): our non-uniform coherence space semantics corresponds to a relaxation of the correctness criterion of linear logic proof-nets (which provide a graphical presentation of proofs). Finding a similar correspondence for non-uniform hypercoherences or multicoherences would certainly be interesting.

1.5. *Structure of the paper*

The next section gives a brief presentation of some useful definitions and properties we will use. The only novelty is the introduction of a convenient framework, P-coherence spaces, for dealing with various static semantics.

In Section 3, we present the bipartite semantics and its uniform version, and compare them. We stress that the bipartite semantics is just given here as a peculiar example of the uniform/non-uniform setting, but does not give an example of the general uniform/non-uniform construction we use further.

Sections 4 and 5 form the core of the paper, where we study the exponentials coming from indexed linear logic, and develop their co-free version. This part is an extended

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version of our conference paper on non-uniform hypercoherences (Boudes 2003). In this part, most of the results will be presented following the order of their discovery.

We begin by presenting K-coherence spaces, which comprise a denotational semantics arising from indexed linear logic and designed to be a generalisation of (hyper)coherence spaces. The semantics is parametrised by a set K, which encompasses both (a kind of) coherence spaces $K = \{2\}$ and (a kind of) hypercoherences $K = \mathbb{N}\backslash\{0,1\}$. We then point out the definability problem at functions types.

We then introduce the co-free exponentials (Subsection 4.4). We show that the co-free exponentials provide the best solution one can expect (Corollary 1). We show that there are still some definability problems (Corollary 2), but only due to non-uniformity.

We then establish (Subsection 4.5) that the K-coherence semantics equipped with the new exponentials gives a deterministic semantics. In fact, the semantics satisfies a property stronger than just determinism. This allows the introduction of a web restriction operation, the neutral functor (Subsection 5.1) giving the uniform version of the K-coherence semantics (Subsection 5.2).

The uniform semantics obtained for $K = \mathbb{N}\backslash\{0,1\}$ is not that of hypercoherences, but a new one, *viz*. multicoherences (Subsection 5.3). Hypercoherences and non-uniform hypercoherences are then obtained through an operation forgetting the multiplicities in the coherence relation (Subsection 5.4).

We conclude this part by relating the extensional collapses of the various semantics (Subsection 5.5).

In the final section, we adopt an interactive viewpoint *à la* Girard to discuss the implications of the existence of deterministic non-uniform semantics (Subsection 6.1).

2. Preliminaries

2.1. *Extensional collapse*

Extensional partial equivalence relations, PER for short, were first introduced by Kreisel in the fifties to deal with higher order partial recursive functions. An extensional PER is meant to relate two algorithms when they implement the same function. The higher order is responsible for the partiality of the equivalence relation.

Simple types are types of the simply typed lambda calculus enriched with basis types in order to form a type system for PCF. They are given by the following grammar:

$$\sigma, \tau := \iota \mid \sigma \to \tau \quad \text{(simple types)}$$

where ι stands for basis types, typically a boolean type **bool** and/or a natural number type **nat**. A product type $\sigma \times \sigma'$ can also be introduced, but we will not bother with this type constructor since it can be obtained by curryfication.

Let $\mathscr{S}$ be a categorical semantics of linear logic and $\mathscr{S}(A)$ denote the interpretation of a type A. We will call semantical agents of type A the elements of the semantics used to interpret proofs of A, that is, morphisms from 1 to $\mathscr{S}(A)$. Suppose an interpretation of basis types is given in $\mathscr{S}$ (usually **bool** is interpreted as the space $1 \oplus 1$ and **nat** is interpreted as an ω-infinite *plus* of 1). Then we extend this intepretation of basis types to

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all simple types by setting $\mathscr{S}(\sigma \to \tau) = !\mathscr{S}(\sigma) \multimap \mathscr{S}(\tau)$. So the function type corresponds to the object of morphisms in the co-Kleisli category. If f and x are semantical agents of types $\sigma \to \tau$ and σ, respectively, then we *apply* f to x by composing in the co-Kleisli category to form a semantical agent $f(x)$ of type τ. For each simple type σ, an extensional PER $\sim_\sigma$ is defined on semantical agents of type σ by chosing the equality on basis types and by setting

$$f \sim_{\sigma\to\tau} g \text{ iff if } x \sim_\sigma y, \text{ then } f(x) \sim_\tau g(y).$$

The *extensional collapse* of the semantics is the set of quotients by extensional PERs of the interpretations of simple types equipped with the following notion of application. If $\overline{f}$ is a class of semantical agents of type $\sigma \to \tau$ (functions) and $\overline{x}$ is a class of semantical agents of type τ (arguments), the application of $\overline{f}$ to $\overline{x}$ is defined by setting $\overline{f}(\overline{x}) = \overline{f(x)}$.

2.2. *Power coherence spaces*

We will now introduce a general notion that will provide us with a very convenient language for describing the various semantics we deal with. A *power* is simply a functor from the category of sets and inclusions to itself. Typical powers relevant for our purposes are:

— The empty power defined by $E \mapsto \varnothing$. This power will simply be denoted $\varnothing$. It can be used to present the relational semantics in terms of power coherence spaces.
— The identity power, id, which will be used to deal with the bipartite relational semantics of Section 3.
— The non-empty finite sets power $\mathscr{P}^*_{\text{fin}}$, which maps each set to the set of its finite non-empty subsets. The power $\mathscr{P}^*_{\text{fin}}$ will be used to deal with hypercoherences.
— Given a subset K of $\mathbb{N} \setminus \{0,1\}$, the power $\mathscr{M}_K$ that maps a set E to the set of all finite multisets over E whose cardinality belongs to K. The power $\mathscr{M}_{\{2\}}$ will be used to deal with coherence spaces. The choice of this power follows the suggestion made at the end of Bucciarelli and Ehrhard (2001) for building non-uniform coherence or hypercoherence like semantics.

Definition 1. Let P be a power. A *P-coherence space* X is a triple $(|X|, \frown_X, \asymp_X)$ where $|X|$ is an at most countable set, the *web* of X, and where $\frown_X$ and $\asymp_X$ are subsets of $P(|X|)$ such that $\frown_X \cup \asymp_X = P(|X|)$. The set $\frown_X$ is called the *coherence* and the set $\asymp_X$ is called the *incoherence*. The intersection of $\frown_X$ and $\asymp_X$ is called the *neutrality*, and denoted by N_X. The strict coherence $\frown_X$ of X is the complementary set of $\asymp_X$ with respect to $P(|X|)$, and the strict incoherence $\smile_X$ is the complementary of $\frown_X$.

Clearly, one can define a P-coherence space X by specifying two well chosen sets among $\frown_X$, $\asymp_X$, N_X, $\frown_X$ and $\smile_X$ subject to obvious constraints (for instance, one must have $\mathsf{N}_X \subseteq \frown_X$ and $\smile_X \cap \frown_X = \varnothing$).

Definition 2. The orthogonal, $X^\perp$, of a P-coherence space $X = (|X|, \frown_X, \asymp_X)$ is the P-coherence space $(|X|, \asymp_X, \frown_X)$. (Orthogonality exchanges coherence and incoherence.)

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Definition 3. Let X be a P-coherence space. A *clique* of X is a subset x of $|X|$ such that $P(x) \subseteq \frown_X$. We use $\mathrm{Cl}(X)$ to denote the set of all cliques of X. An *anti-clique* of X is a clique of $X^\perp$. If for each clique x and each anti-clique y the intersection of x and y contains at most one element, then the P-coherence space X is *deterministic*.

Definition 4. A P-coherence space X is *reflexive* if neutrality corresponds to equality in the sense that

$$\mathsf{N}_X = \cup_{a \in |X|} P(\{a\}). \tag{2}$$

A P-coherence space X is *weakly reflexive* if

$$\mathsf{N}_X \subseteq \cup_{a \in |X|} P(\{a\}). \tag{3}$$

One can define a reflexive P-coherence space by specifying only $\frown_X$ (or $\asymp_X$ or $\frown_X$ or $\smile_X$).

Proposition 1. If the power is strictly monotone and preserves disjointness of sets, then weak reflexivity implies determinism.

Proof. Take a P-coherence space X, and a clique and an anti-clique with at least two points a and b at their intersection. Then $\{a, b\}$ is both a clique and an anti-clique. Thus $P(\{a, b\}) \subseteq \frown_X \cap \asymp_X$. By strict monotonicity and the preservation of disjointness, we have $P(\{a, b\}) \not\subset \cup_{c \in |X|} P(\{c\})$, which contradicts weak reflexivity. □

But weak reflexivity is in general stronger than determinism by itself. For instance, in a $\mathscr{P}^*_{\mathrm{fin}}$-space X one can find a set $\{a, b, c\} \in \mathscr{P}^*_{\mathrm{fin}}(|X|)$ that is both coherent and incoherent (so the space is not weakly reflexive) and still have determinism (take, for instance, $\frown_X = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ and $\smile_X = \varnothing$).

However, weak reflexivity and determinism are equivalent in $\mathscr{M}_{\{2\}}$-spaces.

2.3. *Relational semantics*

In this section we briefly recall the interpretation of linear logic in the category **Rel** of sets and relations.

Recall that composition is given by

$$f \fatsemi g = \{(a, c) \mid \exists b, (a, b) \in f \text{ and } (b, c) \in g\}$$

and identities are given by

$$\mathrm{id}_X = \{(a, a) \mid a \in |X|\}.$$

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Formulae. A formula A is interpreted by a set $|A|$ defined inductively as follows:

$$|0| = |\top| = \varnothing$$
$$|1| = |\bot| = \{*\}$$
$$|A^\perp| = |A|$$
$$|A \oplus B| = |A \& B| = |A| + |B|$$
$$|A \otimes B| = |A \quad B| = |A| \times |B|$$
$$|!A| = |?A| = \mathscr{M}_{\text{fin}}(|A|),$$

where $\mathscr{M}_{\text{fin}}(E)$ is the set of finite multisets on E.

Sequents. We use the right-sided presentation of the linear logic sequent calculus. Up to associativity and commutativity of the Cartesian product, the 'comma' of sequents is safely interpreted as a *par*, that is, by setting

$$|\vdash A_1, \ldots, A_n| = |A_1 \quad \ldots \quad A_n|,$$

which is equal to

$$|A_1| \times \ldots \times |A_n|.$$

Proofs. The interpretation of a proof of a sequent $\vdash \Gamma$ is a subset of $|\vdash \Gamma|$ defined inductively on the proof using cases on the last rule, as shown below.

It is well known that this interpretation is a denotational semantics of linear logic (that is, two proofs of a given sequent have the same interpretation when they are equivalent up to cut-elimination).

Identity group

$$\frac{}{\vdash A, A^\perp : \{(a,a) \mid a \in |A|\}}\ (ax.) \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, A^\perp : g}{\vdash \Gamma, \Delta : \{(\gamma,\delta) \mid \exists a, (\gamma,a) \in f \wedge (\delta,a) \in g\}}\ (cut)$$

Additives

$$\frac{}{\vdash \Gamma, \top : \varnothing}\ (top)$$

$$\frac{\vdash \Gamma, A : f \qquad \vdash \Gamma, B : g}{\vdash \Gamma, A \& B : f \uplus g}\ (with) \qquad \frac{\vdash \Gamma, A_i : f}{\vdash \Gamma, A_1 \oplus A_2 : f}\ i = 1,2\ (plus)$$

Multiplicatives

$$\frac{\vdash \Gamma : f}{\vdash \Gamma, \bot : f \times \{*_\bot\}}\ (bot) \qquad \frac{}{\vdash 1 : \{*_1\}}\ (one)$$

$$\frac{\vdash \Gamma, A, B : f}{\vdash \Gamma, A \parr B : f}\ (par) \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, B : g}{\vdash \Gamma, \Delta, A \otimes B : \{(\gamma,\delta,(a,b)) \mid (\gamma,a) \in f, (\delta,b) \in g\}}\ (tens.)$$

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Exponentials

$$\frac{\vdash ?A_1,\ldots,?A_n,A : f}{\vdash ?A_1,\ldots,?A_n,!A : f^\dagger}\ (prom.) \qquad \frac{\vdash \Gamma,?A,?A : f}{\vdash \Gamma,?A : \{(\gamma,\mu_1+\mu_2) \mid (\gamma,\mu_1,\mu_2) \in f\}}\ (cont.)$$

$$\frac{\vdash \Gamma : f}{\vdash \Gamma,?A : \{(\gamma,[]) \mid \gamma \in f\}}\ (weak.) \qquad \frac{\vdash \Gamma,A : f}{\vdash \Gamma,?A : \{(\gamma,[a]) \mid (\gamma,a) \in f\}}\ (der.)$$

Where $f^\dagger$ is equal to

$$\left\{ \left(\sum_{j\in J} \mu_1^j,\ldots,\sum_{j\in J} \mu_n^j,[a_j \mid j \in J] \right) \mid J \text{ finite and } \forall j \in J, \left(\mu_1^j,\ldots,\mu_n^j,a_j\right) \in f \right\}.$$

The relational semantics is actually a categorical semantics of linear logic, though we shall not give the details of its categorical structure here. The new Seely categorical semantics axiomatisation (Bierman 1995) is appropriate for dealing with the relational semantics and we will use this axiomatic for further semantics. Exponentials are given by a comonad structure $(!, \mathrm{der}, \mathrm{dig})$, as follows. The endofunctor $!$ of **Rel** is defined by $!E = \mathscr{M}_{\mathrm{fin}}(E)$ and

$$!f = \{([a_i \mid i \in I],[b_i \mid i \in I]) \mid I \text{ finite and } \forall i \in I,(a_i,b_i) \in f\}.$$

The natural transformations $\mathrm{der} : !\dot{\to}\mathrm{id}$ and $\mathrm{dig} : !\dot{\to}!!$ are defined by setting

$$\mathrm{der}_E = \{([a],a) \mid a \in E\}$$
$$\mathrm{dig}_E = \{(sum_{i\in I}\mu_i,[\mu_i \mid i \in I]) \mid [\mu_i \mid i \in I] \in !!E\}.$$

2.4. *Coherence spaces*

In this section we briefly recall the coherence spaces semantics of linear logic (Girard 1987).

A coherence space is a reflexive $\mathscr{M}_{\{2\}}$-coherence space.

We define the connectives of linear logic directly on coherence spaces (rather than defining the interpretation of formulae by induction). The web of multiplicatives and additives is the same as in the relational semantics. Coherence is defined as follows. We have $\frown_{X\oplus Y} = \frown_X \uplus \frown_Y$ and $[(a,b),(a',b')] \in \frown_{X\otimes Y}$ if and only if $[a,a'] \in \frown_X$ and $[b,b'] \in \frown_Y$.

