J. Fluid Mech. (2007), *vol.* 575, *pp.* 257–265. © 2007 Cambridge University Press
doi:10.1017/S0022112007004740 Printed in the United Kingdom

Local implications for self-similar turbulent plume models

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(Received 12 October 2006 and in revised form 21 December 2006)

The local implications of the well-known flux conservation equations of Morton *et al.* (*Proc. R. Soc. Lond.* A, vol. 234, 1956, p. 1) for plumes and jets are considered. Given the vertical velocity distributions of a model plume or jet, the divergence-free radial velocity distributions are calculated. It is shown that in general the velocity of the plume boundary is not described by the local total fluid velocity in this way. A two-fluid model tracking the evolution of both 'plume' and 'ambient' fluid is proposed which resolves this apparent inconsistency and also provides a way of explicitly describing the mixing process within a model plume. The plume boundary acts as a phase boundary across which ambient fluid is entrained, and the plume boundary moves at the velocity of the plume fluid. The difference between the plume-fluid radial velocity and the total fluid velocity quantifies in a natural way the purely horizontal entrainment flux of ambient fluid into the plume across the phase boundary at the plume edge.

1. Introduction

The paper of Morton, Taylor & Turner (1956) entitled 'Turbulent gravitational convection from maintained and instantaneous sources' remains one of the most significant publications in the field of buoyancy-driven flows. With far reaching implications for large-scale flows in the atmosphere and oceans, to much smaller scale flows within naturally ventilated buildings, it is no wonder that this paper is still widely cited across many disciplines. The Boussinesq plume model (which we refer to herein as 'the MTT model') developed in their paper is a model for the bulk quantities (assumed to be, on average, steady in time) of a plume and relies solely upon the conservation of fluxes of volume, specific momentum and specific buoyancy.

The MTT model describes vertical variations in volume, specific momentum and specific buoyancy fluxes. With knowledge of these quantities, it is possible to derive expressions for the plume radius, vertical velocity field and buoyancy forces acting on the plume fluid as functions of height above the plume source. The MTT model is often applied to flows in incompressible fluids, and so knowledge of the vertical variations of the bulk quantities immediately implies corresponding radial variations. These required radial counterparts to the vertical motions have, to date, been largely overlooked and it is the local radial properties of the flow which we investigate in the present paper.

https://doi.org/10.1017/S0022112007004740 Published online by Cambridge University Press

The simple, but powerful, assumption that lies at the heart of the MTT model is the entrainment assumption (see Turner 1986 for a detailed review). This assumption requires that the ambient fluid entrained into a plume at a given height and time is proportional to the vertical velocity at the plume centreline at the same height and time. The philosophy behind this assumption is that the entrainment into a plume at a turbulent plume boundary depends on the turbulence intensity of the plume fluid, which can be characterized by the centreline plume velocity.

In the present paper we seek a deeper understanding of the local properties of the MTT model by investigating the radial velocity field of a model plume. In so doing, it will become apparent that there are some potential conceptual difficulties with the MTT model. For example, the velocity of the plume boundary cannot be simply described in terms of the local total fluid velocity. There is no explicit explanation of how ambient fluid is mixed within the plume and relabelled as plume fluid. The entrainment assumption is fundamentally a turbulence closure, and so there is no explicit description of turbulent fluctuations within the MTT model. Therefore, it is not formally consistent, within the MTT framework, to appeal to such processes to provide the desired properties of the model solutions.

We propose a simple two-fluid model to reconcile the apparent difficulties with the MTT model with our physical intuition. We regard the fluid at a given point as comprising ambient fluid with volume fraction a and plume fluid with volume fraction p, where $a + p = 1$. We show that the plume boundary acts as a phase boundary and that there is a continuous flux of ambient fluid across this plume boundary. The velocity of the plume boundary is described by the local velocity of the plume fluid. The process whereby entrained ambient fluid is converted into plume fluid is made explicit and is consistent with the MTT model system.

