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The moduli space of a rational map is Carathéodory hyperbolic

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Abstract. Let f be a rational map of degree $d \geq 2$. The moduli space $\mathcal{M}_f$, introduced by McMullen and Sullivan, is a complex analytic space consisting of all quasiconformal conjugacy classes of f. For f that is not flexible Lattès, we show that there is a normal affine variety X_f of dimension $2d-2$ and a holomorphic injection $i : \mathcal{M}_f \to X_f$ such that $i(\mathcal{M}_f)$ is precompact in X_f. In particular, $\mathcal{M}_f$ is Carathéodory hyperbolic (that is, bounded holomorphic functions separate points in $\mathcal{M}_f$), provided that f is not flexible Lattès. This solves a conjecture of McMullen. When $d \geq 4$, we give a concrete construction of X_f as the normalization of the Zariski closure of the image of the reciprocal multiplier spectrum morphism.

Key words: low-dimensional dynamics, bifurcation theory, arithmetic and algebraic dynamics
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1. *Introduction*

1.1. *The dynamical Teichmüller space and moduli space of a rational map.*

Let $f \in \mathrm{Rat}_d$ be a rational map of degree $d \geq 2$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. McMullen and Sullivan introduced the *Teichmüller space* and *moduli space* for a rational map f [**MS98**]. These two spaces are important in complex dynamics. For example, the hyperbolic conjecture claims that hyperbolic rational maps are dense in Rat_d. Using these two spaces and the Teichmüller theory they developed in [**MS98**], McMullen and Sullivan showed that the hyperbolic conjecture is equivalent to the fact that a non-Lattès rational map carries no invariant line field on its Julia set. Roughly speaking, the moduli space $\mathcal{M}_f$ is a complex analytic space containing all quasiconformal conjugacy classes of f, and the Teichmüller space $\mathcal{T}_f$ is the 'universal cover' of $\mathcal{M}_f$.

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Let us make a precise definition. Let $\mathrm{Bel}(f)$ be the set of L^∞ Beltrami differentials μ invariant under f such that $\|\mu\|_\infty < 1$. Let $\mathrm{QC}(f)$ be the group of quasiconformal homeomorphisms commuting with f and let $\mathrm{QC}_0(f)$ be the normal subgroup of the elements in $\mathrm{QC}(f)$ that are isotopic to identity. The *modular group* of f is defined by the quotient

$$\mathrm{Mod}(f) := \mathrm{QC}(f)/\mathrm{QC}_0(f).$$

The Teichmüller space $\mathcal{T}_f$ is defined by $\mathrm{Bel}(f)$ quotiented by the right action of $\mathrm{QC}_0(f)$ by precomposition. McMullen and Sullivan [**MS98**] showed that the Teichmüller space $\mathcal{T}_f$ is a complex manifold biholomorphic to a contractable bounded domain in $\mathbb{C}^N$, the modular group $\mathrm{Mod}(f)$ acts properly discontinuously on $\mathcal{T}_f$, and the moduli space $\mathcal{M}_f$ is defined as the complex analytic space

$$\mathcal{M}_f := \mathcal{T}_f/\mathrm{Mod}(f),$$

and hence $\mathcal{T}_f$ serves as the 'universal cover' of $\mathcal{M}_f$.

Let $[\mu] \in \mathcal{M}_f$, where μ is a Beltrami differentials invariant under f such that $\|\mu\|_\infty < 1$. By the measurable Riemann mapping theorem, there is a quasiconformal homeomorphism ϕ solving the Beltrami equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu \frac{\partial \phi}{\partial z}.$$

The map $g := \phi^{-1} f \phi$ is a rational map of degree d. Let $[g]$ be the $\mathrm{PGL}_2(\mathbb{C})$ conjugacy class of g in the moduli space $\mathcal{M}_d$ of all degree d rational maps. The map

$$\begin{aligned} \Psi : \mathcal{M}_f &\to \mathcal{M}_d, \\ [\mu] &\mapsto [g], \end{aligned} \tag{1.1}$$

is well defined and is a holomorphic injection [**MS98**], moreover, $\Psi(\mathcal{M}_f)$ is the set of all $\mathrm{PGL}_2(\mathbb{C})$ conjugacy classes that are quasiconformally conjugate to f. We refer the readers to [**Ast17**] for more details.