Morphisms from X to Y in the corresponding (linear) category are just cliques of $X \multimap Y = X^\perp \parr Y$.

The interpretation of proofs in coherence spaces differs from the relational semantics on exponential rules only. Recall that the interpretation of a proof is just a subset of the web of the interpretation of the conclusion sequent. Soundness of the semantics ensures that this set is a clique in the corresponding space.

There are two variants for the web of the exponentials: set-based and multiset-based. In the multiset-based semantics, the web of $!X$ is the set of multisets whose supports are cliques in X:

$$|!X| = \{\mu \in \mathscr{M}_{\mathrm{fin}}(|X|) \mid \mathrm{supp}(\mu) \in \mathrm{Cl}(X)\}. \tag{4}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bot, 1 : \{(*,*)\}}(ax.) \quad \dfrac{}{\vdash \bot, 1 : \{\underline{(*,*)}\}}(ax.)}{\vdash \bot \,\&\, \bot, 1 : \{(\mathtt{t},*), \underline{(\mathtt{f},*)}\}}(with)}{\vdash \bot \,\&\, \bot, 1 \oplus 1 : \{(\mathtt{t},\mathtt{t}), \underline{(\mathtt{f},\mathtt{t})}\}}(plus\ 1) \quad \dfrac{\dfrac{\dfrac{}{\vdash \bot, 1 : \{\underline{(*,*)}\}}(ax.) \quad \dfrac{}{\vdash \bot, 1 : \{(*,*)\}}(ax.)}{\vdash \bot \,\&\, \bot, 1 : \{\underline{(\mathtt{t},*)}, (\mathtt{f},*)\}}(with)}{\vdash \bot \,\&\, \bot, 1 \oplus 1 : \{\underline{(\mathtt{t},\mathtt{f})}, \underline{(\mathtt{f},\mathtt{f})}\}}(plus\ 2)}{\vdash \bot \,\&\, \bot, 1 \oplus 1, \bot : \{(\mathtt{t},\mathtt{t},*), \underline{(\mathtt{f},\mathtt{t},*)}\}}(bot) \quad \dfrac{\dfrac{}{\vdash \bot, 1 : \{\underline{(*,*)}\}}(ax.) \quad \dfrac{}{\vdash \bot, 1 : \{(*,*)\}}(ax.)}{\vdash \bot \,\&\, \bot, 1 : \{\underline{(\mathtt{t},*)}, (\mathtt{f},*)\}}(with)}{\vdash \bot \,\&\, \bot, 1 \oplus 1, \bot : \{\underline{(\mathtt{t},\mathtt{f},*)}, (\mathtt{f},\mathtt{f},*)\}}(bot)}{\vdash \bot \,\&\, \bot, 1 \oplus 1, \bot \,\&\, \bot : \{(\mathtt{t},\mathtt{t},\mathtt{t}), \underline{(\mathtt{f},\mathtt{t},\mathtt{t})}, \underline{(\mathtt{t},\mathtt{f},\mathtt{f})}, (\mathtt{f},\mathtt{f},\mathtt{f})\}}(with)$$

$$\dfrac{\vdash ?(\bot \,\&\, \bot), 1 \oplus 1, \bot \,\&\, \bot : \{([\mathtt{t}],\mathtt{t},\mathtt{t}), \underline{([\mathtt{f}],\mathtt{t},\mathtt{t})}, \underline{([\mathtt{t}],\mathtt{f},\mathtt{f})}, ([\mathtt{f}],\mathtt{f},\mathtt{f})\}}{\vdash ?(\bot \,\&\, \bot), 1 \oplus 1, ?(\bot \,\&\, \bot) : \{([\mathtt{t}],\mathtt{t},[\mathtt{t}]), \underline{([\mathtt{f}],\mathtt{t},[\mathtt{t}])}, \underline{([\mathtt{t}],\mathtt{f},[\mathtt{f}])}, ([\mathtt{f}],\mathtt{f},[\mathtt{f}])\}}(der.)$$

$$\dfrac{\vdash ?(\bot \,\&\, \bot), 1 \oplus 1, ?(\bot \,\&\, \bot) : \{([\mathtt{t}],\mathtt{t},[\mathtt{t}]), \underline{([\mathtt{f}],\mathtt{t},[\mathtt{t}])}, \underline{([\mathtt{t}],\mathtt{f},[\mathtt{f}])}, ([\mathtt{f}],\mathtt{f},[\mathtt{f}])\}}{\vdash ?(\bot \,\&\, \bot), 1 \oplus 1 : \{([\mathtt{t},\mathtt{t}],\mathtt{t}), \cancel{([\mathtt{t},\mathtt{f}],\mathtt{t})}, \cancel{([\mathtt{t},\mathtt{f}],\mathtt{f})}, ([\mathtt{f},\mathtt{f}],\mathtt{f})\}}(cont.)$$

Fig. 1. Linear logic proof of Example 1 (corresponding to the PCF term (1) in the Introduction)

Two elements μ and ν of the web of $!X$ are coherent if and only if the support of $\mu + \nu$ is a clique of X.

The interpretation of exponential rules is the same as in the relational semantics, but restricted to the web of the exponentials. Two rules have their interpretations modified by this restriction. In the contraction rule, the support of $\mu_1 + \mu_2$ has to be an anti-clique of (the space interpreting) A. In the promotion rule:

(i) the support of $\sum_{j \in J} \mu_i^j$ must be an anti-clique of A_i, for each i;
(ii) the support of $[a_j \mid j \in J]$ must be a clique of A.

In fact, it is easy to show that (i) implies (ii), so the only condition to check is (i). This will also be the case in hypercoherences.

In the set-based semantics, the web of $!X$ is the set of finite cliques of X. The interpretation of exponentials follows the last pattern, but with sets and unions instead of mulitsets and sums.

Example 1. In the Introduction we gave the relational interpretation and the uniform coherence space interpretation of a PCF term (term (1)). The proof in Figure 1 is a linear logic version of this term annotated by the relational interpretation of its sub-proofs. In this example, we have denoted $(1, *)$ by $\mathtt{t}$ and $(2, *)$ by $\mathtt{f}$. The two points that are forgotten by the coherence space semantics are shown with a line through the text and the points from which they come are underlined.

Example 2. Another example of the action of uniformity concerns restrictions on the number of times an argument will be copied. This is a bit more subtle than just removing pieces of dead code.

Consider the proofs in Figure 2. The intuitive meaning of π_0 is that we make 1 available *ad libidum*. If we add a dereliction as the final rule, the resulting proof, π_1, is intuitively a program taking as its argument another program that itself requires as its argument an arbitrary number of copies of 1. For instance, the two points $([[*,*]])$ and $([[*]])$ in the interpretation of π_1 correspond to providing the argument of π_1 with two copies or one copy, respectively, of 1. In fact, π_1 provides its argument *once*, but with an arbitrary number of copies of 1.

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$$\pi_0 \left\{ \dfrac{\dfrac{}{\vdash 1}\,(one)}{\vdash !1}\,(prom.) \right. \qquad \pi_1 \left\{ \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(one)}{\vdash !1}\,(prom.)}{\vdash ?!1}\,(der.) \right. \qquad \pi_2 \left\{ \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(one)}{\vdash !1}\,(prom.)}{\vdash ?!1}\,(der.)}{\vdash ?!1, \bot}\,(bot) \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(one)}{\vdash !1}\,(prom.)}{\vdash ?!1}\,(der.)}{\vdash ?!1, ?!1, \bot \otimes \bot}\,(tens.)}{\vdash ?!1, \bot \otimes \bot}\,(cont.) \right.$$

Fig. 2. Proofs for Example 2

Now consider two copies of π_1, which we assemble into a unique proof by a combination of bottom and tensor introductions (we could also have used the *mix* rule if it were available). We contract the two copies of ?!1. The resulting proof π_2 is shown in Figure 2.

In coherence spaces, the uniformity restriction forces each of the two copies of π_1 to receive a program asking for 1 the *same number of times*. For instance, in coherence spaces, $([[*,*],[*,*]],(*,*))$ is in the interpretation of π_2 but $([[*],[*,*]],(*,*))$ is not, but in the relational semantics, both of these points are in the interpretation.

2.5. *Hypercoherences*

A hypercoherence is a reflexive $\mathscr{P}^*_{\text{fin}}$-coherence space. The interpretation of linear logic in hypercoherences (Ehrhard 1993) follows exactly the same pattern as for coherence spaces, and we will just stress a few points.

In a tensor, $x = \{(a_1, b_1), \ldots, (a_n, b_n)\} \in \frown_{X \times Y}$ if and only if $\pi_1 x = \{a_1, \ldots, a_n\} \in \frown_X$ and $\pi_2 x = \{b_1, \ldots, b_n\} \in \frown_Y$. In a *with*, the dual of a *plus*, $\asymp_{A\&B} = \asymp_A \uplus \asymp_B$, so for every $x \in \mathscr{P}^*_{\text{fin}}(|A| + |B|)$, if x intersects both $|A|$ and $|B|$, then $x \in \frown_{A\&B}$ (and conversely).

The two variants for the exponentials, set-based and multiset-based, are also present.

The coherence in $!X$ is defined using a notion of section. If $U = \{x_i \mid i \in I\}$ is a finite set of finite sets or of multisets, we say that s is a *section* of U and write $s \triangleleft U$ when for each $i \in I$ there exists $a_i \in s$ such that $a_i \in x_i$ and $s \subseteq \cup_{i \in I} x_i$. A non-empty finite subset U of $|!X|$ is coherent in $!X$ if and only if each section of U is coherent in X.

Of course, since the notion of coherence differs between coherence spaces and hypercoherence, the notion of cliques and thus, because of uniformity, the interpretations of proofs also differ.

Note that the coherence in hypercoherence may have *holes*: in general, we can have $x \in \frown_X$ and $y \subset x$ without having $y \in \frown_X$.

Property 1. In hypercoherence and coherence space semantics, the intersection of a clique and an anti-clique contains at most one point (determinism). However, in these two semantics, if $(a, c) \in f \fatsemi g$ for $f : A \to B$ and $g : B \to C$, there is only one b such that $(a, b) \in f$ and $(b, c) \in g$.

Recall that one cannot equip the relational semantics with a set-based exponential ($|!X| = \mathscr{P}_{\text{fin}}(|X|)$) similar to the one for coherence spaces and hypercoherences. This will not give a sound interpretation. Consider for instance the diagram setting the naturality

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of dereliction

$$\begin{array}{ccc} !X & \xrightarrow{\mathrm{der}_X} & X \\ {\scriptstyle !f}\downarrow & & \downarrow{\scriptstyle f} \\ !Y & \xrightarrow[\mathrm{der}_Y]{} & Y \end{array}$$

In the particular case where $f = \{(a,b),(a',b)\}$, we have $(\{a,a'\},\{b\}) \in !f$, and thus $(\{a,a'\},b) \in !f \fatsemi \mathrm{der}_Y$ but $(\{a,a'\},b) \notin \mathrm{der}_X \fatsemi f$. Hence the diagram does not commute. Alternatively, we can say without reference to categories that with such set-based exponentials, the elimination of a cut between a promotion and a dereliction will not, in general, leave the interpretation invariant.

3. Bipartite uniform and non-uniform relational semantics

In this section, we introduce a simple semantics of linear logic based on the relational semantics and id-coherence spaces that admits both non-uniform and uniform versions. We use these two semantics to demonstrate that uniform semantics can lose a lot of information about the terms (proofs) they interpret.

Definition 5. A *bipartite space* is just an id-coherence space $(|X|, \frown_X, \asymp_X)$ where N_X is empty. So $\frown_X$, $\asymp_X$ is a bipartition of $|X|$ and every bipartition of $|X|$ defines a bipartite space.

We call the coherence of X, the *positive web*, and use $|X|^+$ to denote it, and we call the incoherence of X the *negative web*, and use $|X|^-$ to denote it. The elements of $|X|^+$ and $|X|^-$ are the positive and negative points of X, respectively.

Following the general definition (Definition 3), in our particular case, a clique is just a set of positive points of the web.

The category **Bip** has as objects bipartite spaces and as morphisms between bipartite spaces X and Y the relations that are cliques in the bipartite space $X \multimap Y$ defined below. The composition and identities and are those of **Rel**. We will now describe the non-uniform bipartite semantics of linear logic.

On formulae the logical connectives are interpreted as follows:

— Linear negation is given, as in the general P-coherence space case, by the exchange of coherence and incoherence, so $X^\perp = (|X|, |X|^-, |X|^+)$.
— Both additive constants, 0 and $\top$, are equal to $(\varnothing, \varnothing, \varnothing)$.
— The bipartite space 1 is equal to $(\{*\}, \{*\}, \varnothing)$, so the bipartite space $\perp$ is equal to $(\{*\}, \varnothing, \{*\})$.
— *With* is given by $X \,\&\, Y = (|X| + |Y|, |X|^+ + |Y|^+, |X|^- + |Y|^-)$ and *plus* is given by $X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp$, which is the same bipartite space as $X \,\&\, Y$.
— The tensor of X and Y, $X \otimes Y$, is the bipartite space $|X| \times |Y|$ whose positive web $|X \times Y|^+$ is equal to $|X|^+ \times |Y|^+$. It follows that $X \,\Gamma\, Y = (X^\perp \otimes Y^\perp)^\perp$ is such that (a,b) is positive in $X \,\Gamma\, Y$ if and only if a or b is positive and that $X \multimap Y = X^\perp \,\Gamma\, Y$ is such that $|X \multimap Y| = |X| \times |Y|$ and $(a,b) \in |X \multimap Y|^+$ if and only if $a \in |X|^+$ implies $b \in |Y|^+$.

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— The *of course* of X, $!X$, is the bipartite space of the web $\mathscr{M}_{\mathrm{fin}}|X|$ and the positive web $|!X|^+ = \mathscr{M}_{\mathrm{fin}}|X|^+$. Thus, a multiset μ that is an element of $|!X|$ is negative if and only if μ contains at least one negative point of X. It follows that $?X = (!X^\perp)^\perp$ is such that an element μ of $|?X|$ is positive if and only if it contains at least one positive point of X.

Remark 1. In contrast to the relational semantics, the bipartite semantics distinguishes A and $A^\perp$. In particular, the multiplicative constants are distinct (this is not the case in coherence and hypercoherence semantics).

As usual, a context $A_1, \ldots, A_n$ is interpreted by the same space as the formula $A_1 \parr \ldots \parr A_n$. Interpretations of proofs are defined as in the relational semantics. It is easy to verify that a proof is interpreted by a set of positive points (a clique). The categorical structure of the bipartite semantics is derived from that of the relational semantics. Morphisms involved in natural transformations of the semantics and morphisms obtained by functorial constructions, seen as sets, are defined in the same way as in the relational semantics and it is straightforward to verify that they actually only contain positive points, so they are cliques.