The outline of the paper is as follows. In § 2.1 we establish our two-fluid model and its notation. The MTT model is usually applied with a top-hat or Gaussian profile imposed on the vertical velocity field of the plume fluid. Therefore, in § 2.2, we introduce a general one-parameter family of plume profiles (of which both top-hat, as a limit, and Gaussian profiles are members) and the corresponding version and solutions of the MTT model. In § 3.1 we apply our two-fluid model to the more physically intuitive top-hat plumes and then generalize this in § 3.2 to our more general continuous shape profiles, including Gaussian profiles as a special case. Finally in § 4 we draw our conclusions.

2. General results

2.1. *Two-fluid method*

We describe the fluid at a given point as a superposition of two fluids, an ambient fluid, and a plume fluid. At a given point in space there exists ambient fluid with volume fraction a and velocity $\boldsymbol{u}_a$, and plume fluid with volume fraction p and velocity $\boldsymbol{u}_p$, where $a + p = 1$. Defining the reduced gravity at a given point as $g' = g(\rho_a - \rho)/\rho_a$, where ρ is the total fluid density, ρ_p is the density of the plume fluid, and ρ_a is the density of the ambient fluid, the fluid velocity and reduced gravity at a given point are given by

$$\boldsymbol{u} = a\boldsymbol{u}_a + p\boldsymbol{u}_p, \quad g' = pg'_p, \tag{2.1}$$

since the reduced gravity of the ambient fluid is zero by definition, i.e. $g'_a \equiv 0$. We impose incompressibility so that

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.2}$$

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It follows from (2.1) and (2.2) that

$$\nabla \cdot (p\boldsymbol{u}_p) = -\nabla \cdot (a\boldsymbol{u}_a). \tag{2.3}$$

Equation (2.3), as we shall see, is an explicit statement of how ambient fluid is converted into plume fluid and it is this essential equation that describes the mixing within the plume. Entrainment of ambient fluid provides a source of plume fluid, with the ambient fluid being relabelled as plume fluid according to (2.3). The generation of plume fluid from ambient fluid is intuitively satisfactory from a physical viewpoint as the vertical volume flux within a plume increases with height. Hence, (2.3) is an explicit statement about mixing within a plume which does not appeal to turbulent fluctuations, and is self-consistent within the MTT model framework.

The application of the two-fluid model is as follows. The MTT model gives an explicit solution of the plume-fluid vertical velocity, w_p. If an appropriate statement can be made about the divergence of the plume fluid, i.e. $\nabla \cdot \boldsymbol{u}_p$, then the radial plume velocity u_p can be written down. The total divergence of the plume fluid at a given height is equal to the amount of fluid being entrained at that height, i.e.

$$2\pi \int_0^\infty \nabla \cdot \boldsymbol{u}_p \, r \, \mathrm{d}r = -2\pi b u_e, \tag{2.4}$$

where b is identified with the plume radius and u_e is the 'entrainment velocity'. Once the velocity of the plume fluid, $\boldsymbol{u}_p$, is known then (2.3) prescribes the ambient fluid velocity $\boldsymbol{u}_a$ and hence (2.1) yields the total incompressible-fluid velocity.

2.2. *The MTT model for a general shape profile*

In this section we will see that, within the two-fluid framework, the volume fraction, p, of plume fluid at a given height can represent the well-known shape profile function, for example top-hat or Gaussian. The vertical velocity, w_p, and the reduced gravity, g'_p, are the vertical velocity and reduced gravity of the plume fluid, respectively.

Using cylindrical polar coordinates where r denotes the horizontal radial distance from the vertical axis and $b(z)$ denotes the plume radius at a given height z, we let the volume fraction of plume fluid, p, at a given point be described by the shape profile function

$$p = \exp\left\{-\left(\frac{r}{b\,(z)}\right)^n\right\}. \tag{2.5}$$

The parameter n is chosen to give a specific shape profile, where in particular $n = 2$ defines a Gaussian shape profile, and the limit $n \to \infty$ represents the top-hat shape profile.