Unlike the Teichmüller space $\mathcal{T}_f$, not much of the complex structure of the moduli space $\mathcal{M}_f$ is known. A complex analytic space X is called *Carathéodory hyperbolic* if bounded holomorphic functions separate points in X, that is, for every $x \neq y$ in X, there is a bounded holomorphic function $\phi : X \to \mathbb{C}$ such that $\phi(x) \neq \phi(y)$. Carathéodory hyperbolicity is a strong hyperbolicity condition which implies Kobayashi hyperbolicity [**Kob98**, Proposition 3.1.7(1)]. As examples, bounded domains in $\mathbb{C}^N$ are Carathéodory hyperbolic.

McMullen [**McM87**, p. 473] made the following conjecture in 1987.

Conjecture 1.1. (McMullen) Let f be a rational map of degree $d \geq 2$ which is not flexible Lattès, then $\mathcal{M}_f$ is Carathéodory hyperbolic.

Here, a rational map of degree $d \geq 2$ is called *Lattès* if it is semi-conjugate to an endomorphism on an elliptic curve. A Lattès map f is called *flexible Lattès* if one can continuously vary the complex structure of the elliptic curve to get a family of Lattès maps passing through f. The structure of flexible Lattès maps is well understood [**Mil06**, §5], and the flexible Lattès locus in $\mathcal{M}_d$ is either empty (when d is not a square) or is an

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algebraic curve with at most two connected components (when d is a square). When f is flexible Lattès, then $V := \Psi(\mathcal{M}_f)$ contains the connected component of FL_d containing $[f]$ (actually, V is equal to this connected component, see Lemma 3.1). Since V is quasi-projective, by Riemann's extension theorem, bounded holomorphic functions on V are constant, and hence $\mathcal{M}_f$ is *not* Carathéodory hyperbolic, so that the condition that f is not flexible Lattès in Conjecture 1.1 cannot be dropped.

1.2. *Main results.* The purpose of this paper is to solve McMullen's conjecture. In fact, we shall prove a stronger statement. We first recall the notion of *structural stability*. Let $(f_t)_{t\in X}$ be a holomorphic family of degree d rational maps parameterized by a complex analytic space X. The family is called structurally stable if periodic points do not change their types (attracting, repelling, or indifferent) in this family, which is equivalent to that f_t are all quasiconformally conjugate on their Julia sets [**McM16**, Theorem 4.2]. Note that structural stability is a local property.

Let X be a complex analytic space and let $\Phi : X \to \mathcal{M}_d$ be a holomorphic map. It is not always possible to lift Φ to a map taking image in Rat_d (the space of degree d rational maps). However, we can always lift Φ locally.

We define the map $\Phi : X \to \mathcal{M}_d$ to be *structurally stable* if every local lift $\tilde{\Phi}$ defines a structurally stable family of rational maps. By definition, the map Ψ in (1.1) is structurally stable. We show the following result which implies Conjecture 1.1.

THEOREM 1.2. *Let $d \geq 2$. Let X be a connected complex analytic space and let $\Phi : X \to \mathcal{M}_d$ be a holomorphic injective map that is structurally stable. Assume that there exists $t \in X$ such that $\Phi(t) \notin \mathrm{FL}_d$. Then, there is a normal affine variety Y of dimension $2d-2$ and a holomorphic injection $i : X \to Y$ such that $i(X)$ is precompact in Y. In particular, X is Carathéodory hyperbolic.*

We now explain the construction of the space Y and the map i in Theorem 1.2, and hence give a sketch of the proof of Theorem 1.2. We will use the *reciprocal multiplier spectrum morphism*. The precise definition is given in §2. Basically, for fixed integers $m \geq n \geq 1$ and $f \in \mathrm{Rat}_d$, we collect all periodic points of f with exact periods j, $n \leq j \leq m$. Assume that f has no super-attracting cycle with exact periods j, $n \leq j \leq m$. Using elementary symmetric polynomials, the reciprocal of the multipliers of these periodic points determine a point in $\mathbb{C}^{N_{n,m}}$, where $N_{n,m}$ is the number of periodic points with exact periods j, $n \leq j \leq m$, counted with multiplicity. Hence, we can define the reciprocal multiplier spectrum morphism $\tau_{n,m} : \mathcal{M}_d \setminus Z_{n,m} \to \mathbb{C}^{N_{n,m}}$, where $Z_{n,m}$ is the locus of f having a super-attracting cycle with exact periods j, $n \leq j \leq m$.