3.1. *Uniform bipartite semantics*

We will now introduce a uniform variant of the bipartite semantics as follows. The uniform interpretation of exponentials is given by a comonad $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ described below, and the other categorical constructions are the same as in the non-uniform bipartite semantics.

Setting $\mathrm{Pos}(X) = (|X|^+, |X|^+, \varnothing)$ for each bipartite space X and $\mathrm{Pos}(f) = f \cap (|X|^+ \times |Y|^+)$ for each $f \in \mathbf{Bip}(X, Y)$ trivially makes Pos a functor.

Since f is a clique in $X \multimap Y$, if $(a, b) \in f$ and $a \in |X|^+$, then $b \in |X|^+$. Hence $\mathrm{Pos}(f)$ is equal as a set to $\mathrm{Pos}(\mathrm{id}_X) \fatsemi f$, where $\mathrm{Pos}(\mathrm{id}_X)$ is viewed by set inclusion as a morphism from $\mathrm{Pos}(X)$ to X. This can be used to verify that Pos commutes with composition: if $f \in \mathbf{Bip}(X, Y)$ and $g \in \mathbf{Bip}(Y, Z)$, we have the following set equalities

$$\begin{aligned}\mathrm{Pos}(f) \fatsemi \mathrm{Pos}(g) &= \mathrm{Pos}(f) \fatsemi \mathrm{id}_{\mathrm{Pos}(\mathrm{id}_Y)} \fatsemi g \\ &= \mathrm{Pos}(f) \fatsemi g \\ &= \mathrm{Pos}(\mathrm{id}_X) \fatsemi f \fatsemi g \\ &= \mathrm{Pos}(f \fatsemi g).\end{aligned}$$

The functor $\underset{u}{!}$ is defined by setting $\underset{u}{!} = \mathrm{Pos}\,!$ (where $!$ is the 'of course' functor of the non-uniform bipartite semantics). Note that $\mathrm{Pos}\,! = !\,\mathrm{Pos}$. The natural transformations $\mathrm{der}_u : \underset{u}{!} \dot{\to} \mathrm{id}$ and $\mathrm{dig}_u : \underset{u}{!} \dot{\to} \underset{u}{!}\,\underset{u}{!}$ are defined by setting $\mathrm{der}_{u,X} = \{[a] \mid a \in |X|^+\}$ (which is equal as a set to $\mathrm{der}_{\mathrm{Pos}(X)}$ and $\mathrm{Pos}\,\mathrm{der}_X$) and $\mathrm{dig}_u = \mathrm{Pos}(\mathrm{dig})$ (we have $\mathrm{Pos}\,!! = \underset{u}{!}\,\underset{u}{!}$). We now have to verify that der_u is actually a natural transformation (for dig_u, this follows from the definition). Let $f \in \mathbf{Bip}(X, Y)$. Then

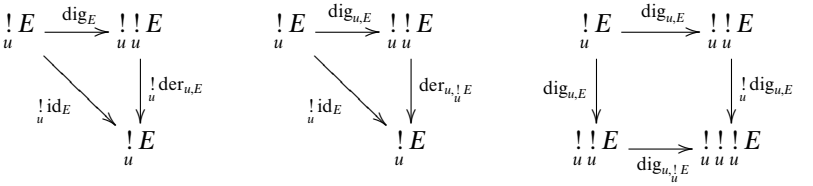
$$\mathrm{der}_{u,X} \fatsemi f = \mathrm{der}_{u,x} \fatsemi \{(a, b) \mid a \in |X|^+\} = \mathrm{der}_{u,X} \fatsemi \mathrm{Pos}(\mathrm{id}_X) \fatsemi f,$$

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which is equal as a set to

$$\mathrm{Pos}(\mathrm{der}_X \fatsemi f) = \mathrm{Pos}(!f \fatsemi \mathrm{der}_Y) = \underset{u}{!} f \fatsemi \mathrm{der}_{u,Y},$$

which concludes the proof. The following commutative diagrams are the image by Pos of the corresponding commutative diagrams for $(!, \mathrm{der}, \mathrm{dig})$ in **Bip**:



Hence $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ is truly a comonad.

In order to verify that this setting forms a new Seely categorical semantics of linear logic, we have to verify that the adjunction induced by the comonad $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ is monoidal. We will not show this in detail but it can be easily derived from the fact that in **Rel** the comonad $(!, \mathrm{der}, \mathrm{dig})$ induces a monoidal adjunction – just note that the isomorphisms $\underset{u}{!}(X \,\&\, Y) \cong \underset{u}{!}(X) \otimes \underset{u}{!}(Y)$ and $\underset{u}{!} \top \cong 1$ hold and are natural (since $\mathrm{Pos}(X') \otimes \mathrm{Pos}(Y') \cong \mathrm{Pos}(X' \otimes Y')$, $\mathrm{Pos}(f) \otimes \mathrm{Pos}(g)$ is the same set as $\mathrm{Pos}(f \otimes g)$ and $\mathrm{Pos}(1) = 1$).

The interpretation of exponential rules in the uniform bipartite semantics is obtained by a set restriction to the uniform web(s), as follows.

The interpretations of the two rules contraction and weakening, are, in fact, unchanged:

$$\frac{\vdash \Gamma, ?A, ?A : f}{\vdash \Gamma, ?A : \{(\gamma, \mu_1 + \mu_2) \mid (\gamma, \mu_1, \mu_2) \in f\}}\ (cont.) \qquad \frac{\vdash \Gamma : f}{\vdash \Gamma, ?A : \{(\gamma, []) \mid (\gamma) \in f\}}\ (weak.)$$

This is because in the sets produced by these two rules there are already only negative points in $?A$. In the contraction rule, μ_1 and μ_2 contain only negative points of the interpretation of A, as does $\mu_1 + \mu_2$. And for the weakening rule, we have that $[]$ is negative in $?A$.

Thus, in contrast to the coherence space situation (for instance), the construction associated with the contraction is the same as in the relational semantics.

The interpretation of the dereliction rule is

$$\frac{\vdash \Gamma, A : f}{\vdash \Gamma, ?A : \{(\gamma, [a]) \mid (\gamma, a) \in f, a \in |A|^-\}}\ (der.)$$

(where $|A|^-$ stands for $|X|^-$ with X interpreting A). Note that this construction forgets some points relative to the non-uniform interpretation (unlike what happens in coherence spaces).

The interpretation of the promotion rule is

$$\frac{\vdash ?A_1, \ldots, ?A_n, A : f}{\vdash ?A_1, \ldots, ?A_n, !A : f^{\dagger_u}}\ (prom.)$$

where $f^{\dagger_u}$ is equal to the restriction to the uniform web of the $f^\dagger$ of the relational semantics. But again (unlike what happens in coherence spaces), there is no need to

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restrict, so

$$f^{\dagger_u} = f^{\dagger} = \left\{ \left(\sum_{j\in J} \mu_1^j, \ldots, \sum_{j\in J} \mu_n^j, [a_j \mid j \in J] \right) \mid J \text{ finite and } \forall j \in J, \left(\mu_1^j, \ldots, \mu_n^j, a_j\right) \in f \right\}.$$

There are two reasons for this:

(i) The sum of multisets of negative points is a multiset of negative points, so, for each i, we have $\sum_{j\in J} \mu_i^j$ is in $|?A_i|^{-}$.

(ii) Since f is a clique, each element $(\mu_1, \ldots, \mu_n, a)$ of f is positive, and since each μ_i is negative in $?A_i$, it follows that a is positive in $!A$, so $[a_j \mid j \in J] \in |!A|^{+}$.

Remark 2. In the uniform bipartite semantics, each proof π of a sequent $\vdash ?A$ is interpreted by the empty set.

To see why this remark holds, simply note that the space interpeting $?A$ contains only negative points and that a clique is a set of positive points.

Example 3. The interpretation of the proof

$$\dfrac{\dfrac{}{\vdash \bot, 1}\,(ax.)}{\vdash ?1, \bot}\,(der.)$$

is the empty set. But the interpretation of the proof of Figure 1 is the same as in the relational semantics.

Curiously, the uniform bipartite semantics maps a lot of proofs to the empty set. (But many other proofs are mapped onto non-trivial subsets of their relational interpretations).

3.2. *Extensional collapses*

The basis types **bool** and **nat** are interpreted by the respective bipartite spaces:

$$\mathbf{bool} = (\{\mathtt{t}, \mathtt{f}\}, \{\mathtt{t}, \mathtt{f}\}, \varnothing)$$
$$\mathbf{nat} = (\mathbb{N}, \mathbb{N}, \varnothing).$$

Simple types are interpreted by the same bipartite spaces in both the uniform and non-uniform bipartite semantics. Moreover, the bipartite spaces interpreting simple types are purely positive (every point of the web is positive), so every subset of the web is a clique. Hence the uniform bipartite semantics, the non-uniform bipartite semantics and the relational semantics have the same extensional collapse. We do not know of any direct expression of this collapse.

4. Non-uniform K-coherence semantics

4.1. *K-coherence spaces*

From now on, we shall assume that a subset K of $\mathbb{N} \setminus \{0, 1\}$ is given and call the corresponding $\mathscr{M}_K$-coherence space a K-coherence space.

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4.2. *Interpreting MALL – nothing new*

The interpretation of the multiplicative additive fragment of linear logic (MALL) follows a standard pattern.

Both additive constants are the empty K-coherence space:

$$0 = \top = (\varnothing, \varnothing, \varnothing).$$

Both multiplicative constants are the reflexive one point K-coherence space:

$$1 = \bot = (\{*\}, \mathscr{M}_K(\{*\}), \mathscr{M}_K(\{*\})).$$

Let X_1 and X_2 be two K-coherence spaces.

— The K-coherence space $X_1 \oplus X_2$ is defined by setting

$$\begin{aligned} |X_1 \oplus X_2| &= |X_1| + |X_2| \\ \mathsf{N}_{X_1 \oplus X_2} &= \mathsf{N}_{X_1} \uplus \mathsf{N}_{X_2} \\ \frown_{X_1 \oplus X_2} &= \frown_{X_1} \uplus \frown_{X_2}. \end{aligned}$$

Of course, $X_1 \& X_2 = ({X_1}^\perp \oplus {X_2}^\perp)^\perp$.

— The space $X_1 \otimes X_2$ is defined as follows. We set $|X_1 \otimes X_2| = |X_1| \times |X_2|$. For $i = 1, 2$, let π_i be the canonical projections:

$$\begin{aligned} \pi_i : \mathscr{M}_K(|X_1 \otimes X_2|) &\to \mathscr{M}_K(|X_i|) \\ [(a_j^1, a_j^2) \mid j \in J] &\mapsto [a_j^i \mid j \in J]. \end{aligned}$$

Then for each $s \in \mathscr{M}_K(|X_1 \otimes X_2|)$, we set

$$\begin{aligned} s \in \mathsf{N}_{X_1 \otimes X_2} &\text{ iff } \pi_1(s) \in \mathsf{N}_{X_1} \text{ and } \pi_2(s) \in \mathsf{N}_{X_2} \\ s \in \smile_{X_1 \otimes X_2} &\text{ iff } \pi_1(s) \in \smile_{X_1} \text{ or } \pi_2(s) \in \smile_{X_2}, \end{aligned}$$

which suffices to determine $\mathsf{N}_{X_1 \otimes X_2}$, $\smile_{X_1 \otimes X_2}$ and $\frown_{X_1 \otimes X_2}$. We also set $X_1 \parr X_2 = ({X_1}^\perp \otimes {X_2}^\perp)^\perp$.

The *linear map* construction $\multimap$ between K-coherence spaces is defined by setting $X \multimap Y = X^\perp \parr Y$. A *linear morphism* from X to Y, where X and Y are two K-coherence spaces, is a clique of $X \multimap Y$. Note that

$$s \in \supset_{X \multimap Y} \text{ iff } \begin{cases} \pi_1(s) \in \supset_X \implies \pi_2(s) \in \supset_Y \\ \pi_1(s) \in \frown_X \implies \pi_2(s) \in \frown_Y \end{cases} \tag{5}$$

or, equivalently,

$$s \in \supset_{X \multimap Y} \text{ iff } \begin{cases} \pi_2(s) \in \smile_Y \implies \pi_1(s) \in \smile_X \\ (\pi_1(s) \in \supset_X \text{ and } \pi_2(s) \in \mathsf{N}_Y) \implies \pi_1(s) \in \mathsf{N}_X \end{cases} \tag{6}$$

or, equivalently,

$$s \in \supset_{X \multimap Y} \text{ iff } \begin{cases} \pi_1(s) \in \supset_X \implies \pi_2(s) \in \supset_Y \\ \pi_2(s) \in \asymp_Y \implies \pi_1(s) \in \asymp_X \end{cases} \tag{7}$$

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We use $\mathbf{NCOH}_K$ to denote the category whose objects are the K-coherence spaces, whose morphisms are the linear morphisms and where compositions and identities are defined as in **Rel** (it is easy to verify that the composition of a clique of $X \multimap Y$ and a clique of $Y \multimap Z$ is a clique of $X \multimap Y$). For every $K' \subseteq K$, the corresponding categories come naturally with forgetful functors $U_{K,K'} : \mathbf{NCOH}_K \to \mathbf{NCOH}_{K'}$, which act as the identity on morphisms.

The *boolean type*, denoted by **bool** and represented by the formula $1 \oplus 1$, will be interpreted in $\mathbf{NCOH}_{\mathbb{N}\setminus\{0,1\}}$ by the uniform $\mathbb{N}\setminus\{0,1\}$-coherence space whose web is $\{\mathtt{t},\mathtt{f}\}$ and whose coherence is $\mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(\{\mathtt{t}\}) \cup \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(\{\mathtt{f}\})$.

Definition 6 (logical functor). Given two categorical semantics of a logical system, a functor between the two categories is *logical* when it commutes with the interpretation of sequents and proofs.

Proposition 2 (semantics of MALL). For each $K \subseteq \mathbb{N}\setminus\{0,1\}$, the category $\mathbf{NCOH}_K$ is a semantics of MALL. And for each $K' \subseteq K$ (in particular, for $K' = \varnothing$) the functor $U_{K,K'}$ is logical with respect to the $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ MALL semantics.

The fact that $\mathbf{NCOH}_K$ is a semantics of MALL means that $\mathbf{NCOH}_K$ is a symmetric monoidal closed category (with $\otimes$ as tensor product and $\multimap$ as function space constructor) that is $*$-autonomous ($\bot$ being the dualising object), and, furthermore, has all finite products and coproducts (see Amadio and Curien (1998) for precise definitions). The proof sketched below is a straightforward verification.