The vertical fluid velocity is given by $w = pw_p + aw_a$. Modelling the process of entrainment as being purely horizontal, we impose the condition that the contribution of the ambient fluid velocity to the total fluid velocity can only be horizontal, i.e. $aw_a \equiv 0$. The vertical velocity of the plume fluid, w_p, does not vary radially, although the contribution to the total fluid velocity may vary radially since $w = pw_p(z)$. Generalizing the method used by Morton *et al.* (1956) we define volume, specific momentum and specific buoyancy fluxes respectively as

$$Q = 2\pi \int_0^\infty w \, r \, \mathrm{d}r = \pi \frac{2\Gamma(2/n)}{n} b^2 w_p, \tag{2.6a}$$

$$M = 2\pi \int_0^\infty w^2 \, r \, \mathrm{d}r = 2^{-2/n} \pi \frac{2\Gamma(2/n)}{n} b^2 w_p^2, \tag{2.6b}$$

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$$F = 2\pi \int_0^\infty w g' \, r \, \mathrm{d}r = 2^{-2/n} \pi \frac{2\Gamma(2/n)}{n} b^2 g'_p w_p. \tag{2.6c}$$

Here the Gamma function (see e.g. Abramowitz & Stegun 1965, § 6.1) arises since

$$\int_0^\infty p \, r \, \mathrm{d}r = \frac{b^2}{n} \Gamma(2/n). \tag{2.7}$$

(This should not be confused with the plume laziness parameter, e.g. Morton 1959; Hunt & Kaye 2001.) It follows from (2.6) therefore that the plume radius, b, plume vertical velocity, w_p, and plume reduced gravity, g'_p, are given by

$$b = \left\{ \frac{2^{-2/n}}{\pi} \frac{n}{2\Gamma(2/n)} \right\}^{1/2} \frac{Q}{M^{1/2}}, \quad w_p = \frac{2^{2/n} M}{Q}, \quad g'_p = \frac{2^{2/n} F}{Q}. \tag{2.8}$$

We make the standard entrainment assumption, $u_e = -\alpha w_p < 0$, where α is the well-known 'entrainment constant' (Morton *et al.* 1956) taking typical values of approximately 0.1. By considering the vertical gradient of the volume flux, momentum flux and buoyancy flux in an unstratified ambient fluid, the MTT model is thus

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = -2\pi b u_e = 2\pi b \alpha w_p = 2\alpha M^{1/2} \left\{ 2^{2/n} \pi \frac{n}{2\Gamma(2/n)} \right\}^{1/2}, \tag{2.9a}$$

$$\frac{\mathrm{d}M}{\mathrm{d}z} = 2\pi \int_0^\infty g' \, r \, \mathrm{d}r = \pi \frac{2\Gamma(2/n)}{n} g'_p b^2 = \frac{QF}{M}, \tag{2.9b}$$

$$\frac{\mathrm{d}F}{\mathrm{d}z} = 0. \tag{2.9c}$$

Assuming power-law behaviour (as shown in Caulfield & Woods 1995, general source conditions converge to such solutions) (2.9) can be straightforwardly solved for Q, M and F, or equivalently b, w_p and g'_p, where (assuming $Q(0) = 0$, $M(0) = 0$ and $F(0) = F_0 > 0$)

$$b = \frac{6\alpha z}{5} \frac{n}{2\Gamma(2/n)}, \tag{2.10a}$$

$$w_p = \frac{5}{6\alpha} \left(\frac{9\alpha}{10} \right)^{1/3} F_0^{1/3} z^{-1/3} \left\{ \frac{2^{4/n}}{\pi} \frac{2\Gamma(2/n)}{n} \right\}^{1/3}, \tag{2.10b}$$

$$g'_p = \frac{5}{6\alpha} \left(\frac{10}{9\alpha} \right)^{1/3} F_0^{2/3} z^{-5/3} \left\{ \frac{2^{1/n}}{\pi} \frac{2\Gamma(2/n)}{n} \right\}^{2/3}, \tag{2.10c}$$

the power-law dependence of which was originally derived by Zeldovich (1937). These solutions are commonly referred to as the ideal plume solutions since they are derived for initial conditions $Q(0) = 0$, $M(0) = 0$, which implies that the plume rises from a point source of buoyancy flux alone. In particular (2.10*b*) gives us the vertical velocity of the plume fluid, which is constant with radius, but decreases with height while the plume radius, b, increases with height.† These general properties have certain implications for the radial velocity distribution, as discussed below.