In the setting of Theorem 1.2, we can choose n large enough such that $\Phi(X)$ is contained in $\mathcal{M}_d \setminus Z_{n,m}$; moreover, $\tau_{n,m}(\Phi(X))$ is contained in a bounded domain in $\mathbb{C}^{N_{n,m}}$ by the structural stability. Let FL_d be the flexible Lattès locus. By a theorem of McMullen [**McM87**], for fixed n, the morphism $\tau_{n,m} : \mathcal{M}_d \setminus (Z_{n,m} \cup \mathrm{FL}_d) \to \mathbb{C}^{N_{n,m}}$ is quasi-finite (which means that every fiber of $\tau_{n,m}$ is a finite set) for m large enough. By a generalized Zariski's main theorem [**Gro66**, Théorème 8.12.6], there exists a normal affine variety Y, an open immersion $\eta : \mathcal{M}_d \setminus (Z_{n,m} \cup \mathrm{FL}_d) \to Y$, and a finite morphism (which means quasi-finite and proper) $\tilde{\tau}_{n,m} : Y \to \mathbb{C}^{N_{n,m}}$, such that $\tau_{n,m} = \tilde{\tau}_{n,m} \circ \eta$. The map i in

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Theorem 1.2 is given by $i := \eta \circ \Phi$. We get that $i(X)$ is precompact in Y by the finiteness of $\tilde{\tau}_{n,m}$.

When $d \geq 4$, using a recent result of the authors about the generic injectivity of a multiplier spectrum morphism [**JX25**], we can concretely construct the space Y in Theorem 1.2.

THEOREM 1.3. *Let $d \geq 4$. Then, the complex analytic space Y in Theorem 1.2 can be chosen to be the normalization of the Zariski closure of the image of the reciprocal multiplier spectrum morphism $\tau_{n,m}$ for some $m \geq n \geq 1$.*

We believe that the restriction $d \geq 4$ in Theorem 1.3 is unnecessary.

1.3. *Structure of the paper.* In §2, we give the definition of the reciprocal multiplier spectrum morphism. The proof of Theorem 1.2 and Conjecture 1.1 is given in §3. The proof of Theorem 1.3 is given in §4.

2. *The reciprocal multiplier spectrum morphism*

In this section, we will define the reciprocal multiplier spectrum morphism τ_n and we will show that τ_n are quasi-finite by using McMullen's rigidity theorem [**McM87**, Theorem 2.2].

Definition 2.1. Let f be a rational map of degree d and let x be a periodic point of f. We say an integer $n \geq 1$ is a formal exact period of x if one of the following holds:

(i) n is the minimal integer such that $f^n(x) = x$;
(ii) $n = mr$ and $df^m(x)$ is a primitive rth root of unity, where m is the minimal integer such that $f^m(x) = x$.

By definition, for every periodic point x of f, it has at most two formal exact periods.

Let $\{x_1, \ldots, x_{N_n}\}$ be the multiset of periodic points of f with formal exact periods n, counted with multiplicity. The multipliers $df^n(x_i)$ of these points determine an element $s_n(f) \in \mathbb{C}^{N_n}/S_{N_n}$, where S_{N_n} is the symmetric group that acts on $\mathbb{C}^{N_n}$ by permuting the coordinates. It was shown in [**Sil07**, Theorem 4.50] that

$$[f] \mapsto s_n(f)$$

defines a morphism on $\mathcal{M}_d$.

Let $m \geq n \geq 1$ be two integers. Let $\rho_{n,m} : \mathcal{M}_d \to \mathbb{C}^{N_n}/S_{N_n} \times \cdots \times \mathbb{C}^{N_m}/S_{N_m}$ be the morphism

$$[f] \mapsto (s_n(f), \ldots, s_m(f)).$$

Let $W_{n,m}$ be the Zariski closed set $\{(f, g) : \rho_{n,m}(f) = \rho_{n.m}(g)\} \subset \mathcal{M}_d \times \mathcal{M}_d$. Then, for fixed n, $W_{n,m}$ is a decreasing sequence with respect to m. By Noetheriality, there exists a minimal $N = N(d, n) \geq n$ such that

$$\bigcap_{m \geq n} W_{n,m} = W_{n,N}.$$

The following is a consequence of McMullen's rigidity theorem [**McM87**, Theorem 2.2].