Proof. The only thing to verify is that all the morphisms of **Rel** that make **Rel** a $*$-autonomous category are cliques, that is, linear morphisms of the category $\mathbf{NCOH}_K$. Provided Y_1 and Y_2 have disjoint webs, $Y_1 \& Y_2$ equipped with the two projections

$$p_i = \{(a,a) \mid a \in |Y_i|\} : (Y_1 \& Y_2) \to Y_i$$

for $i = 1,2$ is the Cartesian product of Y_1 and Y_2, and if $f_1 \in \mathbf{NCOH}_K(X, Y_1)$ and $f_2 \in \mathbf{NCOH}_K(X, Y_2)$, then the *pairing* of f_1 and f_2 is just $(f_1 \cup f_2) \in \mathbf{NCOH}_K(X, Y_1 \& Y_2)$. In fact, it is easy to verify that p_1, p_2 and $f_1 \cup f_2$ are cliques. For each $f \in \mathbf{NCOH}_K(X, Y)$ and $f' \in \mathbf{NCOH}_K(X', Y')$,

$$f \otimes f' = \{((a,a'),(b,b')) \mid (a,b) \in f, (a',b') \in f'\}$$

is obviously a clique of $(X \otimes X') \multimap (Y \otimes Y')$, thus the construction $\otimes$ on K-coherence spaces is functorial. For each $X, Y, Z \in \mathbf{NCOH}_K$, it is easy to show that the isomorphisms $\mathrm{unit}_X : X \otimes 1 \cong X$, $\mathrm{ass}_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and $\mathrm{com}_X : X \otimes X \to X \otimes X$ of **Rel** given by

$$\begin{aligned}
\mathrm{unit}_X &= \{((a,*),a) \mid a \in X\} \\
\mathrm{ass}_{X,Y,Z} &= \{(((a,b),c),(a,(b,c))) \mid a \in X, b \in Y, c \in Z\} \\
\mathrm{com}_X &= \{((a,b),(b,a)) \mid a,b \in X\}
\end{aligned}$$

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are also isomorphisms in $\mathbf{NCOH}_K$. So, taken together with the functor $\otimes$, this gives a symmetric monoidal structure on $\mathbf{NCOH}_K$, which turns out to be closed for the built-in object of morphisms $X \multimap Y$. Finally, the dualising object $\bot$ is clearly such that $X \multimap \bot \cong X^{\bot}$, so $X^{\bot\bot}$ is obviously isomorphic to X. □

Remark 3 (foliation). The coherence relation is *foliated* with respect to the interpretation of MALL, that is, for each formula A of MALL, the coherence relations on multisets of cardinality n in the interpretation of A is totally determined by the coherence relations on multisets of cardinality n in the interpretation of the sub-formulae of A. In fact, it is exactly by constructing independently each coherence relation of level k for $k \in K$ in the Bucciarelli–Ehrhard machinery that the K-coherence spaces semantics has been obtained, so this remark also holds for the linear logic semantics with the exponential provided by this machinery. Anticipating a bit, it will also hold for both the Bucciarelli–Ehrhard exponential and the new exponential construction we present (and in both cases the forgetful functors $U_{K,K'}$ will still be logical in LL).

4.3. *Exponentials*

Using the constructions presented in Bucciarelli and Ehrhard (2001), we can define exponentials for K-coherence spaces.

This gives a semantics that accepts a variant of Berry's well-known example of a stable and non-sequential function from $\mathbf{bool} \times \mathbf{bool} \times \mathbf{bool}$ to $\mathbf{bool}$. Of course, both the standard (set-based) hypercoherence semantics and the multiset-based hypercoherence semantics we use here reject such first-order non-sequential functions.

The corresponding 'of course' operation is denoted by $\underset{\mathrm{BE}}{!}$ and defined as follows. We set $|\underset{\mathrm{BE}}{!} X| = \mathscr{M}_{\mathrm{fin}}(|X|)$. A multiset $[x_i \mid 1 \leqslant i \leqslant k] \in \mathscr{M}_K(|\underset{\mathrm{BE}}{!} X|)$ (such that $k \in K$) is strictly incoherent in $\underset{\mathrm{BE}}{!} X$ if and only if there exists a multiset $[a_j \mid 1 \leqslant j \leqslant k] \in \mathscr{M}_K(|X|)$ that is strictly incoherent in X and satisfies

$$[a_j \mid 1 \leqslant j \leqslant k] \leqslant \sum_{i=1}^{k} x_i.$$

If such a multiset $[a_j \mid 1 \leqslant j \leqslant k]$ does not exist, then $[x_i \mid 1 \leqslant i \leqslant k]$ is coherent, so it is strictly coherent exactly when $\sum_{i=1}^{k} x_i$ is *star-shaped*, that is, when there exists $a \in |\sum_{i=1}^{k} x_i|$ such that

$$\forall (a_j)_{1 \leqslant j \leqslant k} \in |X|^k, \left([a_j \mid 1 \leqslant j \leqslant k] \leqslant \sum_{i \in I} x_i \text{ and } a_k = a \right) \implies [a_j \mid 1 \leqslant j \leqslant k] \in \frown_X .$$

We now describe $\underset{\mathrm{BE}}{!}\mathbf{bool}$. We consider a multiset $[x_i \mid i \in I] \in \mathscr{M}_K(|\underset{\mathrm{BE}}{!} X|)$ and use k to denote its cardinality ($k = \sharp I \in K$). We let $p, q \geqslant 0$ be such that $\sum_{i \in I} x_i = p[\mathtt{t}] + q[\mathtt{f}]$.

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Then,

$$[x_i \mid i \in I] \in \smile_{\underset{\text{BE}}{!\mathbf{bool}}} \text{ iff } p \times q \neq 0 \text{ and } k \leqslant p + q$$

$$[x_i \mid i \in I] \in \frown_{\underset{\text{BE}}{!\mathbf{bool}}} \text{ iff } 1 \leqslant p + q < k$$

$$\text{hence } [x_i \mid i \in I] \in \mathsf{N}_{\underset{\text{BE}}{!\mathbf{bool}}} \text{ iff } \begin{cases} p = q = 0 \text{ or} \\ q = 0 \text{ and } k \leqslant p \text{ or} \\ p = 0 \text{ and } k \leqslant q. \end{cases}$$

Consider the following subset of $|(\underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool}) \multimap \mathbf{bool}|$:

$$\begin{aligned} f = \{\ & ((\mathtt{[]}, & [\mathtt{t},\mathtt{t}], & [\mathtt{f},\mathtt{f}]), & \mathtt{t}), \\ & ((\mathtt{[f,f]}, & \mathtt{[]}, & [\mathtt{t},\mathtt{t}]), & \mathtt{t}), \\ & ((\mathtt{[t,t]}, & \mathtt{[f,f]}, & \mathtt{[]}), & \mathtt{t}) \quad \}. \end{aligned} \tag{8}$$

This is a variant of Berry's well-known example of a stable and non-sequential function from $\mathbf{bool} \times \mathbf{bool} \times \mathbf{bool}$ to $\mathbf{bool}$. This function is not a morphism in the multiset-based hypercoherence semantics, but the $\mathbb{N}\backslash\{0,1\}$-coherence semantics with the $\underset{\text{BE}}{!}$ exponential accepts it[†]. Indeed, for each multiset $s \in \mathscr{M}_K(f)$, we have $\pi_2(s) \in \mathscr{M}_K(\{\mathtt{t}\})$, so $\pi_2(s) \in \mathsf{N}_{\mathbf{bool}}$. With respect to (5), the only way for s to be strictly incoherent in $(\underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool}) \multimap \mathbf{bool}$ is to have $\pi_1(s) \in \frown$ in $\underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool}$. But if m_1, m_2 and m_3 are the respective numbers of occurrences of points of f in s, then:

— If only one of the m_i is non-zero, each of the projection of $\pi_1(s)$ on the three arguments is neutral so $\pi_1(s)$ is neutral.
— If exactly two of the m_i are non-zero (say m_1 and m_2), the empty multiset does not occur in one of the three projections of $\pi_1(s)$ (here the third), so the sum of the multisets occurring in this projection contains enough $\mathtt{t}$ and $\mathtt{f}$ (here $2m_1\mathtt{f}$ and $2m_2\mathtt{t}$) compared with its cardinality (here $m_1 + m_2$) to make it strictly incoherent in $\mathbf{bool}$. Hence $\pi_1(s)$ is certainly strictly incoherent.
— Finally, if none of the m_i is zero, $\pi_1(s)$ is coherent if and only if each of its three projections on $\underset{\text{BE}}{!}\mathbf{bool}$ is coherent. That is, if and only if

$$m_1 + m_2 + m_3 > 2m_2 + 2m_3 \text{ (coherence on the first argument)}$$
$$m_1 + m_2 + m_3 > 2m_1 + 2m_3 \text{ (coherence on the second argument)}$$
$$m_1 + m_2 + m_3 > 2m_1 + 2m_2 \text{ (coherence on the third argument).}$$

But we will then have $3(m_1 + m_2 + m_3) > 4(m_1 + m_2 + m_3)$, which is impossible.

Thus f is definitely a clique of $(\underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool} \otimes \underset{\text{BE}}{!}\mathbf{bool}) \multimap \mathbf{bool}$.

The really surprising fact is that one can easily try to correct this by choosing another definition for the coherence relations of the exponential construction and thereby obtain a new semantics of linear logic. Among these variants for the exponentials, there is a

[†] Note that the same function with $[\mathtt{t}]$ instead of $[\mathtt{t},\mathtt{t}]$ and $[\mathtt{f}]$ instead of $[\mathtt{f},\mathtt{f}]$, is rejected by this semantics.

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most general one in a sense to be made precise in Theorem 1 and Corollary 1. Indeed, the definition is guided by the needs of Theorem 1.

First we adapt the notion of a section of hypercoherences to the K-coherence spaces setting.

Definition 7. If $\mu = [x_i \mid i \in I]$ is a multiset of finite sets or of multisets, and if s is another multiset, we say that s is a *section* of μ and we write $s \lhd \mu$ when $s = [a_i \mid i \in I]$ and $\forall i \in I, a_i \in x_i$. Note that s and μ have the same cardinality.

The notion of a section between *sets* that we have used up to this point for hypercoherences can be rephrased by saying that a set s is a section of a set x if and only if there exists two multisets μ and ν such that $\nu \lhd \mu$, $\mathrm{supp}(\mu) = x$ and $\mathrm{supp}(\nu) = s$. We will use the same name (section), but a different notation for the two notions: $\triangleleft$ between sets and $\lhd$ between multisets.

Definition 8. For each K-coherence space X, we define the K-coherence space $!X$ as follows. Its web is $|!X| = \mathscr{M}_{\mathrm{fin}}(|X|)$ and for each element $[x_i \mid i \in I]$ of $\mathscr{M}_K(|!X|)$, we set

$$[x_i \mid i \in I] \in \smile_{!X} \text{ iff } \exists (a_i)_{i\in I}, [a_i \mid i \in I] \in \smile_X \text{ and } \forall i \in I, a_i \in x_i \tag{9}$$

and

$$[x_i \mid i \in I] \in \mathsf{N}_{!X} \text{ iff } \begin{cases} [x_i \mid i \in I] \notin \smile_{!X} \text{ and} \\ \exists (a_i^j)_{i\in I}^{j\in J}, \begin{cases} \forall i \in I, [a_i^j \mid j \in J] = x_i \text{ and} \\ \forall j \in J, [a_i^j \mid i \in I] \in \mathsf{N}_X. \end{cases} \end{cases} \tag{10}$$

We also define $?X$ by setting $?X = (!X^{\perp})^{\perp}$.

When $K = \varnothing$, the exponential construction on objects is the standard exponential of **Rel**.

Example 4. The coherence in $!\mathbf{bool}$ is as follows. For $\mu \in \mathscr{M}_K(|\mathbf{bool}|)$,

$$\mu \in \frown_{!\mathbf{bool}} \text{ iff } \begin{cases} \mathrm{supp}(\mu) \subset \mathscr{M}_{\mathrm{fin}}(\{\mathtt{t}\}) \text{ or} \\ \mathrm{supp}(\mu) \subset \mathscr{M}_{\mathrm{fin}}(\{\mathtt{f}\}) \text{ or} \\ [] \in \mu \end{cases}$$

$$\mu \in \mathsf{N}_{!\mathbf{bool}} \text{ iff } \mathrm{supp}(\mu) = \{k[\mathtt{t}]\} \text{ or } \{k[\mathtt{f}]\}, k \in \mathbb{N}.$$

Using this exponential, our variant of Berry's example is successfully rejected: if $3 \in K$, then f (as previously defined in (8)) is not a clique. Take $[a, b, c] \in \mathscr{M}_K(f)$ where a, b and c are the three points of f, then each of the three projections of $[a, b, c]$ on $!\mathbf{bool}$ is strictly coherent, and so is the projection on $!\mathbf{bool} \otimes !\mathbf{bool} \otimes !\mathbf{bool}$, but the projection on **bool** is neutral so $[a, b, c] \in \smile_X$.

Example 5. Consider the K-coherence space G with web $|G| = \{a, b, c\}$ and such that, for each $u \in \mathscr{M}_{\mathrm{fin}}(|G|)$: $u \in \mathsf{N}_G$ if and only if $\mathrm{supp}(u)$ is a singleton; $u \in \frown_G$ if and only if $\sharp\,\mathrm{supp}(u) = 2$ and $u \in \smile_G$ if and only if $\mathrm{supp}(u) = \{a, b, c\}$. The space G is in fact isomorphic to the subspace of $\mathbf{bool}^3 \to \mathbf{bool}$ given by the set f of (8).

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Suppose $2 \in K$. All the sections of $[[a],[b,c]]$ are coherent in G, moreover, $[a]$ and $[b,c]$ do not have the same cardinality. So $[[a],[b,c]] \in \frown_{!G}$. Now suppose $3 \in K$. Then $[[a],[b,c],[b,c]]$ admits the strictly incoherent section $[a,b,c]$ but not $[[a],[a],[b,c]]$. Hence $[[a],[b,c],[b,c]] \in \smile_{!G}$ but $[[a],[a],[b,c]] \in \frown_{!G}$. So the coherence relations of $!G$ depends on multiplicities.

For each $k \in K$ such that $k \geqslant 3$, each $m \in \mathscr{M}_{\{k\}}(|G|)$ such that $\mathrm{supp}(m) = \{[a,b],[a,c]\}$ is strictly incoherent in $!G$, but if $2 \in K$, we have $[[a,b],[a,c]] \in \frown_{!G}$ (all the sections of $[[a,b],[a,c]]$ are coherent in G and b is not neutral with any element of $[a,c]$).