† For non-ideal source conditions it is possible for w_p to increase with height, or indeed for b to decrease with height, over some finite distance (see Caulfield 1991; Hunt & Kaye 2001) before converging to a solution with w_p decreasing and b increasing with height.

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3. The two-fluid model

In the following two subsections we apply the general two-fluid method described in § 2.1 to top-hat profile plumes in § 3.1 and then to plumes with general shape profiles in § 3.2, focusing on the implications of incompressibility for the radial velocity distribution.

3.1. *Top-hat profile*

With a top-hat plume (unlike a Gaussian plume for example), a plume boundary is well-defined, outside which there is no plume fluid. Exterior to the top-hat plume we have only ambient fluid and hence $a = 1$ and $p = 0$. Conversely, inside a top-hat plume $p = 1$ and $a = 0$. It is important to stress that in this extreme limit, au_a is not necessarily zero.

The MTT model given in § 2.2 prescribes the plume fluid vertical velocity w_p. In order to apply the two-fluid method a statement about the divergence of the plume fluid must be found. This is straightforward in the case of a top-hat plume. Since the amount of fluid entering the plume at a given height is $-2\pi b u_e$ the only uniform, top-hat, local divergence that can balance this entrainment of ambient fluid is given by

$$\nabla \cdot \boldsymbol{u}_p = \begin{cases} -\dfrac{2\pi b u_e}{\pi b^2} & (r < b) \\ 0 & (r > b) \end{cases} = \begin{cases} -\dfrac{2u_e}{b} & (r < b) \\ 0 & (r > b), \end{cases} \tag{3.1}$$

which satisfies condition (2.4). It follows therefore that

$$pu_p = \begin{cases} -\dfrac{r}{2}\dfrac{\mathrm{d}w_p}{\mathrm{d}z} - \dfrac{u_e r}{b} & (r < b) \\ 0 & (r > b) \end{cases} = \begin{cases} \dfrac{r}{b} w \dfrac{\mathrm{d}b}{\mathrm{d}z} & (r < b) \\ 0 & (r > b), \end{cases} \tag{3.2}$$

where the second form of the $r < b$ solution follows from the conservation-of-volume MTT equation (2.9*a*). It is apparent therefore that the radial velocity of the plume fluid increases linearly with r from zero at the axis, and the velocity of the plume boundary, $w\mathrm{d}b/\mathrm{d}z$ (typically positive as noted above), is given by the local plume fluid velocity.

Inside the plume, (3.1) together with the fact that $p = 1$ implies

$$\nabla \cdot (a\boldsymbol{u}_a) = \frac{2u_e}{b}. \tag{3.3}$$

Since $aw_a = 0$, we find that inside the plume $au_a = u_e r/b$, and so we have that

$$au_a = \begin{cases} \dfrac{u_e r}{b} & (r < b) \\ \dfrac{u_e b}{r} & (r > b), \end{cases} \tag{3.4}$$

noting that au_a is continuous across $r = b$. Since $u_e < 0$, (3.4) implies that $au_a < 0$. The implication of $a \to 0$ inside the plume is that the velocity $u_a \to -\infty$ inside the plume (allowing the self-consistent modelling of instantaneous entrainment and homogenization of plume fluid within a top-hat framework). Hence we have from (3.4) that

$$\boldsymbol{u} = \begin{cases} \left(-\dfrac{r}{2}\dfrac{\mathrm{d}w_p}{\mathrm{d}z}, w_p \right) & (r < b) \\ \left(\dfrac{u_e b}{r}, 0 \right) & (r > b). \end{cases} \tag{3.5}$$