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THEOREM 2.2. *For every $d \geq 2$ and $n \geq 1$, we set $\rho_n := \rho_{n,N}$. Then, ρ_n is quasi-finite on $\mathcal{M}_d \setminus \mathrm{FL}_d$.*

Proof. Assume by contradiction that ρ_n is not quasi-finite on $\mathcal{M}_d \setminus \mathrm{FL}_d$. Then, there exists an algebraic family of rational maps $\phi : V \to \mathrm{Rat}_d$, $t \mapsto f_t$, parameterized by the algebraic curve V such that $\rho_n(f_t)$ is constant, $\pi \circ \phi : V \to \mathcal{M}_d$ is non-constant with $\pi \circ \phi(V) \cap \mathrm{FL}_d = \emptyset$, where $\pi : \mathrm{Rat}_d \to \mathcal{M}_d$ is the canonical projection.

By the definition of ρ_n, $\rho_{n,m}(f_t)$ is constant for every $m \geq n$. This implies the multipliers of periodic points with large exact periods are constant in this family. In particular, the number of attracting periodic points is bounded in this family. This implies $\phi : V \to \mathrm{Rat}_d$ is a structurally stable algebraic family [**McM16**, Theorem 4.2]. By [**McM87**, Theorem 2.2], $\phi : V \to \mathrm{Rat}_d$ is a flexible Lattès family, which is a contradiction. □

We now define the reciprocal multiplier spectrum morphism. Let $n \geq 1$ be an integer. Let $Z_n \subset \mathcal{M}_d$ be the subvariety containing rational map with a super-attracting periodic point with exact periods n. We let $\delta_n : \mathcal{M}_d \setminus Z_n \to \mathbb{C}^{N_n}$ be the morphism

$$[f] \to (\sigma_1(\alpha_1, \ldots, \alpha_n), \ldots, \sigma_{N_n}(\alpha_1, \ldots, \alpha_n)),$$

where $\alpha_i := df^n(x_i)^{-1}$ and σ_i is the ith elementary symmetric polynomial of N_n variables.

Let $m \geq n \geq 1$ be two integers. Let $Z_{n,m} \subset \mathcal{M}_d$ be the subvariety containing rational map with a super-attracting periodic point with exact periods j such that $n \leq j \leq m$. We let $\tau_{n,m} : \mathcal{M}_d \setminus Z_{n,m} \to \mathbb{C}^{N_n} \times \cdots \times \mathbb{C}^{N_m}$ be the morphism given by

$$[f] \mapsto (\delta_n(f), \ldots, \delta_m(f)).$$

Definition 2.3. The nth reciprocal multiplier spectrum morphism is defined as $\tau_n := \tau_{n,N}$, where $N = N(d, n)$ is the same as in Theorem 2.2.

COROLLARY 2.4. *For every $d \geq 2$ and $n \geq 1$, τ_n is quasi-finite on $\mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d)$.*

Proof. This a corollary of Theorem 2.2, since elementary symmetric polynomials give an isomorphism between $\mathbb{C}^N/S_N$ and $\mathbb{C}^N$. □

3. *Proof of Theorem 1.2 and Conjecture 1.1*

We begin with a lemma.

LEMMA 3.1. *Let $d \geq 2$. Let X be a connected complex analytic space and let $\Phi : X \to \mathcal{M}_d$ be a holomorphic map that is structurally stable, such that $\Phi(X) \cap \mathrm{FL}_d \neq \emptyset$. Then, $\Phi(X) \subset \mathrm{FL}_d$.*

Proof. Pick $t_0 \in \Phi(X) \cap \mathrm{FL}_d$. Since Φ is structurally stable, by [**McM16**, Theorem 4.2], for every $t \in X$, $\Phi(t)$ and $\Phi(t_0)$ are quasiconformally conjugate on their Julia sets (which is equal to $\mathbb{P}^1(\mathbb{C})$). In particular, all $\Phi(t)$ are postcritically finite (PCF), which means that the critical orbits are finite) with the same critical orbits relation as $\Phi(t_0)$. Hence, $\Phi : X \to \mathcal{M}_d$ is a holomorphic family of PCF maps. By Thurston's rigidity theorem

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[McM87, Theorem 6.2], either $\Phi(X)$ is a single point or $\Phi(X) \subset \mathrm{FL}_d$. In these two cases, we have $\Phi(X) \subset \mathrm{FL}_d$. □

Proof of Theorem 1.2. Since Φ is structurally stable, by [McM16, Theorem 4.2], we can choose n large enough such that all attracting periodic points have exact periods less than n along the family $\Phi : X \to \mathcal{M}_d$. Let τ_n be the reciprocal multiplier spectrum morphism given in Definition 2.3. By Lemma 3.1, $\Phi(X) \subset \mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d)$, where $N = N(d, n)$ is given in Theorem 2.2. By Corollary 2.4, $\tau_n : \mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d) \to \mathbb{C}^l$ is quasi-finite, $l \geq 1$. By our construction, $\tau_n \circ \Phi(X)$ is contained in a bounded domain in $\mathbb{C}^l$.