Finally, $[[a,b,c],[a,b,c],[a,b,c]]$ is an example of a non-neutral (strictly incoherent, here) multiset in $!G$ with a singleton support.

Proposition 3 (semantics of LL). Any category $\mathbf{NCOH}_K$ with the exponentials of Definition 8 is a semantics of linear logic (see Amadio and Curien (1998) and Bierman (1995)) and for each $K' \subseteq K$ (in particular, for $K' = \varnothing$) the functor $U_{K,K'}$ is logical with respect to the $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ LL semantics.

Proof. We equip $\mathbf{NCOH}_K$ with the comonad structure $(!, \mathrm{der}, \mathrm{dig})$ of **Rel** and exploit the fact that the required commutative diagrams already hold in **Rel** and therefore also in $\mathbf{NCOH}_K$. Hence, to check that $(!, \mathrm{der}, \mathrm{dig})$ is really a comonad, we only need to prove that if f is a clique of $X \multimap Y$ then $!f$ is a clique of $!X \multimap !Y$, that der_X is a clique of $!X \multimap X$ and that dig_X is a clique of $!X \multimap !!X$. The same applies for the monoidality of the adjunction: we only need to check that the **Rel** isomorphisms $!\top \cong 1$ and $!(X \& Y) \cong !X \otimes !Y$ are cliques (in both directions).

Let $[(x_j, y_j) \mid j \in J] \in \mathscr{M}_K(!f)$. If $[b_j \mid j \in J] \lhd [y_j \mid j \in J]$, then, by the construction of $!f$, there exists $[a_j \mid j \in J]$ such that $[(a_j, b_j) \mid j \in J] \in \mathscr{M}_K(f)$ and $[a_j \mid j \in J] \lhd [x_j \mid j \in J]$. Note that since f is a clique, we have $[(a_j, b_j) \mid j \in J] \in \frown_{X \multimap Y}$. In particular, if $[b_j \mid j \in J] \in \smile_Y$, then $[a_j \mid j \in J] \in \smile_X$. Hence, if $[y_j \mid j \in J]$ admits a strict incoherent section, then $(x_j)_{j \in J}$ admits one too. So

$$[y_j \mid j \in J] \in \smile_{!Y} \implies [x_j \mid j \in J] \in \smile_{!X}.$$

Now suppose $[x_j \mid j \in J] \in \frown_{!X}$ and $[y_j \mid j \in J] \in \mathsf{N}_{!Y}$. We must prove that $[x_j \mid j \in J] \in \mathsf{N}_{!X}$. There exists $(b^i_j)_{(i,j) \in I \times J}$ such that

$$\forall j \in J, y_j = [b^i_j \mid i \in I] \text{ and } \forall i \in I, [b^i_j \mid j \in J] \in \mathsf{N}_Y.$$

By the construction of $!f$, there exists $(a^i_j)_{(i,j) \in I \times J}$ such that

$$\forall (i,j) \in I \times J, (a^i_j, b^i_j) \in f \text{ and } \forall j \in J, x_j = [a^i_j \mid i \in I].$$

Since $[x_j \mid j \in J] \in \frown_{!X}$ for each $i \in I$, we have $[a^i_j \mid j \in J] \in \frown_X$. But, for each $i \in I$, $[(a^i_j, b^i_j) \mid j \in J] \in \mathscr{M}_K(f) \subseteq \frown_{X \multimap Y}$ and $[b^i_j \mid j \in J] \in \mathsf{N}_Y$, so $[a^i_j \mid j \in J] \in \mathsf{N}_X$ for each $i \in I$. Finally, $[x_j \mid j \in J] \in \mathsf{N}_{!X}$, which completes the proof that $!f$ is a clique.

The fact that der_X is a clique is straightforward, so we now prove that dig_X is a clique of $!X \multimap !!X$. Let $s = [(\sum_{i \in I_j} x^j_i, [x^j_i \mid i \in I_j]) \mid j \in J] \in \mathscr{M}_K(\mathrm{dig}_X)$. We prove the two parts of the statement on the right-hand side of (6) instantiated by $!X \multimap !!X$.

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Suppose $[[x_i^j \mid i \in I_j] \mid j \in J] \in \smile_{!!X}$. Then this multiset admits a section $[y_j \mid j \in J]$ strictly incoherent in $!X$. Hence this section $[y_j \mid j \in J]$ admits a section $[a_j \mid j \in J]$ strictly incoherent in X. Clearly, this last section is also a section of $[\sum_{i\in I} x_i^j \mid j \in J]$, so this multiset is strictly incoherent in $!X$.

Now suppose $[\sum_{i\in I_j} x_i^j \mid j \in J] \in \frown_{!X}$ and $[[x_i^j \mid i \in I_j] \mid j \in J] \in \mathsf{N}_{!!X}$. We have to prove that $[\sum_{i\in I_j} x_i^j \mid j \in J] \in \mathsf{N}_{!X}$. Since $[[x_i^j \mid i \in I_j] \mid j \in J] \in \mathsf{N}_{!!X}$, there exists a family $(y_i^j)_{i\in I}^{j\in J}$ such that for all $j \in J$, we have $[y_i^j \mid i \in I]$ equals $[x_i^j \mid i \in I_j]$ (so $I = I_j$ and $\sum_{i\in I} y_i^j = \sum_{i\in I_j} x_i^j$), and for all $i \in I$, we have $[y_i^j \mid j \in J] \in \mathsf{N}_{!X}$. Hence, for each $i \in I$, there exists a family $(a_{i,l}^j)_{l\in L_i}^{j\in J}$ such that for all $j \in J$, we have $y_i^j = [a_{i,l}^j \mid l \in L_i]$ and such that for all $l \in L_i$, we have $[a_{i,l}^j \mid j \in J] \in \mathsf{N}_X$. Without any loss of generality, the L_i can be chosen to be pairwise disjoint. Setting $L = \cup_{i\in I} L_i$, we then have $\sum_{l\in L} a_l^j = \sum_{i\in I_j} x_i^j$, and for all $l \in L$, we have $[a_l^j \mid j \in J] \in \mathsf{N}_X$. Hence $[\sum_{i\in I_j} x_i^j \mid j \in J] \in \mathsf{N}_{!X}$.

The set $\{([],*)\}$ is a clique of $!\top \multimap 1$ and the set $\{(*,[])\}$ is a clique of $1 \multimap !\top$ so $!\top \cong 1$. We now prove that $!(X\&Y) \cong !X \otimes !Y$ for each X and Y. The graph f of the bijection map

$$\begin{cases} \mathscr{M}_{\mathrm{fin}}(|X|) \times \mathscr{M}_{\mathrm{fin}}(|Y|) \to \mathscr{M}_{\mathrm{fin}}(|X\&Y|) \\ (x,y) \mapsto x + y \end{cases}$$

is a relational isomorphism. It remains to prove that f is a clique of $!(X\&Y) \multimap (!X \otimes !Y)$ and that its transpose is a clique of $(!X \otimes !Y) \multimap !(X\&Y)$. Consider a multiset $[((x_i,y_i), x_i + y_i) \mid i \in I] \in \mathscr{M}_K(f)$. Since an element of $\smile_{X\&Y}$ is either an element of $\smile_X$ or an element of $\smile_Y$, a section s of $[x_i + y_i \mid i \in I]$ is strictly incoherent in $X\&Y$ if and only if s is a strictly incoherent section of $[x_i \mid i \in I]$ or of $[y_i \mid i \in I]$. It follows that

$$[x_i + y_i \mid i \in I] \in \smile_{!(X\&Y)} \iff [(x_i,y_i) \mid i \in I] \in \smile_{!X\otimes !Y}.$$

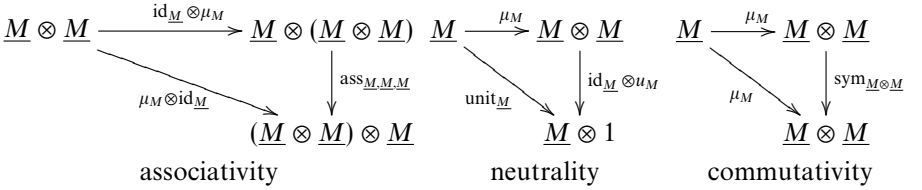
An element of $\mathsf{N}_{X\&Y}$ is either an element of N_X or an element N_Y. Hence, if $[x_i+y_i \mid i \in I]$ is neutral in $!(X\&Y)$, there exists a family $(c_i^j)_{i\in I}^{j\in J}$ such that for each $j \in J$, $[c_i^j \mid i \in I] \in \mathsf{N}_{X\&Y}$ and such that $J = J_X + J_Y$ with, for each $i \in I$, $[c_i^j \mid j \in J_X] = x_i$ and $[c_i^j \mid j \in J_Y] = y_i$, and this family splits into two families, the first corresponding to the neutrality of $[x_i \mid i \in I]$ in $!X$ and the second to the neutrality of $[y_i \mid i \in I]$ in $!Y$. Consequently, the neutrality of $[x_i + y_i \mid i \in I]$ in $!(X\&Y)$ implies the neutrality of $[(x_i,y_i) \mid i \in I]$ in $!X \otimes !Y$. The converse is straightforward. So the required isomorphisms $!\top \cong 1$ and $!(X\&Y) \cong !X \otimes !Y$ hold. Finally, we obtain for free that these two isomorphisms are natural and that the adjunction induced by the comonad is monoidal (Bierman 1995) simply by using the fact that this is already the case in **Rel**. □

4.4. *The* of course *is the co-free commutative* ⊗*-comonoid*

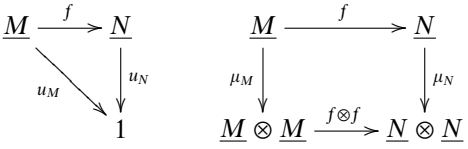
A commutative comonoid M on a symmetric monoidal category $\mathscr{C}$ with respect to a monoidal structure $(\otimes, \mathrm{sym}, \mathrm{ass}, \mathrm{unit})$ is a 3-tuple $(\underline{M}, u_M, \mu_M)$, where $\underline{M} \in \mathscr{C}$, $u_M \in \mathscr{C}(\underline{M}, 1)$

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and $\mu_M \in \mathscr{C}(\underline{M}, \underline{M} \otimes \underline{M})$, such that the following diagrams commute:



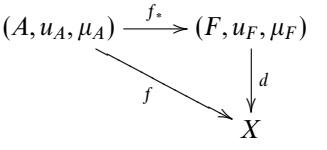
A comonoid morphism f from $(\underline{M}, u_M, \mu_M)$ to $(\underline{N}, u_N, \mu_N)$ is a morphism $f \in \mathscr{C}(M, N)$ such that the following diagrams commute:



In any categorical semantics $\mathscr{C}$ of linear logic, the 'of course' naturally provides a commutative comonoid $(!X, \mathrm{weak}, \mathrm{cont})$ for each object X: weak_X is $!\top_X$ where $\top_X$ is the unique morphism of $\mathscr{C}(X, \top)$ and cont_X is $(!\langle \mathrm{id}_X, \mathrm{id}_X \rangle) \fatsemi e_X$ where $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$ denotes the pairing of the identity with itself and where e_X is the isomorphism $!(X \,\&\, X) \cong !X \otimes !X$. Moreover, for each $f \in \mathscr{C}(X, Y)$, we have $!f$ is a $\otimes$-comonoid morphism between $(!X, \mathrm{weak}_X, \mathrm{cont}_X)$ and $(!Y, \mathrm{weak}_Y, \mathrm{cont}_Y)$.

In $\mathbf{NCOH}_K$, we have $\mathrm{weak}_X = \{([], *)\}$ and $\mathrm{cont}_X = \{(x_1 + x_2, (x_1, x_2)) \mid x_1, x_2 \in |!X|\}$.

A commutative comonoid (F, u_F, μ_F) is said to be co-free over an object X of $\mathscr{C}$ when there exists a morphism $d \in \mathscr{C}(F, X)$ such that for each commutative comonoid (A, u_A, μ_A) and for each $f \in \mathscr{C}(A, X)$ there exists a unique comonoid morphism f_* from (A, u_A, μ_A) to (F, u_F, μ_F) such that $f_* \fatsemi d = f$:



By extension, the 'of course' ! is said to be the co-free commutative $\otimes$-comonoid or, for short, to be co-free, when for each commutative comonoid (A, u_A, μ_A), each $X \in \mathscr{C}$ and each $f \in \mathscr{C}(A, X)$ there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{weak}_X, \mathrm{cont}_X)$ such that

$$f_* \fatsemi \mathrm{der}_X = f.$$

Remark 4. If ! is co-free, then $f_* = \mathrm{id}_* \fatsemi !f$ where id is the identity morphism in $\mathscr{C}(A, A)$.

Lemma 1. In **Rel** the exponential is co-free. Moreover, if (A, u_A, μ_A) is a commutative $\otimes$-comonoid in **Rel**, then $(a, x) \in (\mathrm{id}_A)_*$ if and only if we have if $(a_i)_{1 \leqslant i \leqslant n}$ is such that $[a_1, \ldots, a_n] = x$, then $\exists (b_i)_{0 \leqslant i \leqslant n}$ such that $b_0 = a$, $(b_i, (a_{i+1}, b_{i+1})) \in \mu_A$ for each $i < n$, and $(b_n, *) \in u_A$.

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Theorem 1 (co-free). The 'of course' ! is the co-free commutative $\otimes$-comonoid of $\mathbf{NCOH}_K$, and the forgetful functor $U_{K,\varnothing} : \mathbf{NCOH}_K \to \mathbf{Rel}$ maps this structure to the standard one.

Proof. We prove that for each commutative comonoid (A, u_A, μ_A) of $\mathbf{NCOH}_K$, each $X \in \mathbf{NCOH}_K$ and each $f \in \mathbf{NCOH}_K(A, X)$ there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{der}, \mathrm{cont})$ such that $f_* \fatsemi \mathrm{der} = f$.