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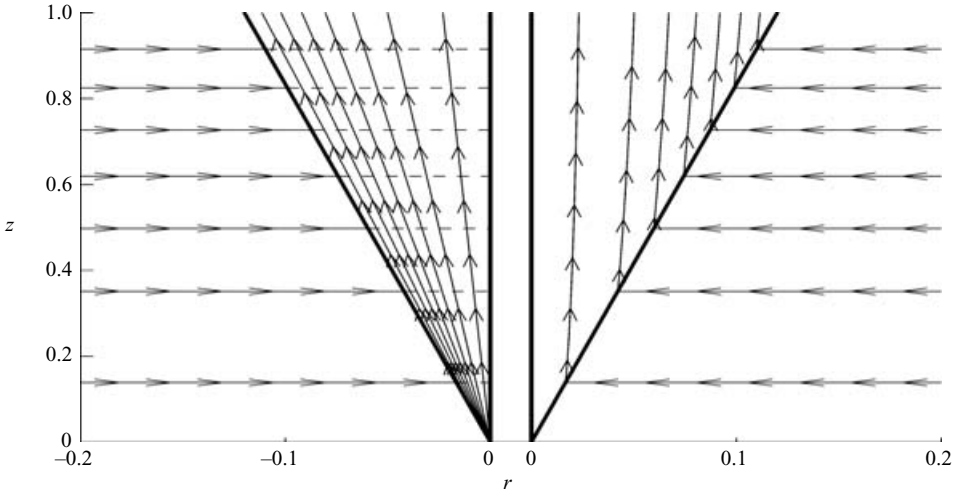


FIGURE 1. The streaklines of the ambient- and plume-fluid velocity fields are shown on the left-hand side for $r<0$. The right-hand side, $r>0$ shows the streamlines of the combined velocity field $\boldsymbol{u}=p\boldsymbol{u}_p+a\boldsymbol{u}_a$.

Typically u is positive inside the plume, dominated by the spreading of the plume due to entrainment of ambient fluid, while u is negative outside the plume, where u is dominated by the inward flow of ambient fluid to supply this entrainment. Therefore there is, in general, a discontinuity in both $\boldsymbol{u}$ and $\boldsymbol{u}_p$, but not $a\boldsymbol{u}_a$, at the plume boundary. So, although there is both a discontinuity and a change of sign in the radial velocity, u, ambient fluid may still cross the boundary into the plume and undergo a relabelling and conversion into plume fluid within the plume. The plume boundary moves with the velocity of the plume fluid, and there is a continual flux of ambient fluid across the plume boundary (i.e. entrainment), so the plume boundary acts as a phase boundary.

Figure 1 shows on the left-hand side the streaklines for the ambient-fluid velocity and plume-fluid velocity in a top-hat plume. The right-hand side shows the streamlines for the total fluid velocity. On the left-hand side we see the ambient fluid remaining horizontal as it propagates in towards the axis. The plume fluid has radial velocity proportional to r for $r<b$ and matches the velocity of the plume boundary at $r=b$.

3.2. *General shape profiles*

As with the top-hat plume, we assume that the amount of turbulent mixing into a plume at a given point is proportional to the local amount of plume fluid, i.e. p. Intuitively it seems reasonable that an absence of plume fluid indicates that there is no turbulent mixing of plume fluid and ambient fluid, and hence conversion of ambient fluid into plume fluid, whereas a high concentration of plume fluid indicates higher amounts of mixing. Hence, noting that the plume fluid divergence in (3.1) has shape profile given by p (in the limit $n\to\infty$) we take

$$\nabla\cdot\boldsymbol{u}_p=-\frac{2u_e}{b}\frac{n}{2\Gamma(2/n)}p, \tag{3.6}$$

such that the plume-fluid divergence for general shape parameter n has shape profile given by p and (2.4) is satisfied. We recover (3.1) in the limit as $n\to\infty$, since p acts

