The second one is Feynman's address to science teachers in 1966 [7], coincidentally three centuries after Newton's pursuit for approximating π. Feynman mentioned,

> … Pi was a thing with circles, and here is pi coming out of …
> Where was the circle?

Echoing and responding to the questions of De Morgan's acquaintance and Feynman, we justify π in the standard normal density function to come out of a semicircle, namely, Newton's semicircle in the history of the approximation of π.

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100.19 A quick evaluation of $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$

Just as there are ‘earworms’ – catchy tunes that lodge in the brain and refuse to budge – so, I believe, there are ‘mathematical earworms’ – problems that niggle us and will not let go. One such for me is the limit in the title which I first encountered some ten years ago in the articles [1, 2, 3]. Michael Spivey's original proof in [1, 2] used the Euler-Maclaurin summation formula to estimate $\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$ and then evaluate the limit; Finbarr Holland in [3] used monotone convergence arguments to find the limit via

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an interchange of the limit and summation operations involved. The purpose of this short Note is to give a quick evaluation of $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$ which just uses the basic exponential inequality $e^x \geqslant 1 + x$, valid for all real x, and its immediate two-sided consequence $1 - x \leqslant e^{-x} \leqslant \frac{1}{1+x}$, valid for all $x > -1$.

For, on the one hand, $\frac{m-i}{m} = 1 - \frac{i}{m} \leqslant e^{-\frac{i}{m}}$ leads to

$$\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m = \sum_{i=0}^{m}\left(\frac{m-i}{m}\right)^m \leqslant \sum_{i=0}^{m} e^{-i} < \sum_{i=0}^{\infty} e^{-i} = \frac{e}{e-1}.$$

However, on the other hand, $e^{-x} \leqslant \frac{1}{1+x}$ with $x = \frac{i}{m-i}$ gives $e^{-\frac{i}{m-i}} \leqslant \frac{m-i}{m}$ for $0 \leqslant i \leqslant m-1$, so that

$$\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m = \sum_{i=0}^{m-1}\left(\frac{m-i}{m}\right)^m \geqslant \sum_{i=0}^{m-1} e^{-\frac{mi}{m-i}} = \sum_{i=0}^{m-1} e^{-i}.e^{\frac{-i^2}{m-i}} \geqslant \left(\sum_{i=0}^{K} e^{-i}\right).e^{-\frac{K^2}{m-K}}$$

for any $0 \leqslant K \leqslant m-1$, since $i \to \frac{i^2}{m-i}$ is increasing for $0 \leqslant i \leqslant m-1$. But if we choose (say) $K = \lfloor \sqrt[3]{m} \rfloor$ then, as $m \to \infty$, $\sum_{i=0}^{K} e^{-i} \to \frac{e}{e-1}$ and $e^{-\frac{K^2}{m-K}} \to 1$ enabling us to conclude from our upper and lower bounds that $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m = \frac{e}{e-1} \approx 1.58198$.

Finally, we note that it is straightforward to give reasonably tight bounds for $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$. By considering upper and lower rectangles enclosing the curve $y = x^m$ between $x = k-1$ and $x = k$, since the area above the curve occupies at least half of the surrounding rectangle, we have

$$\tfrac{1}{2}\left[k^m - (k-1)^m\right] \leqslant k^m - \int_{k-1}^{k} x^m dx \leqslant k^m - (k-1)^m.$$

So, on summing for $1 \leqslant k \leqslant m$, $\frac{1}{2}m^m \leqslant \sum_{k=1}^{m} k^m - \frac{m^{m+1}}{m+1} \leqslant m^m$ whence $\frac{1}{2} + \frac{m}{m+1} \leqslant \sum_{k=0}^{m}\left(\frac{k}{m}\right)^m \leqslant 1 + \frac{m}{m+1}$. Thus if $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$ exists, it is between $\frac{3}{2}$ and 2. Spivey in [1; p. 63] lowers the upper bound to $\frac{13}{8}$ and observes that any upper bound less than 2 implies that the final term of the sum $\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$ contributes more to its value than all of the other terms put together. Finally, and I am grateful to the referee for this observation,

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once we know that the limit is $\frac{e}{e-1}$, relatively crude bounds for e refine these bounds further. For example, $e < \frac{11}{4}$ leads to $\frac{11}{7} < \frac{e}{e-1}$, a nice companion to the upper bound of $\frac{13}{8}$ above.

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100.20 Outside the box

Stephen Lee and Charlie Stripp recently wrote an article [1] about MEI (Mathematics in Education and Industry, a charitable body in the UK concerned with improving mathematical education). It included their *Maths Item of the Month* from September 2013, which was a delightful geometric problem. I stopped to admire the scenery, as it were, and wander around the mathematical neighbourhood of the problem. It is the connections between different parts of mathematics that I find so fascinating, and here, what started out as school trigonometry turned into elementary number theory, which sadly is not in the school syllabus, at least not in England, but is well within the grasp of the interested student.

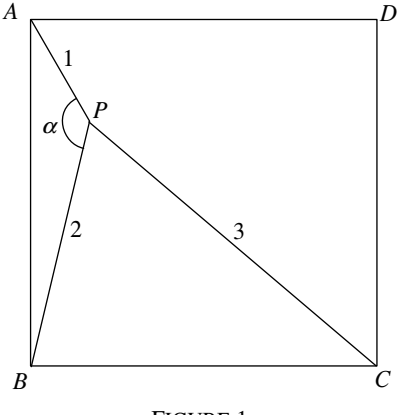


FIGURE 1

The original problem asked 'What is angle α?' in Figure 1, in which *ABCD* is a square. The solution on the MEI website, [2], is pretty and purely geometric, but I dived in and did the general case in Figure 2, which is a great little exercise in school trigonometry.

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