By a generalized version of Zariski's main theorem [Gro66, Théorème 8.12.6], there is a normal affine variety Y, an open immersion $\eta : \mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d) \to Y$, and a finite morphism $\tilde{\tau}_n : Y \to \mathbb{C}^l$, such that $\tau_n = \tilde{\tau}_n \circ \eta$. A morphism $\eta : Z \to X$ is called an open immersion if there exists a Zariski open subset $X' \subset X$ and an isomorphism $\eta' : Z \to X'$ such that $\eta = \iota \circ \eta'$, where $\iota : X' \to X$ is the inclusion map. So, in particular, η is injective. We define $i := \eta \circ \Phi : X \to Y$. Then, i is a holomorphic injection. We need to show that $i(X)$ is precompact in Y. We have $\tilde{\tau}_n(i(X)) = \tilde{\tau}_n \circ \eta \circ \Phi(X) = \tau_n \circ \Phi(X)$, which is precompact in $\mathbb{C}^l$. Since $\tilde{\tau}_n : Y \to \mathbb{C}^l$ is finite (hence, proper), we know that $i(X)$ is precompact in Y, as $i(X) \subset \tilde{\tau}_n^{-1}(\tau_n \circ \Phi(X))$.

Since Y is normal and affine, Y can be embedded in $\mathbb{C}^q$ as a Zariski closed subset for some $q \geq 1$. The coordinates functions $z_1, \ldots, z_q$ define bounded holomorphic injective functions on $i(X)$. Since $i : X \to i(X)$ is biholomorphic, for $1 \leq j \leq q$, $z_j \circ i$ are bounded holomorphic injective functions on X, which clearly separate points in X. This implies that X is Carathéodory hyperbolic. This finishes the proof. □

Proof of Conjecture 1.1. Let Ψ be the holomorphic injection given in (1.1), then $\Psi : \mathcal{M}_f \to \mathcal{M}_d$ is structurally stable since elements in $\Psi(\mathcal{M}_f)$ are quasiconformally conjugate. Since $f \notin \mathrm{FL}_d$, by Theorem 1.2, $\mathcal{M}_f$ is Carathéodory hyperbolic. □

4. *Proof of Theorem 1.3*

We first show the following result, which is a generalization of [JX25, Theorem 1.3]. Here, [JX25, Theorem 1.3] corresponds to the case $n = 1$ in the following result. The proof follows the same line as in the proof in [JX25]. Recall that we have shown that for every $d \geq 2$ and $n \geq 1$, the two morphisms ρ_n and τ_n are quasi-finite (after excluding FL_d), see Theorem 2.2 and Corollary 2.4.

THEOREM 4.1. *For every $d \geq 4$ and $n \geq 1$, ρ_n and τ_n are generically injective, that is, they are injective morphisms when restricted on a non-empty Zariski open subset.*

Proof. Since elementary symmetric polynomials give an isomorphism between $\mathbb{C}^N/S_N$ and $\mathbb{C}^N$, it suffices to show that ρ_n is generically injective.

Two rational maps f and g are called intertwined if there is an irreducible algebraic curve $Z \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ whose projections to both coordinates are onto, such that Z is preperiodic by the map $f \times g$. By [JX25, Theorem 3.3] (building on Pakovich's result [Pak25a]), there is a non-empty Zariski open subset U of $\mathcal{M}_d$ such that for every

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$f, g \in \mathrm{Rat}_d(\mathbb{C})$ with $[f], [g] \in U$, if f and g are intertwined, then $[f] = [g]$. This is the only step that we need the condition $d \geq 4$.