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	As $r \to 0$	As $r \to \infty$	As $n \to \infty$
$\mathscr{I}$	1	$\left(\frac{b}{r}\right)^2 \frac{2\Gamma(2/n)}{n}$	$\left.\begin{matrix} 1 & (r \leqslant b) \\ b^2/r^2 & (r > b) \end{matrix}\right\}$
$\frac{\partial \mathscr{I}}{\partial z}$	$\frac{\partial}{\partial z}\left(\frac{b}{r}\right) \frac{2n}{n+2} \left(\frac{r}{b}\right)^{n+1}$	$\frac{\partial}{\partial z}\left(\frac{b^2}{r^2}\right) \frac{2\Gamma(2/n)}{n}$	$\left.\begin{matrix} 0 & (r \leqslant b) \\ \partial\,(b^2/r^2)/\partial z & (r > b) \end{matrix}\right\}$

TABLE 1. Leading-order asymptotic properties of the integral $\mathscr{I}$ (3.8) in the limits $r \to 0$, $r \to \infty$ and $n \to \infty$, assuming $n > 1$.

like the Heaviside step function and $2\Gamma(2/n)/n \to 1$. It follows from (3.6) that

$$\boldsymbol{u}_p = \left(-\frac{2u_e}{b}\frac{n}{2\Gamma(2/n)}\frac{r}{2}\mathscr{I} - \frac{r}{2}\frac{\mathrm{d}w_p}{\mathrm{d}z}, w_p\right), \tag{3.7}$$

where

$$\mathscr{I} = \frac{2}{r^2}\int_0^r Rp\,\mathrm{d}R \tag{3.8}$$

is the average amount of plume fluid at a given height within a disk of radius r. Hence, combining (3.6) and (3.7), together with the condition that $aw_a \equiv 0$, we obtain

$$a\boldsymbol{u}_a = \left(\frac{2u_e}{b}\frac{n}{2\Gamma(2/n)}p\frac{r}{2}\mathscr{I} + \frac{rp}{2}\frac{\mathrm{d}w_p}{\mathrm{d}z} - \frac{r}{2}\frac{\partial}{\partial z}(w_p\mathscr{I}), 0\right), \tag{3.9}$$

and so it follows that

$$\boldsymbol{u} = \left(-\frac{r}{2}\frac{\partial}{\partial z}(w_p\mathscr{I}), pw_p\right). \tag{3.10}$$

We list some asymptotic properties of the quantity $\mathscr{I}$ in table 1 under the assumption that $n > 1$. In the specific case of a Gaussian shape profile, $n = 2$, then $\mathscr{I} = (1-p)b^2/r^2$. It is clear that in general the total radial velocity, u, behaves in an analogous way to the limiting top-hat case, with u being positive for small r and negative for large r (compared to b).

Figure 2(a) plots the contribution to the total radial velocity field, u, of the plume fluid, pu_p, and the ambient fluid, au_a, normalized by the entrainment velocity. Bold solid lines are the top-hat shape profile, $n \to \infty$, thin lines are the Gaussian shape profiles, $n = 2$, and the dashed lines are an intermediate shape profile given by $n = 20$. As was shown in (3.2), the plume fluid in a top-hat plume has a radial velocity which increases linearly with r inside the plume, capturing the spreading of the plume and the fact that (in general) $\mathrm{d}w_p/\mathrm{d}z < 0$ in a plume. Outside the top-hat plume w_p does not contribute to the total fluid velocity since $p\boldsymbol{u}_p = 0$. The ambient fluid propagates from infinity towards the axis, with constant transport $2\pi ru$ inwards outside the plume, and crosses the plume boundary at $r = b$, capturing the entrainment process.

Figure 2(b) shows the total radial fluid velocity, u, normalized by the entrainment velocity, for the three shape profiles. In particular, inside the top-hat plume the radial velocity again increases linearly with r, but the total radial fluid velocity is less than the velocity of the plume boundary owing to the flux of the ambient fluid crossing the boundary. For all values of $n > 1$, we see that the dominant contribution to the radial velocity in the far field is the inward ambient fluid velocity due to entrainment of ambient fluid into the plume.