But if there is such an f_* in $\mathbf{NCOH}_K$, then $U_{K,\varnothing}(f_*)$ is a comonoid morphism from $(U_{K,\varnothing}(A), U_{K,\varnothing}(u_A), U_{K,\varnothing}(\mu_A))$ to $(U_{K,\varnothing}(!X), U_{K,\varnothing}(\mathrm{der}_X), U_{K,\varnothing}(\mathrm{cont}))$ and

$$U_{K,\varnothing}(f_*) \fatsemi U_{K,\varnothing}(\mathrm{der}_X) = U_{K,\varnothing}(f).$$

As $(U_{K,\varnothing}(!X), U_{K,\varnothing}(\mathrm{der}_X), U_{K,\varnothing}(\mathrm{cont}))$ is the co-free $\otimes$-comonoid in **Rel**, this means that in **Rel**,

$$U_{K,\varnothing}(f_*) = U_{K,\varnothing}(f)_*.$$

Moreover,

$$U_{K,\varnothing}(f)_* = U_{K,\varnothing}(\mathrm{id})_* \fatsemi !U_{K,\varnothing}(f)$$

and

$$!U_{K,\varnothing}(f) = U_{K,\varnothing}(!f).$$

So the only thing to prove is that $U_{K,\varnothing}(\mathrm{id})_*$ is a clique of $\mathrm{Cl}(A \multimap !A)$.

Let $[(a^i, [a^i_1, \ldots, a^i_{n_i}]) \mid i \in I]$ be an element of $\mathscr{M}_K(\mathrm{id}_*)$. Then, using Lemma 1, for each $i \in I$, let $(b^i_j)_{0 \leqslant j \leqslant n_i}$ be a family such that $b^i_0 = a^i$, $(b^i_j, (a^i_{j+1}, b^i_{j+1})) \in \mu_A$ for each $j < n_i$ and $(b^i_{n_i}, *) \in u_A$.

If we assume $[[a^i_1, \ldots, a^i_{n_i}] \mid i \in I] \in \smile_{!A}$, then this multiset admits a strict incoherent section. Up to a choice of an adequate indexation of the multiset $[a^i_1, \ldots, a^i_{n_i}]$, we can suppose without any loss of generality that this section is $[a^i_1 \mid i \in I]$. Note that due to the existence of a section, none of the n_i is zero. We then have $[(a^i, (a^i_1, b^i_1)) \mid i \in I] \in \mathscr{M}_K(\mu_A)$ with $[a^i_1 \mid i \in I] \in \smile_A$. Hence $[(a^i_1, b^i_1) \mid i \in I] \in \smile_{A \otimes A}$. And, since $[(a^i, (a^i_1, b^i_1)) \mid i \in I]$ must be coherent for μ_A to be a clique of $A \multimap (A \otimes A)$, we then have $[a^i \mid i \in I] \in \smile_A$.

Now suppose $[a^i \mid i \in I] \in \frown_A$ and $[[a^i_1, \ldots, a^i_{n_i}] \mid i \in I] \in \mathsf{N}_A$. According to the definition of neutrality in the 'of course', all the n_i are equal, say $n_i = n (\forall i \in I)$, and, up to an appropriate re-indexing, $[a^i_j \mid i \in I] \in \mathsf{N}_A$, for each $1 \leqslant j \leqslant n$. Since $[(b^i_n, *) \mid i \in I] \in \mathscr{M}_K(u_A) \subseteq \frown_{A \multimap 1}$ and $[* \mid i \in I] \in \mathsf{N}_1$, this means that $[b^i_n \mid i \in I] \in \asymp_A$. Now suppose $[b^i_{k+1} \mid i \in I] \in \asymp_A$ for a certain $k < n$. Then, using $[a^i_{k+1} \mid i \in I] \in \mathsf{N}_A$ and $[(b^i_k, (a^i_{k+1}, b^i_{k+1})) \mid i \in I] \in \mathscr{M}_K(\mu_A)$, it follows that $[b^i_k \mid i \in I] \in \asymp_A$. Thus, for all $j \leqslant n$, we have $[b^i_j \mid i \in I] \in \asymp_A$ and, in particular, $[b^i_0 \mid i \in I] = [a^i \mid i \in I]$ is then proved to be both coherent and incoherent, that is, to be neutral. So id_* is a clique. □

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Consider a sub-category $\mathscr{C}$ of $\mathbf{NCOH}_K$ that is a categorical semantics of intuitionistic multiplicative exponential linear logic[†]. Let E be the operation modelling the 'of course' on objects in $\mathscr{C}$. We shall say that this semantics is *multiset-based* if for each $X \in \mathscr{C}$:

— The web of $E(X)$ is made of multisets of points of the web of X (that is, $|E(X)| \subseteq \mathscr{M}_{\text{fin}}(|X|)$).
— The commutative comonoid structure provided with $E(X)$ by the semantics is defined by $\text{weak}'_X = \{([], *)\}$ (of type $E(X) \to 1$) and

$$\text{cont}'_X = \{(x_1 + x_2, (x_1, x_2)) \mid x_1 + x_2 \in |E(X)| \text{ and } x_1, x_2 \in \mathscr{M}_{\text{fin}}(|X|)\}$$

(of type $E(X) \to E(X) \otimes E(X)$).
— The associated dereliction morphism is

$$\text{der}'_X = \{([a], a) \mid a \in |X|\} \text{ (of type } E(X) \to X).$$

Corollary 1 (maximality of the co-free 'of course'). If a sub-monoidal category $\mathscr{C}$ of $\mathbf{NCOH}_K$ is a multiset-based LL semantics of 'of course' E, then for each object $X \in \mathscr{C}$,

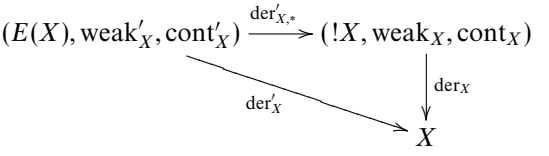
$$\supset_{E(X)} \subseteq \supset_{!X} \tag{11}$$

and

$$\frown_{E(X)} \subseteq \frown_{!X}. \tag{12}$$

Where 'sub-monoidal category' means that $\mathscr{C}$ is a sub-category of $\mathbf{NCOH}_K$ equipped with the same symmetric monoidal structure as $\mathbf{NCOH}_K$.

Proof. Since $\mathscr{C}$ is a semantics of linear logic, $E(X)$ comes with a $\otimes$-comonoid structure $(E(X), \text{weak}'_X, \text{cont}'_X)$ where weak'_X is the weakening morphism and cont'_X is the contraction morphism. Let der'_X be the dereliction morphism for X of $\mathscr{C}$. Using Theorem 1, there exists a morphism $\text{der}'_{X,*}$ of $E(X) \multimap !X$ such that $\text{der}'_{X,*} \fatsemi \text{der}_X = \text{der}'_X$:



We claim that $\text{der}'_{X,*}$ is equal to $\{(x, x) \mid x \in |E(X)|\}$ (the inclusion morphism of $E(X)$ in $!X$). Once we have proved this result, (5) will give us (11) and (12).

It remains to prove our claim. We can work in **Rel** since all the above hypotheses ($\otimes$-comonoid, ...) also hold in **Rel**. Since $\mathscr{C}$ is multiset-based, $\text{der}'_X = \{([a], a) \mid a \in |X|\}$. Hence $!\,\text{der}'_X = \{([[a_1], \ldots, [a_n]], [a_1, \ldots, a_n]) \mid [a_1, \ldots, a_n] \in |E(X)|\}$. We have (Remark 4) $\text{der}'_{X,*} = \text{id}_{E(X),*} \fatsemi \text{der}'_X$. We will use Lemma 1 applied to the $\otimes$-comonoid $(E(X), \text{weak}'_X, \text{cont}'_X)$ to prove that $(a, [[a_1], \ldots, [a_n]]) \in \text{id}_{E(X),*}$ if and only if $a =$

[†] We do not require that $\mathscr{C}$ satisfies more that this, but a typical $\mathscr{C}$ for our purposes will be a new Seely category where the multiplicative additive and orthogonal constructions are the ones of $\mathbf{NCOH}_K$, so one should have a semantics for the full linear logic fragment, where the exponentials are given by a comonad.

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$[a_1,\ldots,a_n]$. This will then allow us to conclude. Up to reindexing, if b_i is a family as described in the Lemma, then $b_0 = [a_1] + b_1$, $b_1 = [a_2] + b_2$, $\ldots$ $b_{n-1} = [a_n] + b_n$ (because $(b_i,([a_{i+1}],b_{i+1}) \in \mathrm{cont}'_X)$) and $b_n = []$ (because $(b_n,*) \in \mathrm{weak}'_X)$), so $a = b_0 = [a_1] + \ldots + [a_n] + [] = [a_1,\ldots,a_n]$. Conversely, it suffices to set $b_i = [a_{i+1},\ldots,a_n]$ for each $0 \leqslant i \leqslant n$ to obtain a family $(b_i)_{0\leqslant i\leqslant n}$ as in the Lemma and such that $(b_0,[[a_1],\ldots,[a_n]]) \in \mathrm{id}_{E(X),*}$. □

We shall say that a multiset-based semantics of LL in a sub-category $\mathscr{C}$ of $\mathbf{NCOH}_K$ is *non-uniform* when the web of the 'of course' E is the whole set of finite multisets (that is, $|E(X)| = \mathscr{M}_{\mathrm{fin}}(|X|), \forall X \in \mathscr{C}$).

Corollary 2 (sequentiality failure). Every non-uniform multiset-based semantics of LL in a sub-monoidal category $\mathscr{C}$ of $\mathbf{NCOH}_K$ fails to reject the morphism

$$\{([\mathtt{t}],\mathtt{t}),([\mathtt{f}],\mathtt{t}),([\mathtt{t},\mathtt{f}],\mathtt{t})\}$$

of type **bool** → **bool**.

Proof. It is easy to verify that $\mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(\{[\mathtt{t}],[\mathtt{f}],[\mathtt{t},\mathtt{f}]\}) \cap \frown_{!\mathbf{bool}} = \varnothing$, so the above morphism is indeed accepted by the $\mathbf{NCOH}_{\mathbb{N}\setminus\{0,1\}}$ semantics. So this set is *a fortiori* a morphism in each $\mathbf{NCOH}_K$ semantics, and, by the maximality of the 'of course', in any multiset-based non-uniform semantics in a sub-monoidal category of $\mathbf{NCOH}_K$. □

This is a strong negative results since this set cannot be included in the interpretation of a term of PCF. Such a term would have to test the value of its boolean input once and terminate with answer true in both cases (input is true or input is false). But it would also have to test its argument again, at least in one of the two cases, *before* answering. This gives a contradiction.

Our feeling is that it will be the same for any reasonably sequential calculus interpretable in our semantics of linear logic. Note that the very similar morphism

$$\{([\mathtt{t},\mathtt{t}],\mathtt{t}),([\mathtt{f},\mathtt{f}],\mathtt{t}),([\mathtt{t},\mathtt{f}],\mathtt{t})\}$$

is the interpretation of

$$\lambda b.\,\mathtt{if}\ b\ \mathtt{then}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{t})\ \mathtt{else}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{t}\ \mathtt{else}\ \mathtt{t}).$$

4.5. *Determinism*

From now on, we will assume that K is a *non-empty* subset of $\mathbb{N}\setminus\{0,1\}$. In that case the power $\mathscr{M}_K$ is strictly monotone and preserves disjointness.

Definition 9. Let $\mathbf{NCoh}_K$ be the full sub-category of $\mathbf{NCOH}_K$ whose objects are the weakly reflexive K-coherence spaces.

Recall that being weakly reflexive for a K−*coherence* space X means that

$$\mathsf{N}_X \subseteq \cup_{a\in|X|}\mathscr{M}_K(\{a\}). \tag{13}$$

Clearly, $\mathbf{NCoh}_K$ is closed under the orthogonal, additive and multiplicative constructions. It is easy to verify that this is also the case for the exponential construction. Indeed,

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assume X is weakly reflexive and consider a neutral multiset $[x_i \mid i \in I]$ in $!X$. Then there exists a family $(a_i^j)_{i\in I}^{1\leqslant j\leqslant p}$ such that for each $i \in I$ we have $x_i = [a_i^j \mid 1 \leqslant j \leqslant p]$ and for each $1 \leqslant j \leqslant p$ we have $[a_i^j \mid i \in I] \in \mathsf{N}_X$. So, using the weak reflexivity of X, we get that there exists a family $(a^j)_{1\leqslant j\leqslant p}$ such that $a_i^j = a^j(\forall i, j)$ and, consequently, all the x_i are equal.

Hence this sub-category is a denotational semantics of propositional linear logic. Each forgetful functor $U_{K,K'}$ between $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ (for $K' \subseteq K$) defines a forgetful functor between $\mathbf{NCoh}_K$ and $\mathbf{NCoh}_{K'}$ having similar properties and for which we use the same notation $U_{K,K'}$.

Proposition 4 (determinism). If $X \in \mathbf{NCoh}_K$ and x is a clique of X and y is an anti-clique of X (that is a clique of $X^\perp$), then $\sharp(x \cap y) \leqslant 1$.

This is a direct consequence of Proposition 1.

Remark 5. It is worth noting that in non-uniform K-coherence semantics we do not have the second part of Property 1 since if $(a, c) \in f \fatsemi g$ for $f : X \to Y$ and $g : Y \to Z$, there exists a b such that $(a, b) \in f$ and $(b, c) \in g$ but b is not necessarily unique. But uniqueness of b holds again if $\mathscr{M}_K(\{(a, c)\}) \subseteq \mathsf{N}_{X\to Z}$ and, moreover, in that case b is such that $\mathscr{M}_K(\{b\}) \subseteq \mathsf{N}_Y$.

In the following we will no longer be interested in non-weakly reflexive K-coherence spaces or the category $\mathbf{NCOH}_K$, and the most general category will be $\mathbf{NCoh}_K$.

5. Relating uniform and non-uniform semantics

We will now define uniform K-coherence semantics and relate them to non-uniform K-coherence semantics.

5.1. *The neutral web*

Proposition 4 can be made more precise since only certain points can be at the intersection of a clique and an anti-clique. These points constitute the *neutral web*.

Definition 10. Let $X \in \mathbf{NCoh}_K$. We call the set $\{a \in |X| \mid \mathscr{M}_K(\{a\}) \subseteq \mathsf{N}_X\}$ the *neutral web* of X and denote it by $|X|_{\mathsf{N},K}$ (or simply by $|X|_{\mathsf{N}}$).

Remark 6. In the relational model case, $K = \varnothing$, the neutral web is the whole web: $|X|_{\mathsf{N},\varnothing} = |X|$.

Example 6. For the K-coherence space G of Example 5 we have $[a, b] \in |!G|_{\mathsf{N},K}$, and $[a, b, c] \in |!G|_{\mathsf{N},K}$ if $K \subseteq \{2\}$ and $[a, b, c] \notin |!G|_{\mathsf{N},K}$ elsewhere.

A key result for the neutral web is its behaviour when an 'of course' construction is performed.