Assume by contradiction that ρ_n is not generically injective. Similar to the construction in the third paragraph in the proof of [**JX25**, Theorem 1.3], after shrinking U, we can construct two algebraic families f_V, g_V of degree d rational maps parameterized by the same irreducible algebraic curve V such that the following holds. There exists $\underline{n} := \{n_1, \ldots, n_{2d-3}\} \in (\mathbb{N}^*)^{2d-3}$ ($\mathbb{N}^*$ stands for the set of positive integers) such that $n_i \geq n$, and we have

$$\Psi_f(V) \subset Y_{\underline{n}} \cap U \quad \text{and} \quad \Psi_g(V) \subset U, \tag{4.1}$$

where $Y_{\underline{n}} \subset \mathcal{M}_d$ is the algebraic curve containing all conjugacy classes having $2d-3$ super-attracting cycles with exact periods $n_1, \ldots, n_{2d-3}$. The map $\Psi_f : V \to \mathcal{M}_d$ (similarly for Ψ_g) is defined by $t \mapsto [f_t]$. Moreover, we have

$$\rho_n \circ \Psi_f = \rho_n \circ \Psi_g, \tag{4.2}$$

finally, for every $t \in V$, we have

$$\Psi_f(t) \neq \Psi_g(t). \tag{4.3}$$

By [**JX25**, Lemma 2.4], there are infinitely many $t \in V$ such that f_t is a PCF map with $2d-2$ distinct super-attracting cycles. We claim that we can further ask that these f_t have no super-attracting cycles with exact periods less than n. Assume by contradiction that this claim is not true. Then, there exists an infinite set $X \subset \mathcal{M}_d$ such that every $g \in X$ has $2d-2$ distinct super-attracting cycles; moreover, the exact periods of these super-attracting cycles are uniformly bounded. Let $\overline{X}$ be the Zariski closure of X. Then, $\dim_{\mathbb{C}}(\overline{X}) \geq 1$. Pick an irreducible component Y of $\overline{X}$ with $\dim_{\mathbb{C}}(Y) \geq 1$. Then, Y is a positive dimensional family of PCF maps. By Thurston's rigidity theorem [**McM87**, Theorem 6.2], $Y \subset \mathrm{FL}_d$, which is a contradiction, since flexible Lattès maps have no super-attracting cycle.

By (4.2), $\rho_n(f_t) = \rho_n(g_t)$, and hence by [**JX25**, Lemma 3.5], for such t, g_t is also a PCF map. By [**JX25**, Theorem 3.4], after shrinking V, f_t and g_t are intertwined for every $t \in V$. By the definition of U and by (4.1), $\Psi_f(t) = \Psi_g(t)$ for every $t \in V$. This contradicts (4.3). We then conclude the proof. □

Proof of Theorem 1.3. In the proof of Theorem 1.2, we have constructed an affine and normal variety Y, an open immersion $\eta : \mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d) \to Y$, and a finite morphism $\tilde{\tau}_n : Y \to \mathbb{C}^l$ such that $\tau_n = \tilde{\tau}_n \circ \eta$. Let $d \geq 4$. By Theorem 4.1, τ_n is generically injective on $\mathcal{M}_d \setminus Z_{n,N}$. This implies that $\tilde{\tau}_n$ is generically injective on Y. Since Y is normal, $\tilde{\tau}_n : Y \to \tilde{\tau}_n(Y)$ is a normalization. It remains to show that $\tilde{\tau}_n(Y)$ is the Zariski closure of $\tau_n(\mathcal{M}_d \setminus Z_{n,N})$. Let Z be the Zariski closure of $\tau_n(\mathcal{M}_d \setminus Z_{n,N})$, which is also the Zariski closure of $\tau_n(\mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d))$. Since $\tilde{\tau}_n(Y)$ is closed and $\tau_n = \tilde{\tau}_n \circ \eta$, we have $\tau_n(\mathcal{M}_d \setminus Z_{n,N}) \subset \tilde{\tau}_n(Y)$. Since $\tilde{\tau}_n(Y)$ is closed, we have $Z \subset \tilde{\tau}_n(Y)$. However, since $\eta(\mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d))$ is dense in Y, then $Z = \tilde{\tau}_n \circ \eta(\mathcal{M}_d \setminus (Z_{n,N} \cup \mathrm{FL}_d))$ is dense in $\tilde{\tau}_n(Y)$. This implies $Z = \tilde{\tau}_n(Y)$. The proof is finished. □

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Added in the final version. After our paper was submitted, Pakovich [**Pak25b**, Theorem 1.2] proved that when $d \geq 2$, there is a non-empty Zariski open subset U of $\mathcal{M}_d$ such that for every $f, g \in \mathrm{Rat}_d(\mathbb{C})$ with $[f], [g] \in U$, if f and g are intertwined, then $[f] = [g]$. By using this result, we can remove the additional condition $d \geq 4$ in Theorems 1.3 and 4.1. Actually, in the proof of Theorems 1.3 and 4.1, we use this result only for $d \geq 4$, and this is the only step where we use the condition $d \geq 4$.

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