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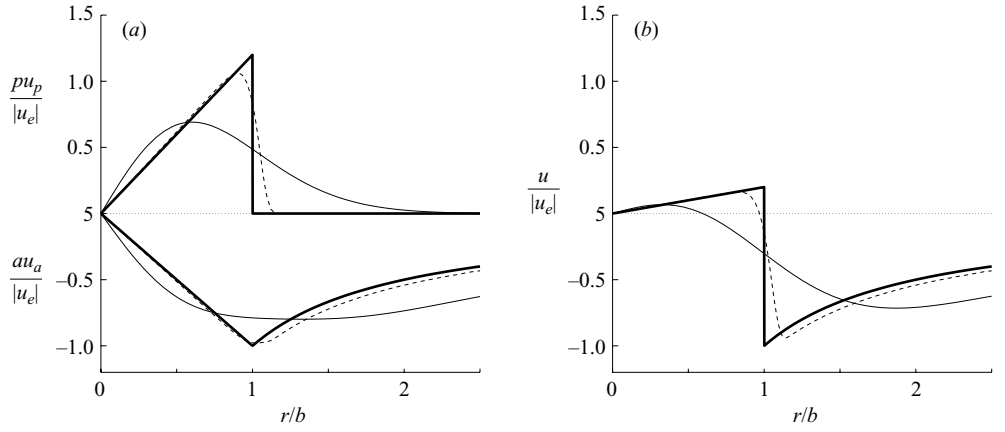


FIGURE 2. (*a*) pu_p and au_a, normalized by the entrainment velocity, for different values of the shape parameter n. Bold lines are for $n \to \infty$, thin lines are $n = 2$ and dashed lines are $n = 20$. pu_p is positive, demonstrating that the plume fluid propagates outwards, whereas au_a is negative, demonstrating that the ambient fluid propagates inwards. (*b*) The combined u field, normalized by the entrainment velocity, showing that as $r \to \infty$, $u \to au_a$, i.e. in the far field virtually all fluid motion is inward flow induced by entrainment, while for small r, u is positive, indicating that the radial velocity of the plume fluid dominates near the axis.

4. Conclusions

We have considered the radial velocity fields of the classical MTT model plumes and have highlighted some apparent difficulties with the model. We have demonstrated that there is a discontinuity, and indeed sign change, in the radial velocity of a top-hat plume and the velocity of the plume boundary does not coincide with the local fluid velocity. These apparent inconsistencies can be rectified by use of a two-fluid model. Within this two-fluid model there is a constant flux of ambient fluid into the top-hat plume and the velocity of the plume boundary is given by the local velocity of the plume fluid which acts as a phase boundary. The two-fluid model also enables us to write down an explicit mechanism for the conversion of ambient fluid into plume fluid, through entrainment across this phase boundary. This fundamental concept can be generalized to continuous profiles. As an example we consider a one-parameter family of self-similar profiles that includes both top-hat and Gaussian profiles as specific cases. The plume fluid rises upwards (owing to its buoyancy) and spreads outwards (owing to the entrainment and relabelling of ambient fluid) while the ambient fluid flows horizontally inwards to supply this entrainment. Near $r/b = 0$, the total radial flow is outwards, dominated qualitatively by the behaviour of the plume fluid, while in the far field ($r/b \to \infty$) it is dominated by the purely horizontal inward flow of the ambient fluid. The vertical flow is naturally always determined by the behaviour of the buoyant plume fluid.

This self-consistent description of the total fluid velocity in terms of contributing ambient- and plume-fluid components is particularly important when considering plumes with temporal variations in the source conditions. If we wish to calculate the appropriate time-dependent bulk generalizations of the MTT model equations (2.9) starting from the pointwise Euler equation, the appropriate measure of radial velocity is u_p together with the corresponding fluid divergence $\nabla \cdot \boldsymbol{u}_p$ (see Scase, Caulfield & Dalziel 2006*a*; Scase *et al.* 2006*b*). For example if we consider the extreme case of a starting Gaussian plume we can see how a Gaussian distribution of plume-fluid

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divergence (due to entrainment) balanced by an appropriate convergence of ambient fluid being entrained, can create and maintain the plume in a self-similar fashion.

The authors would like to thank Professor H. E. Huppert for useful discussions in the early stages of their investigation, and Dr A. J. Aspden for motivating this research. M. M. S. was funded under NERC award NER/A/S/2002/00892.

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