Lemma 2 (key lemma). For $X \in \mathbf{NCoh}_K$,

$$|!X|_{\mathsf{N},K} = \{x \in \mathscr{M}_{\mathrm{fin}}(|X|_{\mathsf{N},K}) \mid \mathrm{supp}(x) \in \mathrm{Cl}(X)\}.$$

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Proof. Let $x \in |!X|_{\mathsf{N},K}$. Then for all $k \in K$, there exists a family $(a_i^j)_{1 \leqslant i \leqslant k}^{j \in J}$ such that $[a_i^j \mid j \in J] = x$ and $[a_i^j \mid 1 \leqslant i \leqslant k] \in \mathsf{N}_X$. From (13), for each $j \in J$, we have $a_1^j = \ldots = a_k^j$. Hence, for all $k \in K$ and all $a \in x$, we have $k.[a] \in \mathsf{N}_X$. So $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. Each $y \in \mathscr{M}_K(\mathrm{supp}(x))$ is a section of the multiset $(\sharp y).[x] \in \mathsf{N}_{!X} \subseteq \supset_{!X}$, so $\mathrm{supp}(x)$ is a clique. Thus the left to right inclusion is proved. Conversely, let $x \in \mathscr{M}_{\mathrm{fin}}(|X|_{\mathsf{N},K})$. If $\mathrm{supp}(x)$ is a clique, $k.[x] \in \supset_{!X}$ for any $k \in K$. Moreover, each of the elements a of x satisfies $k.[a] \in \mathsf{N}_X$, so $k.[x] \in \mathsf{N}_{!X}$ for any $k \in K$, which proves the right to left inclusion. □

Example 7. In $(!G)^\perp$, the set $x = \{[a,b],[a,c]\} \subseteq |(!G)^\perp|_{\mathsf{N},K}$ is not a clique if $2 \in K$ but is a clique if $2 \notin K$. Hence $[[a,b],[a,c]] \notin |!(!G)^\perp|_{\mathsf{N},\{2\}}$ and $[[a,b],[a,c]] \in |!(!G)^\perp|_{\mathsf{N},\{3\}}$.

This lemma teaches us that a restriction to the reflexive subspaces of K-coherence spaces has a good chance of providing us with a new version of the semantics that is comparable to the multiset-based coherence space semantics, Equation (4), when $K = \{2\}$. This will be shown, amongst other things, in the next section. A more direct consequence, which will be shown in Proposition 5, is that such a restriction can be performed at any inductive step of the interpretation. Provided it is performed at the last step, the resulting reflexive object (in the case of the interpretation of a formula) or morphism (in the case of the interpretation of a proof) will be the same.

Definition 11. If $X \in \mathbf{NCoh}_K$, the *neutral restriction* of X is the *sub-space* of X of web $|X|_{\mathsf{N}}$, that is $(|X|_{\mathsf{N}}, \supset_X \cap M, \asymp_X \cap M)$ where $M = \mathscr{M}_K(|X|_{\mathsf{N}})$, and the neutral restriction of a clique x of X is $x \cap |X|_{\mathsf{N}}$. The functor $N_K : \mathbf{NCoh}_K \to \mathbf{NCoh}_K$, which will sometimes be simply denoted by N, associates to objects and morphisms their neutral restrictions.

It is easy to verify that N_K is indeed a functor.

Remark 7. The K-coherence space $|X|_{\mathsf{N}}$ is reflexive. Moreover, it is the maximal reflexive subspace of X.

Proposition 5. The functor N_K commutes with all the multiplicative additive constructions. Moreover, $N_K ! = N_K !N_K$.

Proof. The first statement is an obvious consequence of the corresponding definitions.

On objects, $N_K ! = N_K !N_K$ is a consequence of Lemma 2. Indeed, in the right-hand side of the equality established in this lemma, $\mathrm{Cl}(X)$ can be replaced with $\mathrm{Cl}(N_K X)$ since $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. This gives $|!X|_{\mathsf{N},K} = |!N_K X|_{\mathsf{N},K}$, which is what we required. The equality $N_K ! = N_K !N_K$ on morphisms is then a straightforward consequence of the equality on objects. □

5.2. *Uniform K-coherence semantics*

In this section we define a uniform K-coherence semantics in the full sub-category of $\mathbf{NCoh}_K$ that objects are reflexive K-coherence spaces. So uniform will be synonymous with reflexive for objects of $\mathbf{NCoh}_K$.

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We use $\mathbf{Coh}_K$ to denote the full sub-category of $\mathbf{NCoh}_K$ whose objects are the uniform K-coherence spaces.

A uniform $\{2\}$-coherence space is just an ordinary coherence space.

The functor N_K maps $\mathbf{NCoh}_K$ to $\mathbf{Coh}_K$, and N_K acts like the identity functor on $\mathbf{Coh}_K$.

The additive and multiplicative constructions of $\mathbf{NCoh}_K$ preserve uniform K-coherence spaces. This is not the case for the 'of course' functor. Fortunately, Lemma 2 gives a clear hint as to what should be the correct exponentials for $\mathbf{Coh}_K$.

Definition 12. We define the functor $\underset{u}{!}$ interpreting the 'of course' in $\mathbf{Coh}_K$ by setting $\underset{u}{!} = N_K !$. We use $\underset{u}{?}$ to denote the corresponding 'why not' functor.

The web of $\underset{u}{!} X$, called the *uniform web*, is then

$$|\underset{u}{!} X| = \{x \in \mathscr{M}_{\mathrm{fin}}(|X|) \mid \mathrm{supp}(X) \in \mathrm{Cl}(X)\}$$

and the coherence of $\underset{u}{!} X$ is then given by

$$M \in \frown_{\underset{u}{!} X} \text{ iff } \forall m \lhd M, m \in \frown_X.$$

This definition of the exponentials appears as a *multiplicities aware* version of the hypercoherences exponentials introduced in Ehrhard (1993).

As shown by the following theorem, these definitions give rise to a new class of uniform semantics together with a straightforward way to extract these interpretations from the non-uniform ones.

Theorem 2. For each $K \subseteq \mathbb{N}\backslash\{0,1\}$, $\mathbf{Coh}_K$ equipped with the uniform exponentials and the standard multiplicative additive structures of $\mathbf{NCoh}_K$ is a categorical semantics of linear logic. Moreover:

(1) The functor $N_K : \mathbf{NCoh}_K \to \mathbf{Coh}_K$ is *logical*, which means, in particular, that the *neutral restriction* of the K-coherence space $[A]_K$ is the uniform K-coherent interpretation $[A]^u_K$ of a formula A and that the neutral restriction $[\pi]_K \cap |\vdash \Gamma|_{\mathbb{N},K}$ of the K-coherence interpretation of a proof π of a sequent $\vdash \Gamma$ is the uniform K-coherence interpretation $[\pi]^u_K$ of π.

(2) When $K = \{2\}$, this semantics is exactly the usual multiset-based coherence semantics.

Proof. The multiplicative–additive part of the verification of the fact that $\mathbf{Coh}_K$ is a semantics of linear logic is easy and relies essentially on the fact that N commutes with all the additive and multiplicative constructions.

The exponential part is also reasonably straightforward. By setting $\mathrm{der}_{u,X} = N(\mathrm{der}_X)$ and $\mathrm{dig}_{u,X} = N(\mathrm{dig}_X)$ for each $X \in \mathbf{Coh}_K$, we obtain two natural transformations $\mathrm{der}_u : N! \dot{\to} N\,\mathrm{id}$ and $\mathrm{dig}_u : N! \dot{\to} N!!$ in $\mathbf{Coh}_K$.

But N is the identity functor on $\mathbf{Coh}_K$, $N! = \underset{u}{!}$ and, using Proposition 5 ($N! = N!N$ in $\mathbf{NCoh}_K$), we obtain $N!! = \underset{u\,u}{!\,!}$ and $N!!! = \underset{u\,u\,u}{!\,!\,!}$. So der_u and dig_u are in fact natural transformations $\mathrm{der}_u : \underset{u}{!} \dot{\to} \mathrm{id}$ and $\mathrm{dig}_u : \underset{u}{!} \dot{\to} \underset{u\,u}{!\,!}$.

These two natural transformations endow $\underset{u}{!}$ with a comonad structure. In fact, we can deduce the commutation of the required diagrams from the commutation of the

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corresponding diagrams already holding for the comonad $(!, \mathrm{der}, \mathrm{dig})$ using the functor N. The only non-obvious step is then to prove that for each $X \in \mathbf{Coh}_K$,

$$N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X}$$

and

$$N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}.$$

This can be done as follows. For all $f \in \mathbf{NCoh}_K(N!X, !X)$,

$$!f \fatsemi \mathrm{dig}_{!X} = \mathrm{dig}_{N!X} \fatsemi f,$$

so

$$N(!f \fatsemi \mathrm{dig}_{!X}) = N(\mathrm{dig}_{!N!X} \fatsemi f),$$

so

$$N!f \fatsemi N\,\mathrm{dig}_{!X} = N(\mathrm{dig}_{N!X}) \fatsemi Nf. \tag{14}$$

The set $\mathrm{id}_{N!X}$ is clearly a clique of $N!X \multimap !X$, so it can be seen as an (inclusion) morphism i from $N!X$ to $!X$. We then have the set equalities

$$N!i = \mathrm{id}_{\underset{u}{!}\,\underset{u}{!}X}$$

$$Ni = \mathrm{id}_{\underset{u}{!}X}.$$

So, finally, by taking $f = i$ in Equation (14), we obtain the set equality

$$N\,\mathrm{dig}_{!X} = N\,\mathrm{dig}_{N!X},$$

that is,

$$N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X}.$$

Starting from the equation

$$N(\mathrm{der}_{!X} \fatsemi i) = N(!i \fatsemi \mathrm{der}_{!X}),$$

we can prove

$$N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}$$

in the same way.

Using Proposition 5, we obtain the isomorphisms $\underset{u}{!}A \otimes \underset{u}{!}B \cong \underset{u}{!}(A\&B)$ and $\underset{u}{!}\top \cong 1$.

$\mathbf{Coh}_K$ has been proved to be a categorical semantics of linear logic, and there is nothing more to say to prove that N is logical.

The comonoid structure of the exponential $\underset{u}{!}$ is then the image of the comonoid structure of the exponential $!$ of $\mathbf{NCoh}_K$ through the functor N. The fact that $(\underset{u}{!}, \mathrm{der}_u)$ is co-free relies essentially on the set equality $\mathrm{der}_{u,X} = \mathrm{der}_X$ $(\forall X \in \mathbf{Coh}_K)$, which is just a consequence of the fact that all singletons are cliques in $\mathbf{Coh}_K$. In fact, given a commutative comonoid (A, u_A, μ_A) of $\mathbf{Coh}_K$ and $f \in \mathbf{Coh}_K(A, X)$, we have

$$N(f_*) \fatsemi \mathrm{der}_{u,X} = N(f_* \fatsemi \mathrm{der}_X)$$

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for each $f \in \mathbf{Coh}_K(A, X)$ where f_* is the unique comonoid morphism $A \to !X$ such that $f_* \fatsemi \mathrm{der}_X = f$. But $N(f_*) : A \to \underset{u}{!} X$ is also a comonoid morphism. Note that the inclusion morphism $i : \underset{u}{!} X \to !X$ is a comonoid morphism, so $N(f_*) \fatsemi i : A \to !X$ is a comonoid morphism. We also have the set equalities

$$N(f_*) \fatsemi i = N(f_*)$$

and, because $\mathrm{der}_X = \mathrm{der}_{u,X}$,

$$N(f_*) \fatsemi i \fatsemi \mathrm{der}_X = N(f_*) \fatsemi \mathrm{der}_{u,X} = f.$$

By the uniqueness of f_*, we have $N(f_*) \fatsemi i$ equals f_*, so we finally obtain the set equality $f_* = N(f_*)$, and the co-freeness of $\underset{u}{!}$ then follows.

Finally, $[x, y] \in \multimap_{\underset{u}{!} X}$ if and only if $\forall a \in x, \forall b \in y, [a, b] \in \multimap_X$, that is, in $\mathbf{Coh}_{\{2\}}$, if and only if $\mathrm{supp}(x + y)$ is a clique. So in $\mathbf{Coh}_{\{2\}}$, which is the category of coherence spaces, $\underset{u}{!}$ is the well-known multiset-based exponential of coherence spaces. □

Spelling out the categorical definition of the semantics, the interpretation of linear logic in $\mathbf{Coh}_K$ is now defined as its interpretation in **Rel** for the multiplicative–additive and identity groups and with an exponential group similarly defined but using uniform exponentials and the restriction they induce on the interpretation of proofs.

The promotion and contraction rule cases are subject to the standard restrictions: in the case of the contraction, we take only the $(\gamma, \mu_1 + \mu_2)$ such that $\mathrm{supp}(\mu_1 + \mu_2)$ is a clique of $[A]_K{}^{\perp}$, and for the promotion, in $f^{\dagger}$, we take only the points such that for each $i \leqslant k$ we have $\mathrm{supp}(\sum_{j \in J} \mu_i^j) \in \mathrm{Cl}([A_i]_K{}^{\perp})$. As with the usual coherence spaces and hypercoherences, this condition is sufficient to ensure that $[a_1, \ldots, a_k] \in \mathrm{Cl}([A]_K)$ (under the assumption that f is truly a clique).

As pointed out by an anonymous referee, it would be interesting to compare the equational theories induced by uniform K-coherence semantics for different K. Any uniform K-coherence semantics defines the following equivalence relation on proofs with the same conclusions: $\pi \sim_K \pi'$ if and only if $[\pi]^u_K = [\pi']^u_K$. We believe that for different K, say $K \neq K'$, these equivalences are in general different ($\sim_K \neq \sim_{K'}$). But at the moment, we can only make the following remark.

Remark 8. If $K \subseteq K'$, then $[\pi]^u_K = [\pi']^u_K$ implies $[\pi]^u_{K'} = [\pi']^u_{K'}$.

Proof. If $K \subseteq K'$, then $|X|_{\mathsf{N},K'} \subseteq |X|_{\mathsf{N},K}$, for any weakly reflexive K'-coherence space X. Theorem 2 states that for all $J \subseteq \mathbb{N} \backslash \{0, 1\}$, we have $[\pi]^u_J = [\pi]_J \cap |\vdash \Gamma|_{\mathsf{N},J}$ where $\vdash \Gamma$ is the conclusion of π. And we have the set equality $[\pi]_J = [\pi]_{\varnothing}$. Hence the statement of the remark holds. □

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5.3. *Multicoherences*

We call the categorical semantics based on $\mathbf{Coh}_{\mathbb{N}\setminus\{0,1\}}$ the *multicoherence semantics*[†], we call its objects *multicoherences* and the objects of $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$ *non-uniform multicoherences*. The only difference between hypercoherences and multicoherences is that multicoherences take into account the multiplicity of points for the coherence relation.

Proposition 6 (sequentiality). In the multicoherence semantics, every finite clique of function type $!(\mathbf{bool} \,\&\, \ldots \,\&\, \mathbf{bool}) \multimap \mathbf{bool}$ is sub-definable in PCF.

Proof. The proof follows along the same lines as that for the usual hypercoherence semantics. □

Remark 9. All cliques in the multicoherence semantics are cliques in the coherence semantics (this is a consequence of the foliation property).

5.4. *Non-uniform hypercoherences*

Hypercoherences can be seen as particular multicoherences that are not sensitive to multiplicities: the multicoherences X such that

$$\forall u \in \frown_X, \forall v \in \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(|X|), \mathrm{supp}(u) = \mathrm{supp}(v) \subseteq \Longrightarrow\ v \in \frown_X.$$

If X is a non-uniform multicoherence having this property for both the coherence relation and the incoherence relation, we say that X is a *non-uniform hypercoherence*. So a non-uniform hypercoherence is indeed simply a weakly reflexive $\mathscr{P}^*_{\mathrm{fin}}$-space. But it is more convenient here to present non-uniform hypercoherences as particular non-uniform multicoherences.

If X is a non-uniform multicoherence, $S(X)$ is the non-uniform hypercoherence defined by

$$\frown_{S(X)} = \{u \in \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \forall v \in \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(|X|), \mathrm{supp}(v) = \mathrm{supp}(u) \implies v \in \frown_X\}$$
$$\mathsf{N}_{S(X)} = \{u \in \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \forall v \in \mathscr{M}_{\mathbb{N}\setminus\{0,1\}}(|X|), \mathrm{supp}(v) = \mathrm{supp}(u) \implies v \in \mathsf{N}_X\}.$$

Note that the operation $S\,!_u$ that maps X to $S(!_u X)$ is the hypercoherence multiset-based exponential construction on objects.

Theorem 3.

(1) The sub-category **NHc** of $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$ with objects the non-uniform hypercoherences, equipped with the exponential $S!$ on objects and acting like ! on morphisms is a semantics of linear logic.
(2) The functor N from **NHc** to **Hc**, the category of hypercoherences, is logical (for the multiset-based hypercoherence semantics).
(3) The exponentials $S!$ and $S\,!_u$ are co-free in **NHc** and **Hc**, respectively.

[†] General graph theory lacks a term for such graphs and, unlike the '*hyper*-' situation, where hypercoherences and hypergraphs are the same, multicoherences are not the same as the already existing *multigraphs*.

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Proof. The proof of these statements follows from the proofs of Proposition 3, Theorem 1 and Theorem 2. Just note that some results can be re-used since for each non-uniform multicoherence X, we have $S(!X) = S(!S(X))$ and $N(S(X)) = S(N(X))$. □

Note that Corollary 1 applies to $S!$ and $S\underset{u}{!}$.

Example 8. The K-coherence space G of Example 5 is uniform. And when $K = \mathbb{N}\backslash\{0,1\}$, G is a hypercoherence. The multisets $[a,b]$ and $[a,c]$ are elements of $|!G|_{\mathsf{N}}$. The set $x = \{[a,b],[a,c]\} \subseteq |!G|_{\mathsf{N}}$ is an anti-clique of $S(\underset{u}{!}G)$. But this set is not an anti-clique (or a clique) of $\underset{u}{!}G$. Hence, each finite multiset of support x is an element of $|?S(\underset{u}{!}G)|_{\mathsf{N}}$ but not an element of $|?\underset{u}{!}G|_{\mathsf{N}} = |?!G|_{\mathsf{N}}$.

For the sake of direct usability, we will now spell out the definition of the non-uniform exponential of hypercoherences on objects directly in the $\mathscr{P}^*_{\mathrm{fin}}$-space setting. The definition of this exponential, which is denoted by $\underset{nuh}{!}$ is as follows.

If X is a weakly reflexive $\mathscr{P}^*_{\mathrm{fin}}$-space, then $\underset{nuh}{!}X$ is the $\mathscr{P}^*_{\mathrm{fin}}$-space of web $|\underset{nuh}{!}X| = \mathscr{M}_{\mathrm{fin}}(|X|)$ and such that for each $x \subseteq^*_{\mathrm{fin}} |\underset{nuh}{!}X|$

$$x \in \smile_{\underset{nuh}{!}X} \text{ iff } \exists s \in \smile_X, s \lhd x \tag{15}$$

$$x \in \mathsf{N}_{\underset{nuh}{!}X} \text{ iff } \exists \mu, x = \{\mu\} \text{ and } \forall a \in \mu, \{a\} \in \mathsf{N}_X. \tag{16}$$

Of course, $\underset{nuh}{!}X$ is weakly reflexive.

5.5. *Extensional collapses*

Consider the situation where the same symmetric monoidal closed category has two different exponentials defining two different semantics of linear logic.

Melliès has shown that if there is a *coercion* between the two exponentials that preserves some structure, the two semantics will have the same extensional collapse (Melliès 2004). He used this result to prove that the extensional collapse of the multiset-based hypercoherence semantics is the set-based hypercoherence semantics, and also proved the same thing for the coherence spaces semantics (the latter result had already been proved in Barreiro and Ehrhard (1997)).

This result applies easily to our situation. We then have that the multiset-based coherence semantics and the non-uniform coherence space semantics have the same extensional collapse, which is the set-based coherence space semantics. The same applies for hypercoherences and multicoherences, which we equip with the set-based exponential $\underset{s}{!}$ defined by

$$|\underset{s}{!}X| = \{x \in \mathscr{P}_{\mathrm{fin}}(|X|) \mid x \in \mathrm{Cl}(X)\}$$

$$M \in \frown_{\underset{s}{!}X} \text{ iff } \{m \mid m \lhd M\} \subseteq \frown_X.$$

Of course this last exponential also provides a semantics of linear logic.

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We can give a more precise characterisation of the relation between the extensional collapses of uniform and non-uniform semantics.

Let $\mathscr{M}$ and $\mathscr{M}'$ be the non-uniform and uniform semantics, respectively, of coherence spaces or hypercoherence or multicoherence semantics. Let $\approx$ and $\sim$ be the extensional PERs on $\mathscr{M}$ and $\mathscr{M}'$, respectively. In the following, we use N to denote the neutral restriction functor.

Lemma 3. Let σ and τ be simple types. If f is a clique of $\mathscr{M}(\sigma \to \tau)$ and x is a clique of $\mathscr{M}(\sigma)$, then the clique $N(f(x))$ of $\mathscr{M}'(\tau)$ is equal to $N(f)(N(x))$ and also to $N(f)(x)$.

Proof. The equality $N(f)(N(x)) = N(f)(x)$ is trivial, so we only need to prove $N(f(x)) = N(f)(N(x))$. Since $Nf \subset f$ and $Nx \subset x$, we have $N(f)(N(x)) \subset f(x)$, and since N is monotone, we have $N(f)(N(x)) = N(N(f)(N(x))) \subset N(f(x))$. Conversely, let $b \in N(f(x))$. Then there exists a $\mu \in \mathscr{M}_{\text{fin}}(x)$ such that $(\mu, b) \in f$ and $k[b]$ is neutral for all $k \in K$ ($K = \{2\}$ or $K = \mathbb{N}\backslash\{0,1\}$). Since $k[b]$ is neutral and f is a clique, $k[\mu]$ is incoherent in $!\mathscr{M}(\sigma)$. But since x is a clique, $k[\mu]$ is also coherent in $!\mathscr{M}(\sigma)$. Thus $\mu \in N!\mathscr{M}(\sigma)$, so $(\mu, b) \in Nf$ and $\mu \in \mathscr{M}_{\text{fin}}(Nx)$. This completes the proof by establishing $b \in N(f)(N(x))$. □

Lemma 4. If σ is a simple type and f and g are cliques of $\mathscr{M}(\sigma)$, then

$$f \approx_\sigma g \text{ iff } Nf \sim_\sigma Ng.$$

Proof. This is trivially true on basis types (N acts as the identity functor on basis types). Suppose this is true for types σ and τ. We prove the property for the type $\sigma \to \tau$. Let $f \approx_{\sigma\to\tau} g$ and $x \sim_\sigma y$. Since $Nx = x$ and $Ny = y$, by the induction hypothesis, $x \approx_\sigma y$. Hence $f(x) \approx_\tau g(y)$, so, by the induction hypothesis, $(Nf)(x) \sim_\tau (Ng)(y)$. So $f \approx_{\sigma\to\tau} g$ implies $Nf \sim_{\sigma\to\tau} Ng$. Conversely, let f and g be two cliques of $\mathscr{M}(\sigma \to \tau)$ such that $Nf \sim_{\sigma\to\tau} Ng$. Let $x \approx_\sigma y$. Then, by the induction hypothesis, $Nx \sim_\sigma Ny$. Hence $Nf(Nx) \sim_\tau Ng(Ny)$ and $N(f(x)) \sim_\tau N(g(y))$. And, by the induction hypothesis, $f(x) \approx_\tau g(y)$. This completes the proof by establishing that $Nf \sim_{\sigma\to\tau} Ng$ implies $f \approx_{\sigma\to\tau} g$. □

Theorem 4. The neutral functor N defines a one-to-one correspondence between the extensional collapse of $\mathscr{M}$ and the extensional collapse of $\mathscr{M}'$.

This is a direct consequence of the last two lemmas.

Finally, using the fact that the set-based hypercoherence and multicoherence semantics are both extensional, we show that hypercoherences and multicoherences are extensionally different by exhibiting, in Example 9, a relation at a functional type that is a clique in one of the semantics but not the other (this example was originally designed to exhibit a set that is a clique in the set-based hypercoherence semantics but not in the coherence spaces semantics).

Example 9. For the hypercoherence G of the previous examples, $\{a, b\}$ and $\{c\}$ are elements of $|\underset{s}{!}G| = |S(\underset{s}{!}G)|$. Moreover, $(\{a, b\}, \mathtt{t})$ and $(\{c\}, \mathtt{f})$ are elements of $|\underset{s}{!}G \multimap \mathbf{bool}| = |S(\underset{s}{!}G) \multimap \mathbf{bool}|$. The relation $F = \{(\{a, b\}, \mathtt{t}), (\{c\}, \mathtt{f})\}$ is a clique of the hypercoherence

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$S(\underset{s}{!}G) \multimap \mathbf{bool}$. But F is not a clique of the multicoherence $\underset{s}{!}G \multimap \mathbf{bool}$ since $[\{a,b\},\{c\}]$ is coherent but $[\mathtt{t},\mathtt{f}]$ is strictly incoherent.

6. Conclusions and further work

6.1. *Static interactivity*

The current paper gives some strong evidence that static uniformity is a matter of restriction to possible results of computation through interactions, and especially through interactions in a closed case: between a *proof* of A and a *proof* of $A^{\perp}$. We further argue this point by adopting basic ideas from the interactive point of view of computation developed in Girard's ludics (Girard 2001).

The main idea is to consider the linear logic system extended with new rules (such that the system still enjoys cut-elimination). Then a formula A and its linear negation $A^{\perp}$ can both be provable. Hence we can provoke interactions through cut elimination between proofs of A and proofs of $A^{\perp}$. If we further require that the extended logical system admits one of the deterministic non-uniform semantics we have presented here, then the consequence of determinism is that such an interaction involves at most one point.

Let suppose that we add two para-rules to linear logic:

$$\frac{}{\vdash}\,(\textit{give up}) \qquad \frac{}{\vdash \Gamma}\,(\textit{divergence})$$

These are a *give up* rule (which corresponds roughly to the *daemon* of Girard's Ludics (Girard 2001)) and a *divergence* rule, whose respective interpretations in the relational semantics are a singleton (the unique point in the unit context $\perp$) and the empty set. It is easy to extend the cut-elimination procedure for these two rules and to check that the relational semantics extends into a semantics of the strongly normalising calculus we then obtain. In this setting, a formula A and its dual $A^{\perp}$ are always both provable.

When we apply a cut rule between a proof π' of A and a proof π'' of $A^{\perp}$, the proof π we obtain normalises into a proof of the empty sequent. And there are only two cut-free proofs of the empty sequent: one is an instance of the *give up* rule and one is an instance of the *divergence* rule, with an empty context. It is easy to verify that the resulting cut-free proof is a *give up* or a *divergence* if and only if the relational interpretation of π' and π'' have a non-empty or empty intersection, respectively. If the interpretation is not empty, we say that π' and π'' interact. From the point of view of the bipartite relational semantics, the *give up* rule is not valid (since it introduces a sequent interpreted by a negative point), and there can be no interaction between a proof of A and a proof of $A^{\perp}$. We think that this is the fundamental reason allowing a uniform semantics in which so many proofs have empty interpretations: uniformity is a restriction to the possible interactions, and if no interaction is possible, uniformity empties things.

Note that *give up* and *divergence* are valid rules in the other coherence-like semantics we have presented in this paper. In particular, the property of determinism of non-uniform semantics tell us that when π' and π'' interact, there is only one result of this interaction (there is only one point in the intersection). This seems difficult to prove directly in the relational semantics without introducing non-uniform coherence relations. Moreover, the

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result of interaction is always in the neutral webs of the various semantics (coherence spaces, hypercoherences, multicoherences), hence, in the extended linear logic we use here, a part of the web is never visited by closed interactions. This will be the case in any extension of linear logic that admits one of these semantics.

In fact, one can imagine new para-rules such that there can be interactions at any point of the relational semantics. We still need to check whether this is useful, but as Curien suggested to us, adding a *sum* rule like

$$\frac{\vdash \Gamma \qquad \vdash \Gamma}{\vdash \Gamma}\,(sum)$$

interpreted by a union in the relational semantics will certainly give a semantics of this kind. However, determinism will be lost.

6.2. *Extending non-uniform static semantics*

Using the co-free exponential for non-uniform static semantics has led to a comfortable situation where non-uniform semantics are deterministic and strongly related to the uniform semantics. A (still) open question is whether the general construction introduced in Bucciarelli and Ehrhard (2001) can be modified to give the co-free exponentials directly and in a general way. Another related issue concerns full completeness for static semantics. Ehrhard has proved a completeness theorem in an indexed linear logic framework (Ehrhard 2004). A better understanding of the co-freeness issue in indexed linear logic may help in connecting his result with the usual static semantics (hypercoherences and coherence spaces).

In an unpublished work, Winskel has introduced a generalisation of hypercoherences, and it may be interesting to adapt this generalisation to a non-uniform framework, including the semantics we have presented here.

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